

Model Reduction Comparison for the Elastic Crankshaft Mechanism

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Abstract

Model reduction methods (Static, Dynamic, IRS, SEREP, CMS, Krylov) are implemented and applied to an elastic crankshaft. Eigenvectors and eigenfrequencies of each reduced model are compared using various Modal Correlation Criteria (MCC). Minimization of reduction error is ascertained offering the best reduction candidate for model validation. Numerical aspects for the efficient solution (simulation time, storage space requirements) of large sparse linear systems $\mathbf{Ax} = \mathbf{b}$, obtained by the reduction, are discussed and a new method for the fast iterative solution of such systems is introduced. Program packages: ANSYS, MATLAB.

Keywords: Model reduction, crankshaft, modal correlation criteria, large sparse linear systems.

1 Introduction

Model reduction is an essential procedure for the analysis and simulation of increasingly large models obtained from different research areas, e.g. dynamical systems, circuit simulation, structural mechanics, etc. Various discipline-oriented techniques have been developed aiming to reduce simulation time needs, while keeping intact the dynamical behaviour of the model. Concerning the case of linear time-invariant systems, most of these techniques share some common characteristics and therefore can be divided into three categories [2]:

1. Modal truncation, sub-structuring and static condensation.
2. Padé and Padé-type approximations.
3. Balancing-related truncation techniques.

1.1 Modal truncation - sub-structuring - static condensation

As far as mechanical Multi Body Systems (MBS) are concerned, reduction approaches belonging to this category have been widely applied and utilized by commercial Finite Element Method (FEM) software packages, e.g. ANSYS, NASTRAN, and ABAQUS. Their main functionality as modal methods is to catch the dominant modes of the system in the reduced order model.

Modal truncation tries to accumulate specific eigenvalues (eigenfrequencies and eigenvectors), which dominate the long-term dynamics of the Ordinary Differential Equation (ODE) solution. A common criterion is to select a small set of the lowest eigenvalues. Sub-structuring demands the

total structure to be split up into smaller structures, while static condensation neglects the effect of inertia terms. The system matrices of the FEM discretised linear-time invariant (LTI) 2nd-order ODE $\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{C}\dot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) = \mathbf{f}$ (1) are partitioned into sub-blocks, originated by the theory of master/external (m) and slave/internal (s) Degrees of Freedom (DOF), as shown in (2) and (3):

$$\tilde{\mathbf{M}}\ddot{\tilde{\mathbf{x}}}(t) + \tilde{\mathbf{C}}\dot{\tilde{\mathbf{x}}}(t) + \tilde{\mathbf{K}}\tilde{\mathbf{x}}(t) = \tilde{\mathbf{f}} \quad (2)$$

$$[\] := \begin{pmatrix} []_{mm} & []_{ms} \\ []_{sm} & []_{ss} \end{pmatrix}, [\] = \{\tilde{\mathbf{M}}, \tilde{\mathbf{C}}, \tilde{\mathbf{K}}\}$$

$$\tilde{\mathbf{x}} := \begin{pmatrix} \mathbf{x}_m \\ \mathbf{x}_s \end{pmatrix}, \tilde{\mathbf{f}} := \begin{pmatrix} \mathbf{f}_m \\ \mathbf{f}_s \end{pmatrix}$$

$$m \cup s = n, n = \text{DOF}_{\text{total}}, m \cap s = \emptyset \quad (3)$$

where $\mathbf{M}, \mathbf{C}, \mathbf{K} \in \mathbb{R}^{n \times n}$, $\mathbf{x} \in \mathbb{R}^{n \times 1}$, $\mathbf{f} \in \mathbb{R}^{n \times 1}$ are the mass-, damping-, stiffness matrix, displacement and load vector respectively.

The methods developed in this category are:

- Guyan (static) reduction [6]
- Dynamic reduction [5]
- Component Mode Synthesis (CMS) [4]
- Improved Reduction System Method (IRS) and its iterative variant [7, 9]
- System Equivalent Expansion Reduction Process (SEREP) [8]

From the above mentioned techniques, only the Guyan and the CMS reduction are implemented in FEM software packages, with CMS being the best available one.

1.2 Padé and Padé-type approximations

This category corresponds to techniques developed for obtaining a reduced order model based on the theory of moment matching ($\lambda_o < \infty$) or partial realization ($\lambda_o = \infty$). By considering a LTI system ($\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$, $\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$) and $\lambda_o \notin \Lambda(\mathbf{A}, \mathbf{E})$, where $\Lambda(\mathbf{A}, \mathbf{E})$ is the eigenvalue set of the matrix pencil $\mathbf{A} - \lambda\mathbf{E}$, the transfer function $\mathbf{G}(s)$ can be expanded into a Laurent series, as shown in (4):

$$\mathbf{G}(s) = \mathbf{M}_0 + \mathbf{M}_1(s - \lambda_o) + \mathbf{M}_2(s - \lambda_o)^2 + \dots \quad (4)$$

If $\lambda_o = 0$ the coefficient matrices in the Laurent series are given by $\mathbf{M}_j := \mathbf{C}(\mathbf{A}^{-1}\mathbf{E})^j \mathbf{A}^{-1}\mathbf{B}$ and are called *moments* and if $\lambda_o = \infty$ they are given by $\mathbf{M}_j := \mathbf{C}(\mathbf{E}^{-1}\mathbf{A})^j \mathbf{E}^{-1}\mathbf{B}$ and called *Markov parameters*. For the reduced order model the i -th Padé approximation is used and the reduced transfer function $\tilde{\mathbf{G}}$ is defined by:

$$\mathbf{G}(s) = \tilde{\mathbf{G}}(s) + O((s - \lambda_0)^{2i}) \Leftrightarrow \mathbf{M}_j = \tilde{\mathbf{M}}_j \quad \forall j \in [0, 2i - 1] \quad (5)$$

The solution of (5) is computed via Asymptotic Waveform Evaluation and thereafter using projection on certain Krylov subspaces [10] with the Arnoldi process (advantage of producing orthogonal subspaces and thus good numerical stability properties) or the Lanczos method [2]. The method developed in this category and applied in various large FEM models is:

- Krylov Subspace Method [9]

As it will be shown, this method, although not implemented in commercial FEM software packages, offers up to a point qualitatively better results than the already standardized CMS reduction.

1.3 *Balancing-related truncation techniques*

Balanced truncation is an approach that uses the idea of computing a special realization of a LTI system, called balanced realization. By this approach the state-space transformation matrix for the reduced order system is obtained via $\mathbf{T} = \Sigma^{1/2} \mathbf{U}^T \mathbf{R}^{-T}$, where \mathbf{R} is a Cholesky factor of the controllability Grammian and $\mathbf{U} \Sigma^2 \mathbf{V}$ is a singular value decomposition of the positive definite matrix $\mathbf{R} \mathbf{W}_o \mathbf{R}^T$ with \mathbf{W}_o being the observability Grammian.

This method has not been implemented in FEM software packages and is rarely applied to FEM models, because of the vast simulation time and storage space requirements. Recent developments, though, in the field of numeric linear algebra [2, 3] constitute this technique quite promising, since the result quality it offers exceeds the bounds of the above mentioned reduction methods.

2 Transformation matrices

The concept of model reduction, applied on the system (1), is to find a low dimensional subspace $\mathbf{T} \in \mathbb{R}^{n \times m}$, $m \ll n$ in order to approximate the state/displacement vector \mathbf{x} as good as possible, i.e.: $\mathbf{x} = \mathbf{T} \mathbf{z} + \boldsymbol{\varepsilon}$. The reduction's quality depends on the value of $\boldsymbol{\varepsilon}$. By projecting (1) on this subspace, a lower dimension 2^{nd} -order ODE is obtained:

$$\begin{aligned} \mathbf{M}_z \ddot{\mathbf{z}}(t) + \mathbf{C}_z \dot{\mathbf{z}}(t) + \mathbf{K}_z \mathbf{z}(t) &= \mathbf{f}_z \\ \mathbf{M}_z = \mathbf{T}^T \mathbf{M} \mathbf{T}, \mathbf{C}_z &= \mathbf{T}^T \mathbf{C} \mathbf{T}, \mathbf{K}_z = \mathbf{T}^T \mathbf{K} \mathbf{T}, \mathbf{f}_z = \mathbf{T}^T \mathbf{f} \end{aligned} \quad (6)$$

In this section, the transformation matrices of the above mentioned reduction methods (Guyan, Dynamic, IRS, CMS, SEREP, Krylov) are presented for the case of an undamped system of the form $\mathbf{M} \ddot{\mathbf{x}} + \mathbf{K} \mathbf{x} = \mathbf{f}$ and the DOF-partition as shown in (2), (3). For detailed information on how to obtain the matrices as well as on the specific nature of each reduction approach, the reader is prompted to [9].

$$\text{Static reduction: } \mathbf{T}_{\text{static}} = \begin{pmatrix} \mathbf{E} \\ -\mathbf{K}_{ss}^{-1} \mathbf{K}_{sm} \end{pmatrix} \quad (7)$$

$$\text{Dynamic reduction: } \mathbf{T}_{\text{dynamic}} = \begin{pmatrix} \mathbf{E} \\ -\mathbf{B}(\omega)_{ss}^{-1} \mathbf{B}(\omega)_{sm} \end{pmatrix}, \mathbf{B}(\omega)_{i,j} := \mathbf{M}_{i,j} \omega^2 + \mathbf{K}_{i,j}, i, j \in \{s, m\} \quad (8)$$

$$\underline{\text{IRS}}: \mathbf{T}_{IRS} = \mathbf{T}_{static} + \mathbf{PMT}_{static} \mathbf{M}_R^{-1} \mathbf{K}_R^{-1}, \mathbf{P} = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{K}_{ss}^{-1} \end{pmatrix} \quad (9)$$

$$\underline{\text{CMS}}: \mathbf{T}_{CMS} = \begin{pmatrix} \mathbf{E} & \mathbf{0} \\ \mathbf{\Gamma} & \mathbf{\Phi} \end{pmatrix}, \mathbf{\Gamma} = -\mathbf{K}_{ss}^{-1} \mathbf{K}_{sm}, \mathbf{\Phi}: \text{Craig-Bampton set} \quad (10)$$

$$\underline{\text{SEREP}}: \mathbf{T}_{SEREP} = \begin{pmatrix} \mathbf{\Phi}_m \\ \mathbf{\Phi}_s \end{pmatrix} (\mathbf{\Phi}_m^T \mathbf{\Phi}_m)^{-1} \mathbf{\Phi}_m^T, \mathbf{\Phi}: \text{modal matrix} \quad (11)$$

$$\underline{\text{Krylov}}: \mathbf{T}_{krylov} = \mathbf{K}_q (\mathbf{K}^{-1} \mathbf{M}, \mathbf{K}^{-1} \mathbf{f}) := \text{span}\{\mathbf{K}^{-1} \mathbf{f}, (\mathbf{K}^{-1} \mathbf{M}) \mathbf{K}^{-1} \mathbf{f}, \dots, (\mathbf{K}^{-1} \mathbf{M})^{q-1} \mathbf{K}^{-1} \mathbf{f}\} \quad (12)$$

Only the Krylov Subspace Method is master/slave-DOF-free: the user only has to define the Krylov parameter q , thus minimizing the reduction's preparation time. The system matrices do not have to be reordered, maintaining their band-diagonal property as produced by the FEM software programs, shown in Fig.1 and Fig.2 for the mass matrix of the elastic crankshaft.

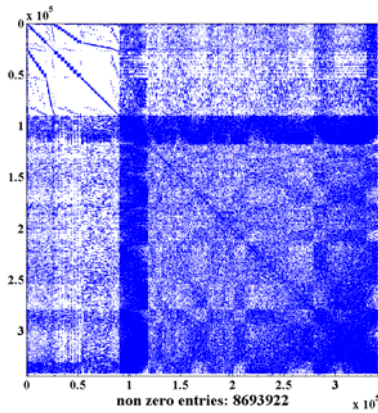


Figure 1 Reordered mass matrix used in Guyan-, Dynamic-, IRS-, SEREP- and CMS reduction

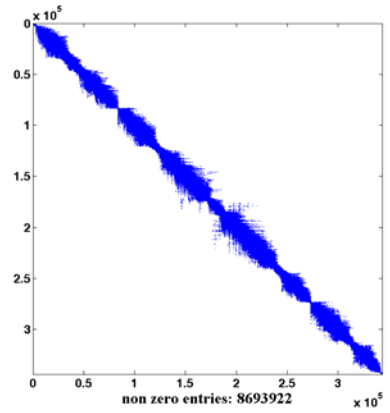


Figure 2 Mass Matrix used in Krylov Subspace Method

3 Elastic Crankshaft

The reduction methods were implemented in MATLAB and tested on the elastic crankshaft mechanism [1]. The original model (Fig. 3) was discretized with FE in ANSYS (Fig. 4). The number of elements (tetrahedron *SOLID95*) produced is $n_{elem} = 73141$ and the number of nodes $n_{nodes} = 114608$. Each node was appointed with 3 DOF (UX, UY, and UZ). The system matrices $\mathbf{M}, \mathbf{K} \in \mathbb{R}^{n \times n}, n = 3 \cdot n_{nodes}$ are real positive definite and positive semi-definite respectively. The set of master DOF is selected according to standard criteria, restricted though, to a relative low number $m_{node} = 11, m_{DOF} = 3 \cdot m_{node}$ as depicted in Fig. 4 (black points). The purpose of this restriction is to check whether the master nodes selection (number and position) affects the result's

quality or not. The Krylov Subspace Method being m_{node} -free (only the number of Krylov modes q need to be defined) appears to be advantageous concerning this restriction.

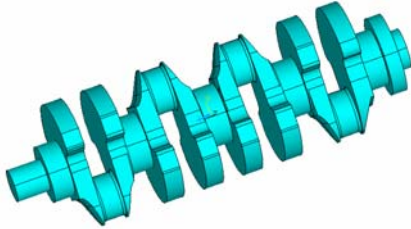


Figure 3 Elastic Crankshaft - Geometry



Figure 4 Elastic Crankshaft – FEM discretized – master nodes as black points

4 Modal Correlation Criteria

In order to check the validity of a reduction method a comparison between the original (full) and the reduced model has to be done. A useful approach is to compare the eigenvalues (eigenfrequencies and eigenvectors) for both models during a modal analysis.

Since FEM software packages offer the eigenfrequency information immediately, most of the comparisons made are based only on this feature of modal analysis. A good eigenfrequency correlation does not always imply the adequate eigenvector correlation. Thus, criteria for eigenvector comparison should be used. This is only possible either by expanding the reduced model's eigenvector to the dimension of the full model's eigenvector or the opposite. The knowledge of the transformation matrix (6), (7)-(12) is therefore necessary (not directly available by the FEM software programs).

Here the following criteria are used [9] and applied to the elastic crankshaft (their results will be discussed in the conclusion's section) for the above mentioned reduction methods:

- Normalized Relative Eigenfrequency Difference (*NRFD*)
- Modified Modal Assurance Criterion (*modMAC*)
- Mass Normalized Vector Difference (*MNVD*)
- Stiffness Normalized Vector Difference (*SNVD*)
- Normalized Modal Difference (*NMD*)

4.1 Normalized Relative Eigenfrequency Difference (*NRFD*)

The lower the value of this criterion is $reig_{diff} := |eig_{sub} - eig_{full}|_2 / |eig_{sub}|_2$ the better the reduction method used is.

4.2 Modified Modal Assurance Criterion (*modMAC*)

This criterion gives the information about the eigenvector's angle. The eigenvectors are mass normalized and expanded or reduced to the same dimension.

$$\text{mod } MAC_{i,j} = \frac{(\Phi_i^T \mathbf{M} \Psi_j)^2}{(\Phi_i^T \mathbf{M} \Phi_i)(\Psi_j^T \mathbf{M} \Psi_j)} \quad (13)$$

Φ_i : i -th eigenvector (full model)

Ψ_j : j -th expanded eigenvector

A value of $\text{mod } MAC = 100\%$ implies a perfect correlation of the eigenvectors; the less this value becomes, the worse the correlation is. A thumb rule assures that a $\text{mod } MAC$ correlation of a magnitude larger than 80% implies qualitatively successful reduction. In Fig. 6, 7 and 8 the first six eigenvectors (rigid body motion) are of no interest and therefore not depicted.

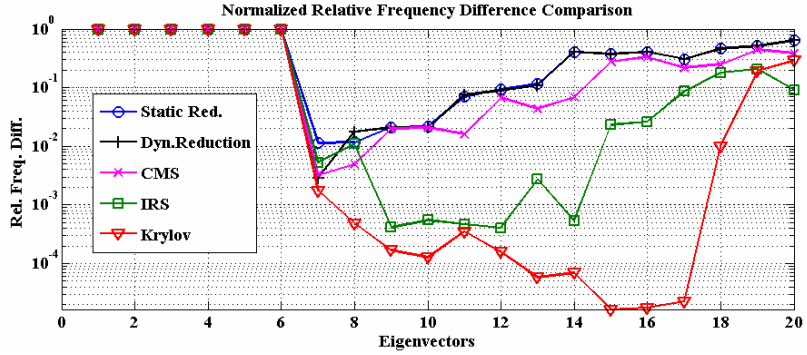


Figure 5 Elastic Crankshaft – $NRFD$ Comparison

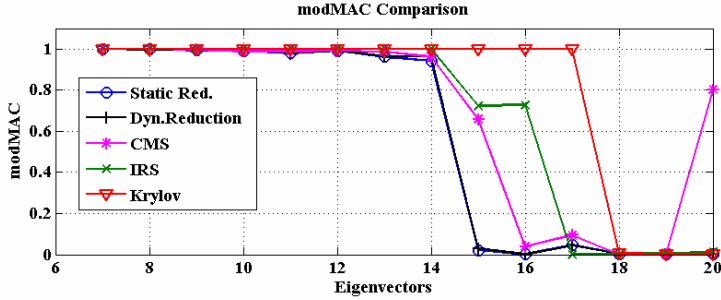


Figure 6 Elastic Crankshaft – $\text{mod } MAC$ Comparison

4.3 Mass Normalized Vector Difference (MNVD)

$MNVD$ gives the information about the relative vector difference of mass-normalized eigenvectors. It utilizes the characteristic of $\Phi^T \mathbf{M} \Phi = \mathbf{E}$, with Φ being the modal matrix of the full model and \mathbf{E} the identity matrix and measures the normalized deviation for the analogous relation of the reduced model $\Psi^T \mathbf{M}_R \Psi = \tilde{\mathbf{E}}$. The smaller $MNVD$ value depicts the best result.

$$MNVD_{i,j} = \frac{|\tilde{\mathbf{M}}_{sub} - \tilde{\mathbf{M}}_{full}|_2}{|\tilde{\mathbf{M}}_{sub}|_2} \quad (14)$$

$\tilde{\mathbf{M}}_{full}$: modal mass of the full model

$\tilde{\mathbf{M}}_{sub}$: modal mass of the reduced model

4.4 Stiffness Normalized Vector Difference (SNVD)

Analogously to *MNVD*, *SNVD* gives the information about the characteristic $\Phi^T \mathbf{K} \Phi = \text{diag}(\omega^2)$ for the full model and the deviation from it of $\Psi^T \mathbf{K}_R \Psi = \text{diag}(\tilde{\omega}^2)$. This criterion resembles the normalized relative difference criterion. The smallest *SNVD* value depicts the best result.

$$SNVD_{i,j} = \frac{|\tilde{\mathbf{K}}_{sub} - \tilde{\mathbf{K}}_{full}|_2}{|\tilde{\mathbf{K}}_{sub}|_2} \quad (15)$$

$\tilde{\mathbf{K}}_{full}$: modal stiffness of the full model

$\tilde{\mathbf{K}}_{sub}$: modal stiffness of the reduced model

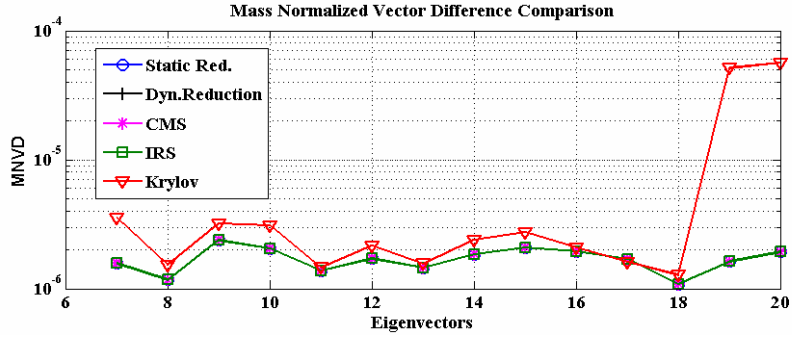


Figure 7 Elastic Crankshaft – *MNVD* Comparison

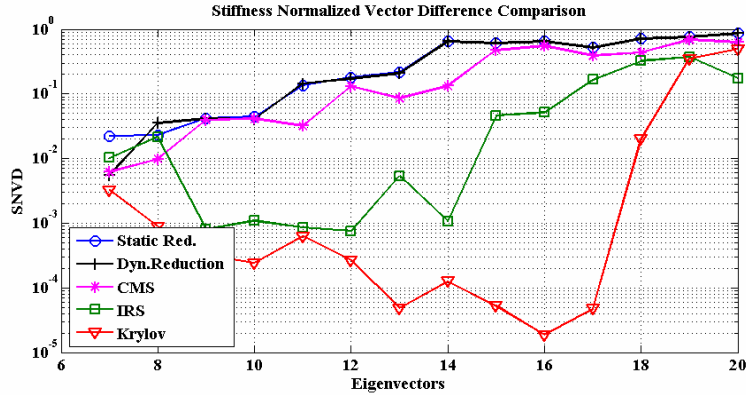


Figure 8 Elastic Crankshaft – *SNVD* Comparison

4.5 Normalized Modal Difference (NMD)

NMD gives information about the deviation of single DOFs. Its computation is based on the principle of least-square error scale factor (Modal Scale Factor - *MSF*).

$$NMD_{k,r} = \frac{|\Psi_k(r) - MSF \cdot \Phi_k(r)|_2}{\Psi_k(r)}, MSF_{i,j} = \frac{\Psi_i^T \Phi_j}{\Psi_i^T \Psi_i} \quad (16)$$

$\Phi_k(r)$ and $\Psi_k(r)$: r-th coordinate of the k-th full and expanded eigenvector respectively.

For this criterion the 7th and the 17th eigenvectors are randomly chosen to be used, as shown in Fig. 9 and 10.

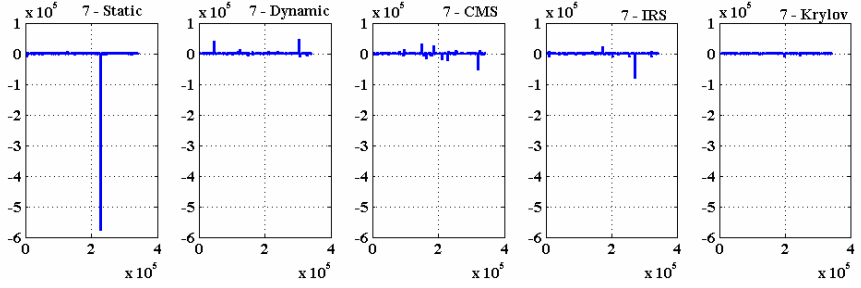


Figure 9 Elastic Crankshaft – *NMD* Comparison – 7th Eigenvector

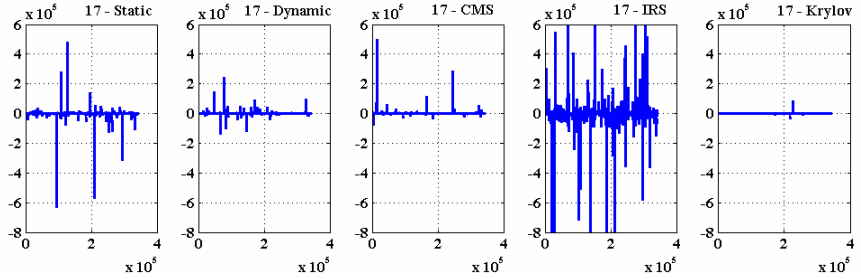


Figure 10 Elastic Crankshaft – *NMD* Comparison – 17th Eigenvector

5 Diagonal Perturbation

Independent of the reduction approach applied, at the end a linear equation system of the form $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{A} \in \mathbb{R}^{n \times n}$, $\mathbf{b} \in \mathbb{R}^{m \times 1}$, $m \ll n$, must be solved in order to compute the equivalent transformation matrix. Depending on the hardware profile of the computer two different approaches could be followed:

- *Direct calculation*: factorization of \mathbf{A} by applying known decomposition methods, e.g. Cholesky, QR, LU, ILU, Dulmage-Mendelsohn, etc. and then solution of the decomposed sub-systems. The calculation time is equivalent to the static-analysis time needed by common FEM software packages. This approach, though, is highly dependent on the hardware architecture and for constantly increasing large FEM models it could not always be feasible.
- *Iterative calculation*: application of iterative algorithms [11]. Always feasible, but could lead to vast computation time in presence of ill-conditioned matrices, which appear in case of tolerance failure during the FEM discretization. Special techniques have been developed that deflate the smallest eigenvalues λ_{\min} , decreasing the matrix condition number $C(\mathbf{A}) = \lambda_{\max} / \lambda_{\min}$ and thus achieving a faster convergence rate.

Here the *diagonal perturbation* method [9] is proposed. Instead of solving the original linear system $\mathbf{Ax} = \mathbf{b}$, we solve the perturbed system as shown below:

$$(\mathbf{A} + \alpha \mathbf{A}_d) \mathbf{x} = \mathbf{b} \quad (17)$$

$$\alpha := 10^{-(n+k)}$$

$$n = \max_{j \in \mathbb{N}} \{f(j), \forall i \in (1, \dim(\mathbf{A}))\}, k \in \mathbb{Z}$$

$$f(j) := 10^{\pm j} \cdot \alpha_{ii} \in \mathbf{A}, \text{ floating-point number form}$$

$$k \geq \min(j) - \max(j)$$

$$\mathbf{A}_d := \text{diag}(\text{diag}(\mathbf{A})), \text{ diagonal matrix } \mathbf{A}$$

By this small perturbation of \mathbf{A} the eigenvalues are vastly affected reducing the condition number and thus the convergence rate. This method produces the following numeric error

$$\|(\alpha \mathbf{A}_d) \mathbf{x}\|_2 = 10^{-(n+k)} \|\mathbf{A}_d \mathbf{x}\|_2. \quad (18)$$

Values of $0 \leq k \leq 2$ normally deliver small numeric error. The algorithm has been tested on the crankshaft model and the simulation time as well as the quality of results is depicted in the next figures (Fig. 11, 12, 13) for three variations of $k = 0, 1, 2$.

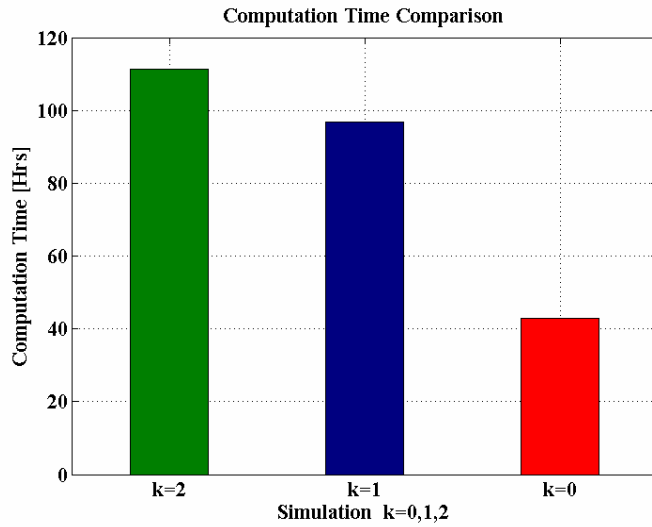


Figure 11 Diagonal perturbation – Simulation time comparison for different values of k

Diagonal perturbation can be applied to any of the above mentioned reduction methods. The results depicted here concern the Krylov Subspace Method. An upper limit of allowed iteration steps is set to 5000.

The value $k = 2$ represents a small perturbation for the diagonal entries of the stiffness matrix. The condition number is almost unaffected, thus the simulation time is vast. The iteration step limit is

reached and the solution converges not according the applied tolerance of 10^{-3} (Fig. 12.). By choosing the values $k = 0,1$ the simulation time is radically reduced, i.e. the convergence rate is faster (Fig. 12): for $k = 1$ it is reduced by **13.2%** and for $k = 0$ by **61.5%** (Fig. 11).

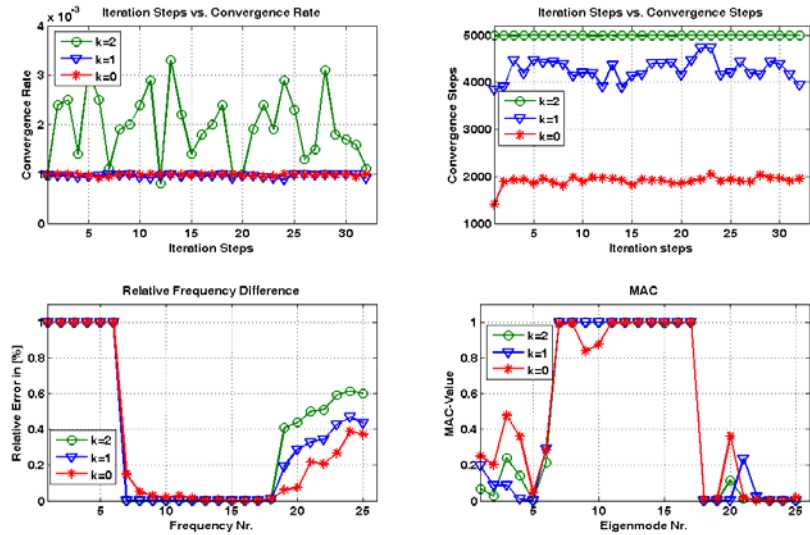


Figure 12 Iteration vs. Convergence & Rel. Freq. Diff. & MAC for $k=0,1,2$

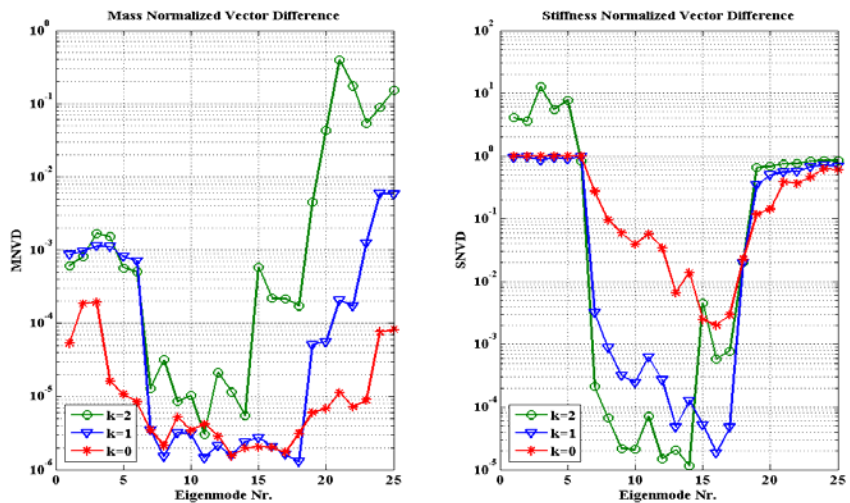


Figure 13 Iteration vs. Convergence & Rel. Freq. Diff. & MAC for $k=0,1,2$

The result's quality for all these k -cases is depicted in Fig. 12 and 13: $k = 0$ offers the best outcome in comparison to the simulation time needed. The small deviations of the MAC value for the 9th and

10th eigenvectors can be neglected, since the 80% *MAC*-thumb-rule is satisfied. Also, the deviations depicted in *SNVD* are to be expected, since the kern of *diagonal perturbation* is to deviate the diagonal entries of the stiffness matrix from their original values. Thus, the method proposed reduces the simulation time, without seriously affecting the quality of results.

6 Results and Conclusion

The MCC comparison for the elastic crankshaft clearly indicates that the Krylov Subspace Method (KSM) is a promising reduction approach. Without having to specify the master-DOF-set KSM delivers the best results in eigenfrequency as well as eigenvector level. While for all the other methods the correlation obtained is very good until the 14th eigenvector, KSM offers an absolute correlation up to the 17th eigenvector, as assured by *NRFD*, *modMAC* and *SNVD* (Fig. 5, 6 and 8). The minor deviation in *MNVD* of KSM (Fig. 7) is due to the application of *diagonal perturbation*. *NMD* depicts the master-DOF independence of KSM. A bad selection of the master-DOF-set could lead to severe deviation of single DOFs for high-frequency motion, as depicted in Fig. 9 and 10.

By increasing the number of Craig-Bampton (CB) modes in CMS the quality of results is considerably improved. By an adequate increase of the Krylov *q*-set, though, KSM still remains a better option (Fig. 14 and 15).

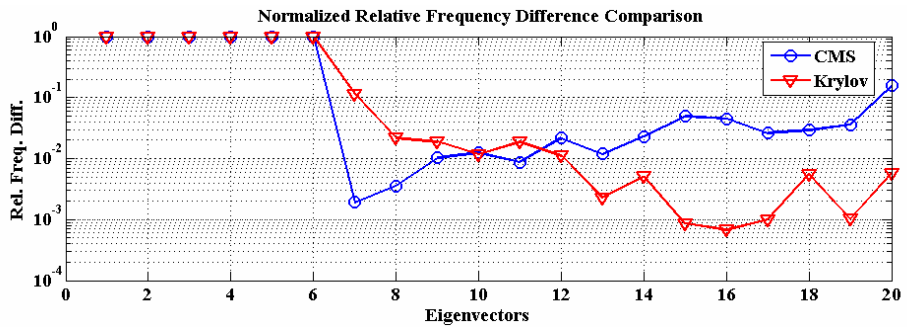


Figure 14 Elastic Crankshaft – *NRFD* Comparison of CMS and Krylov for 15 extra CB and Krylov modes respectively

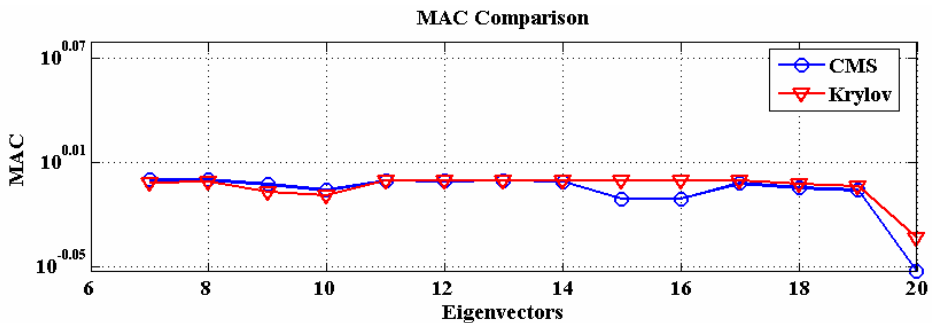


Figure 15 Elastic Crankshaft – *MAC* Comparison of CMS and Krylov for 15 extra CB and Krylov modes respectively

In case of having to solve the large sparse linear systems produced by the reduction approaches iteratively, *diagonal perturbation* contributes in speeding up the process without essentially harming the result's quality.

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