FrontUQ 2017
Frontiers of Uncertainty Quantification in Engineering
5-8 September, Munich, Germany

Bayesian updating of rare events with meta-model-based reliability methods

Max Ehre | Dr. Iason Papaioannou | Prof. Daniel Straub
Engineering Risk Analysis Group, Technische Universität München. Arcisstraße 21, 80333 München, Germany

1. Motivation

The uncertain input parameters of a system $\boldsymbol{\theta}$, described by their joint PDF $f_{\boldsymbol{\theta}}$, can be reduced by means of Bayesian updating. Since, typically, not every parameter in $\boldsymbol{\theta}$ can be learned, $\boldsymbol{\theta}$ is split as $\boldsymbol{\theta} = [\boldsymbol{\theta}_A, \boldsymbol{\theta}_B]^T$, where $\boldsymbol{\theta}_B$ represents all parameters included in the updating process. The data enters the Bayesian formulation in shape of the likelihood function $L(\boldsymbol{\theta}_B) = f(\mathcal{D}|\boldsymbol{\theta}_B)$ which measures how well a realization of $\boldsymbol{\theta}_B$ may explain the obtained data $\mathcal{D}$ through comparing the data with the output of a numerical model. Formally, the posterior distribution $f(\boldsymbol{\theta}'_B) = f(\boldsymbol{\theta}_B|\mathcal{D})$ reads

$$f(\boldsymbol{\theta}'_B) = \frac{L(\boldsymbol{\theta}_B)\, f(\boldsymbol{\theta}_B)}{\int_{\Omega_{\boldsymbol{\theta}_B}} L(\boldsymbol{\theta}_B)\, f(\boldsymbol{\theta}_B)\, \mathrm{d}\boldsymbol{\theta}_B}.$$

This results in a refined prediction of system responses and therefore more accurate predictions of any quantities of interest (Figure 1).

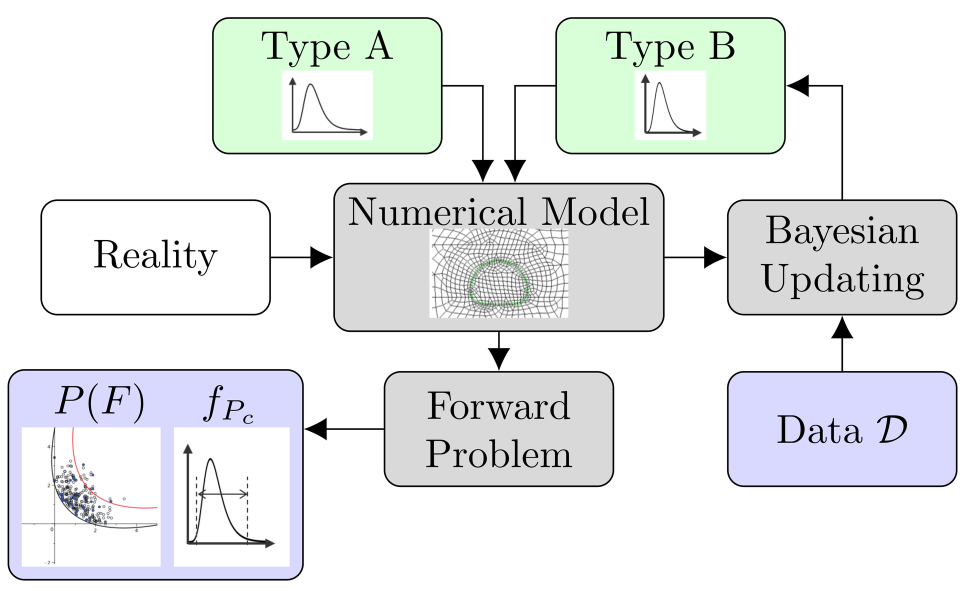

Figure 1 : *Bayesian updating and reliability with different types of uncertainty.*

In complex models, the large number of required model evaluations often renders Bayesian updating computationally intractable. Meta-models (MM), which mimic the numerical model at a small fraction of the computational cost, can be used the enhance the computational efficiency of Bayesian updating.

Theory

Meta-models

We compare the performance of two meta-modeling techniques, which, using a set of training points (or experimental design) $\mathcal{E} = \{\boldsymbol{X}_{\mathcal{E}}, Y_{\mathcal{E}}\}$ obtained from the original model (OM) $\mathcal{G}$: $Y = \mathcal{G}(\boldsymbol{X})$), $\boldsymbol{X} \in \mathbb{R}^{d\times 1}$, mimic $\mathcal{G}$. Orthogonal polynomial bases $\{\psi_j^{(i)}(X_i), 0 \leq j \leq p_i\, i \leq d\}$ of order p_i in the i-th input are used to construct the meta-models.

- **Sparse polynomial chaos expansion** (PCE, Sudret et al., 2013):

$$\mathcal{G}^{PCE}(\boldsymbol{X}) = \sum_{j=1}^{P} a_j \prod_{i=1}^{d} \psi^{(i)}_{\alpha_{ij}}(X_i),$$

The model parameters $\boldsymbol{a}$ are determined via the ordinary least squares method. $\boldsymbol{\alpha}$ is a truncated set of d-dimensional multi-indices, in which P elements have been retained. In sparse PCEs, P is minimized e.g. by using model selection algorithms like least-angle regression to find an optimal set of regressors $\prod_{i=1}^{d} \psi^{(i)}_{\alpha_{ij}}(X_i)$ to best describe $\mathcal{G}$.

- **Low-rank approximation** (LRA, Konakli and Sudret, 2016)

$$\mathcal{G}^{LRA}(\boldsymbol{X}) = \sum_{k=1}^{R} b_k \prod_{i=1}^{d} \sum_{j=1}^{p_i} z_{ijk} \psi_j^{(i)}(X_i).$$

The model parameters $\boldsymbol{z}$ and $\boldsymbol{b}$ are computed via an alternating least squares method. All LRAs have been computed at $R = 1$.

Updating

After having built the meta-models, updating is performed with BUS and subset simulation (Straub and Papaioannou, 2015). BUS uses an acceptance-rejection formulation of the updating problem to recast it as a reliability problem with an associated limit-state function (LSF) $h(\boldsymbol{\theta}_B) = \mathcal{U}_{(0,1)} - cL(\boldsymbol{\theta}_B)$. Posterior samples are obtained by solving the reliability problem with subset simulation.

Prediction

The considered predictive QoI will be the distribtion of the failure probabilitiy conditioned on the posterior vector of reducible parameters $\boldsymbol{\theta}_B$. The probability of failure given $\boldsymbol{\theta}_B$ is defined as:

$$\boldsymbol{P}_c(\boldsymbol{\theta}_B) = \mathbb{E}_{\boldsymbol{\theta}_A}\left[\mathrm{I}\left(g(\boldsymbol{\theta}_A, \boldsymbol{\theta}'_B) \leq 0\right) |\boldsymbol{\theta}_B\right],$$

where g is the failure event LSF and I is the indicator function. P_c is a function of the reducible uncertain parameters $\boldsymbol{\theta}_B$ and therefore is itself a random variable. The estimated density $\hat{f}_{P_c}$ then expresses the uncertainty about the predicitive quantity $\mathrm{P}(F)$ associated with the stochastic model of $\boldsymbol{\theta}_B$.

Methodology

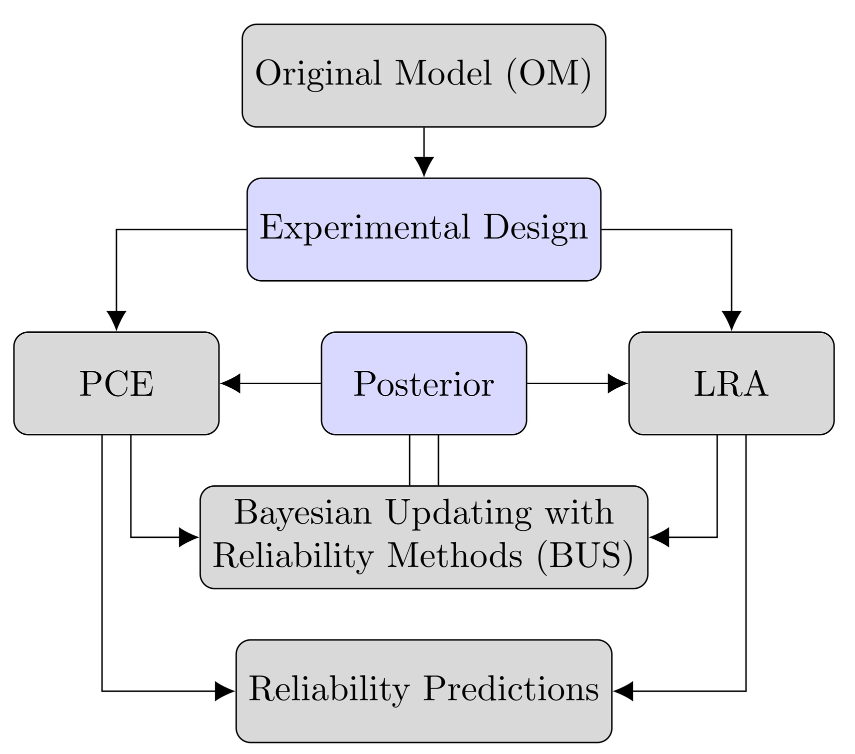

Figure 2 : *Flow diagram of the proposed methodology.*

Example

A one-dimensional beam under constant Gumbel-distributed line load q with $\mu_q = 3kN$ is considered (Figure 3).

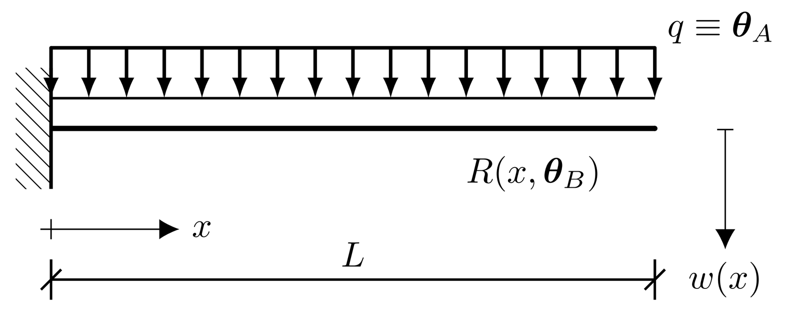

Figure 3 : *Cantilever beam under constant line load.*

The beam stiffness $R = EI$ ($\mu_R = 10^7 N/m$, $\sigma_R = 3 \cdot 10^6 N/m$) is modelled through a lognormal random field (RF) represented by a Karhunen-Loeve-expansion:

$$R(x, \boldsymbol{\theta}_B) = \exp\left\{\mu_{\ln R} + \sigma_{\ln R} \sum_{k=1}^{n_{KL}} \sqrt{\lambda_k}\phi_k(x)\, \theta_B^k\right\},$$

where (λ_k, ϕ_k) are the eigenpairs of the correlation kernel of the underlying Gaussian RF $\Gamma_{\mathcal{N}}(\Delta x) = 1/\sigma_{\ln R} \ln[1 + \exp\{-|\Delta x|/l_R\}]$, in which $\mu_{\ln R}$ and $\sigma_{\ln R}$ are its mean and standard deviation and θ_B^k are standard-normal random numbers ($n_{KL} = 10$). For $n_{\mathcal{E}} = 10^3$ points, which is obtained by latin hypercube sampling, the optimal polynomial orders are found to be $p^{PCE} = 9$ (and $p^{LRA} = 6$. A single observation at $R^*(x)$ and $q^* = 3\mu_q$ results in a true deflection $w^*(x)$ and the noisy deflection observation $\tilde{w}(x)$. The observation error ϵ_w is assumed as exponentially correlated ($l_{\epsilon_w} = 1$m) Gaussian ($\mu_{\epsilon_w} = 0$, $\sigma_{\epsilon_w} = 1$mm) additive error, such that $\tilde{w}(x) = w^*(x) + \epsilon_w(x)$.

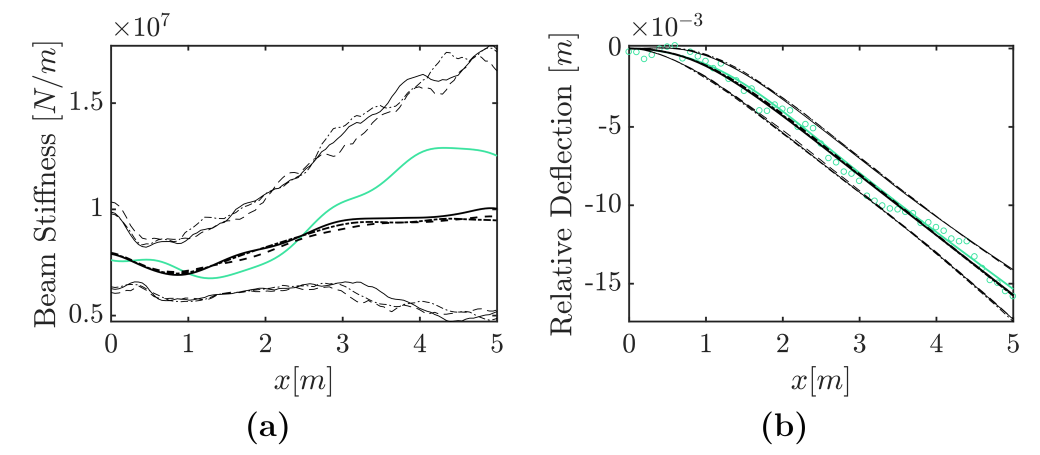

Figure 4 : *solid - OM, dashed: PCE, dot-dashed: LRA | (a): bold - Posterior stiffness mean $\mu'_R(x)$, thin - 95% credibility intervals, green - true R^* | (b): bold - deflection mean relative to prior deflection mean conditional on $q*$ $\mu_{w'|q^*}(x) - \mu_{w|q^*}(x)$, thin - 95% credibility intervals, green & solid - true w^*, green & circles - observed $\tilde{w}$.*

Figure 4 shows, LRA and PCE perform comparably in the updating (note that the for the updating with PCE/LRA, 10^3 model calls are required compared to approximately $6 \cdot 10^4$ for the OM) while Figure 5 provides evidence that the LRA model has an advantage in predicting $\hat{f}_{P_c}$.

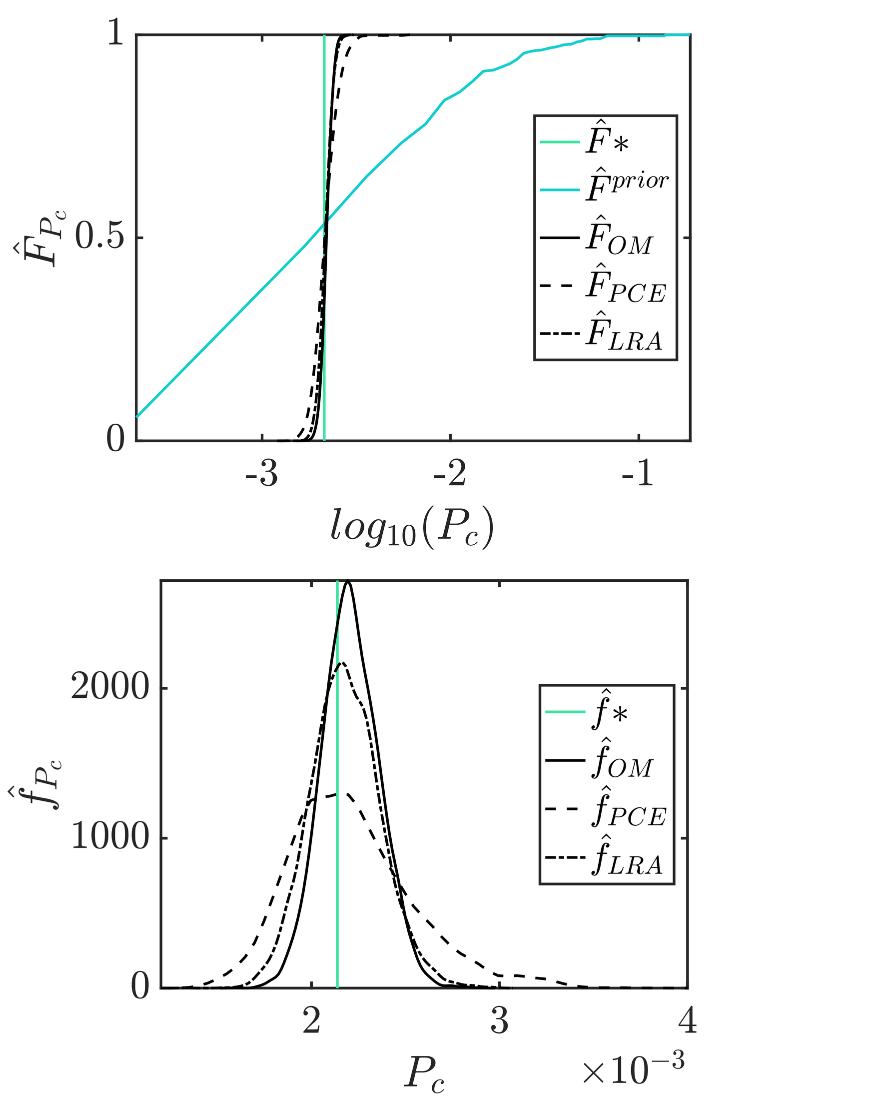

Figure 5 : *PDF and CDF of P_c. $*$ denotes true values (obtained with R^*).*

References

Konakli, Katerina and Bruno Sudret (2016). "Polynomial meta-models with canonical low-rank approximations: Numerical insights and comparison to sparse polynomial chaos expansions". In: *Journal of Computational Physics* 321, pp. 1144–1169.

Straub, Daniel and Iason Papaioannou (2015). "Bayesian Updating with Structural Reliability Methods". In: *Journal of Engineering Mechanics* 141.3.

Sudret, Bruno, Géraud Blatman, and Marc Berveiller (2013). "Response surfaces based on polynomial chaos expansions". In: *Construction reliability: safety, variability and sustainability*, pp. 147–167.