

Gravitational waves and small-field astrometry

Robin Geyer^{ID*} and Sven Zschocke^{ID}, Michael Soffel, Sergei Klioner^{ID}
Lohrmann Observatory, Technische Universität Dresden, Helmholtzstraße 10, 01069 Dresden, Germany

Lennart Lindegren^{ID}
Lund Observatory, Division of Astrophysics, Department of Physics, Lund University, Box 43, 22100 Lund, Sweden

Uwe Lammers^{ID}
European Space Agency (ESA), European Space Astronomy Centre (ESAC), Camino bajo del Castillo, s/n, Urbanización Villafranca del Castillo, Villanueva de la Cañada, 28692 Madrid, Spain

Astrometric observations can, in principle, be used to detect gravitational waves. In this paper we give a practical overview of the gravitational wave effects which can be expected specifically in small-field astrometric data. Particular emphasis is placed on the differential effect between pairs of sources within a finite field of view. We also present several general findings that are not restricted to the small-field case. A detailed theoretical derivation of the general astrometric effect of a plane gravitational wave is provided. Numerical simulations, which underline our theoretical findings, are presented.

We find that small-field missions suffer from significant detrimental properties, largely because their relatively small fields only allow the measurement of small differential effects which can be expected to be almost totally absorbed by standard plate calibrations.

January 12, 2026

I. INTRODUCTION

Over the past several years, we have witnessed a growing interest in designing space-based small-field astrometric telescopes of ultimate sub- μ s accuracy and their possible scientific applications (see e.g. [1–3]). Moreover, the idea has emerged that even non-astrometric, imaging space telescopes could be used to make high-precision astrometric measurements [4, 5]. On the other hand, space-based global astrometry, like *Gaia*, has already shown that it is able to deliver a remarkable observational accuracy [6–8]. At the same time, with gravitational wave (GW) astronomy now a routine reality [9–11], there is a certain interest in also using astrometric measurements to detect GW signals [12–15]. The effects of GWs on astrometric measurements have been known for a long time [16–18].

While it has already been discussed in the literature that all-sky astrometric missions like *Gaia* are, in principle, sensitive to GWs [19, 20], a practical yet succinct discussion for small-field missions is still lacking.

In this paper, we present a practical overview of the GW effects that can be expected in small-field astrometry. In Section II, we discuss various upper estimates of the astrometric GW effect both for global astrometry and for small-field astrometry with a given field of view (FoV) size. Section III contains the results from numerical simulations which validate and illustrate the

theoretical formulas. There, we also discuss the appearance of the astrometric GW effects in a given small FoV. A concluding discussion can be found in Section IV. Appendix A contains a concise theoretical derivation of the astrometric GW effect from the basic principles. Several important aspects of the effect are elucidated there.

In the whole paper, we will discuss GWs with an effectively plane wavefront at the observer and at the observed astrometric source. For all simulations, and discussions of the maximum measurable signal, we use Eq. (A48), or, if angles between sources are concerned, Eq. (A38). Finally, a note with respect to terminology: for the rest of the paper, we will refer to the source of the gravitational wave itself as “GW emitter(s)”, never as “source(s)”. Conversely, the sources which we astrometrically observe we refer to as “astrometric source(s)” or just “source(s)”.

II. MAXIMAL CHANGE OF THE ANGULAR DISTANCE BETWEEN SOURCES DUE TO A GW

In this Section, we use the theoretical formulation of the astrometric GW effect given in Appendix A of [20] to compute theoretical upper estimates of the magnitude of the astrometric GW effect in various situations. Appendix A.1 of [20] gives the basic formulas for the effect, while Appendix A.2 gives an alternative representation of the formulas given in Appendix A.1, which is especially useful for the astrometric discussion of the effect. Appendix A of this work contains a basic derivation of the formulas from Appendix A.1 of [20].

Equation (A.11) of [20] gives an expression for the angular variation of the observed position of a source due to a GW. The components of the two-dimensional displacement vector ($\delta\alpha^*$, $\delta\delta$) are given relative to the local

* Contact author: robin.geyer@tu-dresden.de

coordinate system on the sphere attached to the undisturbed position of the source, represented by the local dyad $(\mathbf{e}_\alpha, \mathbf{e}_\delta)$.

One can show that the maximal change of the source position

$$\sqrt{(\delta\alpha^*)^2 + (\delta\delta)^2} \leq \Delta_{\max} \sin \theta, \quad (1)$$

where θ is the angle between the source with coordinates (α, δ) and the direction of GW propagation $(\alpha_{\text{GW}}, \delta_{\text{GW}})$, $\Delta_{\max} = (2 - e^2)^{-1/2} h/2$ is the maximal astrometric effect, and $h = \left((h_+^c)^2 + (h_+^s)^2 + (h_\times^c)^2 + (h_\times^s)^2 \right)^{1/2}$ is given by the strain and phase parameters of the GW as in Eqs. (A.19)–(A.21) of [20]—see the reasoning in Appendix A.2 of [20]. Here, e is the eccentricity of the elliptic astrometric signal caused by the GW. This parameter is defined by Eq. (A.17) of [20] and depends only on the strain and phase parameters of the GW. We note that $0 \leq \theta \leq \pi$ and $0 \leq \sin \theta \leq 1$. In all equations here, we consider a linear model for the GW effect, so that terms $\mathcal{O}(\Delta_{\max}^2)$ are always neglected.

The general discussion in [20] is relevant for absolute astrometric observations like those of *Gaia*. However, for any kind of differential observations—that is, observations of the angular distances between pairs of sources—this discussion immediately gives an upper estimate of observable variations of the angular distance due to a GW. We denote the undisturbed observable angular distance between source A and source B as ψ_{AB} . This is the angular distance in the absence of the GW and can be computed from the unperturbed positions (α_i, δ_i) , $i = \text{A, B}$, e.g. as $\cos \psi_{\text{AB}} = \sin \delta_A \sin \delta_B + \cos \delta_A \cos \delta_B \cos(\alpha_B - \alpha_A)$. We denote the disturbed observable angular distance between these sources in the presence of the GW as $\psi_{\text{AB}}^{\text{gw}}$. The observable change of the angular distance, $\delta\psi_{\text{AB}} = \psi_{\text{AB}}^{\text{gw}} - \psi_{\text{AB}}$, can obviously be estimated as

$$|\delta\psi_{\text{AB}}| \leq \Delta_{\max} (\sin \theta_A + \sin \theta_B) \quad (2)$$

for a given GW with a known maximal astrometric effect Δ_{\max} and for any pair of sources. Here, θ_i are the angles between the source i ($i = \text{A, B}$) and the direction of GW propagation. This simply means that the differential effects cannot exceed the sum of the absolute effects for the two involved sources. Equation (2) implies that

$$|\delta\psi_{\text{AB}}| \leq 2\Delta_{\max}, \quad (3)$$

which gives the upper limit for the GW-induced changes in angular separations of arbitrary pairs of sources.

While Eq. (2) is correct for any pair of sources, one can derive a better estimate for source pairs with a maximal angular distance below a certain limit, ε , so that $\psi_{\text{AB}} \leq \varepsilon$. Here, we consider ε to be sufficiently small, so that the effect of order ε^2 can be neglected. Interestingly, one can demonstrate that for arbitrarily small ε the estimate given by Eq. (2) is almost reachable (this

happens when the center of the field of view coincides with the direction of the GW propagation, or the opposite direction). Nevertheless, Eq. (2) is, in most cases, overly coarse and a considerably better estimate can be derived.

The GW-disturbed positions read $(\alpha_i + \delta\alpha_i, \delta_i + \delta\delta_i)$, $i = \text{A, B}$, where $\delta\alpha_i$ and $\delta\delta_i$ are given by Eq. (A.11) of [20]. Using the standard formula for the angular distance between two points and applying it to both the disturbed and undisturbed source positions, we arrive at a first-order approximation in Δ_{\max} (implying also first order in $\delta\psi_{\text{AB}}$):

$$\begin{aligned} \delta\psi_{\text{AB}} &= -(\sin \psi_{\text{AB}})^{-1} \\ &\times \left[(\cos \delta_A \sin \delta_B - \sin \delta_A \cos \delta_B \cos(\alpha_B - \alpha_A)) \delta\delta_A \right. \\ &\quad + (\sin \delta_A \cos \delta_B - \cos \delta_A \sin \delta_B \cos(\alpha_B - \alpha_A)) \delta\delta_B \\ &\quad \left. - \cos \delta_A \cos \delta_B \sin(\alpha_B - \alpha_A) (\delta\alpha_B - \delta\alpha_A) \right]. \quad (4) \end{aligned}$$

Then, considering the variations to first order in ε , one finds the upper estimate

$$\begin{aligned} |\delta\psi_{\text{AB}}| &\leq \varepsilon \Delta_{\max} \\ &\times \left[1 + \sqrt{(1 - e^2 \sin^2(2\bar{\alpha}_r - \phi))(1 - \cos \theta_r)} \right] + \mathcal{O}(\varepsilon^2). \quad (5) \end{aligned}$$

Note that, since one has $\sin \psi_{\text{AB}} = \mathcal{O}(\varepsilon)$ in the denominator of Eq. (4), the terms of second order in ε have been considered in its numerator. Here, ϕ is the position angle of the elliptic astrometric signal caused by the GW defined by Eq. (A.18) of [20]. Furthermore, θ_r is the angular distance between one of the sources, A or B, used as the reference point with coordinates (α_r, δ_r) and the propagation direction of the GW

$$\cos \theta_r = \sin \delta_r \sin \delta_{\text{GW}} + \cos \delta_r \cos \delta_{\text{GW}} \cos(\alpha_r - \alpha_{\text{GW}}) \quad (6)$$

and $\bar{\alpha}_r$ is the right ascension of the reference point in the coordinate system in which the GW propagates toward the north pole:

$$\begin{pmatrix} \cos \bar{\alpha}_r \cos \bar{\delta}_r \\ \sin \bar{\alpha}_r \cos \bar{\delta}_r \\ \sin \bar{\delta}_r \end{pmatrix} = \mathbf{P}^T \begin{pmatrix} \cos \alpha_r \cos \delta_r \\ \sin \alpha_r \cos \delta_r \\ \sin \delta_r \end{pmatrix}, \quad (7)$$

where \mathbf{P}^T is the transposed matrix \mathbf{P} given by Eq. (A.10) of [20]. Since Eq. (5) neglects terms quadratic in ε , the reference point (α_r, δ_r) can be formally taken as the coordinates of the first source (α_A, δ_A) .

The upper estimate (5) cannot be improved for $e = 0$, and $e = 1$, or when $\sin(2\bar{\alpha} - \phi) = 0$ for any value of e : for a given GW with such parameters and for a given (α_A, δ_A) and ε , one can find a moment of time and a position (α_B, δ_B) for which $|\delta\psi_{\text{AB}}|$ is exactly given by the right-hand side of Eq. (5). Expectedly, one can also demonstrate that the estimate given by Eq. (5) is reached for pairs with maximal allowed angular distance $\psi_{\text{AB}} = \varepsilon$.

For $0 < e < 1$ and $\sin(2\bar{\alpha} - \phi) \neq 0$, Eq. (5) cannot be exactly attained and can, in principle, be improved. For

the *particular* case of $\sin^2(2\bar{\alpha} - \phi) = 1$ one can derive the following reachable estimate:

$$|\delta\psi_{AB}|_{\sin^2=1} \leq \varepsilon \Delta_{\max} \sqrt{1 + (e^{-2} - 1)(1 - \cos\theta_r)^2}, \quad (8a)$$

and

$$|\delta\psi_{AB}|_{\sin^2=1} \leq \varepsilon \Delta_{\max} \sqrt{1 - e^2(2 - \cos\theta_r)}, \quad (8b)$$

where Eq. (8a) is valid in case of $(e^{-2} - 1)(1 - \cos\theta_r) \leq 1$, while Eq. (8b) is valid otherwise. We note that Eqs. (8a) and (8b) give exactly the same estimate for $|\delta\psi_{AB}|$ as Eq. (5) for $e = 0$ and $e = 1$ (and $\sin^2(2\bar{\alpha} - \phi) = 1$). A combined analytical and numerical investigation shows that Eq. (5) overestimates the real maximal value of $|\delta\psi_{AB}|$ by at most a factor of 1.4. Even if a better general estimate could be given as a complicated function of e , $\cos\theta_r$ as well as the sine and cosine of $2\bar{\alpha}_r - \phi$, we prefer to use Eq. (5) because of its simplicity.

We note that one can further simplify Eq. (5) as

$$|\delta\psi_{AB}| \leq \varepsilon \Delta_{\max} (2 - \cos\theta_r) \quad (9)$$

to make it valid for any eccentricity e (this is exactly Eq. (5) for $e = 0$) and, finally, as

$$|\delta\psi_{AB}| \leq 3\varepsilon \Delta_{\max}. \quad (10)$$

This gives the upper estimate of the GW-induced variation of the angular distance for any e and for any position on the sky. Since ε is considered to be sufficiently small (in principle, $\varepsilon \ll 1$) this latter estimate does not contradict Eq. (3). Both Eq. (9) and Eq. (10) are reachable in the respective parameter space.

Finally, we point out two interesting aspects of $|\delta\psi_{AB}|$. First, since the absolute astrometric effect of a GW is proportional to $\sin\theta$ (e.g. it is maximal at the angular distance of $\theta = \pi/2$ from the GW propagation direction) one could, naively, expect similar dependence of the differential effect $\delta\psi_{AB}$ also for source pairs with small angular distance ε . The estimate given by Eq. (2) seems to support this expectation. However, we see from Eq. (9) that the maximal value of $|\delta\psi_{AB}|$ is proportional to $2 - \cos\theta$. This means that the maximal differential effect is, in fact, minimal in the direction of the GW propagation, where it reaches $\varepsilon \Delta_{\max}$. It then continuously increases up to $3\varepsilon \Delta_{\max}$ towards the direction of the GW source ($\theta = \pi$). This is also illustrated by Figure 2.

Another remarkable aspect of $|\delta\psi_{AB}|$ is the existence of a flower-like pattern with four “petals” for GWs with $e > 0$, as one can see in Figure 1. This pattern is related to the term in Eq. (5) that depends on $\bar{\alpha}_A$. For four values of $\bar{\alpha}_A$ for which $\sin(2\bar{\alpha}_A - \phi) = 0$ the maximal differential effect is given by Eq. (9), while for the other values of the sine the maxima of $|\delta\psi_{AB}|$ for $\theta = \pi$ become shallower and reach their minimal values $\varepsilon \Delta_{\max} (1 + 2\sqrt{1 - e^2})$ for $\sin(2\bar{\alpha}_A - \phi) = \pm 1$. We note that for $e = 1$ and $\sin(2\bar{\alpha}_A - \phi) = \pm 1$ one gets $|\delta\psi_{AB}| \leq \varepsilon \Delta_{\max}$ independently of θ .

This four-petalled pattern in the differential astrometric effect has an important consequence for the choice of the reference point in Eq. (5). The linear approximation in ε used in Eq. (5) is sufficient when the effect is not changing much on the scale of ε across the sky. However, if the sources are located close to the GW source, the four-petalled pattern shown in the two lower plots of Fig. 1 changes very quickly, and the factor $(1 - e^2 \sin^2(2\bar{\alpha} - \phi))^{1/2}$ that is computed for the reference point (α_r, δ_r) for $r = A$ in Eq. (5) can be significantly different when computed for the other point with $r = B$. Our numerical studies show that in some extreme cases, when the distances of both sources from the GW source are comparable to ε , and if (α_A, δ_A) is chosen for (α_r, δ_r) as discussed above, the estimate from Eq. (5) can give a slightly lower value than the actual $|\delta\psi_{AB}|$. To cover these cases, the reference point (α_r, δ_r) should be chosen to be either (α_A, δ_A) or (α_B, δ_B) , whichever gives the smaller value of $(1 - e^2 \sin^2(2\bar{\alpha} - \phi))^{1/2}$ in Eq. (5).

Eq. (5) gives a reasonable upper estimate for pairs of sources with maximal angular distance ε . For all possible pairs of sources within a round field of view with angular diameter ϱ , Eq. (5) remains valid for $\varepsilon = \varrho$. Similarly, for all pairs of sources within a square field of view of angular size $\varrho \times \varrho$, Eq. (5) is valid for $\varepsilon = \sqrt{2} \varrho$.

Overall, we conclude that the differential effect $\delta\psi_{AB}$ in the angular distance between two sources remains, as expected, of the same order of magnitude as the absolute effect discussed e.g. by [12, 20]: $|\delta\psi_{AB}| \leq 2\Delta_{\max}$ for arbitrary pairs of stars. However, for limited FoVs the differential effect is limited to $|\delta\psi_{AB}| \leq 3\varepsilon \Delta_{\max}$ for any pairs of sources at the angular distance of ε or lower.

The authors of [21] and [22] predicted an increasingly large astrometric effect from GWs at small separations. This is not confirmed by our analysis. The technical reasons for the flaw in their work are described in the Appendix (see also [23], where this flaw is discussed as well).

III. NUMERICAL SIMULATIONS

In order to verify and visualize the findings from Section II we conducted a series of numerical simulations in which we explicitly compute the differential astrometric GW effect $\delta\psi_{AB}$ between two stars A and B for specific angular distances.

As the theoretical discussion above suggests, and our numerical simulations confirm, the particular values of the maximum possible angular separation, ε , of source pairs, as well as the maximal GW amplitude, Δ_{\max} , can be chosen arbitrarily as long as they remain sufficiently small: the effects are always proportional to $\varepsilon \Delta_{\max}$. In particular, this remains true unless we consider very large ε of many degrees, for which the second-order effects neglected in Section II become numerically important. To be realistic for small-field astrometry, for all simulations, we used $\varepsilon = 0.1^\circ$ and $\Delta_{\max} = 10 \text{ mas} \approx 4.8 \times 10^{-8}$. The

values are small enough that second-order effects can be neglected, and at the same time, the magnitude of the effect is large enough so that numerical noise is not an issue. It is clear that in reality Δ_{\max} will certainly be much smaller, most likely in the region of nano-arcseconds and below.

All results presented below are given as a normalized angular change, $\mathcal{F} = \delta\psi_{AB}/(\varepsilon \Delta_{\max})$. In the linear approximation, which we consider, the value of \mathcal{F} is independent of ε and Δ_{\max} . Conversely, the corresponding magnitude of the angular change can be restored as $\delta\psi_{AB} = \mathcal{F} \varepsilon \Delta_{\max}$ for a particular FoV size ε and a GW with a maximal astrometric effect of Δ_{\max} .

A. Basic statistics of a typical differential signal

To get a first coarse overview, we computed some statistics of \mathcal{F} as one might expect from a random selection of stars and GW parameters. This, in a way, also reflects our lack of a priori knowledge about specific GW emitters. To this end, we simulated 10^9 sets of randomly selected GW parameters. After random selection, the strain parameters were always scaled in such a way that Δ_{\max} remained constant. For each GW, we selected 100 random source pairs across the celestial sphere; each source pair had a randomly selected angular separation of $0 < \psi_{AB} \leq \varepsilon$ and a random orientation. For each of the pairs, we computed the change in angular distance due to a GW at a random time. This gives 10^8 overall samples of \mathcal{F} , the basic statistics of which are given in Table I. First, we see that our simulations confirm that $|\mathcal{F}| \leq 3$ as suggested by Eq. (10). We also see that a typical value of $|\mathcal{F}|$ is 0.18, which means that a typical value of $|\delta\psi_{AB}|$ in the random small-field astrometric observations is about $0.18 \varepsilon \Delta_{\max}$. For a FoV with, e.g., $\varepsilon = 0.1^\circ$, this implies that a typical sensitivity of only 0.03% of the already minuscule Δ_{\max} would be necessary to detect such a typical signal.

Table I. Statistics of normalized absolute angular changes \mathcal{F} from the simulated data using random star pairs and random GW parameters. The values are, from top to bottom, the mean, the standard deviation, and a series of quantiles: the minimum, the 0.1, 0.5, and 0.9 quantiles and the maximum of the absolute value of \mathcal{F} .

Parameter	Value
mean ($ \mathcal{F} $)	0.30
std (\mathcal{F})	0.46
min ($ \mathcal{F} $)	0.00
$Q_{0.1}$ ($ \mathcal{F} $)	0.02
median ($ \mathcal{F} $)	0.18
$Q_{0.9}$ ($ \mathcal{F} $)	0.76
max ($ \mathcal{F} $)	2.98

B. Spatial distribution of the differential astrometric GW signal

Next, we consider the spatial distribution of the variations in angular distances $|\delta\psi_{AB}|$ for a given GW. The spatial distribution of the signal is especially relevant for small-field astrometry, given that only a limited number of sky regions can typically be observed. If, for instance, a GW emitter candidate is identified beforehand, observations might be optimally directed towards the area of the sky with the highest probability of detecting the GW from this emitter.

We note that the propagation direction of the GW can be chosen arbitrarily, since any other GW direction is equivalent to a different orientation of the coordinate system. Since we are looking for the maximal values of $|\delta\psi_{AB}|$ over an extended period of time, the GW frequency can also be selected arbitrarily, provided that the tested time interval covers at least one GW period. Thus, only four strain parameters are important in this study. As shown in the Appendix A of [20], the magnitude of the astrometric GW effect can alternatively be described by the maximal astrometric effect Δ_{\max} , the eccentricity e , and the position angle ϕ of the ellipse representing the astrometric GW signal.

One can show that a change of the position angle ϕ is equivalent to a rotation of the reference system around the propagation direction \mathbf{p} of the GW.

Δ_{\max} can also be fixed to a constant for the reasons explained above, and it then remains to examine the distributions for various values of e .

To investigate this, we simulated the signals from three GWs with eccentricities e equal to 0, 0.7, and 1. For each GW, we generated 10^8 randomly selected and oriented source pairs, each possessing a constant (unperturbed) angular separation $\psi_{AB} = \varepsilon$. We deliberately chose ψ_{AB} to be the maximum pair separation, ε , since we are mainly interested in the maximal angular change observable in a given area of the sky. Subsequently, for each source pair, we computed the change in angular separation $\delta\psi_{AB}$ induced by the GW at randomly selected observation times (over a time span considerably larger than the GW period). These data were then used to visualize the sky distribution of the maximal value of $|\delta\psi_{AB}|$, and to validate the dependence of this quantity on θ_r and $\bar{\alpha}_r$ as in Eqs. (5) and (9).

Figures 1 and 2 show the results of the simulations. The sky maps in Fig. 1 show the maximal value of $|\mathcal{F}|$ per HEALPix [24] on the sky, for all simulated source pairs falling into the respective pixel. These maps clearly illustrate the expected dependence on θ_r and $\bar{\alpha}_r$, which form the four-petalled pattern on the sky for $e > 0$. The respective scatter plots of simulated angular changes over $\bar{\alpha}_r$ in the right column of Fig. 1 further elucidate the relationship with e . It is important to note that these scatter plots also unambiguously demonstrate that the actual measured angular change of the source pair can be zero at any $\bar{\alpha}_r$ because the change in angular separation

due to a GW depends on the orientation of the source pair. The differential effect can be zero even at $\bar{\alpha}_r$, where a maximum in $|\mathcal{F}|$ is reached.

Figure 2 illustrates the dependence of the simulated angular changes on θ_r , the angular distance to the GW propagation direction. This result agrees with Eq. 9 for the maximum values. We note that in this case, too, the actual measured angular change depends on the orientation of the source pair, and it can be zero for any θ_r .

C. Differential astrometric GW signal inside a FoV

While considering the change in the angular distance of individual pairs of sources is important, it provides limited insight into the distribution of the differential astrometric GW signal over the area of a FoV. To elucidate that distribution, we simulated a GW signal using a purely $+$ -polarized GW, sampled at the time of maximal magnitude inside a FoV for different points on the sky. This GW configuration is also representative of the \times -polarisation, although the specific examples would look different, the conclusions are the same.

Figure 3 shows four examples of these simulations. Each row of plots corresponds to a FoV area at a different location on the sky. In this way, the variation of the astrometric GW effect depending on the position can be seen. The plots again show normalized values of both absolute (right column) and differential (left column) GW effect.

A number of effects can be seen in Fig. 3. Generally, the absolute astrometric GW effect inside a FoV appears basically uniform at first glance, as can be seen in the left column. An exception is observed in the top-left panel of Fig. 3, where the FoV is precisely directed towards the GW emitter ($\bar{\alpha} = 0^\circ$; $\theta = 180^\circ$).

At this specific location, the overall GW-induced effect is minimal (see e.g. Eq. 1), but lacks any dominant overall shift. Consequently, the GW signal in this case is entirely differential, as evident when compared to its corresponding plot in the right column. A similar picture would be visible for the opposite point in the sky, in the GW propagation direction ($\bar{\alpha} = 0^\circ$; $\theta = 0^\circ$). Even a modest 5° displacement of the FoV from the GW emitter, as shown in the second row with ($\bar{\alpha} = 0^\circ$; $\theta = 175^\circ$), results in all absolute positional changes being overwhelmingly dominated by an average shift. The differential effect, however, is strongest there, inside one of the four petals of the patterns discussed before. At other areas on the sky (two bottom rows) the overall shift changes direction and the differential changes are generally weaker. Notably, at the point ($\bar{\alpha} = 90^\circ$; $\theta = 90^\circ$) (row 3 in Fig. 3), where the absolute GW effect is maximal, the differential GW effect is significantly smaller compared to, e.g., the case of ($\bar{\alpha} = 0^\circ$; $\theta = 175^\circ$) (row 2 in Fig. 3), where the maximum absolute GW effect is only 8% of the maximal possible for this GW. A further effect is dis-

cernible in the bottom row, with the randomly selected center position of ($\bar{\alpha} = 305.7^\circ$; $\theta = 133.4^\circ$), where the differential GW effect in the FoV exhibits a significant rotational component. This leads to an interesting consequence that the angular changes compared to the center are negligible (note the relatively large dark blue arrows). Angular separation between the center and other sources would only change in the two corners of the FoV, where a linear component in the differential shifts is present. Such rotation-like and shear-like, differential patterns exist at many positions on the sky.

We stress that the differential GW effects shown in the right column of Fig. 3 are computed with respect to the central point of the FoV. If the reference point is selected differently (e.g. in one of the FoV's corners), the resulting plots can appear significantly different, and the maximum absolute normalized angular change, $\max(|\mathcal{F}|)$, may vary considerably.

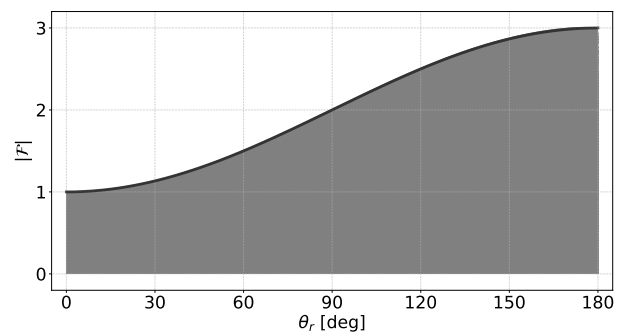


Figure 2. The normalized changes of the angular distance from the numerical simulations, $|\mathcal{F}|$, as a function of θ_r . This plot is valid for all eccentricities e . The differences between the three cases shown in Fig. 1 are negligible. The black line represents the value given by Eq. 9.

D. GW-induced astrometric signal and the plate solutions for small-field astrometry

Standard plate corrections, like linear affine plate models (e.g. translation, rotation, scale, and shear) or polynomial corrections, must certainly be applied for small-field astrometric observations. For high-accuracy astrometric solutions, the correction parameters may even need to be time dependent. Fig. 3 highlights a crucial finding in this respect: the variation of the differential effect over the FoV is very smooth, and even very simple plate corrections, like a quadratic correction in each axis, would absorb most of the differential effect in the FoV. In this way, the standard small-field calibrations will most likely dramatically reduce the GW imprint on the astrometric results.

To test this, we took the simulated differential GW effects from Fig. 3 and applied a basic plate correction. We fitted 2D bivariate Legendre polynomials up to or-

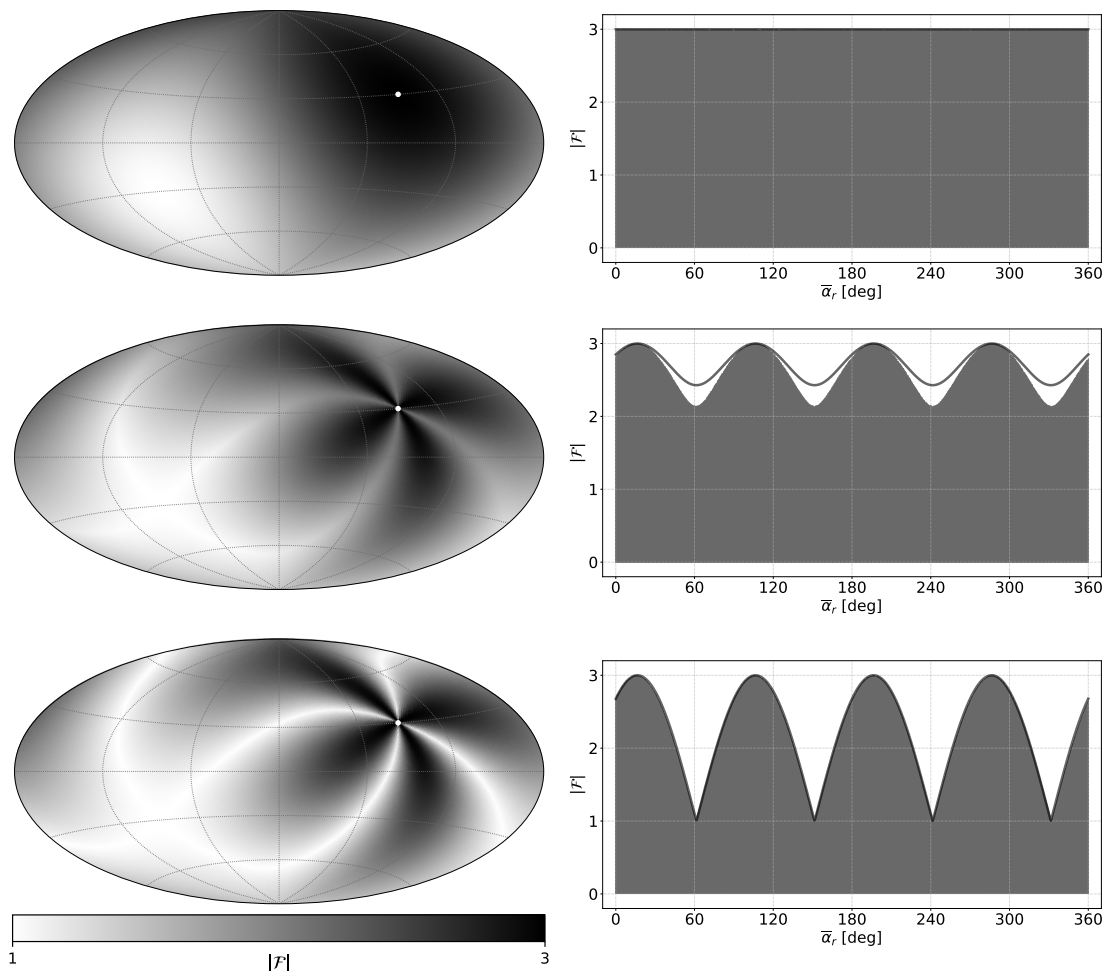


Figure 1. Resulting normalized angular changes represented by $|\mathcal{F}|$ from numerical simulations using random star pairs and three GWs with fixed parameters. The top row corresponds to a GW signal with an eccentricity of $e = 0$, the middle row to $e = 0.7$, and the bottom row to $e = 1$. The left column presents sky maps showing the maximum absolute normalized change of angular distance, $\max(|\mathcal{F}|)$, per HEALPix of level 6. The small white dot in the sky maps marks the position of the GW emitter. In the right column, all simulated normalized angular changes are displayed as a function of $\bar{\alpha}_r$. A black line in these plots marks the maximal achievable $|\mathcal{F}|$ according to Eq. (2). The sky maps use the Hammer-Aitoff projection in equatorial coordinates, with $\alpha = \delta = 0$ at the center, north up, and α increasing from right to left.

der three for each component of the shift (separately in both coordinates) and subtracted the result from the differential effect shown in Fig. 3. The maximal differential effect for the example with the highest differential magnitude, with $(\bar{\alpha} = 0^\circ; \theta = 175^\circ)$, decreased by a factor of more than 20 000, leaving a completely negligible residual signal. For $(\bar{\alpha} = 90^\circ; \theta = 90^\circ)$ and $(\bar{\alpha} = 305.7^\circ; \theta = 133.4^\circ)$ the differential GW signal is virtually completely absorbed by the calibration polynomials. Even in the exact direction of the GW emitter $(\bar{\alpha} = 0^\circ; \theta = 180^\circ)$ a simple third-order correction attenuates the differential signal by more than a factor of 2.

IV. CONCLUDING REMARKS

In this article, we investigated the effect of the astrometric GW signal in small-field astrometry. The astrometric effect generated by GWs will be tiny in any case. A detection of even the strongest conceivable signals will require instruments at the absolute forefront of technology. Practical detections of GWs with astrometry should hence rely on long series of observations and a large number of observed sources. Given the relatively small FoV (ε) and the correspondingly small number of sources observed by such instruments, the prospects for detecting GWs with small-field astrometry projects are rather bleak. For realistic FoVs (e.g., up to some degrees in extent), the measurable effects introduced by GWs are tiny compared to the magnitude of absolute GW effects,

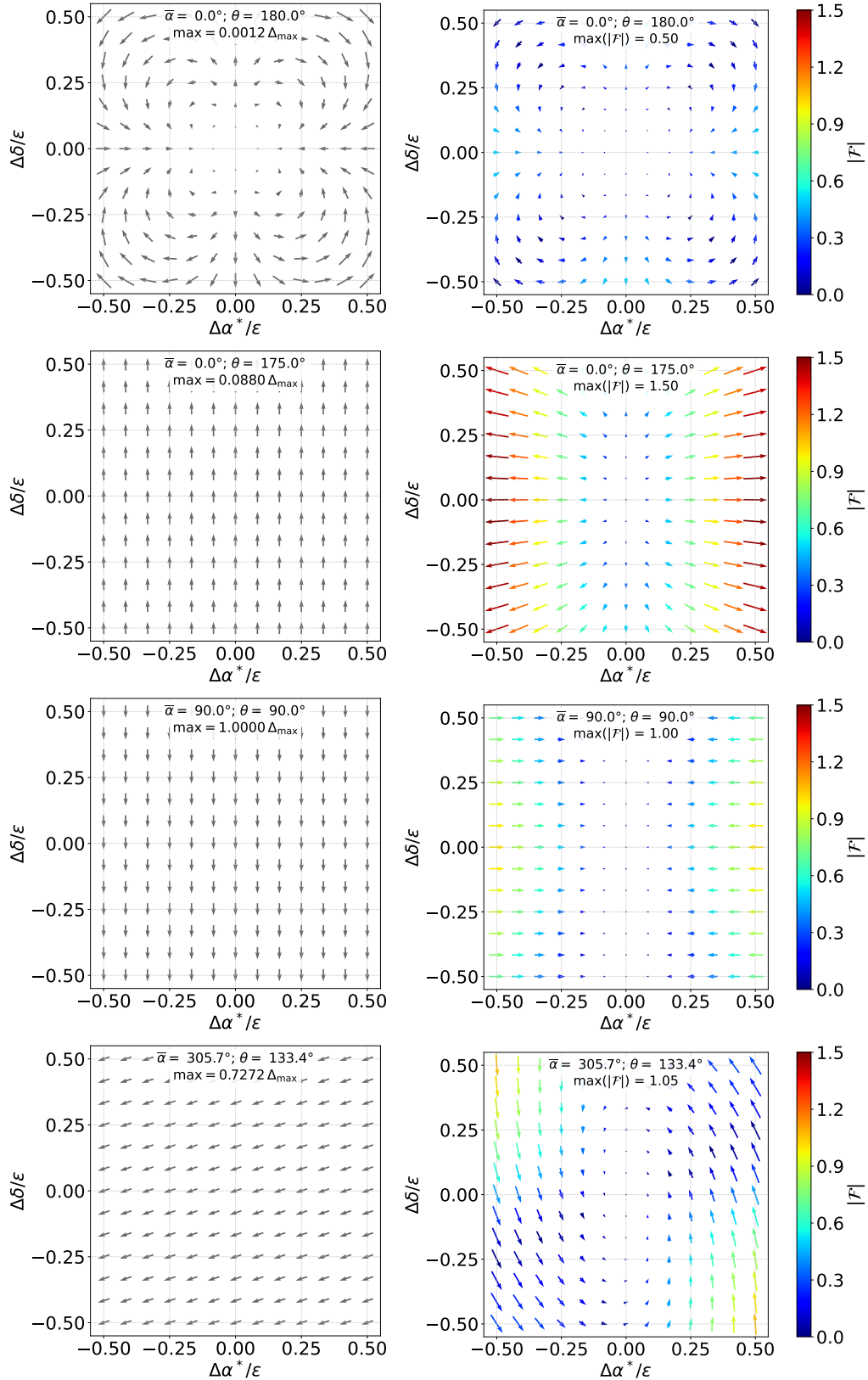


Figure 3. Vector field visualizations of the absolute (left column) and differential (right column) astrometric GW effects within an $(\epsilon \times \epsilon)$ -sized FoV, for different positions on the sky relative to the GW propagation direction, as indicated at the top of each plot with the $(\bar{\alpha}, \theta)$ of the center. In the left column, the absolute GW effect is plotted, i.e., the displacement of every point due to the GW, normalized to the maximum shift Δ_{\max} for this GW. The gray arrows in this column are scaled independently to optimize visibility. At the top of each plot in this column, the maximum overall displacement, normalized to the maximal GW effect $\sqrt{(\delta\alpha^*)^2 + (\delta\delta)^2}$ in terms of Δ_{\max} , is indicated. The right column shows the differential GW effect referenced to the central point with $(\Delta\alpha^*, \Delta\delta) = (0, 0)$. The arrow length in this column is determined by the difference between the displacement at the point and the displacement at the center. The colors in the plots indicate the normalized angular change $|\mathcal{F}|$ with respect to the center point. The asterisk in $\Delta\alpha^*$ means that the difference in right ascension is a true arc, thus: $\Delta\alpha^* \equiv (\Delta\alpha) \cos \delta$. All plots have been created using the gnomonic projection. It should be noted that all arrow lengths are significantly exaggerated for illustrative purposes compared to typical GW signal magnitudes. A detailed discussion can be found in the text.

even over extended periods of time. This is the fundamental difference from global astrometry, such as *Gaia* [25] and *Gaia*NIR [26], where angular changes of pairs of stars separated by a large angle (e.g. the basic angle of 106.5° for *Gaia*) are observed. With such a large angular distance, a significant number of observations will contain a signal equivalent to the full magnitude of the absolute GW effect. Since global scanning astrometry like *Gaia* observes the whole sky multiple times, billions of observed objects and long duration of observations can be used, increasing the chances of detecting GWs.

ACKNOWLEDGMENTS

This work is financially supported by ESA grant 4000115263/15/NL/IB and the German Aerospace Agency (Deutsches Zentrum für Luft- und Raumfahrt e.V., DLR) under grants 50QG1402 and 50QG2202. We also thank the Center for Information Services and High Performance (ZIH) at TU Dresden for providing a considerable amount of computing time. One of the authors (S.Z.) acknowledges support of the German Research Foundation (Deutsche Forschungsgemeinschaft DFG) under grant number 447922800. The work contributed by L. Lindegren is supported by the Swedish National Space Agency. Finally, we also thank Dr. Enrico Gerlach for fruitful discussions about some problems discussed here.

DATA AVAILABILITY

The data that support the findings of Figs. 2 to 1, and Table I this article are openly available [27]. Other data are available from the authors upon reasonable request.

APPENDICES

Appendix A: Theoretical derivation of astrometric effects of a plane gravitational wave

In this Appendix, we provide a detailed theoretical discussion of the calculation of astrometric observable effects that are influenced by a gravitational wave. As before, we restrict the discussion to a plane monochromatic GW. In line with the notation used in Sect. II we denote the observable angle between two incident light rays originating from sources A and B in the presence of a GW as ψ_{AB}^{gw} . As is well known, this angle can be computed in different ways within the framework of linear gravity used here. A standard way is to project the null tangent vectors to the two light rays at the moment of observation into the rest-space of the observer in a suitably chosen coordinate system, with a well-defined space-time metric tensor employed to compute ψ_{AB}^{gw} by means of scalar products (see below).

In the tetrad formalism, these null tangent vectors are projected onto the co-moving tetrad system of the observer by means of the full metric tensor, and ψ_{AB}^{gw} is computed with the Euclidean metric in the observer's 3-space.

If the observer has the possibility to operationally realize such co-moving tetrads (e.g., by means of some mechanical structure), then astrometric observables involving a single light ray can be defined (e.g., [20]).

In addition, our derivations of the so-called *source terms* as well as our treatment of a moving observer might be of general interest. To the best of our knowledge, these issues have not been considered elsewhere in such detail.

Below we use fairly standard notations. Greek indices α, β, \dots running from 0 to 3 indicate all four space-time components of the corresponding variable. Latin indices a, b, \dots run from 1 to 3 and refer to three spatial components of the corresponding variable. The Kronecker delta is $\delta^{ij} = \text{diag}(1, 1, 1)$. We use Einstein's summation convention for both types of indices, independent of the position of repeated indices: e.g., $x^i x^i \equiv (x^1)^2 + (x^2)^2 + (x^3)^2$. A dot over any quantity designates the total derivative with respect to the coordinate time of the corresponding reference system: e.g., $\dot{a} = \frac{da}{dt}$. The 3-dimensional coordinate quantities ("3-vectors") referred to the spatial axes of the corresponding reference system are set in boldface: $\mathbf{a} = a^i$. The absolute value (Euclidean norm) of a "3-vector" \mathbf{a} is denoted as $|\mathbf{a}|$ and can be computed as $|\mathbf{a}| = (a^1 a^1 + a^2 a^2 + a^3 a^3)^{1/2}$. The scalar product of any two "3-vectors" \mathbf{a} and \mathbf{b} with respect to the Euclidean metric δ_{ij} is denoted by $\mathbf{a} \cdot \mathbf{b}$ and can be computed as $\mathbf{a} \cdot \mathbf{b} = \delta_{ij} a^i b^j = a^i b^i$. The vector product of any two "3-vectors" \mathbf{a} and \mathbf{b} is denoted by $\mathbf{a} \times \mathbf{b}$ and can be computed as $(\mathbf{a} \times \mathbf{b})^i = \varepsilon_{ijk} a^j b^k$, where $\varepsilon_{ijk} = (i-j)(j-k)(k-i)/2$ is the fully antisymmetric Levi-Civita symbol.

1. The metric tensor

The calculations will be performed in the framework of linear gravity, using harmonic coordinates $x^\mu = (ct, x^1, x^2, x^3)$, where, in the space-time region of interest, the metric tensor is of the form

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}, \quad \text{with} \quad |h_{\alpha\beta}| \ll 1, \quad (\text{A1})$$

$\eta_{\alpha\beta} = \text{diag}(-1, +1, +1, +1)$ and terms of order $|h_{\alpha\beta}|^2$ will be neglected. In (harmonic) TT-coordinates, where $h_{00} = h_{0i} = 0$, h_{ij} is assumed to be of the form [19, 20]:

$$h_{ij}(t, \mathbf{x}) = C_{ij} \cos \Phi + S_{ij} \sin \Phi, \quad (\text{A2})$$

with the phase

$$\Phi = \frac{2\pi\nu}{c} (ct - \mathbf{p} \cdot \mathbf{x}), \quad (\text{A3})$$

where ν denotes the frequency of the GW and \mathbf{p} the Euclidean unit vector ($p^i p^i = 1$) in the GW's propagation

direction. The tensorial coefficients in (A2) are given by $C_{ij} = p_{ij}^+ h_c^+ + p_{ij}^\times h_c^\times$ and $S_{ij} = p_{ij}^+ h_s^+ + p_{ij}^\times h_s^\times$, where h_c^+ , h_s^+ , h_c^\times , h_s^\times are four independent strain parameters. The matrices p_{ij}^+ and p_{ij}^\times can be written in the form

$$p_{ij}^+ = (\mathbf{P} \mathbf{e}^+ \mathbf{P}^T)_{ij} \quad \text{and} \quad p_{ij}^\times = (\mathbf{P} \mathbf{e}^\times \mathbf{P}^T)_{ij} \quad (\text{A4})$$

with

$$\mathbf{e}_{ij}^+ = \begin{pmatrix} +1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{e}_{ij}^\times = \begin{pmatrix} 0 & +1 & 0 \\ +1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (\text{A5})$$

Here \mathbf{P} is the rotational matrix between the reference system in which the gravitational wave propagates in the z-direction and the coordinate system in which the direction of gravitational wave propagation is \mathbf{p} [19, 20]. The + and \times parts of h_{ij} correspond to the two polarization modes of the GW.

2. The geodesic equation and the null condition

In the following, we will first consider a single light ray emitted at the event (t_0, \mathbf{x}_0) by some light source (star).

The geodesic equation and the null condition ($ds^2 = 0$) for light rays in linear gravity have been given in the literature, e.g., [28–31]. In TT-gauge, using coordinate time t as parameter, they take the form

$$\frac{\ddot{x}^i(t)}{c^2} = -h_{ij,0} \mu^j + \frac{1}{2} h_{jk,i} \mu^j \mu^k - h_{ij,k} \mu^j \mu^k + \frac{1}{2} h_{jk,0} \mu^i \mu^j \mu^k, \quad (\text{A6})$$

$$\frac{|\dot{\mathbf{x}}(t)|}{c} = 1 - \frac{1}{2} h_{ij} \mu^i \mu^j, \quad (\text{A7})$$

where the dot indicates the time derivative, a comma indicates a partial derivative ($f_{,i} \equiv \partial f / \partial x^i$ and $f_{,0} \equiv c^{-1} \partial f / \partial t$) and

$$\boldsymbol{\mu} = \frac{\dot{\mathbf{x}}(t)}{|\dot{\mathbf{x}}(t)|} \Big|_{t=t_0} \quad (\text{A8})$$

is an Euclidean unit vector ($\mu^i \mu^i = 1$) that points in the spatial coordinate direction of the light ray at the moment of emission.

The solution of the homogeneous equation, $\ddot{x}^i(t) = 0$, is given by the ‘unperturbed light ray’ (e.g., Eq. (C24) in [31]),

$$\mathbf{x}_N(t) = \mathbf{x}_0 + c(t - t_0) \boldsymbol{\mu}. \quad (\text{A9})$$

Note, that the Euclidean ‘tangent vector’ $\boldsymbol{\mu}$ is a free parameter so far. It will be chosen later so that the perturbed light ray goes through the event of observation (t_1, \mathbf{x}_1) . Formally, this choice then shows first-order terms explicitly

a. The first integration of geodesic equation

The function $\dot{x}^i(t)$ is obtained by integrating the geodesic equation (A6) over the time coordinate from t_0 to $t > t_0$ along the unperturbed light ray, i.e., by writing $\mathbf{x} = \mathbf{x}_N(t)$ in (A2) and in particular in the phase Φ , which takes the form

$$\begin{aligned} \Phi(t) &= \Phi_N(t) = 2\pi\nu (t - c^{-1} \mathbf{p} \cdot \mathbf{x}_N(t)) \\ &= \Phi_N(t_0) + 2\pi\nu (1 - \mathbf{p} \cdot \boldsymbol{\mu}) (t - t_0). \end{aligned} \quad (\text{A10})$$

In the special case where $\boldsymbol{\mu} = \mathbf{p}$, the phase $\Phi(t)$ is constant along the light ray, and the gravitational wave, due to its transversal character, does not influence the propagation of the light-ray. In the following, we will assume that $\boldsymbol{\mu} \neq \mathbf{p}$, but the limit $\boldsymbol{\mu} \rightarrow \mathbf{p}$ is discussed after Eq. (A18) below. The integrands are then pure functions of t and one gets

$$\frac{\dot{x}^i(t)}{c} = \frac{\dot{x}^i(t_0)}{c} + \frac{\Delta \dot{x}^i(t, t_0)}{c}, \quad (\text{A11})$$

where

$$\frac{\dot{x}^i(t_0)}{c} = \mu^i - \frac{1}{2} h_{jk}(t_0, \mathbf{x}_0) \mu^j \mu^k \mu^i \quad (\text{A12})$$

that follows from the null condition (A7) and

$$\begin{aligned} \frac{\Delta \dot{x}^i(t, t_0)}{c} &= c \int_{t_0}^t dt \frac{\ddot{x}^i(t)}{c^2} \\ &= \frac{\Delta \dot{x}^i(t)}{c} - \frac{\Delta \dot{x}^i(t_0)}{c} \end{aligned} \quad (\text{A13})$$

and (here and below $\mathbf{x}_N = \mathbf{x}_N(t)$ with the corresponding time argument)

$$\frac{\Delta \dot{x}^i(t)}{c} = +\frac{1}{2} h_{jk}(t, \mathbf{x}_N) \mu^j \mu^k \frac{\mu^i - p^i}{1 - \mathbf{p} \cdot \boldsymbol{\mu}} - h_{ij}(t, \mathbf{x}_N) \mu^j. \quad (\text{A14})$$

b. The second integration of geodesic equation

Considering the result (A11) and the initial condition $x^i(t_0) = x_0^i$ the second integration leads to

$$\begin{aligned} x^i(t) &= x_0^i + c(t - t_0) \mu^i \\ &\quad - \frac{1}{2} c(t - t_0) h_{jk}(t_0, \mathbf{x}_0) \mu^j \mu^k \mu^i + \Delta x^i(t, t_0), \end{aligned} \quad (\text{A15})$$

where

$$\begin{aligned} \Delta x^i(t, t_0) &= c \int_{t_0}^t dt \left(\frac{\Delta \dot{x}^i(t)}{c} - \frac{\Delta \dot{x}^i(t_0)}{c} \right) \\ &= \Delta x^i(t) - \Delta x^i(t_0) - c(t - t_0) \frac{\Delta \dot{x}^i(t_0)}{c}. \end{aligned} \quad (\text{A16})$$

The term $\Delta x^i(t)$ can formally be written in the form

$$\Delta x^i(t) = -\frac{\lambda}{2\pi} \frac{1}{1 - \mathbf{p} \cdot \boldsymbol{\mu}} \times \left[\frac{1}{2} \bar{h}_{jk}(t, \mathbf{x}_N) \mu^j \mu^k \frac{\mu^i - p^i}{1 - \mathbf{p} \cdot \boldsymbol{\mu}} - \bar{h}_{ij}(t, \mathbf{x}_N) \mu^j \right], \quad (\text{A17})$$

$$\bar{h}_{ij}(t, \mathbf{x}_N) = -C_{ij} \sin \Phi_N + S_{ij} \cos \Phi_N. \quad (\text{A18})$$

Here $\lambda = c/\nu$ is the wavelength of the GW. We note that, by using $C_{ij}p^j = S_{ij}p^j = 0$, one can see that $\Delta \dot{x}^i(t)$ for any t in Eq. (A14) tends to zero in the case $\boldsymbol{\mu} \rightarrow \mathbf{p}$ even though the denominator with $1 - \mathbf{p} \cdot \boldsymbol{\mu}$ itself tends to zero. The situation is trickier for $\Delta x^i(t)$ in Eq. (A17) which diverges for $\boldsymbol{\mu} \rightarrow \mathbf{p}$. However, as one can see from Eq. (A10) the phase difference $\Phi_N(t) - \Phi_N(t_0)$ for a given $2\pi\nu(t - t_0)$ tends to zero when $\boldsymbol{\mu} \rightarrow \mathbf{p}$. This allows one to see that $\Delta x^i(t) - \Delta x^i(t_0)$ in Eq. (A16) tends to zero when $\boldsymbol{\mu}$ tends to \mathbf{p} .

3. The boundary value problem

In the previous Sections, we have solved the initial value problem for a single light ray that is emitted from \mathbf{x}_0 , at coordinate time t_0 , in a direction given by the Euclidean 3-vector $\boldsymbol{\mu}$ from (A8). Now we imagine an observer that observes this light ray at coordinate position \mathbf{x}_1 at some coordinate time t_1 :

$$\mathbf{x}_1 = \mathbf{x}(t) \Big|_{t=t_1}. \quad (\text{A19})$$

We note that while \mathbf{x}_1 can be chosen arbitrarily and, given \mathbf{x}_0 and t_0 , defines $\boldsymbol{\mu}$, the moment of time t_1 is itself defined by \mathbf{x}_0 , \mathbf{x}_1 , $\boldsymbol{\mu}$, t_0 and the metric tensor.

Inserting (A19) into the equation for the light trajectory (A15), after some rewriting, leads to the relation

$$\boldsymbol{\mu} = \mathbf{k} - \frac{1}{R} \mathbf{k} \times \left([\Delta \mathbf{x}(t_1) - \Delta \mathbf{x}(t_0)] \times \mathbf{k} \right) + \mathbf{k} \times \left(\frac{\Delta \dot{\mathbf{x}}(t_0)}{c} \times \mathbf{k} \right), \quad (\text{A20})$$

where

$$\mathbf{k} = \frac{\mathbf{x}_1 - \mathbf{x}_0}{|\mathbf{x}_1 - \mathbf{x}_0|} \quad (\text{A21})$$

is the Euclidean spatial unit vector ($k^i k^i = 1$) that points from the emission point to the point of observation, and $R = |\mathbf{x}_1 - \mathbf{x}_0|$ is the Euclidean spatial coordinate distance between the light source and the observer.

By inserting (A20) into (A11) one obtains

$$\frac{\dot{\mathbf{x}}(t_1)}{c} = \mathbf{k} + \frac{\Delta \dot{\mathbf{x}}(t_1)}{c} - \frac{1}{R} \mathbf{k} \times \left([\Delta \mathbf{x}(t_1) - \Delta \mathbf{x}(t_0)] \times \mathbf{k} \right). \quad (\text{A22})$$

The part of the $1/R$ -term with $\Delta \mathbf{x}(t_0)$ depends on the GW-field at the source and is often called “source

term”. Effectively, the terms with $\Delta \mathbf{x}$ in Eq. (A22) are proportional to λ/R . Considering GWs with periods $P_{\text{GW}} = 1/\nu < 30$ years (see [20] for a discussion) we have $\lambda < 9.2$ pc. Except for observations of the relatively small number of nearby stars within a distance of e.g. 100 pc, we have $\lambda/R \lesssim 0.1$.

We see two lines of argument for why the λ/R terms can be neglected. First, normally we are interested in the case where a large number of astrometric sources are observed. In this case, the terms proportional to λ/R are not correlated with each other for different stars. Moreover, for most of the stars these terms cannot be computed with sufficient accuracy, because the distances are not known so precisely, even in the Gaia era. Therefore, as is also often argued in the literature, those terms can be considered as an additional stochastic noise in the data. The second line of argument is related to the magnitude of the λ/R terms for individual observations. For $\lambda/R \lesssim 0.1$ one can see that the term $\Delta \dot{\mathbf{x}}(t)$ dominates the signal as soon as the overall GW effect is comparable to Δ_{max} introduced in [20]. The λ/R terms can become comparable to, or even larger than, the effect of $\Delta \dot{\mathbf{x}}(t)$ only in cases where the overall GW effect is negligibly small. However, if astrometric observations of nearby stars such as the α Centauri system are used, the λ/R term may be of interest.

In the following we will neglect this term and continue with the following first-order expression:

$$\frac{\dot{\mathbf{x}}(t_1)}{c} = \mathbf{k} + \frac{\Delta \dot{\mathbf{x}}(t_1)}{c}, \quad (\text{A23})$$

where

$$\frac{\Delta \dot{x}^i(t_1)}{c} = \frac{1}{2} h_{jk}(t_1, \mathbf{x}_1) k^j k^k \frac{k^i - p^i}{1 - \mathbf{p} \cdot \mathbf{k}} - h_{ij}(t_1, \mathbf{x}_1) k^j. \quad (\text{A24})$$

4. The worldline of an observer

In any realistic observational setup (like e.g., *Gaia* or *GaiaNIR*) the observer (satellite) will undergo a complex motion described by some ephemeris in a suitably chosen coordinate system. Because of aberration, the problem of a moving observer is by no means academic for the central problem discussed here. Accordingly, we consider an observer in TT coordinates with worldline $x_{\text{obs}}^\mu(\tau)$, where τ is the observer’s proper time related to the fundamental length element ds along their worldline. For the construction of astrometric observables for such an observer, one needs the observer’s (normalized) four-velocity

$$u^\mu = \frac{1}{c} \frac{dx_{\text{obs}}^\mu(\tau)}{d\tau} \quad (\text{A25})$$

with $g_{\mu\nu} u^\mu u^\nu = -1$. Effects of the observer’s motion will now be considered with \mathbf{v} being the observer’s TT

coordinate velocity. The normalized 4-velocity is then given by

$$u^\alpha = \gamma_h (+1, \beta^i), \quad (\text{A26})$$

where $\beta^i = v^i/c$, $v^i = dx_{\text{obs}}^i/dt$ is the coordinate velocity of observer, and

$$\gamma_h = (1 - \beta^2 - h_{ij}\beta^i\beta^j)^{-1/2}. \quad (\text{A27})$$

5. The observed angle between two incident light rays

The well-known formula for the angle $\psi_{\text{AB}}^{\text{gw}}$ between two incident light rays (A and B) as measured by an observer reads:

$$\cos \psi_{\text{AB}}^{\text{gw}} = g_{\alpha\beta} \frac{\bar{l}_A^\alpha}{|\bar{l}_A|} \frac{\bar{l}_B^\beta}{|\bar{l}_B|}, \quad (\text{A28})$$

where the quantities of the right-hand side refer to the event of observation (t_1, \mathbf{x}_1) . \bar{l}^α is the tangent null vector to the light trajectory A or B , projected into the rest-space of the observer and

$$|\bar{l}| = \left(g_{\mu\nu} \bar{l}^\mu \bar{l}^\nu \right)^{1/2}. \quad (\text{A29})$$

For our purposes it is sufficient to parametrize a light trajectory with coordinate time t . Then, the null tangent vector to a light ray takes the form

$$l^\alpha = \frac{1}{c} \frac{dx^\alpha}{dt} = \left(1, \frac{\dot{\mathbf{x}}(t)}{c} \right), \quad (\text{A30})$$

where $\dot{\mathbf{x}}(t)$ is given by (A23).

The projected tangent vector \bar{l}^α is then given by

$$\bar{l}^\alpha = P_\beta^\alpha l^\beta, \quad (\text{A31})$$

where

$$P_\beta^\alpha = \delta_\beta^\alpha + g_{\beta\gamma} u^\alpha u^\gamma \quad (\text{A32})$$

projects vectors into the rest-space of the observer orthogonal to his four velocity u^α . Then

$$\bar{l}^\alpha = l^\alpha - \mathcal{E} u^\alpha, \quad (\text{A33})$$

where

$$\mathcal{E} = -g_{\mu\nu} l^\mu u^\nu = |l^\mu u_\mu|. \quad (\text{A34})$$

The four-vector in (A33) is space-like and one finds for the norm

$$|\bar{l}| = \mathcal{E} = \gamma_h \left(1 - \mathbf{k} \cdot \boldsymbol{\beta} - \frac{1}{2} h_{jk} k^j k^k \frac{\boldsymbol{\beta} \cdot \mathbf{k} - \boldsymbol{\beta} \cdot \mathbf{p}}{1 - \mathbf{p} \cdot \mathbf{k}} \right). \quad (\text{A35})$$

We finally obtain ($\eta = 1/\mathcal{E}$)

$$\frac{\bar{l}^\alpha}{|\bar{l}|} = \left(\eta - \gamma_h, \eta \frac{\dot{\mathbf{x}}}{c} - \gamma_h \boldsymbol{\beta} \right). \quad (\text{A36})$$

With this expression, applied to the two light rays, A and B , and the scalar product taken with the space-time metric as shown in Eq. (A28) one obtains $\cos \psi_{\text{AB}}^{\text{gw}}$ as seen by the observer with coordinate velocity v^i and located at \mathbf{x}_1 at $t = t_1$. As we have neglected the source terms in the right-hand side of (A28) is completely determined by the event of observation.

To compare our formulation with those existing already in the literature we derive here the explicit formula for the case of an observer at rest in our coordinate system, where $\boldsymbol{\beta} = 0$ and $\gamma_h = 1$, so that the observer's four-velocity reads $u^\mu = (1, 0, 0, 0)$. For this special case we get $\mathcal{E} = 1$ and

$$\frac{\bar{l}^i}{|\bar{l}|} = k^i + \frac{1}{2} h_{jk} k^j k^k \frac{k^i - p^i}{1 - \mathbf{p} \cdot \mathbf{k}} - h_{ij} k^j. \quad (\text{A37})$$

Note, that the time-component vanishes here ($\bar{l}^0 = 0$). By using this relation in Eq. (A28) for our two light rays A and B , we get our final result for the observed angle between these two light rays in the form:

$$\begin{aligned} \cos \psi_{\text{AB}}^{\text{gw}} &= \mathbf{k}_A \cdot \mathbf{k}_B \\ &+ \frac{1}{2} h_{jk} \left[k_A^j k_A^k \frac{\mathbf{k}_A \cdot \mathbf{k}_B - \mathbf{k}_B \cdot \mathbf{p}}{1 - \mathbf{k}_A \cdot \mathbf{p}} + k_B^j k_B^k \frac{\mathbf{k}_A \cdot \mathbf{k}_B - \mathbf{k}_A \cdot \mathbf{p}}{1 - \mathbf{k}_B \cdot \mathbf{p}} \right] \\ &- h_{ij} k_A^i k_B^j, \end{aligned} \quad (\text{A38})$$

where the right-hand side refers to the event of observation. From (A38) one sees that in the limit when two sources A and B get closer and closer to each other one has

$$\lim_{A \rightarrow B} \cos \psi_{\text{AB}}^{\text{gw}} = \lim_{A \rightarrow B} \mathbf{k}_A \cdot \mathbf{k}_B = 1 \quad (\text{A39})$$

as one can expect from a continuous vector field of the astrometric GW signal [12, 19, 20]. One can see that the same limit holds true also for a moving observer.

6. The observed angle $\psi_{\text{AB}}^{\text{gw}}$ in the tetrad formalism

The angle $\psi_{\text{AB}}^{\text{gw}}$ in Eq. (A28) represents an observable that, theoretically, is described by a scalar, i.e., a coordinate independent quantity. The quantities on the right-hand side of (A28) have been expressed in terms of tensor components with respect to the TT coordinates at the event of observation.

In the context of our problem, local proper coordinates are often employed that have a direct physical meaning. Such coordinates are usually constructed as tetrad-induced quantities, e.g., [32–34]. These tetrads form a set of four orthonormal basis vectors, $e_{(\alpha)}^\mu$, that act as tangent vectors to the local coordinate lines. The indices (α) label the tetrad components, while the indices μ are tensor indices. These tetrads are defined along the worldline $x_{\text{obs}}^\mu(\tau)$ of the observer (defining the origin of local coordinates), and obey the orthonormality condition (e.g., [28, 32, 35] and more specifically [36, 37]):

$$g_{\mu\nu} e_{(\alpha)}^\mu e_{(\beta)}^\nu = \eta_{\alpha\beta}. \quad (\text{A40})$$

The zeroth time-like tetrad vector $e_{(0)}^\mu$ is chosen as normalized 4-velocity of the observer

$$e_{(0)}^\mu = u^\mu, \quad (\text{A41})$$

so that the vectors $e_{(i)}^\mu, i = 1, 2, 3$ span the 3-space of the observer at a certain event on the worldline $x_{\text{obs}}^\mu(\tau)$ of the observer.

For an arbitrary vector A^μ on the observer's worldline, we can then write

$$A^\mu = A^{(\alpha)} e_{(\alpha)}^\mu \quad (\text{A42})$$

and

$$g_{\mu\nu} A^\mu e_{(\beta)}^\nu = \eta_{\alpha\beta} A^{(\alpha)}. \quad (\text{A43})$$

Considering two such vectors A^μ and B^ν one gets

$$g_{\mu\nu} A^\mu B^\nu = \eta_{\alpha\beta} A^{(\alpha)} B^{(\beta)}. \quad (\text{A44})$$

We may assume that in high-precision astrometric satellite missions the “observer” has the possibility to operationally realize such space-like basic vectors (by means of “quasi-rigid” mechanical structures). The tetrad formalism can be used to compute the observables in astrometry in several different ways (see an overview in [37]). Here, we prefer the following approach. For a tangent vector to some light ray projected into the rest space of an observer, \bar{l}^α as in Eq. (A31), its tetrad components are $\bar{l}^{(0)} = 0$ and $\bar{l}^{(i)} = l^{(i)}$ (since $l^{(i)}$ lies in the observer's instantaneous 3-space). The spatial components $l^{(i)}$ can be considered as components of an Euclidean 3-vector \mathbf{l} with Euclidean norm $|\mathbf{l}| = (l^{(i)} l^{(i)})^{1/2}$. We note also that $|\mathbf{l}| = |\bar{\mathbf{l}}|$, where $|\bar{\mathbf{l}}|$ is defined by Eqs. (A29) and (A35). Then the Cartesian components $l^{(i)}$ of a (null) tangent vector l^α to some incident light ray are observable (coordinate-independent quantities). The corresponding (negative) Euclidean unit vector

$$s^i = -\frac{l^{(i)}}{|\mathbf{l}|} \quad (\text{A45})$$

then has observable components that can be formulated as directional angles (α, δ) of a single light ray towards the astrometric source as seen by the observer, i.e., $\mathbf{s} = (\cos \alpha \cos \delta, \sin \alpha \cos \delta, \sin \delta)$ (e.g., [20]). The observed angle between two incoming light rays (or equivalently between two observed directions \mathbf{s}_A and \mathbf{s}_B) can then be computed as

$$\cos \psi_{AB}^{\text{gw}} = \delta_{ij} \frac{l_A^{(i)} l_B^{(j)}}{|\mathbf{l}_A| |\mathbf{l}_B|}, \quad (\text{A46})$$

where the right-hand side has to be taken at the event of observation. Eq. (A46) is equivalent to Eq. (A28), but is written using the tetrad components of the corresponding vectors.

For an observer at rest in TT coordinates, the tetrad vectors are given by [28, 37]

$$e_{(0)}^0 = 1, \quad e_{(0)}^i = 0, \quad e_{(i)}^0 = 0, \quad e_{(i)}^j = \delta_{ij} - \frac{1}{2} h_{ij} \quad (\text{A47})$$

and one finds the components of s^i in the form

$$-s^i = \frac{l^{(i)}}{|\mathbf{l}|} = k^i + \frac{1}{2} h_{jk}(t_1, \mathbf{x}_1) k^j k^k \frac{k^i - p^i}{1 - \mathbf{p} \cdot \mathbf{k}} - \frac{1}{2} h_{ij}(t_1, \mathbf{x}_1) k^j. \quad (\text{A48})$$

This result is in agreement with Eq. (58) in [12] as well as Eq. (A1) in [20]. Our final result (A38) for $\cos \psi_{AB}^{\text{gw}}$ can then be recovered by substituting Eq. (A48) into Eq. (A46).

Eq. (A46) is valid for a moving observer as well. However, the tetrad for a moving observer is more complicated and related to (A47) by a Lorentz boost, as discussed in [37].

We note that in [21, 22] obviously the tetrad components of (A48) were used in expression (A28) instead of (A46). This fatal flaw eventually leads to their incorrect final result for $\cos \psi_{AB}^{\text{gw}}$. Another discussion of this flaw can be found in [23].

-
- [1] F. Malbet, C. Boehm, A. Krone-Martins, *et al.*, Faint objects in motion: the new frontier of high precision astrometry, *Experimental Astronomy* **51**, 845 (2021), arXiv:2111.08709 [astro-ph.IM].
 - [2] F. Malbet, L. Labadie, A. Sozzetti, *et al.*, Theia: science cases and mission profiles for high precision astrometry in the future, in *Space Telescopes and Instrumentation 2022: Optical, Infrared, and Millimeter Wave*, Society of Photo-Optical Instrumentation Engineers (SPIE) Conference Series, Vol. 12180, edited by L. E. Coyle, S. Matsuura, and M. D. Perrin (2022) p. 121801F, arXiv:2207.12540 [astro-ph.IM].
 - [3] D. Kawata, H. Kawahara, N. Gouda, *et al.*, JASMINE: Near-infrared astrometry and time-series photometry sci-

- ence, Publications of the Astronomical Society of Japan **76**, 386 (2024), arXiv:2307.05666 [astro-ph.IM].
- [4] D. Nardiello, L. R. Bedin, M. Griggio, *et al.*, Photometry and astrometry with JWST - III. A NIRC2-Gaia DR3 analysis of the open cluster NGC 2506, *Monthly Notices of the Royal Astronomical Society* **525**, 2585 (2023), arXiv:2308.10575 [astro-ph.SR].
- [5] M. Libralato, L. R. Bedin, M. Griggio, *et al.*, Euclid: High-precision imaging astrometry and photometry from Early Release Observations: I. Internal kinematics of NGC6397 by combining Euclid and Gaia data, *Astronomy & Astrophysics* **692**, A96 (2024), arXiv:2411.02487 [astro-ph.SR].
- [6] L. Lindegren, S. A. Klioner, J. Hernández, *et al.*, Gaia

- Early Data Release 3. The astrometric solution, *Astronomy & Astrophysics* **649**, A2 (2021), arXiv:2012.03380 [astro-ph.IM].
- [7] Gaia Collaboration, S. A. Klioner, L. Lindegren, F. Mignard, *et al.*, Gaia Early Data Release 3. The celestial reference frame (Gaia-CRF3), *Astronomy & Astrophysics* **667**, A148 (2022), arXiv:2204.12574 [astro-ph.IM].
- [8] Gaia Collaboration, S. A. Klioner, F. Mignard, L. Lindegren, *et al.*, Gaia Early Data Release 3. Acceleration of the Solar System from Gaia astrometry, *Astronomy & Astrophysics* **649**, A9 (2021), arXiv:2012.02036 [astro-ph.GA].
- [9] B. P. Abbott, R. Abbott, T. D. Abbott, LIGO Scientific Collaboration, and Virgo Collaboration, Observation of Gravitational Waves from a Binary Black Hole Merger, *Phys. Rev. Lett.* **116**, 061102 (2016), arXiv:1602.03837 [gr-qc].
- [10] The LIGO Scientific Collaboration, the Virgo Collaboration, the KAGRA Collaboration, A. G. Abac, *et al.*, GWTC-4.0: An Introduction to Version 4.0 of the Gravitational-Wave Transient Catalog, arXiv e-prints, arXiv:2508.18080 (2025), arXiv:2508.18080 [gr-qc].
- [11] K. Grunthal, R. S. Nathan, E. Thrane, *et al.*, The MeerKAT Pulsar Timing Array: Maps of the gravitational wave sky with the 4.5-yr data release, *Monthly Notices of the Royal Astronomical Society* **536**, 1501 (2025), arXiv:2412.01214 [astro-ph.HE].
- [12] L. G. Book and É. É. Flanagan, Astrometric effects of a stochastic gravitational wave background, *Phys. Rev. D* **83**, 024024 (2011), arXiv:1009.4192 [astro-ph.CO].
- [13] C. J. Moore, D. P. Mihaylov, A. Lasenby, and G. Gilmore, Astrometric Search Method for Individually Resolvable Gravitational Wave Sources with Gaia, *Phys. Rev. Lett.* **119**, 261102 (2017), arXiv:1707.06239 [astro-ph.IM].
- [14] D. P. Mihaylov, C. J. Moore, J. R. Gair, A. Lasenby, and G. Gilmore, Astrometric effects of gravitational wave backgrounds with non-Einsteinian polarizations, *Phys. Rev. D* **97**, 124058 (2018), arXiv:1804.00660 [gr-qc].
- [15] L. O’Beirne and N. J. Cornish, Constraining the polarization content of gravitational waves with astrometry, *Phys. Rev. D* **98**, 024020 (2018), arXiv:1804.03146 [gr-qc].
- [16] V. B. Braginsky, N. S. Kardashev, A. G. Polnarev, and I. D. Novikov, Propagation of electromagnetic radiation in a random field of gravitational waves and space radio interferometry., *Nuovo Cimento B Serie* **105**, 1141 (1990).
- [17] T. Pyne, C. R. Gwinn, M. Birkinshaw, T. M. Eubanks, and D. N. Matsakis, Gravitational Radiation and Very Long Baseline Interferometry, *Astrophys. J.* **465**, 566 (1996), arXiv:astro-ph/9507030 [astro-ph].
- [18] C. R. Gwinn, T. M. Eubanks, T. Pyne, M. Birkinshaw, and D. N. Matsakis, Quasar Proper Motions and Low-Frequency Gravitational Waves, *Astrophys. J.* **485**, 87 (1997), arXiv:astro-ph/9610086 [astro-ph].
- [19] S. A. Klioner, Gaia-like astrometry and gravitational waves, *Classical and Quantum Gravity* **35**, 045005 (2018), arXiv:1710.11474 [astro-ph.HE].
- [20] R. Geyer, S. A. Klioner, L. Lindegren, and U. Lammers, Influence of a continuous plane gravitational wave on Gaia-like astrometry, *Astronomy & Astrophysics* **695**, A172 (2025), arXiv:2412.15770 [astro-ph.IM].
- [21] M. Crosta, M. G. Lattanzi, C. Le Poncin-Lafitte, *et al.*, Pinpointing gravitational waves via astrometric gravitational wave antennas, *Scientific Reports* **14**, 5074 (2024).
- [22] F. Santucci, M. Crosta, and M. G. Lattanzi, Pattern functions of the Astrometric Gravitational Wave Antenna, *Scientific Reports* **15**, 32908 (2025).
- [23] M. Vaglio, M. Falxa, G. Mentasti, *et al.*, Searching for Gravitational Waves with Gaia and its Cross-Correlation with PTA: Absolute vs Relative Astrometry, arXiv e-prints, arXiv:2507.18593 (2025).
- [24] K. M. Górski, E. Hivon, A. J. Banday, B. D. Wandelt, F. K. Hansen, M. Reinecke, and M. Bartelmann, HEALPix: A Framework for High-Resolution Discretization and Fast Analysis of Data Distributed on the Sphere, *Astrophysical Journal* **622**, 759 (2005).
- [25] Gaia Collaboration, T. Prusti, J. H. J. de Bruijne, A. G. A. Brown, *et al.*, The Gaia mission, *Astronomy & Astrophysics* **595**, A1 (2016), arXiv:1609.04153 [astro-ph.IM].
- [26] D. Hobbs, A. Brown, E. Høg, *et al.*, All-sky visible and near infrared space astrometry, *Experimental Astronomy* **51**, 783 (2021), arXiv:1907.12535 [astro-ph.IM].
- [27] https://gaia.geo.tu-dresden.de/RGY_GW_Small_Field_Data/.
- [28] V. A. Brumberg, *Essential relativistic celestial mechanics*. (CRC Press, 1991).
- [29] S. M. Kopeikin, G. Schäfer, C. R. Gwinn, and T. M. Eubanks, Astrometric and timing effects of gravitational waves from localized sources, *Phys. Rev. D* **59**, 084023 (1999), arXiv:gr-qc/9811003 [gr-qc].
- [30] S. M. Kopeikin and G. Schäfer, Lorentz covariant theory of light propagation in gravitational fields of arbitrary-moving bodies, *Phys. Rev. D* **60**, 124002 (1999), arXiv:gr-qc/9902030 [gr-qc].
- [31] S. A. Klioner and M. Peip, Numerical simulations of the light propagation in the gravitational field of moving bodies, *Astronomy & Astrophysics* **410**, 1063 (2003), arXiv:astro-ph/0305204 [astro-ph].
- [32] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Macmillan Education, 1973).
- [33] M. H. Soffel, *Relativity in Astrometry, Celestial Mechanics and Geodesy* (Springer-Verlag, 1989).
- [34] T. Damour, M. Soffel, and C. Xu, General-relativistic celestial mechanics. I. Method and definition of reference systems, *Phys. Rev. D* **43**, 3273 (1991).
- [35] S. Kopeikin, M. Efroimsky, and G. Kaplan, *Relativistic Celestial Mechanics of the Solar System* (Wiley-VCH, 2011).
- [36] S. A. Klioner and S. M. Kopeikin, Microarcsecond Astrometry in Space: Relativistic Effects and Reduction of Observations, *The Astronomical Journal* **104**, 897 (1992).
- [37] S. A. Klioner, Physically adequate proper reference system of a test observer and relativistic description of the GAIA attitude, *Phys. Rev. D* **69**, 124001 (2004), arXiv:astro-ph/0311540 [astro-ph].