

# Light deflection in the gravitational field of a Solar System body with finite distance of source and observer for sub-micro-arcsecond astrometry

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The effect of deflection of a light signal which propagates through the gravitational field of a Solar System body at rest is considered in the case that the source and the observer are at a finite distance from the body. The observer is assumed to be located somewhere nearby the Earth, for instance at Lagrange point  $L_2$  of the Sun-Earth system. The gravitational fields in the exterior of the bodies are described by the full set of mass-multipoles and spin-multipoles of these bodies. So the gravitating bodies can be of arbitrary shape, inner structure and rotational motion. The unit tangent vector of the light trajectory at the position of the observer is determined in the 1PN and 1.5PN approximation of the post-Newtonian (PN) scheme. A simplified expression for the unit tangent vector is obtained, where all terms are neglected which together contribute less than 10 nano-arcseconds in light deflection for all astrometric configurations between source, body, and observer. It is shown that in case of an axi-symmetric body the unit tangent vector of the light ray as well as the light deflection are given in terms of Chebyshev polynomials. This fact allows for determining the effect of light deflection in the gravitational fields of the bodies up to any order of the mass-multipoles and spin-multipoles. It is shown that the total light deflection, that is the angle of light deflection where source and observer are located at infinite spatial distance to the body, represents an upper limit of the effect of light deflection. These investigations are aiming at astrometric measurements on the sub-micro-arcsecond level of accuracy.

## I. INTRODUCTION AND MOTIVATION

The primary aim of astrometric measurements concerns the determination of spatial positions of celestial objects, like stars, quasars, or Solar System objects. The determination of these spatial positions is based on the observed direction of light signals which are emitted by these celestial light sources. This means that the primary aim of high precision astrometry is directly related to the measurement of the unit tangent vector of these light trajectories at the position of the observer.

Today's precision of astrometric measurements has achieved at the micro-arcsecond ( $\mu\text{as}$ ) scale of accuracy. In particular, the astrometry mission *Gaia* of the European Space Agency (ESA) has measured the parallax for approximately 2 billion ( $2 \times 10^9$ ) stars of our galaxy, mapping them in three dimensions with an accuracy of up to  $10 \mu\text{as}$  [1, 2] for bright stellar sources. Meanwhile, there are several astrometry missions proposed to ESA, which are aiming at the sub-micro-arcsecond (sub- $\mu\text{as}$ ) level of accuracy. The most promising candidates of such mission proposals are the astrometry missions *GaiaNIR* [3] and *Theia* [4]. For instance, *GaiaNIR*, a successor mission of *Gaia*, aims to map about 50 billion ( $50 \times 10^9$ ) stars in near infrared, rather than the 2 billion stars mapped by *Gaia* in the optical band.

An accuracy on the sub- $\mu\text{as}$  scale requires a highly precise modeling of light trajectories in the curved space-time of the Solar System, in order to reduce astrometric measurements correctly and to trace the light rays back to the celestial light sources. But numerical integrations of the geodesic equation of light trajectories are unusable in view of the extensive calculations necessary for astrometry missions like *Gaia* [1, 2] or *GaiaNIR* [3]. In-

stead, analytical solutions for the light trajectories are very necessary in order to treat such a huge amount of astrometric observations. However, the higher the accuracy requirements are, the more complex the analytical expressions of light trajectories through the Solar System become.

Already in case of the first few multipoles, like mass-quadrupole and spin-dipole, the mathematical expressions for the light trajectories become rather cumbersome. One reason for this complexity is lying in the fact, that the solution of light trajectory and, therefore, of the unit tangent vector of light trajectory, contain many terms which are not significant on the sub-micro-arcsecond level. In view of the huge amount of stellar objects for missions like *GaiaNIR*, it is an inevitable assignment to simplify these analytical solutions by taking account of those terms only which are relevant for a given accuracy. Afterwards one has to find compact expressions for these remaining terms of the analytical solutions, which allow for a highly effective reduction of observational data.

The term *sub-micro-arcsecond level* refers to astrometric precisions in positional measurements of about  $0.1 \mu\text{as}$  for bright celestial light sources. Such precisions require a relativistic model of light propagation which is about 10 times more accurate than the end-of-mission accuracy. Accordingly, the given threshold of a sub- $\mu\text{as}$  model of light propagation is 0.01 micro-arcseconds. That means, in a theoretical model of light propagation on the *sub-micro-arcsecond level of precision* one may neglect all those terms which contribute in total less than 10 nano-arcsecond to the angle of light deflection.

In reality both the celestial light sources as well as the observer are located at finite distances from the mas-

sive Solar System bodies. In particular, the observer can be located somewhere nearby the Earth, for instance at Lagrange point  $L_2$  of the Sun-Earth system, like in the mission *Gaia* [1, 2] as well as in the missions *GaiaNIR* [3] and *Theia* [4] proposed to ESA. The light sources are assumed to be located completely arbitrarily: they can be far stellar light sources as well as near solar system objects. That means, the spatial distances between the light sources and the observer can be arbitrarily large and arbitrarily small. In this investigation the solution of the mentioned problems is presented for such general astrometric configurations. The manuscript is organized as follows:

In Section II the metric tensor of an isolated body in terms of the full set of its mass-multipoles and spin-multipoles, the geodesic equation of light propagation, and the light trajectory in the post-Newtonian scheme are given in the 1.5PN approximation of the Post-Newtonian (PN) scheme. The unit tangent vector of the light ray in the field of an arbitrary body at rest and at the observers position is presented in Section III. The fact that the individual terms of the unit tangent vector are of qualitatively different magnitude is considered in Section IV. The reduced unit tangent vector of a light ray is derived in Section V, which is a step in between in order to arrive at the simplified tangent vector of the light trajectories. In Section VI we consider two differential operations which are necessary in order to get the simplified unit tangent vector. The simplified unit tangent vector of a light ray is derived in Section VII. The mass-multipoles and spin-multipoles of an axi-symmetric body are given in Section VIII. The simplified tangent vector of the light trajectory in the gravitational field of an axi-symmetric body is presented in Section IX. In Sections IX and X the unit tangent vector in case of mass-monopole, spin-dipole, and mass-quadrupole are compared with results in the literature. The angle of light deflection, their upper limits, and numerical values are presented in Section XI. Finally, a brief view on 2PN and 3PN terms and their magnitude is given in Section XII. A Summary and Outlook can be found in Section XIII. The notations, conventions, and some details of the calculations are relegated to a set of several Appendices.

## II. THE LIGHT TRAJECTORY IN THE GRAVITATIONAL FIELD OF A BODY

### A. Metric tensor

For weak gravitational fields, the metric tensor can be separated as follows [5–11],

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}, \quad (1)$$

where  $\eta_{\alpha\beta} = \text{diag}(-1, +1, +1, +1)$  is the Minkowski metric. The perturbations  $h_{\alpha\beta}$  are assumed to be small,  $|h_{\alpha\beta}| \ll 1$ , and behave mathematically like symmetric

tensorial fields, which propagate in the flat background space-time [6, 8, 10]. The Minkowski metric in (1) implies that the flat background space-time is covered by Cartesian four-coordinates,  $x^\mu = (x^0, x^1, x^2, x^3)$ , where  $x^0 = ct$  with coordinate time  $t$ , while  $x^1, x^2, x^3$  are the spatial coordinates. In linearized gravity, the field equations of general relativity (GR) in harmonic gauge ( $\partial_\nu \bar{h}^{\mu\nu} = 0$ ) read [5–11]

$$\square \bar{h}_{\alpha\beta} = -\frac{16\pi G}{c^4} T_{\alpha\beta}, \quad (2)$$

where  $\bar{h}_{\alpha\beta} = h_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta}h$  with  $h = \eta^{\alpha\beta}h_{\alpha\beta}$ , the flat d'Alembert operator is  $\square = \partial_\mu \partial^\mu$  with the partial derivatives  $\partial_\mu = (\frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3})$ , and  $T_{\alpha\beta}$  is the energy-momentum tensor of matter. In linearized gravity  $h_{\alpha\beta} = \bar{h}_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta}\bar{h}$  with  $\bar{h} = \eta^{\alpha\beta}\bar{h}_{\alpha\beta}$ . Hence, from  $\bar{h}_{\alpha\beta}$  in (2) one gets  $h_{\alpha\beta}$  in (1) up to terms of order  $\mathcal{O}(G^2)$ .

If the source of matter is isolated (compact support and no incoming radiation), then the integration of the linearized field equations (2) leads to metric perturbations in the exterior of the body, which depend on a set of 10 source-multipoles [11–14]. A detailed proof of that has been presented in [15]. These multipoles are integrals over the energy-momentum tensor of the body. In the pioneering investigations [11–14] it has been worked out, that the energy-momentum conservation of matter reduces the number of independent source-multipoles and leads to metric perturbations, which depend finally on a set of only 6 source-multipoles:  $I_L, J_L, W_L, X_L, Y_L, Z_L$ .

In case of weak gravitational fields and slow motions of matter, the metric perturbation in (1) can further be series expanded in inverse powers of the speed of light, which is called the post-Newtonian (PN) scheme [6, 7, 9–11, 16]. In the 1.5PN approximation this series expansion reads

$$h_{\alpha\beta} = h_{\alpha\beta}^{(2)} + h_{\alpha\beta}^{(3)} + \mathcal{O}(c^{-4}), \quad (3)$$

where the perturbations are of the order  $h_{\alpha\beta}^{(2)} = \mathcal{O}(c^{-2})$  and  $h_{\alpha\beta}^{(3)} = \mathcal{O}(c^{-3})$ . These perturbations can be written as sum of individual perturbations caused by  $N$  individual Solar system bodies. That is why we can consider the influence of one body on light propagation. Finally one may sum over all these  $N$  bodies.

The harmonic gauge condition does not uniquely determine the metric tensor, but leaves freedom of a residual gauge. In the so-called canonical harmonic gauge, the metric perturbations (3) depend finally on a set of only 2 multipoles: mass-multipoles,  $M_L = I_L + \mathcal{O}(c^{-5})$  [17], which account for shape and inner structure of the body, and spin-multipoles,  $S_L = J_L + \mathcal{O}(c^{-5})$  [17], which account for rotational motions and inner currents of the body [11–14]. In the stationary case, the non-vanishing

metric perturbations are given by [9, 11–13, 18]

$$h_{00}^{(2)} = \frac{2}{c^2} \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \hat{M}_L \hat{\partial}_L \frac{1}{r}, \quad (4)$$

$$h_{0i}^{(3)}(\mathbf{x}) = \frac{4}{c^3} \sum_{l=1}^{\infty} \frac{(-1)^l}{(l+1)!} \epsilon_{iab} \hat{S}_{bL-1} \hat{\partial}_{aL-1} \frac{1}{r}, \quad (5)$$

and  $h_{ij}^{(2)} = \delta_{ij} h_{00}^{(2)}$  (in GR), where  $r = |\mathbf{x}|$  and

$$\hat{\partial}_L = \text{STF}_{i_1 \dots i_L} \frac{\partial}{\partial x^{i_1}} \dots \frac{\partial}{\partial x^{i_L}}, \quad (6)$$

where  $\text{STF}_{i_1 \dots i_L}$  denotes an operation, which makes a Cartesian tensor symmetric and trace-free (STF) with respect to its spatial indices  $L = i_1 \dots i_L$ . Further details of STF operations are summarized in Appendix B. The mass-multipoles and spin-multipoles for a stationary source of matter are given by [13]

$$\hat{M}_L = \int d^3x \hat{x}_L \frac{T^{00} + T^{kk}}{c^2} + \mathcal{O}(c^{-4}), \quad (7)$$

$$\hat{S}_L = \int d^3x \epsilon_{jk < i_l} \hat{x}_{L-1 >} x^j \frac{T^{0k}}{c} + \mathcal{O}(c^{-4}), \quad (8)$$

where the integration runs over the volume of the body. In what follows, the metric tensor of Eqs. (4) - (5), where the multipoles are given by Eqs. (7) and (8), will be called the metric of an arbitrary body. In (4) - (5) the orthogonal three-vectors of the principal axes  $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$  of the body are arbitrarily oriented with respect to the Cartesian coordinate system. In what follows it is assumed that the origin of the spatial coordinates is located at the center-of-mass of that body. This circumstance implies that the mass-dipole  $l = 1$  vanishes:  $\hat{M}_a = 0$  (cf. Eq. (8.14c) in [11]).

## B. The geodesic equation

In flat (Minkowskian) space-time, a light signal which is emitted at some initial time,  $t_0$ , into some three-direction,  $\boldsymbol{\sigma}$ , propagates along a straight trajectory,

$$\mathbf{x}_N = \mathbf{x}_0 + c(t - t_0) \boldsymbol{\sigma}. \quad (9)$$

One may define the impact vector of this unperturbed light ray,

$$\mathbf{d}_\sigma = \boldsymbol{\sigma} \times (\mathbf{x}_N \times \boldsymbol{\sigma}) = \boldsymbol{\sigma} \times (\mathbf{x}_0 \times \boldsymbol{\sigma}). \quad (10)$$

This impact vector is a three-vector which points from the center-of-mass of the body towards the unperturbed light ray at their closest distance, as elucidated in Figure 1. The absolute value of this impact vector is the impact parameter  $d_\sigma = |\mathbf{d}_\sigma|$ . In general relativity the propagation of light signals is governed by the geodesic

equation, which in 1.5PN approximation, i.e. up to terms of order  $\mathcal{O}(c^{-4})$ , reads [19, 20]

$$\frac{\ddot{x}^i(t)}{c^2} = \frac{\partial h_{00}^{(2)}}{\partial x^i} - 2 \frac{\partial h_{00}^{(2)}}{\partial x^j} \sigma^i \sigma^j - \frac{\partial h_{0i}^{(3)}}{\partial x^j} \sigma^j + \frac{\partial h_{0j}^{(3)}}{\partial x^i} \sigma^j - \frac{\partial h_{0j}^{(3)}}{\partial x^k} \sigma^i \sigma^j \sigma^k. \quad (11)$$

The double-dot on the left-hand side in (11) means twice of the total differentiation with respect to the coordinate time, and  $h_{ij}^{(2)} = h_{00}^{(2)} \delta_{ij}$  has been taken into account.

## C. The light trajectory

The geodesic equation (11) is a differential equation of second order, thus a unique solution of (11) requires two initial conditions (initial value problem) [19–25]: the unit-direction  $\boldsymbol{\sigma}$  of the light ray at past infinity and the spatial position of light source  $\mathbf{x}_0$  at the moment of emission of the light signal,

$$\boldsymbol{\sigma} = \frac{\dot{\mathbf{x}}(t)}{c} \Big|_{t \rightarrow -\infty}, \quad (12)$$

$$\mathbf{x}_0 = \mathbf{x}(t) \Big|_{t=t_0}, \quad (13)$$

with  $\boldsymbol{\sigma} \cdot \boldsymbol{\sigma} = 1$ . The geodesic equation (11) can be solved by iteration, and the solution of first integration reads formally

$$\begin{aligned} \frac{\dot{\mathbf{x}}(t)}{c} &= \boldsymbol{\sigma} + \sum_{l=0}^{\infty} \frac{\Delta \dot{\mathbf{x}}_{1\text{PN}}^{M_L}(t)}{c} + \sum_{l=0}^{\infty} \frac{\Delta \dot{\mathbf{x}}_{1.5\text{PN}}^{S_L}(t)}{c}, \quad (14) \\ \mathbf{x}(t) &= \mathbf{x}_N + \sum_{l=0}^{\infty} \Delta \mathbf{x}_{1\text{PN}}^{M_L}(t, t_0) + \sum_{l=0}^{\infty} \Delta \mathbf{x}_{1.5\text{PN}}^{S_L}(t, t_0), \quad (15) \end{aligned}$$

up to terms of the order  $\mathcal{O}(c^{-4})$ .

In reality, the light source as well as the observer are located at finite distances  $\mathbf{x}_0$  and  $\mathbf{x}_1$  from the gravitating Solar System body. This fact implies to solve the geodesic equation in terms of these two boundary values (boundary value problem),

$$\mathbf{x}_0 = \mathbf{x}(t) \Big|_{t=t_0}, \quad (16)$$

$$\mathbf{x}_1 = \mathbf{x}(t) \Big|_{t=t_1}. \quad (17)$$

These equations state, that the spatial positions of light signal at the time of emission,  $t_0$ , and at the time of reception,  $t_1$ , are in coincidence with the spatial position of the light source,  $\mathbf{x}_0$ , and observer,  $\mathbf{x}_1$ , respectively. In the boundary value problem one introduces the unit-vector

$$\mathbf{k} = \frac{\mathbf{x}_1 - \mathbf{x}_0}{|\mathbf{x}_1 - \mathbf{x}_0|}, \quad (18)$$

with  $\mathbf{k} \cdot \mathbf{k} = 1$  and which is pointing from the light source towards the observer. The introduction of this three-vector in (18) implicates a new impact vector,

$$\mathbf{d}_k = \mathbf{k} \times (\mathbf{x}_0 \times \mathbf{k}) = \mathbf{k} \times (\mathbf{x}_1 \times \mathbf{k}), \quad (19)$$

which points from the center-of-mass of the body towards the closest point of the coordinate line between  $\mathbf{x}_0$  and  $\mathbf{x}_1$ ; for a graphical elucidation see Figure 1. The absolute value of this impact vector is the impact parameter  $d_k = |\mathbf{d}_k|$ . The transformation from  $\mathbf{k}$  to  $\boldsymbol{\sigma}$  in the 1.5PN approximation reads [26]

$$\boldsymbol{\sigma} = \mathbf{k} - \frac{1}{R} \left[ \mathbf{k} \times \left( \Delta \mathbf{x}_{\text{1PN}}^{M_L}(t_1, t_0) \times \mathbf{k} \right) \right] - \frac{1}{R} \left[ \mathbf{k} \times \left( \Delta \mathbf{x}_{\text{1.5PN}}^{S_L}(t_1, t_0) \times \mathbf{k} \right) \right]. \quad (20)$$

In order to define and to determine the light deflection, the unit tangent vector at the moment of reception needs to be introduced

$$\mathbf{n} = \frac{\dot{\mathbf{x}}(t_1)}{|\dot{\mathbf{x}}(t_1)|}. \quad (21)$$

By inserting the transformation (20) into (14) one obtains for the unit tangent vector (21) the expression [26]

$$\mathbf{n} = \mathbf{k} + \Delta \mathbf{n}_{M_L} + \Delta \mathbf{n}_{S_L} \quad (22)$$

with

$$\Delta \mathbf{n}_{M_L} = \mathbf{k} \left[ \times \left( \frac{\Delta \dot{\mathbf{x}}_{\text{1PN}}^{M_L}(t_1)}{c} - \frac{\Delta \mathbf{x}_{\text{1PN}}^{M_L}(t_1, t_0)}{R} \right) \times \mathbf{k} \right], \quad (23)$$

$$\Delta \mathbf{n}_{S_L} = \mathbf{k} \left[ \times \left( \frac{\Delta \dot{\mathbf{x}}_{\text{1.5PN}}^{S_L}(t_1)}{c} - \frac{\Delta \mathbf{x}_{\text{1.5PN}}^{S_L}(t_1, t_0)}{R} \right) \times \mathbf{k} \right], \quad (24)$$

up to terms beyond 1.5PN approximation.

In [21] advanced integration methods have been developed, which allow for integrating of the geodesic equations (11). These integration methods will be considered below. Afterwards, the expressions of the light ray perturbations in (14) and (15) will be presented in their explicit form by Eqs. (40) - (41) and Eqs. (44) - (45).

#### D. The light trajectory in terms of new variables

In the approach in [21] advanced integration methods were introduced, basing on the new parameters,

$$c\tau = \boldsymbol{\sigma} \cdot \mathbf{x}_N, \quad (25)$$

$$\hat{\boldsymbol{\xi}}^i = P_j^i x_N^j, \quad (26)$$

which are independent of each other in the 3-dimensional sub-space-time orthogonal to  $\boldsymbol{\sigma}$ . Then, the unperturbed light ray is given by Eq. (9) and the operator

$$P^{ij} = \delta^{ij} - \sigma^i \sigma^j \quad (27)$$

is a projection operator onto the plane perpendicular to vector  $\boldsymbol{\sigma}$ . Clearly, by inserting the unperturbed light ray (9) and the projector (27) into (26), one may identify the auxiliary variable  $\hat{\boldsymbol{\xi}}$  as impact vector  $\mathbf{d}_\sigma$  of the unperturbed light ray defined by Eq. (10).

The unperturbed light ray  $\mathbf{x}_N$  and its absolute value  $r_N = |\mathbf{x}_N|$  can be parameterized in terms of these new variables and take the form

$$\mathbf{x}_N = \hat{\boldsymbol{\xi}} + c\tau \boldsymbol{\sigma}, \quad (28)$$

$$r_N = \sqrt{\hat{\boldsymbol{\xi}}^2 + c^2\tau^2}. \quad (29)$$

The three-vector  $\hat{\boldsymbol{\xi}}$  is laying in the plane perpendicular to  $\boldsymbol{\sigma}$  and, therefore, only two of its three components are independent. This fact leads to the following partial derivatives (cf. Eq. (23) in [21], see also Eq. (11.2.12) in [27])

$$\frac{\partial \hat{\xi}^i}{\partial \hat{\xi}^j} = P_j^i = P^{ij} = P_{ij}. \quad (30)$$

Then, the spatial derivatives in (4), when transformed in terms of these new variables, are given by (cf. Eq. (20) in [21])

$$\frac{\partial}{\partial x^i} = \frac{\partial}{\partial \hat{\xi}^i} + \sigma_i \frac{\partial}{\partial c\tau}. \quad (31)$$

In practical calculations it is often more appropriate to consider the three spatial components of three-vector  $\hat{\boldsymbol{\xi}}$  as independent of each other (cf. text above Eq. (31) in [20]). This three-vector is denoted by  $\boldsymbol{\xi}$ . The parameters  $c\tau$  and  $\boldsymbol{\xi}$  are independent of each other in the 4-dimensional space-time. Using  $\boldsymbol{\xi}$  one gets the familiar result (see also Eq. (11.2.13) in [27])

$$\frac{\partial \xi^i}{\partial \xi^j} = \delta_j^i = \delta^{ij} = \delta_{ij}, \quad (32)$$

with subsequent projection into the two-dimensional plane perpendicular to three-vector  $\boldsymbol{\sigma}$ ,

$$P_j^k \frac{\partial \xi^i}{\partial \xi^k} = P_j^i = P^{ij} = P_{ij}. \quad (33)$$

That means, for the spatial derivatives, when transformed into derivatives expressed in terms of these new variables, one obtains

$$\frac{\partial}{\partial x^i} = P_i^j \frac{\partial}{\partial \xi^j} + \sigma_i \frac{\partial}{\partial c\tau}. \quad (34)$$

This relation coincides with Eq. (33) in [20] in case of time-independent functions. It is noticed that (34) in combination with (32) is identical with (31) in combination with (30). Here, we will prefer this procedure in (34), that means we will consider the spatial components of  $\boldsymbol{\xi}$  as three independent components with a subsequent projection onto the two-dimensional plane perpendicular

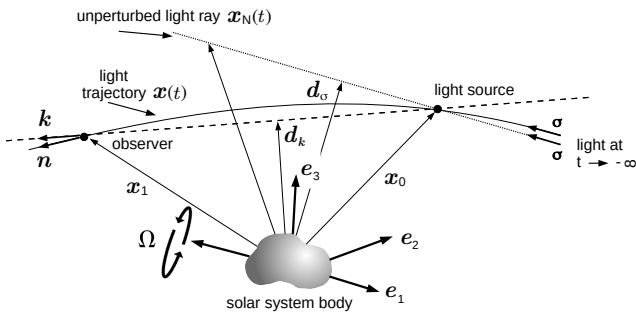


Figure 1: A geometrical representation of the propagation of a light signal through the gravitational field of a body at rest of arbitrary shape, inner structure and rotational motion around an arbitrarily oriented axis with uniform angular velocity  $\Omega$ . The origin of the spatial coordinates  $(x^1, x^2, x^3)$  is assumed to be located at the center-of-mass of the body. The orientation of the orthogonal principal axes  $e_1, e_2, e_3$  of the body with respect to the coordinate system is arbitrary. The light signal is emitted by the light source at  $x_0$  and propagates along the exact light trajectory  $x(t)$ . The unperturbed light ray  $x_N(t)$  is given by Eq. (9) and propagates in the direction of  $\sigma$  along a straight line through the position of the light source at  $x_0$ . The impact vector  $d_\sigma$  is given by Eq. (10) and the impact vector  $d_k$  is given by Eq. (19).

to the three-vector  $\sigma$ . Then, using (34) and the binomial theorem,

$$(a+b)^l = \sum_{p=0}^l \binom{l}{p} a^{l-p} b^p, \quad (35)$$

where

$$\binom{l}{p} = \frac{l!}{p!(l-p)!} \quad (36)$$

are the binomial coefficients, one finds for these  $l$  partial derivatives in (6), when expressed in terms of these new variables, the following expression,

$$\begin{aligned} \widehat{\partial}_L^\tau &= \text{STF}_{i_1 \dots i_l} \sum_{p=0}^l \binom{l}{p} \sigma_{i_1} \dots \sigma_{i_p} P_{i_{p+1}}^{j_{p+1}} \dots P_{i_l}^{j_l} \\ &\times \frac{\partial}{\partial \xi^{j_{p+1}}} \dots \frac{\partial}{\partial \xi^{j_l}} \left( \frac{\partial}{\partial c\tau} \right)^p, \end{aligned} \quad (37)$$

where the upper index  $\tau$  in  $\widehat{\partial}_L^\tau$  refers to the differentiation with respect to the parameter  $\tau$  on the right-hand side of (37). The wide hat in  $\widehat{\partial}_L$  indicates the STF operation with respect to the indices  $L = i_1 \dots i_l$ . In addition, the wide hat also indicates the difference between the differential operator of Eq. (37) from the differential operator of Eq. (6) [28]

The solution of the first integration (14) and second in-

tegration (15) in terms of these new variables reads [21]

$$\frac{\dot{x}(\tau)}{c} = \sigma + \sum_{l=0}^{\infty} \frac{\Delta \dot{x}_{1\text{PN}}^{M_L}(\tau)}{c} + \sum_{l=1}^{\infty} \frac{\Delta \dot{x}_{1.5\text{PN}}^{S_L}(\tau)}{c}, \quad (38)$$

$$x(\tau) = x_N + \sum_{l=0}^{\infty} \Delta x_{1\text{PN}}^{M_L}(\tau, \tau_0) + \sum_{l=1}^{\infty} \Delta x_{1.5\text{PN}}^{S_L}(\tau, \tau_0), \quad (39)$$

up to terms of the order  $\mathcal{O}(c^{-4})$ , and where the unperturbed light ray  $x_N$  in (39) is given by Eq. (28). The light ray perturbations of the mass-multipole and spin-multipole terms of the first integration in (38) are given by [21]

$$\begin{aligned} \frac{\Delta \dot{x}_{1\text{PN}}^{i M_L}(\tau)}{c} &= -\frac{2G\hat{M}_L}{c^2} \frac{(-1)^l}{l!} \widehat{\partial}_L^\tau \frac{\sigma^i}{r_N} \\ &\quad - \frac{2G\hat{M}_L}{c^2} \frac{(-1)^l}{l!} \widehat{\partial}_L^\tau \frac{\hat{\xi}^i}{r_N - c\tau} \frac{1}{r_N}, \end{aligned} \quad (40)$$

$$\begin{aligned} \frac{\Delta \dot{x}_{1.5\text{PN}}^{i S_L}(\tau)}{c} &= -\frac{4G\hat{S}_{bL-1}}{c^3} \epsilon_{iab} \frac{(-1)^l l}{(l+1)!} \widehat{\partial}_{aL-1} \frac{1}{r_N} \\ &\quad - \frac{4G\hat{S}_{bL-1}}{c^3} \epsilon_{abc} \sigma^c \frac{(-1)^l l}{(l+1)!} \widehat{\partial}_{aL-1} \frac{\hat{\xi}^i}{r_N - c\tau} \frac{1}{r_N}. \end{aligned} \quad (41)$$

The light ray perturbations of the mass-multipole and spin-multipole terms of the second integration in (39) are given by [21]

$$\Delta x_{1\text{PN}}^{M_L}(\tau, \tau_0) = \Delta x_{1\text{PN}}^{M_L}(\tau) - \Delta x_{1\text{PN}}^{M_L}(\tau_0), \quad (42)$$

$$\Delta x_{1.5\text{PN}}^{S_L}(\tau, \tau_0) = \Delta x_{1.5\text{PN}}^{S_L}(\tau) - \Delta x_{1.5\text{PN}}^{S_L}(\tau_0), \quad (43)$$

with

$$\begin{aligned} \Delta x_{1\text{PN}}^{i M_L}(\tau) &= +\frac{2G\hat{M}_L}{c^2} \sigma^i \frac{(-1)^l}{l!} \widehat{\partial}_L^\tau \ln(r_N - c\tau) \\ &\quad - \frac{2G\hat{M}_L}{c^2} \frac{(-1)^l}{l!} \widehat{\partial}_L^\tau \frac{\hat{\xi}^i}{r_N - c\tau}, \end{aligned} \quad (44)$$

$$\begin{aligned} \Delta x_{1.5\text{PN}}^{i S_L}(\tau) &= +\frac{4G\hat{S}_{bL-1}}{c^3} \epsilon_{iab} \frac{(-1)^l l}{(l+1)!} \widehat{\partial}_{aL-1} \ln(r_N - c\tau) \\ &\quad - \frac{4G\hat{S}_{bL-1}}{c^3} \epsilon_{abc} \sigma^c \frac{(-1)^l l}{(l+1)!} \widehat{\partial}_{aL-1} \frac{\hat{\xi}^i}{r_N - c\tau}. \end{aligned} \quad (45)$$

These equations (40) - (45) are given in terms of auxiliary variables as defined by equations (25) and (26). After performing all differentiations in these expressions with respect to the auxiliary variables, one obtains the solutions of the first and second integration of geodesic equation of Eqs. (14) and (15) just by replacing  $c\tau$  by  $\sigma \cdot x_N$  and  $P_j^i \xi^j$  by  $d_\sigma^i$ .

### III. THE UNIT TANGENT VECTOR

The unit tangent vector of a light ray at the moment of reception at the spatial position of the observer in terms

of these new variables is given by

$$\mathbf{n} = \frac{\dot{\mathbf{x}}(\tau_1)}{|\dot{\mathbf{x}}(\tau_1)|}. \quad (46)$$

From Eq. (22) with the expressions in (23) and (24) follows the unit tangent vector

$$\mathbf{n} = \mathbf{k} + \sum_{l=0}^{\infty} \Delta \mathbf{n}_{M_L} + \sum_{l=1}^{\infty} \Delta \mathbf{n}_{S_L}, \quad (47)$$

with  $\mathbf{n} \cdot \mathbf{n} = 1$  and where

$$\Delta \mathbf{n}_{M_L} = \mathbf{k} \times \left[ \left( \frac{\Delta \dot{\mathbf{x}}_{1\text{PN}}^{M_L}(\tau_1)}{c} - \frac{\Delta \mathbf{x}_{1\text{PN}}^{M_L}(\tau_1, \tau_0)}{R} \right) \times \mathbf{k} \right], \quad (48)$$

$$\Delta \mathbf{n}_{S_L} = \mathbf{k} \times \left[ \left( \frac{\Delta \dot{\mathbf{x}}_{1.5\text{PN}}^{S_L}(\tau_1)}{c} - \frac{\Delta \mathbf{x}_{1.5\text{PN}}^{S_L}(\tau_1, \tau_0)}{R} \right) \times \mathbf{k} \right], \quad (49)$$

up to terms beyond 1.5PN approximation. The distance between source and observer,  $R$ , is given by

$$R = |\mathbf{x}_1 - \mathbf{x}_0| = \mathbf{k} \cdot \mathbf{x}_1 - \mathbf{k} \cdot \mathbf{x}_0 \\ = \sqrt{r_1^2 + r_0^2 - 2r_0r_1 \cos \delta(\mathbf{x}_0, \mathbf{x}_1)}, \quad (50)$$

where  $\delta(\mathbf{x}_0, \mathbf{x}_1)$  is the angle between the three-vectors  $\mathbf{x}_0$  and  $\mathbf{x}_1$ , while  $r_0 = |\mathbf{x}_0|$  and  $r_1 = |\mathbf{x}_1|$  are the distances between body and source and body and observer, respectively. The perturbations in (48) and (49) are given by Eqs. (40) - (45), where the differential operator is given by Eq. (37). In view of relation (20) we have

$$\boldsymbol{\sigma} = \mathbf{k} + \mathcal{O}(c^{-2}), \quad (51)$$

$$\boldsymbol{\sigma} \cdot \mathbf{k} = 1 + \mathcal{O}(c^{-4}), \quad (52)$$

where the three-vector  $\boldsymbol{\sigma}$  is defined by Eq. (12) and the three-vector  $\mathbf{k}$  is defined by Eq. (18). Therefore, in the perturbations (40) and (44) one may replace three-vector  $\boldsymbol{\sigma}$  by three-vector  $\mathbf{k}$  in line with the 1.5PN approximation. In particular,  $\boldsymbol{\sigma}$  is replaced by  $\mathbf{k}$  in the projector (27), in the new parameters (25) and (26), and in the distance  $r_N$  in (29). Furthermore, we will specify these quantities at the spatial positions of source and observer,  $\mathbf{x}_0$  and  $\mathbf{x}_1$ , as well as at the moment of emission and reception of the light signal,  $t_0$  and  $t_1$  by using

$$\mathbf{x}_0 = \mathbf{x}_N(t_0), \quad (53)$$

$$\mathbf{x}_1 = \mathbf{x}_N(t_1) + \mathcal{O}(c^{-2}), \quad (54)$$

where the unperturbed light ray is given by Eq. (9) and in terms of new variables by Eq. (28). In this way we get the following expressions, which will from now on be

used in the subsequent considerations:

$$P^{ij} = \delta^{ij} - k^i k^j, \quad (55)$$

$$c\tau_0 = \mathbf{k} \cdot \mathbf{x}_0, \quad (56)$$

$$c\tau_1 = \mathbf{k} \cdot \mathbf{x}_1, \quad (57)$$

$$\hat{\xi}^i = P_j^i x_0^j = P_j^i x_1^j, \quad (58)$$

$$r_0 = \sqrt{\hat{\xi}^2 + c^2 \tau_0^2}, \quad (59)$$

$$r_1 = \sqrt{\hat{\xi}^2 + c^2 \tau_1^2}, \quad (60)$$

where the projector (55) is an operator which projects a three-vector onto the plane perpendicular to vector  $\mathbf{k}$  and (59) and (60) are the spatial distances between the center-of-mass of the body and the position of source and observer, respectively. In view of relation (51), these expressions in Eqs. (55) - (60) differ from the previously defined quantities of Eqs. (25) - (29) by terms of the order  $\mathcal{O}(c^{-2})$ . In favor of a simpler notation, we are not introducing a new label for the projector of Eq. (55). By these replacements and for these specific values, the differential operator in (37) becomes

$$\widehat{\partial}_L^{\tau_0} = \widehat{\partial}_L + \widehat{\partial}_L^{\tau_0, p \neq 0}, \quad (61)$$

$$\widehat{\partial}_L^{\tau_1} = \widehat{\partial}_L + \widehat{\partial}_L^{\tau_1, p \neq 0}, \quad (62)$$

where the differential operator term with  $p = 0$  reads

$$\widehat{\partial}_L = \text{STF}_{i_1 \dots i_l} P_{i_1}^{j_1} \dots P_{i_l}^{j_l} \frac{\partial}{\partial \xi^{j_1}} \dots \frac{\partial}{\partial \xi^{j_l}}, \quad (63)$$

and the differential operator terms with  $p \neq 0$  are

$$\widehat{\partial}_L^{\tau_0, p \neq 0} = \text{STF}_{i_1 \dots i_l} \sum_{p=1}^l \binom{l}{p} k_{i_1} \dots k_{i_p} P_{i_{p+1}}^{j_{p+1}} \dots P_{i_l}^{j_l} \\ \times \frac{\partial}{\partial \xi^{j_{p+1}}} \dots \frac{\partial}{\partial \xi^{j_l}} \left( \frac{\partial}{\partial c\tau_0} \right)^p, \quad (64)$$

$$\widehat{\partial}_L^{\tau_1, p \neq 0} = \text{STF}_{i_1 \dots i_l} \sum_{p=1}^l \binom{l}{p} k_{i_1} \dots k_{i_p} P_{i_{p+1}}^{j_{p+1}} \dots P_{i_l}^{j_l} \\ \times \frac{\partial}{\partial \xi^{j_{p+1}}} \dots \frac{\partial}{\partial \xi^{j_l}} \left( \frac{\partial}{\partial c\tau_1} \right)^p. \quad (65)$$

Similarly, because of relation (51), these differential operators of Eqs. (61) and (62) differ from the previously defined differential operator of Eq. (37) by terms of the order  $\mathcal{O}(c^{-2})$ , which is in line with the 1.5PN approximation.

The unit tangent vector of the light trajectory at the observers position is given by Eq. (47) with the mass-multipole and spin-multipole terms of Eqs. (48) and (49). The mass-multipole term in the first lines of Eqs. (40) and (44) do not contribute to this unit tangent vector, as one can see by inserting these terms into Eq. (48) and taking account of Eq. (51). Accordingly, by inserting (40) and (44) into (48) one obtains three mass-multipole terms,

$$\Delta \mathbf{n}_{M_L} = \Delta \mathbf{n}_{M_L}^1 + \Delta \mathbf{n}_{M_L}^2 + \Delta \mathbf{n}_{M_L}^3, \quad (66)$$

with the spatial components ( $i = 1, 2, 3$ )

$$\Delta n_{M_L}^{1i} = -\frac{2G\hat{M}_L}{c^2} \frac{(-1)^l}{l!} \hat{\partial}_L^{\tau_1} \frac{\hat{\xi}^i}{r_1 - c\tau_1} \frac{1}{r_1}, \quad (67)$$

$$\Delta n_{M_L}^{2i} = +\frac{2G\hat{M}_L}{c^2} \frac{(-1)^l}{l!} \frac{1}{R} \hat{\partial}_L^{\tau_1} \frac{\hat{\xi}^i}{r_1 - c\tau_1}, \quad (68)$$

$$\Delta n_{M_L}^{3i} = -\frac{2G\hat{M}_L}{c^2} \frac{(-1)^l}{l!} \frac{1}{R} \hat{\partial}_L^{\tau_0} \frac{\hat{\xi}^i}{r_0 - c\tau_0}, \quad (69)$$

where the differential operators  $\hat{\partial}_L^{\tau_0}$  and  $\hat{\partial}_L^{\tau_1}$  are given by Eqs. (61) and (62). The vector triple product in (48) is identical with Eqs. (67) - (69), because the vectorial expressions in the parenthesis are orthogonal to three-vector  $\mathbf{k}$ .

Similarly, by inserting (41) and (45) into (49) one obtains 6 spin-multipole terms,

$$\Delta \mathbf{n}_{S_L} = \sum_{k=1}^6 \Delta \mathbf{n}_{S_L}^k, \quad (70)$$

where the spatial components ( $i = 1, 2, 3$ ) of the first three terms read

$$\Delta n_{S_L}^{1i} = -\frac{4G\hat{S}_{bL-1}}{c^3} \epsilon_{abc} k^c \frac{(-1)^l l}{(l+1)!} \hat{\partial}_{aL-1}^{\tau_1} \frac{\hat{\xi}^i}{r_1 - c\tau_1} \frac{1}{r_1}, \quad (71)$$

$$\Delta n_{S_L}^{2i} = +\frac{4G\hat{S}_{bL-1}}{c^3} \epsilon_{abc} k^c \frac{(-1)^l l}{(l+1)!} \frac{1}{R} \hat{\partial}_{aL-1}^{\tau_1} \frac{\hat{\xi}^i}{r_1 - c\tau_1}, \quad (72)$$

$$\Delta n_{S_L}^{3i} = -\frac{4G\hat{S}_{bL-1}}{c^3} \epsilon_{abc} k^c \frac{(-1)^l l}{(l+1)!} \frac{1}{R} \hat{\partial}_{aL-1}^{\tau_0} \frac{\hat{\xi}^i}{r_0 - c\tau_0}, \quad (73)$$

while the spatial components ( $i = 1, 2, 3$ ) of the last three terms are given by

$$\Delta n_{S_L}^{4i} = -\frac{4G\hat{S}_{bL-1}}{c^3} \hat{\epsilon}_{iab} \frac{(-1)^l l}{(l+1)!} \left( \hat{\partial}_{aL-1}^{\tau_1} \frac{1}{r_1} \right), \quad (74)$$

$$\Delta n_{S_L}^{5i} = -\frac{4G\hat{S}_{bL-1}}{c^3} \hat{\epsilon}_{iab} \frac{(-1)^l l}{(l+1)!} \frac{1}{R} \hat{\partial}_{aL-1}^{\tau_1} \ln(r_1 - c\tau_1), \quad (75)$$

$$\Delta n_{S_L}^{6i} = +\frac{4G\hat{S}_{bL-1}}{c^3} \hat{\epsilon}_{iab} \frac{(-1)^l l}{(l+1)!} \frac{1}{R} \hat{\partial}_{aL-1}^{\tau_0} \ln(r_0 - c\tau_0), \quad (76)$$

where we use the abbreviation  $\hat{\epsilon}_{iab} = \epsilon_{iab} - k_i \epsilon_{jab} k^j$ , originating from the double cross product in (49). The differential operators were given by Eqs. (61) and (62). These equations (67) - (69) and (71) - (76) are valid for an arbitrary body.

#### IV. ON THE MAGNITUDE OF TERMS

The unit tangent vector of a light trajectory in the gravitational field of an arbitrary body is given by

Eq. (47), with the mass-multipole perturbations (48) and spin-multipole perturbations (49). These perturbations are given in their explicit form by Eqs. (66) - (69) and Eqs. (70) - (76). In the next Section it will be demonstrated that these individual terms of the tangent vector of light trajectory are of qualitatively different magnitude. In particular, for the mass-multipole terms of Eqs. (67) - (69) we may distinguish three kind of terms,

$$\epsilon_1^{M_L} = A^{M_L} \frac{GM}{c^2} \frac{1}{d_k} J_l \left( \frac{P}{d_k} \right)^l \sim \mu\text{as}, \quad (77)$$

$$\epsilon_2^{M_L} = B^{M_L} \frac{GM}{c^2} \frac{1}{r_1} J_l \left( \frac{P}{d_k} \right)^l \sim \text{nas}, \quad (78)$$

$$\epsilon_3^{M_L} = C^{M_L} \frac{GM}{c^2} \frac{1}{r_1} J_l \left( \frac{P}{r_1} \right)^l \sim \text{pas}, \quad (79)$$

where  $\mu\text{as}$ ,  $\text{nas}$ , and  $\text{pas}$  are micro-, nano-, and pico-arcseconds; see also Appendix A. Similarly, for the spin-multipole terms of Eqs. (71) - (76) we may also distinguish between three kind of terms,

$$\epsilon_1^{S_L} = A^{S_L} \frac{GM}{c^3} \Omega J_{l-1} \left( \frac{P}{d_k} \right)^{l+1} \sim \mu\text{as}, \quad (80)$$

$$\epsilon_2^{S_L} = B^{S_L} \frac{GM}{c^3} \Omega J_{l-1} \left( \frac{P}{r_1} \right) \left( \frac{P}{d_k} \right)^l \sim \text{nas}, \quad (81)$$

$$\epsilon_3^{S_L} = C^{S_L} \frac{GM}{c^3} \Omega J_{l-1} \left( \frac{P}{r_1} \right)^{l+1} \sim \text{pas}. \quad (82)$$

The coefficients  $A^{M_L}, B^{M_L}, C^{M_L}$  and  $A^{S_L}, B^{S_L}, C^{S_L}$  will be determined in the next Sections. The parameter  $M$ ,  $P$ ,  $J_l$ , and  $\Omega$  are the mass, equatorial radius, zonal harmonic coefficient, and angular velocity of the solar system body, while  $d_k$  is the impact parameter of the unperturbed light ray with respect to the body and  $r_1$  is the distance between body and observer; a graphical elucidation is shown in Figure 1. Because the impact of spin-multipoles of order  $l$  on light deflection is a about 100 times smaller than the impact of mass-multipoles of order  $l+1$  on light deflection, relation (81) means actually less than 1 nas.

The terms of the unit tangent vector of the light ray, which are of the order (77) and (80), are the relevant terms for astrometry on the sub- $\mu\text{as}$  level. They are given below by Eqs. (136) - (143) and their upper limits are presented by Eqs. (201) - (203).

However, the unit tangent vector of the light ray contains also many terms which are of the order of Eqs.(78) - (79) and Eqs. (81) - (82), which are suppressed by the ratio of equatorial radius of the body and the distance between body and observer. For instance, this ratio is about  $5 \times 10^{-3}$  in case of the Sun and about  $10^{-4}$  in case of Jupiter. Because of these small prefactors, the numerical magnitude of these terms to the effect of light deflection is at most a few nano-arcseconds or even less. Especially, for astrometry on the sub-micro-arcsecond level one may neglect all those terms which contribute only a

few nano-arcseconds (nas) or a few pico-arcseconds (pas) to the angle of light deflection. Thus, by keeping only terms of the magnitude  $\epsilon_1^{M_L}$  and  $\epsilon_1^{S_L}$  and neglecting all terms of the magnitudes  $\epsilon_2^{M_L}, \epsilon_3^{M_L}$  and  $\epsilon_2^{S_L}, \epsilon_3^{S_L}$  one arrives at a considerably simplified unit tangent vector of the light ray. This simplified tangent vector is important for highly effective data reduction of future astrometric measurements on the sub-micro-arcsecond level of accuracy and will be derived in the subsequent Sections V and VII.

## V. THE REDUCED TANGENT VECTOR

The unit tangent vector of the light ray in 1.5PN approximation is given by Eq. (47) with the mass-multipole terms given by Eq. (66) with the terms in (67) - (69) and the spin-multipole terms of the unit tangent vector are given by Eq. (70) with the terms in (71) - (76). As asserted in the previous Section, these expressions are much too complicated for an effective treatment of astrometric data reduction, because they contain many terms which are below the given threshold of  $0.01 \mu\text{as}$  of sub-micro-arcsecond astrometry. In order to identify these tiny terms we separate our considerations into two Sections: in this Section V we will arrive at the reduced tangent vector, which is a step in between towards the simplified tangent vector which is finally obtained in Section VII. The mass-multipole and spin-multipole terms are considered separately in the following two Subsections.

### A. The mass-multipole terms

The mass-multipole terms of the unit tangent vector are given by Eq. (66) with the terms in (67) - (69). According to (61) and (62), we may separate these perturbations into terms with parameter  $p = 0$  and with parameter  $p \neq 0$ . Then, the mass-multipole perturbations read

$$\begin{aligned} \Delta \mathbf{n}_{M_L} = & \Delta \mathbf{n}_{M_L}^{1,p=0} + \Delta \mathbf{n}_{M_L}^{2,p=0} + \Delta \mathbf{n}_{M_L}^{3,p=0} \\ & + \Delta \mathbf{n}_{M_L}^{1,p \neq 0} + \Delta \mathbf{n}_{M_L}^{2,p \neq 0} + \Delta \mathbf{n}_{M_L}^{3,p \neq 0}. \end{aligned} \quad (83)$$

The terms in the first line of (83) read

$$\Delta \mathbf{n}_{M_L}^{1,p=0} = -\frac{2G\hat{M}_L}{c^2} \frac{(-1)^l}{l!} \hat{\partial}_L \frac{\hat{\boldsymbol{\xi}}}{r_1 - c\tau_1} \frac{1}{r_1}, \quad (84)$$

$$\Delta \mathbf{n}_{M_L}^{2,p=0} = +\frac{2G\hat{M}_L}{c^2} \frac{(-1)^l}{l!} \frac{1}{R} \hat{\partial}_L \frac{\hat{\boldsymbol{\xi}}}{r_1 - c\tau_1}, \quad (85)$$

$$\Delta \mathbf{n}_{M_L}^{3,p=0} = -\frac{2G\hat{M}_L}{c^2} \frac{(-1)^l}{l!} \frac{1}{R} \hat{\partial}_L \frac{\hat{\boldsymbol{\xi}}}{r_0 - c\tau_0}, \quad (86)$$

where  $l \geq 0$  and the differential operator is given by

Eq. (63). The terms in the second line of (83) read

$$\Delta \mathbf{n}_{M_L}^{1,p \neq 0} = -\frac{2G\hat{M}_L}{c^2} \frac{(-1)^l}{l!} \hat{\partial}_L^{\tau_1, p \neq 0} \frac{\hat{\boldsymbol{\xi}}}{r_1 - c\tau_1} \frac{1}{r_1}, \quad (87)$$

$$\Delta \mathbf{n}_{M_L}^{2,p \neq 0} = +\frac{2G\hat{M}_L}{c^2} \frac{(-1)^l}{l!} \frac{1}{R} \hat{\partial}_L^{\tau_1, p \neq 0} \frac{\hat{\boldsymbol{\xi}}}{r_1 - c\tau_1}, \quad (88)$$

$$\Delta \mathbf{n}_{M_L}^{3,p \neq 0} = -\frac{2G\hat{M}_L}{c^2} \frac{(-1)^l}{l!} \frac{1}{R} \hat{\partial}_L^{\tau_0, p \neq 0} \frac{\hat{\boldsymbol{\xi}}}{r_0 - c\tau_0}, \quad (89)$$

where  $l \geq 2$  and the differential operators are given by Eqs. (64) and (65). In Appendices I - J it is shown that the upper limit of the term of Eq. (87) and the upper limit of the sum of Eqs. (88) and (89) is given by

$$\left| \delta \left( \mathbf{k}, \Delta \mathbf{n}_{M_L}^{1,p \neq 0} \right) \right| = C_1^{M_L} \frac{GM}{c^2 r_1} |J_l| \left( \frac{P}{r_1} \right)^l, \quad (90)$$

$$\left| \delta \left( \mathbf{k}, \Delta \mathbf{n}_{M_L}^{2+3,p \neq 0} \right) \right| = B_1^{M_L} \frac{GM}{c^2 r_1} |J_l| \left( \frac{P}{d_k} \right)^l. \quad (91)$$

The coefficients of these equations have been determined in the Appendices I and J. They are given in Table VII of Appendix I and in Table X of Appendix J.

The term (90) is of the order of Eq. (79) and contributes much less than 1 nas,

$$\left| \delta \left( \mathbf{k}, \Delta \mathbf{n}_{M_L}^{1,p \neq 0} \right) \right| \ll 1 \text{ nas}, \quad (92)$$

while the term (91) is of the order of Eq. (78). The terms of Eqs. (88) and (89) are much larger than the term of Eq. (87). The reason is that the distance  $R$  between source and observer appears in the denominator of these terms. This distance can in principle be arbitrarily small. Therefore, one has to estimate the sum of Eqs. (88) and (89), which is a finite quantity, but considerably larger than (87). Nevertheless, according to (91) these terms contribute at most a few nas.

Numerical values of these terms of Eq. (91) are presented in Table I for the case of grazing rays at Jupiter and Saturn. For all other Solar System bodies the magnitude of these terms is even less. Such tiny effects are negligible for astrometry missions like *GaiaNIR* [3] or *Theia* [4], aiming at astrometry on the sub- $\mu\text{as}$  scale of accuracy, that means aiming at an accuracy of about  $0.1 \mu\text{as}$ . Clearly, even for future astrometry missions aiming at an accuracy of about  $0.01 \mu\text{as}$  these terms remain negligible.

### B. The spin-multipole terms

The spin-multipole terms of the unit tangent vector are given by Eq. (70) with the terms (71) - (76). Before we proceed further, it is useful to consider the magnitude of the terms of Eqs. (74) - (76) at this stage, because these terms turn out to be negligibly small. The upper limit of

Table I: The contribution of the terms of Eq. (91) to the angle of light deflection in case of a grazing ray at Jupiter and Saturn. For all other solar system bodies these terms are even smaller. All values are given in nano-arcseconds (nas). A blank entry means less than 1 nas.

$l$	$ \delta_{\hat{\eta}}(\mathbf{k}, \Delta \mathbf{n}_{M_L}^{2+3, p \neq 0}) $	$ \delta_{\hat{\eta}}(\mathbf{k}, \Delta \mathbf{n}_{M_L}^{2+3, p \neq 0}) $
2	20.3	6.7
4	6.7	3.2
6	1.7	1.3
8	0.5	0.6
10	0.2	0.5

the term of Eq. (74) and the sum of Eqs. (75) and (76) to the angle of light deflection are given by

$$|\delta(\mathbf{k}, \Delta \mathbf{n}_{S_L}^4)| = C_1^{S_L} \frac{GM}{c^3} \Omega |J_{l-1}| \left(\frac{P}{r_1}\right)^{l+1}, \quad (93)$$

$$|\delta(\mathbf{k}, (\Delta \mathbf{n}_{S_L}^{5+6}))| = B_1^{S_L} \frac{GM}{c^3} \Omega |J_{l-1}| \left(\frac{P}{r_1}\right)^l \left(\frac{P}{d_k}\right)^l. \quad (94)$$

These terms of Eqs. (93) and (94) are terms of the order (82) and (81), respectively. A proof of these relations is obtained by similar considerations as shown in Appendices I and J for the mass-multipole terms. But actually, only the very first few spin-multipoles are relevant on the sub- $\mu$ as-level: the spin-dipole term  $l = 1$  and the spin-hexapole term  $l = 3$ . In order to get an idea about the magnitude of these terms we consider the case of spin-dipole, which is the most dominating term. Inserting the spin-dipole (150) and the differential operators (61) and (62) into Eqs. (74) - (76) yields the coefficients for the case of spin-dipole:  $C_1^{S_1} = 2\kappa^2$  and  $B_1^{S_1} = 2\kappa^2$ ; for a proof see Appendix C. By inserting the parameters of Table III into (93) and (94), one obtains for grazing light rays at any Solar System body, that these upper limits contribute much less than 1 nas to the angle of light deflection. It is clear that the contribution of higher spin-multipoles is less than the spin-dipole term. Thus, the contribution of these terms of Eqs. (74) - (76) to the angle of light deflection is much less than 1 nas to any spin-multipole order,

$$|\delta(\mathbf{k}, \Delta \mathbf{n}_{S_L}^4)| \ll 1 \text{ nas}, \quad (95)$$

$$|\delta(\mathbf{k}, (\Delta \mathbf{n}_{S_L}^{5+6}))| \ll 1 \text{ nas}. \quad (96)$$

Therefore, these terms of Eqs. (74) - (76) can be neglected for astrometric measurements on the sub-micro-arcsecond level. Accordingly, only the following three spin-multipole terms are considered,

$$\Delta \mathbf{n}_{S_L} = \Delta \mathbf{n}_{S_L}^1 + \Delta \mathbf{n}_{S_L}^2 + \Delta \mathbf{n}_{S_L}^3, \quad (97)$$

with the expressions of Eqs. (71) - (73). According to (61) and (62), we may separate these perturbations into

terms with parameter  $p = 0$  and with parameter  $p \neq 0$ . Then, the spin-multipole perturbations read

$$\begin{aligned} \Delta \mathbf{n}_{S_L} &= \Delta \mathbf{n}_{S_L}^{1,p=0} + \Delta \mathbf{n}_{S_L}^{2,p=0} + \Delta \mathbf{n}_{S_L}^{3,p=0} \\ &+ \Delta \mathbf{n}_{S_L}^{1,p \neq 0} + \Delta \mathbf{n}_{S_L}^{2,p \neq 0} + \Delta \mathbf{n}_{S_L}^{3,p \neq 0}. \end{aligned} \quad (98)$$

The terms in the first line of (98) read

$$\Delta \mathbf{n}_{S_L}^{1,p=0} = -\frac{4G\hat{S}_{bL-1}}{c^3} \epsilon_{abc} k^c \frac{(-1)^l l}{(l+1)!} \hat{\partial}_{aL-1} \frac{\hat{\xi}}{r_1 - c\tau_1} \frac{1}{r_1}, \quad (99)$$

$$\Delta \mathbf{n}_{S_L}^{2,p=0} = +\frac{4G\hat{S}_{bL-1}}{c^3} \epsilon_{abc} k^c \frac{(-1)^l l}{(l+1)!} \frac{1}{R} \hat{\partial}_{aL-1} \frac{\hat{\xi}}{r_1 - c\tau_1}, \quad (100)$$

$$\Delta \mathbf{n}_{S_L}^{3,p=0} = -\frac{4G\hat{S}_{bL-1}}{c^3} \epsilon_{abc} k^c \frac{(-1)^l l}{(l+1)!} \frac{1}{R} \hat{\partial}_{aL-1} \frac{\hat{\xi}}{r_0 - c\tau_0}, \quad (101)$$

where  $l \geq 1$  and the differential operator is given by Eq. (63). The terms in the second line of (98) read

$$\Delta \mathbf{n}_{S_L}^{1,p \neq 0} = -\frac{4G\hat{S}_{bL-1}}{c^3} \epsilon_{abc} k^c \frac{(-1)^l l}{(l+1)!} \hat{\partial}_{aL-1}^{\tau_1, p \neq 0} \frac{\hat{\xi}}{r_1 - c\tau_1} \frac{1}{r_1}, \quad (102)$$

$$\Delta \mathbf{n}_{S_L}^{2,p \neq 0} = +\frac{4G\hat{S}_{bL-1}}{c^3} \epsilon_{abc} k^c \frac{(-1)^l l}{(l+1)!} \frac{1}{R} \hat{\partial}_{aL-1}^{\tau_1, p \neq 0} \frac{\hat{\xi}}{r_1 - c\tau_1}, \quad (103)$$

$$\Delta \mathbf{n}_{S_L}^{3,p \neq 0} = -\frac{4G\hat{S}_{bL-1}}{c^3} \epsilon_{abc} k^c \frac{(-1)^l l}{(l+1)!} \frac{1}{R} \hat{\partial}_{aL-1}^{\tau_0, p \neq 0} \frac{\hat{\xi}}{r_0 - c\tau_0}, \quad (104)$$

where  $l \geq 1$  and the differential operators are given by Eqs. (64) and (65).

For the upper limit of the term of Eq. (102) and for the upper limit of the sum of the terms of Eqs. (103) and (104) one obtains

$$|\delta(\mathbf{k}, \Delta \mathbf{n}_{S_L}^{1,p \neq 0})| = C_2^{S_L} \frac{GM}{c^3} \Omega |J_{l-1}| \left(\frac{P}{r_1}\right)^{l+1}, \quad (105)$$

$$|\delta(\mathbf{k}, \Delta \mathbf{n}_{S_L}^{2+3, p \neq 0})| = B_2^{S_L} \frac{GM}{c^3} \Omega |J_{l-1}| \left(\frac{P}{r_1}\right)^l \left(\frac{P}{d_k}\right)^l. \quad (106)$$

The proof of these relations is similar to the considerations of Appendices I and J for the mass-multipole terms. But, as mentioned above, only the very first few spin-multipoles are relevant on the sub- $\mu$ as-level: the spin-dipole term  $l = 1$  and the spin-hexapole term  $l = 3$ . In particular, for the spin-dipole one obtains  $C_2^{S_1} = 0$  and  $B_2^{S_1} = 0$ , which can be deduced by very similar steps as presented in Appendix C. From these relations (105) and (106) it is obvious that the contributions of these terms

to the angle of light deflection is much less than 1 nas in higher orders of the spin-multipoles,

$$\left| \delta \left( \mathbf{k}, \Delta \mathbf{n}_{S_L}^{1,p \neq 0} \right) \right| \ll 1 \text{ nas}, \quad (107)$$

$$\left| \delta \left( \mathbf{k}, \Delta \mathbf{n}_{S_L}^{2+3,p \neq 0} \right) \right| \ll 1 \text{ nas}, \quad (108)$$

while in case of spin-dipole these terms vanish. Therefore, these terms (102) - (102) can be neglected in astrometric measurements on the sub-micro-arcsecond level.

These facts allow us to simplify the unit tangent vector (47) of a light ray considerably by neglecting the terms of Eqs. (87) - (89) as well as of Eqs. (74) - (76) and of Eqs. (102) - (104). In other words, for sub- $\mu$ as astrometry it is sufficient to take into account only the terms of Eqs. (84) - (86) and of Eqs. (99) - (101).

### C. The reduced tangent vector

As discussed in the previous Subsections, for astrometry on the sub- $\mu$ as level one needs to account only for the terms in the first line of Eq. (83) and in the first line of Eq. (98). Then one arrives at the reduced unit tangent vector of the light ray at the position of the observer, given by

$$\tilde{\mathbf{n}} = \mathbf{k} + \sum_{l=0}^{\infty} \Delta \tilde{\mathbf{n}}_{M_L} + \sum_{l=1}^{\infty} \Delta \tilde{\mathbf{n}}_{S_L}. \quad (109)$$

The individual terms of the reduced unit tangent vector of Eq. (109) are given by

$$\Delta \tilde{\mathbf{n}}_{M_L} = \Delta \mathbf{n}_{M_L}^{1,p=0} + \Delta \mathbf{n}_{M_L}^{2,p=0} + \Delta \mathbf{n}_{M_L}^{3,p=0}, \quad (110)$$

$$\Delta \tilde{\mathbf{n}}_{S_L} = \Delta \mathbf{n}_{S_L}^{1,p=0} + \Delta \mathbf{n}_{S_L}^{2,p=0} + \Delta \mathbf{n}_{S_L}^{3,p=0}, \quad (111)$$

with the mass-multipole terms of Eqs. (84) - (86) and spin-multipole terms of Eqs. (99) - (101). In the expression of the reduced unit tangent vector (109) terms have been neglected which contribute at most a few nano-arcseconds to the angle of light deflection. The reduced unit tangent vector (109) carries a wide tilde in order to distinguish it from the unit tangent vector of Eq. (47) which contains the complete set of terms in the 1.5PN approximation. In Section VII the reduced tangent vector (109) will further be simplified. For that we need two differential operations which are considered in the subsequent Section.

## VI. TWO DIFFERENTIAL OPERATIONS

By considering the mass-multipole terms of Eqs. (110) with (84) - (86) and the spin-multipole terms (111) with (99) - (101) one encounters two typical differential oper-

ations:

$$\mathbf{U}_L(c\tau, r) = \hat{\partial}_L \frac{\hat{\xi}}{r - c\tau} \frac{1}{r}, \quad (112)$$

$$\mathbf{V}_L(c\tau, r) = \hat{\partial}_L \frac{\hat{\xi}}{r - c\tau}. \quad (113)$$

For these differential operations we are using the following two relations, which are shown in Appendix G,

$$\mathbf{U}_L(c\tau, r) = \mathbf{U}_L^1(c\tau, r) + \mathbf{U}_L^2(c\tau, r), \quad (114)$$

with

$$\begin{aligned} \mathbf{U}_L^1(c\tau, r) = & (-1)^{l+1} \text{STF}_{i_1 \dots i_l} \frac{\partial}{\partial \hat{\xi}} \left[ \sum_{n=0}^{[l/2]} G_n^l P_{i_1 i_2} \dots P_{i_{2n-1} i_{2n}} \right. \\ & \left. \times \left( 1 + \frac{c\tau}{r} \right) \frac{\hat{\xi}_{i_{2n+1}} \dots \hat{\xi}_{i_l}}{\left( \hat{\xi} \right)^{2l-2n}} \right], \end{aligned} \quad (115)$$

$$\begin{aligned} \mathbf{U}_L^2(c\tau, r) = & (-1)^{l+1} \text{STF}_{i_1 \dots i_l} \frac{\partial}{\partial \hat{\xi}} \left[ \sum_{n=0}^{[l/2]} G_n^l P_{i_1 i_2} \dots P_{i_{2n-1} i_{2n}} \right. \\ & \left. \times \hat{F}_n^l(c\tau, r) \frac{\hat{\xi}_{i_{2n+1}} \dots \hat{\xi}_{i_l}}{\left( \hat{\xi} \right)^{2l-2n}} \right], \end{aligned} \quad (116)$$

and

$$\mathbf{V}_L(c\tau, r) = \mathbf{V}_L^1(c\tau, r) + \mathbf{V}_L^2(c\tau, r), \quad (117)$$

with

$$\begin{aligned} \mathbf{V}_L^1(c\tau, r) = & (-1)^{l+1} \text{STF}_{i_1 \dots i_l} \frac{\partial}{\partial \hat{\xi}} \left[ \sum_{n=0}^{[l/2]} G_n^l P_{i_1 i_2} \dots P_{i_{2n-1} i_{2n}} \right. \\ & \left. \times c\tau \left( 1 + \frac{c\tau}{r} \right) \frac{\hat{\xi}_{i_{2n+1}} \dots \hat{\xi}_{i_l}}{\left( \hat{\xi} \right)^{2l-2n}} \right], \end{aligned} \quad (118)$$

$$\begin{aligned} \mathbf{V}_L^2(c\tau, r) = & (-1)^{l+1} \text{STF}_{i_1 \dots i_l} \frac{\partial}{\partial \hat{\xi}} \left[ \sum_{n=0}^{[l/2]} G_n^l P_{i_1 i_2} \dots P_{i_{2n-1} i_{2n}} \right. \\ & \left. \times \left( c\tau \hat{F}_n^l(c\tau, r) + r W_n^l \left( \frac{\hat{\xi}}{r} \right)^{2l-2n} \right) \frac{\hat{\xi}_{i_{2n+1}} \dots \hat{\xi}_{i_l}}{\left( \hat{\xi} \right)^{2l-2n}} \right]. \end{aligned} \quad (119)$$

The dimensionless scalar function  $\hat{F}_n^l(c\tau, r)$  is defined by Eq. (F23). Let us notice here that this scalar function exists only for  $l > 1$  and vanishes for  $l = 1$ . Below it is shown that these terms of Eqs. (116) and (119) can be neglected on the sub-micro-arcsecond level. In Eqs. (115) - (116) and Eqs. (118) - (119) we have used the abbreviation [21]

$$\frac{\partial}{\partial \hat{\xi}^i} = P^{ij} \frac{\partial}{\partial \xi^j} \quad (120)$$

for the spatial components ( $i = 1, 2, 3$ ) of the operation  $\partial/\partial\hat{\xi}$ , where the projector is defined by Eq. (55). In these relations (114) and (117) the coefficients

$$G_n^l = (-1)^n 2^{l-2n-1} \frac{l!}{n!} \frac{(l-n-1)!}{(l-2n)!}, \quad (121)$$

$$W_n^l = \frac{(2l-2n-3)!!}{(2l-2n-2)!!} = \frac{1}{2^{2l-2n-2}} \binom{2l-2n-2}{l-n-1}, \quad (122)$$

have been introduced. The coefficient (121) is identical with the coefficient given by Eq. (55) in [30].

A further comment is in order here. Because the STF operation of any tensor vanishes if it contains a Kronecker symbol (cf. Eqs (B13) - (B15) in Appendix B), one may replace the projectors (55) in these relations (114) and (117) by

$$P_{i_{p+1}i_{p+2}} \dots P_{i_{p+2n-1}i_{p+2n}} \rightarrow (-1)^n k_{i_{p+1}} \dots k_{i_{p+2n}}. \quad (123)$$

For later purposes we give here a slightly more general version of that replacement, while in these relations (114) and (117) one only needs the case with  $p = 0$ . In case of the spin-multipoles (99) - (101) one would seemingly encounter  $\hat{\partial}_{aL-1}$  instead of  $\hat{\partial}_L$  of Eqs. (112) and (113). But by changing the dummy index from  $a$  to  $i_l$  one arrives at the very same expressions as given by Eqs. (112) and (113). That means, one may apply the same replacement (123) also in case of spin-multipole terms.

## VII. THE SIMPLIFIED TANGENT VECTOR

The reduced tangent vector of the light ray is given by Eq. (109) with (110) and (111), where the individual mass-multipole terms are given by Eqs. (84) - (86) and the individual spin-multipole terms are given by Eqs. (99) - (101). The reduced tangent vector is a step in between to arrive at the simplified tangent vector, which is the primary aim of this investigation and which is obtained in this Section.

### A. Two further simplifications

The expressions (114) and (117) have to be implemented into Eqs. (84) - (86) and into Eqs. (99) - (101). In doing so one obtains expressions, which are still much too cumbersome for an effective treatment of astrometric data. In particular, they do contain many terms which contribute less than a few nas to the effect of light deflection. Therefore, two further simplifications are performed:

#### 1. First simplification

The term (116) is neglected both for the mass-multipole and spin-multipole terms. In order to demon-

strate that (116) can be neglected on the sub- $\mu\text{as}$  level, we consider the mass-multipole terms. That means, we insert (116) into Eq. (84) and obtain

$$\mathbf{U}_2^{ML} (c\tau_1, r_1) = -\frac{2G\hat{M}_L}{c^2} \frac{(-1)^l}{l!} \mathbf{U}_L^2 (c\tau_1, r_1), \quad (124)$$

where  $\mathbf{U}_2^{ML}$  is a three-vector in space. The tensorial indices of multi-index  $L$  of  $\mathbf{U}_L^2$  and of the STF mass-multipoles  $\hat{M}_L$  are contracted with each other. In Appendix K its contribution to the angle of light deflection (183) is determined and given by

$$|\delta(\mathbf{k}, \mathbf{U}_2^{ML})| = C_2^{ML} \frac{GM}{c^2 r_1} \left(\frac{P}{r_1}\right) |J_l| \left(\frac{P}{d_k}\right)^{l-1}. \quad (125)$$

The coefficients  $C_2^{ML}$  are presented in Table XI of Appendix K. This term (125) is of the order (79) and contributes less than 1 nas to the effect of light deflection at any body of the Solar System,

$$|\delta(\mathbf{k}, \mathbf{U}_2^{ML})| \ll 1 \text{ nas}. \quad (126)$$

The same statement is valid for the spin-multipole terms, that means we insert (116) into Eq. (99) and obtain

$$\mathbf{U}_2^{SL} (c\tau_1, r_1) = -\frac{4G\hat{S}_{bL-1}}{c^3} \epsilon_{abc} k^c \frac{(-1)^l l}{(l+1)!} \mathbf{U}_{aL-1}^2 (c\tau_1, r_1), \quad (127)$$

where  $\mathbf{U}_2^{SL}$  is a three-vector in space. By implementing the spin-multipoles (151) into (127) one finds that the contribution of (127) to the angle of light deflection for  $l > 1$  is given by

$$|\delta(\mathbf{k}, \mathbf{U}_2^{SL})| = C_3^{SL} \frac{GM}{c^3} \Omega |J_{l-1}| \left(\frac{P}{r_1}\right)^2 \left(\frac{P}{d_k}\right)^{l-1}. \quad (128)$$

For the spin-dipole  $l = 1$  one obtains  $C_3^{S1} = 0$ ; see also the corresponding remark below Eq. (119). The proof of the step from (127) to (128) goes similar as the approach of Appendix L for the mass-multipoles. From these relations it is clear that these terms contribute much less than 1 nas to the angle of light deflection in higher orders of the spin-multipoles. By inserting the parameters of Table III into (128), one obtains for grazing light rays at any solar system body

$$\left| \delta(\mathbf{k}, \mathbf{U}_2^{SL}) \right| \ll 1 \text{ nas}. \quad (129)$$

Therefore, these terms (127) can be neglected for astrometry on the sub-micro-arcsecond level of accuracy.

#### 2. Second simplification

The term (119) is neglected both for the mass-multipole and spin-multipole terms. In order to demonstrate that (119) can be neglected on the sub- $\mu\text{as}$  level,

Table II: The contribution of the terms of Eq. (131) to the angle of light deflection in case of a grazing ray at Jupiter and Saturn. For all other solar system bodies these terms are even smaller. All values are given in nano-arcseconds (nas). A blank entry means less than 1 nas.

$l$	$ \delta_{\gamma_4}(\mathbf{k}, \mathbf{V}_2^{M_L}) $	$ \delta_{\gamma_7}(\mathbf{k}, \mathbf{V}_2^{M_L}) $
2	5.0	1.7
4	1.4	0.6
6	0.3	0.2
8	0.1	0.1
10	—	—

we consider the mass-multipole terms. That means, we insert (119) into Eqs. (85) and (86) and obtain

$$\mathbf{V}_2^{M_L} = \frac{2G\hat{M}_L}{c^2} \frac{(-1)^l}{l!} \frac{1}{R} (\mathbf{V}_L^2(c\tau_1, r_1) - \mathbf{V}_L^2(c\tau_0, r_0)), \quad (130)$$

where  $\mathbf{V}_2^{M_L}$  is a three-vector in space. The tensorial indices of multi-index  $L$  of  $\mathbf{V}_L^2$  and of the STF mass-multipoles  $\hat{M}_L$  are contracted with each other. In Appendix L its contribution to the angle of light deflection (183) is determined and given by

$$|\delta(\mathbf{k}, \mathbf{V}_2^{M_L})| = B_2^{M_L} \frac{GM}{c^2 r_1} |J_l| \left(\frac{P}{d_k}\right)^l. \quad (131)$$

The coefficients  $B_2^{M_L}$  are presented in Table XII of Appendix L. This term is of the order (78). Numerical values of this term are presented in Table II. It is noticed here, that the term of Eq. (130) is much larger than the term of Eq. (124). The reason is that the distance  $R$  between source and observer appears in the denominator of Eq. (130). The distance  $R$  can in principle be arbitrarily small. A similar observation has been made above; see text below Eq. (92). Nevertheless, the term of Eq. (130) is finite and, according to (131), contributes at most a few nas.

The same statement is valid for the spin-multipole terms, that means we insert (119) into Eqs. (100) and (101) and obtain

$$\mathbf{V}_2^{S_L} = \frac{4G\hat{S}_{bL-1}}{c^3} \epsilon_{abc} k^c \frac{(-1)^l}{(l+1)!} \frac{1}{R} \times (\mathbf{V}_{aL-1}^2(c\tau_1, r_1) - \mathbf{V}_{aL-1}^2(c\tau_0, r_0)), \quad (132)$$

where  $\mathbf{V}_2^{S_L}$  is a three-vector in space. By implementing the spin-multipoles (151) into (132) one finds that the contribution of (132) to the angle of light deflection is given by

$$|\delta(\mathbf{k}, \mathbf{V}_2^{S_L})| = B_3^{S_L} \frac{GM}{c^3} \Omega |J_{l-1}| \left(\frac{P}{r_1}\right) \left(\frac{P}{d_k}\right)^l. \quad (133)$$

The proof of the step from (132) to (133) proceeds in the same way as the approach presented in Appendix L for the mass-multipoles. As mentioned in the text below Eq. (94) only the spin-dipole term  $l = 1$  and the spin-hexapole term  $l = 3$  are relevant on the sub- $\mu\text{as}$  level. In order to get an idea about the magnitude of these terms in (133) we consider the case of spin-dipole, which is the most dominating term. By using very similar steps as done in Appendix C, one obtains  $B_3^{S_1} = 2\kappa^2$ . By inserting the parameters of Table III into (133) one finds that this term contributes much less than 1 nas to the angle of light deflection in case of spin-dipole. The coefficients  $B_3^{S_L}$  of Eq. (133) are not given here explicitly. But from that relation it is obvious that the contributions of higher spin-multipoles to the angle of light deflection is much less than 1 nas to any order of the spin-multipole index,

$$\left| \delta(\mathbf{k}, \mathbf{V}_2^{S_L}) \right| \ll 1 \text{ nas}, \quad (134)$$

for grazing light rays at any Solar System body. Therefore, these terms (132) can be neglected for astrometry on the sub-micro-arcsecond level of accuracy.

## B. The simplified tangent vector

The reduced tangent vector was given by Eq. (109) with the individual terms of Eqs. (110) and (111), where the mass-multipole terms are given by Eqs. (84) - (86) and the spin-multipole terms by Eqs. (99) - (101). By taking into account these two simplifications of the previous Subsection, one arrives from the reduced tangent vector the simplified tangent vector of a light ray. This simplified tangent vector is written in the form

$$\hat{\mathbf{n}} = \mathbf{k} + \sum_{l=0}^{\infty} \Delta\hat{\mathbf{n}}_{M_L} + \sum_{l=1}^{\infty} \Delta\hat{\mathbf{n}}_{S_L}. \quad (135)$$

The unit tangent vector (135) and the perturbations are denoted by a *hat*, in order to distinguish them from the unit tangent vector of Eq. (47) and its perturbations. These mass-multipole and spin-multipole terms in (135) are given in the next two Subsections. This simplified tangent vector (135) of a light ray in the gravitational field of an arbitrary Solar System body contains all those terms which contribute at least  $0.01 \mu\text{as}$  to the angle of light deflection. It will be shown below in Section XI that only the first few mass-multipoles with multipole index  $0 \leq l \leq 8$  and the first two spin-multipoles with multipole index  $1 \leq l \leq 3$  are relevant on the sub- $\mu\text{as}$  level of accuracy.

### 1. The mass-multipole terms

The mass-multipole terms in (135) read

$$\Delta\hat{\mathbf{n}}_{M_L} = \Delta\hat{\mathbf{n}}_{M_L}^1 + \Delta\hat{\mathbf{n}}_{M_L}^2 + \Delta\hat{\mathbf{n}}_{M_L}^3, \quad (136)$$

with the individual terms

$$\Delta \hat{\mathbf{n}}_{ML}^1 = + \frac{2G\hat{M}_L}{c^2} \frac{1}{l!} \left( 1 + \frac{\mathbf{k} \cdot \mathbf{x}_1}{r_1} \right) \times \frac{\partial}{\partial \hat{\xi}} \sum_{n=0}^{[l/2]} (-1)^n G_n^l k_{i_1} \dots k_{i_{2n}} \frac{\hat{\xi}_{i_{2n+1}} \dots \hat{\xi}_{i_l}}{(\hat{\xi})^{2l-2n}}, \quad (137)$$

$$\Delta \hat{\mathbf{n}}_{ML}^2 = - \frac{2G\hat{M}_L}{c^2} \frac{1}{l!} \frac{\mathbf{k} \cdot \mathbf{x}_1}{R} \left( 1 + \frac{\mathbf{k} \cdot \mathbf{x}_1}{r_1} \right) \times \frac{\partial}{\partial \hat{\xi}} \sum_{n=0}^{[l/2]} (-1)^n G_n^l k_{i_1} \dots k_{i_{2n}} \frac{\hat{\xi}_{i_{2n+1}} \dots \hat{\xi}_{i_l}}{(\hat{\xi})^{2l-2n}}, \quad (138)$$

$$\Delta \hat{\mathbf{n}}_{ML}^3 = + \frac{2G\hat{M}_L}{c^2} \frac{1}{l!} \frac{\mathbf{k} \cdot \mathbf{x}_0}{R} \left( 1 + \frac{\mathbf{k} \cdot \mathbf{x}_0}{r_0} \right) \times \frac{\partial}{\partial \hat{\xi}} \sum_{n=0}^{[l/2]} (-1)^n G_n^l k_{i_1} \dots k_{i_{2n}} \frac{\hat{\xi}_{i_{2n+1}} \dots \hat{\xi}_{i_l}}{(\hat{\xi})^{2l-2n}}. \quad (139)$$

## 2. The spin-multipole terms

The spin-multipole terms in (135) read

$$\Delta \hat{\mathbf{n}}_{SL} = \Delta \hat{\mathbf{n}}_{SL}^1 + \Delta \hat{\mathbf{n}}_{SL}^2 + \Delta \hat{\mathbf{n}}_{SL}^3, \quad (140)$$

with the individual terms

$$\Delta \hat{\mathbf{n}}_{SL}^1 = + \frac{4G\hat{S}_{bL-1}}{c^3} \epsilon_{i_1bc} k^c \frac{l}{(l+1)!} \left( 1 + \frac{\mathbf{k} \cdot \mathbf{x}_1}{r_1} \right) \times \frac{\partial}{\partial \hat{\xi}} \sum_{n=0}^{[l/2]} (-1)^n G_n^l k_{i_1} \dots k_{i_{2n}} \frac{\hat{\xi}_{i_{2n+1}} \dots \hat{\xi}_{i_l}}{(\hat{\xi})^{2l-2n}}, \quad (141)$$

$$\Delta \hat{\mathbf{n}}_{SL}^2 = - \frac{4G\hat{S}_{bL-1}}{c^3} \epsilon_{i_1bc} k^c \frac{l}{(l+1)!} \frac{\mathbf{k} \cdot \mathbf{x}_1}{R} \left( 1 + \frac{\mathbf{k} \cdot \mathbf{x}_1}{r_1} \right) \times \frac{\partial}{\partial \hat{\xi}} \sum_{n=0}^{[l/2]} (-1)^n G_n^l k_{i_1} \dots k_{i_{2n}} \frac{\hat{\xi}_{i_{2n+1}} \dots \hat{\xi}_{i_l}}{(\hat{\xi})^{2l-2n}}, \quad (142)$$

$$\Delta \hat{\mathbf{n}}_{SL}^3 = + \frac{4G\hat{S}_{bL-1}}{c^3} \epsilon_{i_1bc} k^c \frac{l}{(l+1)!} \frac{\mathbf{k} \cdot \mathbf{x}_0}{R} \left( 1 + \frac{\mathbf{k} \cdot \mathbf{x}_0}{r_0} \right) \times \frac{\partial}{\partial \hat{\xi}} \sum_{n=0}^{[l/2]} (-1)^n G_n^l k_{i_1} \dots k_{i_{2n}} \frac{\hat{\xi}_{i_{2n+1}} \dots \hat{\xi}_{i_l}}{(\hat{\xi})^{2l-2n}}. \quad (143)$$

In Eqs. (137) - (139) and Eqs. (141) - (143) the new variables  $c\tau_0$  and  $c\tau_1$  have been replaced by  $\mathbf{k} \cdot \mathbf{x}_0$  and  $\mathbf{k} \cdot \mathbf{x}_1$ , respectively, in line with the statements below Eq. (45) and by taking account of relations (53) and (54).

There are two factors  $(-1)^n$  in these expressions: one factor is contained in the coefficients  $G_n^l$  and one factor appears in each of these equations. Below it becomes clear, why it is more appropriate not to cancel these terms against each other, but to keep them explicitly as is, namely in the step from (166) to (167) and in the step from (175) to (176).

## C. The complete sum of all neglected terms

As noticed in the introductory Section, the term *sub-micro-arcsecond astrometry* refers to an astrometric precision of 0.1 micro-arcseconds in positional measurements. A relativistic model of light propagation should be 10 times more accurate than this accuracy. Thus, in a theoretical model of light propagation on the sub- $\mu$ as level of precision one may neglect terms which contribute less than 10 nano-arcsecond to the angle of light deflection. The effect of light deflection is an integral effect of all multipoles of the concrete massive body under consideration. Accordingly, the simplified tangent vector (135) should differ from the tangent vector (47), which is exact in the 1.5PN approximation, only by terms whose total sum over all multipoles contributes less than 10 nano-arcsecond to the angle of light deflection.

The neglected spin-multipole terms were given by Eqs. (74) - (76) and by Eqs. (102) - (104) as well as by Eqs. (127) and (132). All these neglected terms contribute much less than 1 nas to the angle of light deflection, as indicated by Eqs. (95), (96), (107), (108), (129), and (134). Therefore, these neglected spin-multipole terms need not to be further discussed here.

The neglected mass-multipole terms were given by Eqs. (87) - (89) as well as Eqs. (124) and (130). The numerical magnitude of the terms of Eqs. (87) and (124) are much less than 1 nas, as indicated by Eqs. (92) and (126), and need also not to be further discussed here.

So we are left with the neglected terms which were given by Eqs. (88) - (89) as well as by Eq. (130). The numerical magnitude of these terms are given by Table I and Table II for grazing light rays at Jupiter and Saturn, that is the impact vector  $d_k$  equals the radius  $P$  of the body. These numerical values for  $d_k = P$  sum up to a total amount of

$$\sum_{l=2}^{10} \left[ |\delta_{\gamma_k}(\mathbf{k}, \Delta \mathbf{n}_{ML}^{2+3,p \neq 0})| + |\delta_{\gamma_k}(\mathbf{k}, \mathbf{V}_2^{ML})| \right] \leq 36.2 \text{ nas}, \quad (144)$$

$$\sum_{l=2}^{10} \left[ |\delta_{\gamma_k}(\mathbf{k}, \Delta \mathbf{n}_{ML}^{2+3,p \neq 0})| + |\delta_{\gamma_k}(\mathbf{k}, \mathbf{V}_2^{ML})| \right] \leq 14.9 \text{ nas}, \quad (145)$$

for Jupiter (144) and Saturn (145). For all other solar system bodies the complete sum of neglected terms over all multipoles would be smaller than the given threshold of 10 nas for all astrometric configurations including grazing light rays.

The neglected terms for grazing ray at Saturn are very nearby the given threshold of 10 nas. In case of Jupiter, the neglected terms sum up to 36.2 nas. However, these values are given for the case of the minimal distance between Earth and Jupiter. Furthermore, in case the impact vector is slightly larger than the radius of Jupiter,

$d_k \geq 1.7 \times P$ , then we get in total only

$$\sum_{l=2}^{10} \left[ |\delta_{\mathbb{4}}(\mathbf{k}, \Delta \mathbf{n}_{M_L}^{2+3, p \neq 0})| + |\delta_{\mathbb{4}}(\mathbf{k}, \mathbf{V}_2^{M_L})| \right] \leq 9.8 \text{ nas}, \quad (146)$$

which is also below the given threshold of 10 nas.

A further comment is made about the total sum of neglected terms (144) in case of grazing rays at Jupiter. The dominant contribution of neglected terms comes from the mass-quadrupole. For extreme configurations in the very vicinity of Jupiter, where the impact parameter is smaller than  $1.7 \times P$ , one may, of course, replace the simplified mass-quadrupole term (i.e. the term  $l = 2$  of Eqs. (137) - (139)) by the exact mass-quadrupole term of the unit tangent vector in 1PN approximation (i.e. the term  $l = 2$  of Eqs. (67) - (69)). That quadrupole term of the tangent vector, which is exact in the 1PN approximation, has also been given in its explicit form by Eq. (14) in [42] or by Eq. (47) in [43]. In this way one would arrive at a simplified unit tangent vector which differs from the exact unit tangent vector (47) in 1.5PN approximation only by terms, which in total contribute less than 11 nas for all multipoles with  $l \geq 0$  and for all possible astrometric configurations, including the case of grazing light rays. The same procedure can, of course, be applied in case of grazing ray at Saturn.

In summary of this Section: the simplified unit tangent vector (135) differs from the unit tangent vector (47), which is exact in the 1.5PN approximation, by a series of neglected terms, which in total contribute less than 10 nas for all Solar System bodies and all possible astrometric configurations. Only in the very vicinity of Jupiter, where the light rays of bright sources have an impact parameter of smaller than 1.7 times the radius of Jupiter, one has to implement the full mass-quadrupole term in the 1PN approximation, in explicit form given in [42, 43].

## VIII. MULTIPOLES OF AXI-SYMMETRIC BODY

In each of the expressions of Eqs. (137) - (139) there appears the mass-multipole,  $\hat{M}_L$ , for an arbitrarily shaped body, as given by Eq. (7), while in each of the expressions of Eqs. (141) - (143) there appears the spin-multipole,  $\hat{S}_L$ , for an arbitrarily shaped body, as given by Eq. (8). In order to determine the magnitude of the individual terms in these equations, one needs a model for these Solar System bodies to determine these multipoles. To get an idea about the magnitude of these individual terms, one may approximate the Sun and the giant planets by a rigid axi-symmetric body with radial dependent mass distribution, having the shape

$$\frac{(x^1)^2}{A^2} + \frac{(x^2)^2}{B^2} + \frac{(x^3)^2}{C^2} = 1, \quad (147)$$

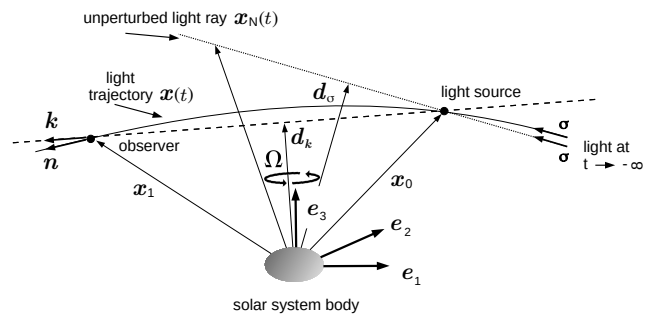


Figure 2: A geometrical representation of the propagation of a light signal through the gravitational field of an axi-symmetric body at rest. The origin of the spatial Cartesian coordinates  $(x^1, x^2, x^3)$  is assumed to be located at the center-of-mass of the body. The unit vectors  $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$  point in the spatial directions of the principal axes of the massive body, which are orthogonal to each other. The body is in uniform rotational motion around its symmetry axis  $\mathbf{e}_3$  with angular velocity  $\Omega$ . The rotational axis of the massive body is aligned with the  $x^3$ -axis of the coordinate system,  $\mathbf{e}_3 = (0, 0, 1)$ . The light signal is emitted by the light source at  $\mathbf{x}_0$  and propagates along the exact light trajectory  $\mathbf{x}(t)$ . The unperturbed light ray  $\mathbf{x}_N(t)$  is given by Eq. (9) and propagates in the direction of  $\boldsymbol{\sigma}$  along a straight line through the position of the light source at  $\mathbf{x}_0$ . The impact vector  $\mathbf{d}_\sigma$  is given by Eq. (10) and the impact vector  $\mathbf{d}_k$  is given by Eq. (19).

where  $A = B \neq C$  are the principal axes of the body. Furthermore, the body can be assumed to be in uniform rotational motion around the symmetry axis,  $\mathbf{e}_3$ , of the body; see Figure 2.

If the coordinate system is chosen such that the rotational axis of the massive body is aligned with the  $x^3$ -axis of the coordinate system,  $\mathbf{e}_3 = (0, 0, 1)$ , then the mass-multipoles (7) and spin-multipoles (8) for such a body are given by [7, 11, 30, 31]

$$\hat{M}_0 = -M (P)^0 J_0, \quad (148)$$

$$\hat{M}_L = -M (P)^l J_l \delta_{<i_1}^3 \dots \delta_{i_l}^3, \quad (149)$$

$$\hat{S}_a = -M \Omega (P)^2 J_0 \kappa^2 \delta_a^3, \quad (150)$$

$$\hat{S}_L = -M \Omega (P)^{l+1} J_{l-1} \frac{l+1}{l+4} \delta_{<i_1}^3 \dots \delta_{i_l}^3, \quad (151)$$

where (149) is valid for a natural number of  $l \geq 2$ , while (151) is valid for a natural number of  $l \geq 3$ , and  $\delta_a^3$  is the  $a$ -th component of the unit vector  $\mathbf{e}_3$  of the principal axis of the body, which points in the  $z$ -direction of the coordinate system:  $\delta_a^3 = (0, 0, 1)$ . Here,  $M$  is the Newtonian mass of the body,  $P$  its equatorial radius of the body, and  $\Omega$  is the angular velocity of the rotating body. The parameter  $\kappa^2$  in (150) is defined by [32] (see also Eqs. (B60) - (B62) in [31])

$$\kappa^2 = \frac{I}{M P^2}, \quad (152)$$

where  $I$  is the moment of inertia of the real Solar System body under consideration, which is related to the body's angular momentum via  $|\mathbf{S}| = I\Omega$ . For a spherically symmetric body with uniform density  $\kappa^2 = 2/5$ , while for real Solar System bodies  $\kappa^2 < 2/5$  because the mass densities are increasing towards the center of the massive bodies. The values of  $\kappa^2$  are given in Table III for the Sun and giant planets of the Solar System bodies.

The parameter  $J_l$  are the actual zonal harmonic coefficients of index  $l$ , defined by [7, 33]

$$J_l = -\frac{1}{M(P)^l} \int d^3x r^l \frac{T^{00} + T^{kk}}{c^2} P_l(\cos\theta), \quad (153)$$

where  $P_l$  are the Legendre polynomials, while the angle  $\theta$  is the colatitude. The zonal harmonic coefficients (153) are defined for an axi-symmetric body  $A = B \neq C$  [7, 33]. They are model-depend in the sense that they depend on assumptions made for the mass distribution in the interior of the Solar System bodies. Therefore, it is preferable to use actual zonal harmonics which are deduced from measurements of the gravitational fields of the Sun and giant planets. Their numerical values are given in Table III for the Sun and the giant planets of the Solar System.

These expressions in (149) and (151) have been derived in Appendix B in [31] for an axi-symmetric body. The zonal harmonic coefficient of the mass-monopole term is  $J_0 = -1$ , as it follows from (153). The mass-dipole term, that is  $l = 1$  in (7) vanishes in case the origin of coordinate system is located at the center-of-mass of the body:  $J_1 = 0$  [7, 9, 11] (e.g. Eq. (8.14c) in [11]) and will, therefore, not be considered in what follows.

The STF tensor  $\text{STF}_{i_1 \dots i_l} \delta_{i_1}^3 \dots \delta_{i_l}^3$  in relations (149) and (151) denotes products of Kronecker symbols which are symmetric and traceless with respect to their indices  $i_1 \dots i_l$ . They are given by the formula [7, 12, 30, 31]

$$\text{STF}_{i_1 \dots i_l} \delta_{i_1}^3 \dots \delta_{i_l}^3 = \sum_{s=0}^{\lfloor l/2 \rfloor} H_s^l \delta_{\{i_1 i_2 \dots i_{2s-1} i_{2s} \}} \delta_{i_{2s+1}}^3 \dots \delta_{i_l}^3 \quad (154)$$

where the coefficients are given by

$$H_s^l = (-1)^s \frac{(2l - 2s - 1)!!}{(2l - 1)!!}. \quad (155)$$

The total number  $N$  of terms under the sum in (154) for a given value of  $l$  and  $s$  is given by [7]

$$N_s^l = \frac{l!}{(l - 2s)!(2s)!!}. \quad (156)$$

The multipoles in (148) - (151) are valid for a coordinate system,  $(x^1, x^2, x^3)$ , where the symmetry axis  $\mathbf{e}_3$  of the massive body is aligned with the  $x^3$ -axis. One may transform the multipoles into another coordinate

system,  $(x'^1, x'^2, x'^3)$ , which has the same origin of spatial coordinates, but which is rotated to the previous one [6, 7, 9, 51],

$$x^a = R_b^a x'^b, \quad (157)$$

where the orthogonal matrix of rotation  $R_b^a$  can be parameterized, for instance, by three Euler angles, and is given, for example, by Eq. (3.94) in [51]. Then, the multipoles in (148) - (151) in coordinate system  $\{x^a\}$  and the multipoles in coordinate system  $\{x'^a\}$  are related to each other by the standard transformation of tensors in three-space [6, 7, 9, 51],

$$\hat{M}_{i_1 \dots i_l} = \hat{M}'_{j_1 \dots j_l} R_{i_1}^{j_1} \dots R_{i_l}^{j_l}, \quad (158)$$

$$\hat{S}_{i_1 \dots i_l} = \hat{S}'_{j_1 \dots j_l} R_{i_1}^{j_1} \dots R_{i_l}^{j_l}. \quad (159)$$

For an explicit example in case of mass-quadrupole we refer to Eqs. (48) - (53) in [26]. These relations (158) and (159) allow to switch the mass-multipoles and spin-multipole multipoles from one coordinate system to the other.

## IX. SIMPLIFIED TANGENT VECTOR IN CASE OF AN AXI-SYMMETRIC BODY

According to Eq. (135), the simplified unit tangent vector of light trajectory is given by

$$\hat{\mathbf{n}} = \mathbf{k} + \sum_{l=0}^{\infty} \Delta \hat{\mathbf{n}}_{M_L} + \sum_{l=1}^{\infty} \Delta \hat{\mathbf{n}}_{S_L}, \quad (160)$$

with the mass-multipole terms of Eqs. (137) - (139) and the spin-multipole terms in Eqs. (141) - (143). In order to find the unit tangent vector of the light ray in the gravitational field of an axi-symmetric body, as elucidated by Figure 2, we have to insert the mass-multipoles (148) - (149) and the spin-multipoles (150) - (151) into these equations.

### A. The mass-multipole terms

According to Eq. (136) the mass-multipole terms in (160) are

$$\Delta \hat{\mathbf{n}}_{M_L} = \Delta \hat{\mathbf{n}}_{M_L}^1 + \Delta \hat{\mathbf{n}}_{M_L}^2 + \Delta \hat{\mathbf{n}}_{M_L}^3, \quad (161)$$

with the individual terms of Eqs. (137) - (139).

#### 1. The mass-monopole term: $l = 0$

By inserting the mass-monopole (148) into these equations (137) - (139) one obtains for  $l = 0$ ,

$$\Delta \hat{\mathbf{n}}_{M_{l=0}} = -\frac{2GM}{c^2} \frac{\mathbf{d}_k}{(d_k)^2} \frac{r_0 r_1 - \mathbf{x}_1 \cdot \mathbf{x}_1}{R r_1}, \quad (162)$$

Table III: Numerical parameter for mass  $M$ , equatorial radius  $P$ , actual zonal harmonic coefficients  $J_l$ , angular velocity  $\Omega = 2\pi/T$  (with rotational period  $T$ ), dimensionless moment of inertia  $\kappa^2$  of the Sun and the giant planets of the Solar System. The values for  $GM/c^2$  and  $P$  are taken from [32]. The value for  $J_l$  of the Sun are taken from [34]. The values  $J_l$  with  $l = 2, 4, 6$  of Jupiter and Saturn are taken from [35], while  $J_l$  with  $l = 8, 10$  of Jupiter and Saturn are taken from [36] and [37], respectively. The values  $J_l$  with  $l = 2, 4, 6$  of Uranus and Neptune are taken from [38], while  $J_8$  of Uranus and Neptune is taken from [39]. The angular velocities  $\Omega$  are taken from NASA planetary fact sheets. The values for the dimensionless moment of inertia  $\kappa^2$  are taken from [32]. The minimal distance between body and observer,  $r_1^{\min}$ , is computed under the assumption that the observer is located at Lagrange point  $L_2$ , i.e.  $1.5 \times 10^9$  m from the Earth's orbit. A blank entry means the values are not known.

Parameter	Sun	Jupiter	Saturn	Uranus	Neptune
$GM/c^2$ [m]	1476.8	1.410	0.422	0.064	0.076
$P$ [m]	$696 \times 10^6$	$71.49 \times 10^6$	$60.27 \times 10^6$	$25.56 \times 10^6$	$24.76 \times 10^6$
$r_1^{\min}$ [m]	$0.147 \times 10^{12}$	$0.59 \times 10^{12}$	$1.20 \times 10^{12}$	$2.57 \times 10^{12}$	$4.35 \times 10^{12}$
$J_2$	$+2.21 \times 10^{-7}$	$+14.696 \times 10^{-3}$	$+16.291 \times 10^{-3}$	$+3.341 \times 10^{-3}$	$+3.408 \times 10^{-3}$
$J_4$	$-4.46 \times 10^{-9}$	$-0.587 \times 10^{-3}$	$-0.936 \times 10^{-3}$	$-0.031 \times 10^{-3}$	$-0.031 \times 10^{-3}$
$J_6$	$-2.80 \times 10^{-10}$	$+0.034 \times 10^{-3}$	$+0.086 \times 10^{-3}$	$+0.444 \times 10^{-6}$	$+0.433 \times 10^{-6}$
$J_8$	$+1.49 \times 10^{-11}$	$-2.5 \times 10^{-6}$	$-10.0 \times 10^{-6}$	$-0.008 \times 10^{-6}$	$-0.007 \times 10^{-6}$
$J_{10}$	–	$+0.21 \times 10^{-6}$	$+2.0 \times 10^{-6}$	–	–
$\Omega$ [sec $^{-1}$ ]	$2.865 \times 10^{-6}$	$1.758 \times 10^{-4}$	$1.638 \times 10^{-4}$	$1.012 \times 10^{-4}$	$1.083 \times 10^{-4}$
$\kappa^2$	0.059	0.254	0.210	0.225	0.240

where we have used  $R = \mathbf{k} \cdot \mathbf{x}_1 - \mathbf{k} \cdot \mathbf{x}_0$  and the parameter  $P^{ij}\xi_j$  has been replaced by the impact vector  $d_k^i$ . The same result can be obtained from the mass-monopole terms (67) - (69) which are exact in the 1PN approximation. That means, the mass-monopole terms of the simplified tangent vector (160) are identical with the mass-monopole terms of the tangent vector (47), which is exact in the 1.5PN approximation. The monopole term (162) coincides with the monopole term of Eq. (24) in [40] for Parameterized Post-Newtonian parameter  $\gamma = 1$ .

## 2. The mass-multipole terms: $l \geq 2$

As mentioned above, the mass-dipole term vanishes, so we need to consider only higher mass-multipoles with  $l \geq 2$ . By inserting the mass-multipoles (149) of an axisymmetric body into these equations (137) - (139) one obtains for the higher mass-multipoles

$$\Delta \hat{\mathbf{n}}_{ML}^1 = -\frac{2GM}{c^2} \frac{J_l}{l} \left(1 + \frac{\mathbf{k} \cdot \mathbf{x}_1}{r_1}\right) \frac{\partial}{\partial \hat{\xi}} \left(\frac{P}{\hat{\xi}}\right)^l F_M^l, \quad (163)$$

$$\Delta \hat{\mathbf{n}}_{ML}^2 = +\frac{2GM}{c^2} \frac{J_l}{l} \frac{\mathbf{k} \cdot \mathbf{x}_1}{R} \left(1 + \frac{\mathbf{k} \cdot \mathbf{x}_1}{r_1}\right) \frac{\partial}{\partial \hat{\xi}} \left(\frac{P}{\hat{\xi}}\right)^l F_M^l, \quad (164)$$

$$\Delta \hat{\mathbf{n}}_{ML}^3 = -\frac{2GM}{c^2} \frac{J_l}{l} \frac{\mathbf{k} \cdot \mathbf{x}_0}{R} \left(1 + \frac{\mathbf{k} \cdot \mathbf{x}_0}{r_0}\right) \frac{\partial}{\partial \hat{\xi}} \left(\frac{P}{\hat{\xi}}\right)^l F_M^l, \quad (165)$$

which is valid for  $l \geq 2$ . The scalar function reads

$$F_M^l = \frac{1}{(l-1)!} \sum_{n=0}^{[l/2]} G_n^l \left( \text{STF}_{i_1 \dots i_l} \delta_{i_1}^3 \dots \delta_{i_l}^3 \right) \times (-1)^n k_{i_1} \dots k_{i_{2n}} \frac{\hat{\xi}^{i_{2n+1}} \dots \hat{\xi}^{i_l}}{(\hat{\xi})^{l-2n}}, \quad (166)$$

where the STF operation in the second line in (166) can be omitted in view of relation (B16). The structure of this function of Eq. (166) is identical with the function of Eq. (D3) in our previous investigation [30], where it has been shown that (166) can be written in the following form,

$$F_M^l = \frac{1}{(l-1)!} \sum_{n=0}^{[l/2]} G_n^l \left( \frac{\mathbf{d}_k \cdot \mathbf{e}_3}{d_k} \right)^{l-2n} |\mathbf{k} \times \mathbf{e}_3|^{2n}, \quad (167)$$

where we have used  $1 - (\mathbf{k} \cdot \mathbf{e}_3)^2 = |\mathbf{k} \times \mathbf{e}_3|^2$  and  $\hat{\xi}$  has been replaced by the impact vector  $\mathbf{d}_k$ . It was one of the primary results in our previous investigation [30], that this function can be expressed in terms of Chebyshev polynomials of first kind (D2),

$$F_M^l = |\mathbf{k} \times \mathbf{e}_3|^l T_l(x), \quad (168)$$

where the argument of these Chebyshev polynomials is given by [30]

$$x = |\mathbf{k} \times \mathbf{e}_3|^{-1} \left( \frac{\mathbf{d}_k \cdot \mathbf{e}_3}{d_k} \right), \quad (169)$$

which is a real number of the closed interval  $x \in [-1, +1]$ .

## B. The spin-multipole terms

According to Eq. (140) the spin-multipole terms in (160) are

$$\Delta \widehat{\mathbf{n}}_{S_L} = \Delta \widehat{\mathbf{n}}_{S_L}^1 + \Delta \widehat{\mathbf{n}}_{S_L}^2 + \Delta \widehat{\mathbf{n}}_{S_L}^3, \quad (170)$$

with the individual terms of Eqs. (141) - (143).

### 1. The spin-dipole term: $l = 1$

By inserting the spin-dipole (150) into these equations (141) - (143) one obtains for  $l = 1$ ,

$$\begin{aligned} \Delta \widehat{\mathbf{n}}_{S_{l=1}} = & -\frac{2GM}{c^3} \Omega \kappa^2 \left( \frac{P}{d_k} \right)^2 \left( 1 + \frac{\mathbf{k} \cdot \mathbf{x}_1}{r_1} \right) \\ & \times \left[ 2 \frac{(\mathbf{k} \times \mathbf{d}_k) \cdot \mathbf{e}_3}{d_k} \frac{\mathbf{d}_k}{d_k} + (\mathbf{k} \times \mathbf{e}_3) \right] \\ & + \frac{2GM}{c^3} \Omega \kappa^2 \left( \frac{P}{d_k} \right)^2 \left( 1 + \frac{(\mathbf{k} \cdot \mathbf{x}_1)^2}{R r_1} - \frac{(\mathbf{k} \cdot \mathbf{x}_0)^2}{R r_0} \right) \\ & \times \left[ 2 \frac{(\mathbf{k} \times \mathbf{d}_k) \cdot \mathbf{e}_3}{d_k} \frac{\mathbf{d}_k}{d_k} + (\mathbf{k} \times \mathbf{e}_3) \right], \quad (171) \end{aligned}$$

where we have used  $R = \mathbf{k} \cdot \mathbf{x}_1 - \mathbf{k} \cdot \mathbf{x}_0$  and the parameter  $\widehat{\xi}$  has been replaced by the impact vector  $\mathbf{d}_k$ . In the limit of spatial infinity of the source the second term of Eq. (171) vanishes. Then, in the limit of spatial infinity of the observer, the first term of Eq. (171) reduces to Eq. (60) in [22], Eq. (72) in [41], and Eq. (98) in [30].

### 2. The spin-multipole terms: $l \geq 3$

By inserting the spin-multipoles (151) of an axisymmetric body into these equations (141) - (143) one obtains for the higher spin-multipoles

$$\Delta \widehat{\mathbf{n}}_{S_L}^1 = -\frac{4GM}{c^3} \Omega P \frac{J_{l-1}}{l+4} \left( 1 + \frac{\mathbf{k} \cdot \mathbf{x}_1}{r_1} \right) \frac{\partial}{\partial \widehat{\xi}} \left( \frac{P}{\widehat{\xi}} \right)^l F_S^l, \quad (172)$$

$$\begin{aligned} \Delta \widehat{\mathbf{n}}_{S_L}^2 = & +\frac{4GM}{c^3} \Omega P \frac{J_{l-1}}{l+4} \frac{\mathbf{k} \cdot \mathbf{x}_1}{R} \left( 1 + \frac{\mathbf{k} \cdot \mathbf{x}_1}{r_1} \right) \\ & \times \frac{\partial}{\partial \widehat{\xi}} \left( \frac{P}{\widehat{\xi}} \right)^l F_S^l, \quad (173) \end{aligned}$$

$$\begin{aligned} \Delta \widehat{\mathbf{n}}_{S_L}^3 = & -\frac{4GM}{c^3} \Omega P \frac{J_{l-1}}{l+4} \frac{\mathbf{k} \cdot \mathbf{x}_0}{R} \left( 1 + \frac{\mathbf{k} \cdot \mathbf{x}_0}{r_0} \right) \\ & \times \frac{\partial}{\partial \widehat{\xi}} \left( \frac{P}{\widehat{\xi}} \right)^l F_S^l, \quad (174) \end{aligned}$$

which is valid for  $l > 1$ . The pseudo-scalar function reads

$$\begin{aligned} F_S^l = & \frac{1}{(l-1)!} \epsilon_{i_1 b c} k^c \left( \text{STF}_{b \dots i_{l-1}} \delta_b^3 \delta_{i_1}^3 \dots \delta_{i_{l-1}}^3 \right) \\ & \times \text{STF}_{i_1 \dots i_l} \left[ \sum_{n=0}^{\lfloor l/2 \rfloor} G_n^l (-1)^n k_{i_1} \dots k_{i_{2n}} \frac{\widehat{\xi}^{i_{2n+1}} \dots \widehat{\xi}^{i_l}}{(\widehat{\xi})^{l-2n}} \right]. \quad (175) \end{aligned}$$

The structure of this function of Eq. (175) is identical with the function of Eq. (E3) in our previous investigation [30], where it has been shown that (175) can be written in the following form,

$$\begin{aligned} F_S^l = & \frac{1}{(l-1)!} \frac{(\mathbf{k} \times \mathbf{d}_k) \cdot \mathbf{e}_3}{d_k} \\ & \times \sum_{n=0}^{\lfloor l/2 \rfloor} G_n^l \frac{l-2n}{l} \left( \frac{\mathbf{d}_k \cdot \mathbf{e}_3}{d_k} \right)^{l-2n-1} |\mathbf{k} \times \mathbf{e}_3|^{2n}. \quad (176) \end{aligned}$$

It was one of the primary results in our investigation, that this function can be expressed in terms of Chebyshev polynomials of second kind (D3),

$$F_S^l = \frac{(\mathbf{k} \times \mathbf{d}_k) \cdot \mathbf{e}_3}{d_k} |\mathbf{k} \times \mathbf{e}_3|^l U_{l-1}(x), \quad (177)$$

where the argument of these Chebyshev polynomials has been given by Eq. (169).

## X. COMPARISON WITH THE LITERATURE

The mass-monopole and spin-dipole terms of the unit tangent vector of a light ray in the field of an axisymmetric body, as presented by Eqs. (162) and (171), have already been compared with the literature.

Results for the unit tangent vector of light rays in case of finite distances of source and observer for higher multipoles beyond the mass-monopole and spin-dipole are very rare. So we are restricted here to consider only the next term of the multipole expansion, that is the mass-quadrupole term ( $l = 2$ ) of the unit tangent vector, with existing results in the literature. From Eqs. (163) - (165) we get for  $l = 2$ :

$$\begin{aligned} \Delta \widehat{\mathbf{n}}_{M_{l=2}} = & -\frac{GM}{c^2} J_2 \left( \frac{P}{d_k} \right)^2 \mathcal{A} \\ & \times \left[ \frac{\mathbf{d}_k}{d_k} - (\mathbf{k} \cdot \mathbf{e}_3)^2 \frac{\mathbf{d}_k}{d_k} - 4 \left( \frac{\mathbf{d}_k \cdot \mathbf{e}_3}{d_k} \right)^2 \frac{\mathbf{d}_k}{d_k} \right. \\ & \left. + 2 \left( \frac{\mathbf{d}_k \cdot \mathbf{e}_3}{d_k} \right) \mathbf{e}_3 - 2 (\mathbf{k} \cdot \mathbf{e}_3) \left( \frac{\mathbf{d}_k \cdot \mathbf{e}_3}{d_k} \right) \mathbf{k} \right] \quad (178) \end{aligned}$$

with the dimensionless scalar function

$$\begin{aligned} \mathcal{A} = & + \left( 1 + \frac{\mathbf{k} \cdot \mathbf{x}_0}{r_0} \right) \frac{r_0 + \mathbf{k} \cdot \mathbf{x}_0}{R} \\ & - \left( 1 + \frac{\mathbf{k} \cdot \mathbf{x}_1}{r_1} \right) \frac{r_1 + \mathbf{k} \cdot \mathbf{x}_1}{R} + 2 \left( 1 + \frac{\mathbf{k} \cdot \mathbf{x}_1}{r_1} \right), \quad (179) \end{aligned}$$

where the parameter  $P^{ij}\xi_j$  has been replaced by  $d_k^i$ , in line with the statements below Eq. (45) and taking account of relations (53) and (54). The scalar function of Eq. (179) is, up to a conventional constant  $(d_k)^3$ , identical to the scalar function of Eq. (A.10) in [42] and to the scalar function of Eq. (48) in [43], up to a term of the order  $(P/r_1)^2$  which can safely be neglected for sub-micro-arcsecond astrometry. That means, the expression in (178) further simplifies the previous results in our articles [42, 43] for the mass-quadrupole. Similarly, the term in the square brackets of Eq. (178) is, up to a conventional constant  $d_k$ , identical to the term in the square brackets of Eq. (57) in our article [43].

## XI. THE ANGLE OF LIGHT DEFLECTION

### A. Light deflection at finite spatial distances

The light deflection is defined as angle between three-vector  $\mathbf{k}$  of Eq. (18) and three-vector  $\mathbf{n}$  of Eq. (21),

$$\delta(\mathbf{k}, \mathbf{n}) = \arcsin |\mathbf{k} \times \mathbf{n}|. \quad (180)$$

A graphical elucidation is presented by Figures. 1 and 2. By inserting (47) into (180) the light deflection can be written in the following form,

$$\delta(\mathbf{k}, \mathbf{n}) = \left| \sum_{l=0}^{\infty} \mathbf{k} \times \Delta \mathbf{n}_{M_L} \right| + \left| \sum_{l=1}^{\infty} \mathbf{k} \times \Delta \mathbf{n}_{S_L} \right| + \mathcal{O}(c^{-4}), \quad (181)$$

where  $\arcsin x = x + \mathcal{O}(x^3)$  for  $x \ll 1$  has been used. This equation can also be compared with Eq. (59) in [30], which was only valid for the total light deflection, that means for infinite distances between source and observer from the Solar System body, while (181) is the angle of light deflection at finite distances between source and observer from the Solar System body. The individual terms of Eq. (181) are given by Eqs. (48) and (49).

The effect of the monopole term is by far the most dominant term of light deflection and several orders larger than the higher multipole terms in (181). Accordingly, it is appropriate to take the monopole term, given by Eq. (162), in front of the right-hand side of Eq. (181) and to perform a series expansion. In this way one obtains the following form for the light deflection,

$$\delta(\mathbf{k}, \mathbf{n}) = \sum_{l=0}^{\infty} \delta(\mathbf{k}, \Delta \mathbf{n}_{M_L}) + \sum_{l=1}^{\infty} \delta(\mathbf{k}, \Delta \mathbf{n}_{S_L}) + \mathcal{O}(c^{-4}), \quad (182)$$

where the individual multipole terms are given by

$$\delta(\mathbf{k}, \Delta \mathbf{n}_{M_L}) = -\Delta \mathbf{n}_{M_L} \cdot \frac{\hat{\boldsymbol{\xi}}}{\hat{\xi}}, \quad (183)$$

$$\delta(\mathbf{k}, \Delta \mathbf{n}_{S_L}) = -\Delta \mathbf{n}_{S_L} \cdot \frac{\hat{\boldsymbol{\xi}}}{\hat{\xi}}. \quad (184)$$

A similar step has been used in [21, 22, 30]. One may compare Eqs. (183) and (184) with Eqs. (61) and (62) in [30], respectively, which were valid for the case of source and observer at infinite spatial distances from the body, while here we consider the case of source and observer at finite distances from the gravitating Solar System body.

Now we will use the mass-multipole and spin-multipole terms of the simplified unit tangent vector (135) and obtain for the light deflection angle (182),

$$\delta(\mathbf{k}, \hat{\mathbf{n}}) = \sum_{l=0}^{\infty} \delta(\mathbf{k}, \Delta \hat{\mathbf{n}}_{M_L}) + \sum_{l=1}^{\infty} \delta(\mathbf{k}, \Delta \hat{\mathbf{n}}_{S_L}) + \mathcal{O}(c^{-4}), \quad (185)$$

where the individual multipole terms are given by

$$\delta(\mathbf{k}, \Delta \hat{\mathbf{n}}_{M_L}) = -\left( \Delta \hat{\mathbf{n}}_{M_L}^1 + \Delta \hat{\mathbf{n}}_{M_L}^2 + \Delta \hat{\mathbf{n}}_{M_L}^3 \right) \cdot \frac{\hat{\boldsymbol{\xi}}}{\hat{\xi}}, \quad (186)$$

$$\delta(\mathbf{k}, \Delta \hat{\mathbf{n}}_{S_L}) = -\left( \Delta \hat{\mathbf{n}}_{S_L}^1 + \Delta \hat{\mathbf{n}}_{S_L}^2 + \Delta \hat{\mathbf{n}}_{S_L}^3 \right) \cdot \frac{\hat{\boldsymbol{\xi}}}{\hat{\xi}}, \quad (187)$$

with the individual terms of Eqs. (163) - (165) and Eqs. (172) - (174). The difference between these equations for the light deflection angle is, that Eq. (182) contains all terms of the unit tangent vector, while Eq. (185) contains only those terms which are relevant for the given goal accuracy on the level of  $0.01 \mu\text{as}$ .

In order to get the angle of Eqs. (186) and (187) one would have to perform the differentiation  $\partial/\partial \hat{\boldsymbol{\xi}}$  of Eqs. (163) - (165) as well as Eqs. (172) - (174). A compact form for these individual terms can be achieved by means of the following relations

$$\frac{\hat{\boldsymbol{\xi}}}{\hat{\xi}} \cdot \left( \frac{\partial}{\partial \hat{\boldsymbol{\xi}}} f(\hat{\boldsymbol{\xi}}) \right) = \frac{\partial}{\partial \hat{\xi}} f(\hat{\boldsymbol{\xi}}), \quad (188)$$

which is valid for scalar functions which depend on the absolute value  $\xi = |\boldsymbol{\xi}|$ , and

$$\frac{\hat{\boldsymbol{\xi}}}{\hat{\xi}} \cdot \left( \frac{\partial}{\partial \hat{\boldsymbol{\xi}}} \hat{\xi}_{i_n} \right) = \frac{\partial}{\partial \hat{\xi}} \hat{\xi}_{i_n} = \hat{n}_{i_n}, \quad (189)$$

which is valid for vectorial components of  $\boldsymbol{\xi}$  and where  $\hat{n}_{i_n} = \hat{\xi}_{i_n}/\hat{\xi}$  is the unit direction of the impact vector; we also note that

$$\frac{\partial}{\partial \hat{\xi}} \hat{n}_{i_n} = \frac{\partial}{\partial \hat{\xi}} \frac{\hat{\xi}_{i_n}}{\hat{\xi}} = 0 \quad (190)$$

which reflects the fact that the unit direction of a three-vector is independent of the spatial length of this three-vector. Here we refer to the abbreviation which has been introduced by Eq. (120). Then, using relations (188) and (189), the calculation of these individual terms simplifies

considerably and yield for the mass-multipole terms

$$\Delta \hat{\mathbf{n}}_{ML}^1 \cdot \frac{\hat{\boldsymbol{\xi}}}{\hat{\xi}} = -\frac{2GM}{c^2} \frac{J_l}{l} \left( 1 + \frac{\mathbf{k} \cdot \mathbf{x}_1}{r_1} \right) \times \frac{\partial}{\partial \hat{\xi}} \left( \frac{P}{\hat{\xi}} \right)^l |\mathbf{k} \times \mathbf{e}_3|^l T_l(x), \quad (191)$$

$$\Delta \hat{\mathbf{n}}_{ML}^2 \cdot \frac{\hat{\boldsymbol{\xi}}}{\hat{\xi}} = +\frac{2GM}{c^2} \frac{J_l}{l} \frac{\mathbf{k} \cdot \mathbf{x}_1}{R} \left( 1 + \frac{\mathbf{k} \cdot \mathbf{x}_1}{r_1} \right) \times \frac{\partial}{\partial \hat{\xi}} \left( \frac{P}{\hat{\xi}} \right)^l |\mathbf{k} \times \mathbf{e}_3|^l T_l(x), \quad (192)$$

$$\Delta \hat{\mathbf{n}}_{ML}^3 \cdot \frac{\hat{\boldsymbol{\xi}}}{\hat{\xi}} = -\frac{2GM}{c^2} \frac{J_l}{l} \frac{\mathbf{k} \cdot \mathbf{x}_0}{R} \left( 1 + \frac{\mathbf{k} \cdot \mathbf{x}_0}{r_0} \right) \times \frac{\partial}{\partial \hat{\xi}} \left( \frac{P}{\hat{\xi}} \right)^l |\mathbf{k} \times \mathbf{e}_3|^l T_l(x), \quad (193)$$

and for the spin-multipole terms

$$\Delta \hat{\mathbf{n}}_{SL}^1 \cdot \frac{\hat{\boldsymbol{\xi}}}{\hat{\xi}} = -\frac{4GM}{c^3} \Omega P \frac{J_{l-1}}{l+4} \left( 1 + \frac{\mathbf{k} \cdot \mathbf{x}_1}{r_1} \right) \times \frac{\partial}{\partial \hat{\xi}} \left( \frac{P}{\hat{\xi}} \right)^l \frac{(\mathbf{k} \times \hat{\boldsymbol{\xi}}) \cdot \mathbf{e}_3}{\hat{\xi}} |\mathbf{k} \times \mathbf{e}_3|^l U_{l-1}(x), \quad (194)$$

$$\Delta \hat{\mathbf{n}}_{SL}^2 \cdot \frac{\hat{\boldsymbol{\xi}}}{\hat{\xi}} = +\frac{4GM}{c^3} \Omega P \frac{J_{l-1}}{l+4} \frac{\mathbf{k} \cdot \mathbf{x}_1}{R} \left( 1 + \frac{\mathbf{k} \cdot \mathbf{x}_1}{r_1} \right) \times \frac{\partial}{\partial \hat{\xi}} \left( \frac{P}{\hat{\xi}} \right)^l \frac{(\mathbf{k} \times \hat{\boldsymbol{\xi}}) \cdot \mathbf{e}_3}{\hat{\xi}} |\mathbf{k} \times \mathbf{e}_3|^l U_{l-1}(x), \quad (195)$$

$$\Delta \hat{\mathbf{n}}_{SL}^3 \cdot \frac{\hat{\boldsymbol{\xi}}}{\hat{\xi}} = -\frac{4GM}{c^3} \Omega P \frac{J_{l-1}}{l+4} \frac{\mathbf{k} \cdot \mathbf{x}_0}{R} \left( 1 + \frac{\mathbf{k} \cdot \mathbf{x}_0}{r_0} \right) \times \frac{\partial}{\partial \hat{\xi}} \left( \frac{P}{\hat{\xi}} \right)^l \frac{(\mathbf{k} \times \hat{\boldsymbol{\xi}}) \cdot \mathbf{e}_3}{\hat{\xi}} |\mathbf{k} \times \mathbf{e}_3|^l U_{l-1}(x), \quad (196)$$

where the argument  $x$  is defined by Eq. (169) and  $T_l$  and  $U_l$  are Chebyshev polynomials of first and second kind, given by Eqs. (D1) - (D2) and (D3), respectively. Eqs. (191) - (193) and (194) - (196) have to be inserted into (186) and (187) and yield the angle of light deflection at the observers position in terms of the global coordinate system.

According to Eqs. (186) and (187) one has to build the sum of these expressions in Eqs. (191) - (193) as well as of Eqs. (194) - (196) in order to get the angle of light deflection of Eq. (182). By performing the derivative one gets

$$\delta(\mathbf{k}, \Delta \hat{\mathbf{n}}_{ML}) = -\frac{2GM}{c^2 d_k} J_l F(\mathbf{x}_0, \mathbf{x}_1) \times \left( \frac{P}{d_k} \right)^l |\mathbf{k} \times \mathbf{e}_3|^l T_l(x), \quad (197)$$

for the mass-multipoles and

$$\delta(\mathbf{k}, \Delta \hat{\mathbf{n}}_{SL}) = -\frac{4GM}{c^3} \Omega J_{l-1} \frac{l}{l+4} F(\mathbf{x}_0, \mathbf{x}_1) \left( \frac{P}{d_k} \right)^{l+1} \times \frac{(\mathbf{k} \times \mathbf{d}_k) \cdot \mathbf{e}_3}{d_k} |\mathbf{k} \times \mathbf{e}_3|^l U_{l-1}(x), \quad (198)$$

for the spin-multipoles, where the parameter  $\hat{\boldsymbol{\xi}}$  has been replaced by the impact vector  $\mathbf{d}_k$ . The dimensionless scalar function reads

$$F(\mathbf{x}_0, \mathbf{x}_1) = \frac{\mathbf{k} \cdot \mathbf{x}_1}{r_1} - \frac{1}{R} \left( \frac{(\mathbf{k} \cdot \mathbf{x}_1)^2}{r_1} - \frac{(\mathbf{k} \cdot \mathbf{x}_0)^2}{r_0} \right). \quad (199)$$

The upper limit of this function is

$$|F(\mathbf{x}_0, \mathbf{x}_1)| \leq 2. \quad (200)$$

We note that the distance between the light source and the observer can be arbitrarily small and even be zero, i.e. extreme configurations with  $R \rightarrow 0$  are entirely possible. By taking account of the upper limits (D9) and (D11) of Chebyshev polynomials of first and second kind, respectively, we obtain for the upper limits of (197) and (198)

$$|\delta(\mathbf{k}, \Delta \hat{\mathbf{n}}_{ML})| \leq \frac{4GM}{c^2 d_k} |J_l| \left( \frac{P}{d_k} \right)^l, \quad (201)$$

$$|\delta(\mathbf{k}, \Delta \hat{\mathbf{n}}_{SL})| \leq \frac{8GM}{c^3} \Omega \frac{l^2}{l+4} |J_{l-1}| \left( \frac{P}{d_k} \right)^{l+1}, \quad (202)$$

$$|\delta(\mathbf{k}, \Delta \hat{\mathbf{n}}_{S1})| \leq \frac{4GM}{c^3} \Omega \kappa^2 \left( \frac{P}{d_k} \right)^2, \quad (203)$$

where (201) is valid for  $l \geq 0$  and (202) for  $l > 1$ , while the upper limit of spin-dipole  $l = 1$  has been obtained by inserting (171) into (184).

The relations (201) - (202) and (203) represent strict upper limits of light deflection for the case of finite distances of source and observer from the massive Solar System bodies, where the observer is assumed to be located somewhere nearby the Earth (e.g. at Lagrange point  $L_2$ ), while the sources can be located arbitrarily. Numerical values of these upper limits of light deflection for the Sun and the giant planets of the Solar System are presented by Table IV and V. These numerical results show that astrometric measurements on the sub-micro-arcsecond level of accuracy needs to account mass-multipoles of the orders  $0 \leq l \leq 8$  and spin-multipoles of the orders  $1 \leq l \leq 3$  in the unit tangent vector (160).

## B. Light deflection at infinite spatial distances

In this Subsection we will consider how these upper limits for the effect of light deflection of Eqs. (201) - (202) and (203), where the source and the observer are located

Table IV: The upper limit of light deflection at the Sun and the giant planets of the Solar System caused by their mass-multipole structure according to Eq. (201). For the numerical value we assume an grazing light ray,  $d_k = P$ , that means the impact parameter equals the radius of the body. All values are given in micro-arcsecond ( $\mu\text{as}$ ). A blank entry indicates that the light deflection is less than 1 nano-arcsecond (nas).

Light deflection	Sun	Jupiter	Saturn	Uranus	Neptune
$ \delta(\mathbf{k}, \Delta\hat{\mathbf{n}}_{M_0}) $	$1.75 \times 10^6$	$16.3 \times 10^3$	$5.8 \times 10^3$	$2.1 \times 10^3$	$2.5 \times 10^3$
$ \delta(\mathbf{k}, \Delta\hat{\mathbf{n}}_{M_2}) $	0.35	239	94	6.9	8.6
$ \delta(\mathbf{k}, \Delta\hat{\mathbf{n}}_{M_4}) $	1.72	9.6	5.41	0.06	0.08
$ \delta(\mathbf{k}, \Delta\hat{\mathbf{n}}_{M_6}) $	0.07	0.55	0.50	0.001	0.001
$ \delta(\mathbf{k}, \Delta\hat{\mathbf{n}}_{M_8}) $	0.007	0.04	0.06	–	–
$ \delta(\mathbf{k}, \Delta\hat{\mathbf{n}}_{M_{10}}) $	–	0.003	0.01	–	–

Table V: The upper limit of light deflection at the Sun and the giant planets of the Solar System caused by their spin-multipole structure to Eqs. (203) - (202). For the numerical value we assume an grazing light ray,  $d_k = P$ , that means the impact parameter equals the radius of the body. All values are given in micro-arcsecond ( $\mu\text{as}$ ). A blank entry indicates that the light deflection is less than 1 nano-arcsecond (nas).

Light deflection	Sun	Jupiter	Saturn	Uranus	Neptune
$ \delta(\mathbf{k}, \Delta\hat{\mathbf{n}}_{S_1}) $	0.7	0.17	0.04	0.004	0.005
$ \delta(\mathbf{k}, \Delta\hat{\mathbf{n}}_{S_3}) $	–	0.026	0.008	–	–
$ \delta(\mathbf{k}, \Delta\hat{\mathbf{n}}_{S_5}) $	–	0.001	–	–	–

at finite spatial distances, are related to previously obtained results for the case of infinite spatial distances of source and observer [30]. In the limit of infinite spatial distances of source and observer the light deflection at finite distances of Eq. (180) becomes

$$\delta(\boldsymbol{\sigma}, \boldsymbol{\nu}) = \arcsin |\boldsymbol{\sigma} \times \boldsymbol{\nu}|, \quad (204)$$

where  $\boldsymbol{\sigma}$  and  $\boldsymbol{\nu}$  are the unit tangent vectors of the light ray at minus and plus infinity. The term (204) is called *total light deflection* [21, 30]. The total light deflection (204) has also been defined in the text above by Eq. (44) in [21] and also by Eq. (58) in [30].

In fact, in the limit of infinite spatial distances of source and observer,  $R \rightarrow \infty$ , we get for the function (199)

$$\lim_{\substack{r_0 \rightarrow +\infty \\ r_1 \rightarrow +\infty}} F(\mathbf{x}_0, \mathbf{x}_1) = 1 - \cos \delta(\mathbf{x}_0, \mathbf{x}_1) \rightarrow +2, \quad (205)$$

where in the last step of (205) we have used that in the limit of spatial infinity  $\cos \delta(\mathbf{x}_0, \mathbf{x}_1) \rightarrow -1$  if source and observer are located in opposite direction as seen from the body. Hence, by inserting (205) into Eqs. (197) and (198) one finds that these equations pass over to Eqs. (114) and (121) in [30] in this limit. The remarkable fact, that these equations (114) and (121) in [30] are related to Chebyshev polynomials has, at the very first time, allowed for determining the numerical magnitude of light deflection in the gravitational field of an axi-symmetric body at rest up to any multipole order. They were given by Eqs. (117)

and (124) - (125) in our previous work [30] and read

$$|\delta(\boldsymbol{\sigma}, \Delta\boldsymbol{\nu}_{M_L})| \leq \frac{4GM}{c^2 d_k} |J_l| \left(\frac{P}{d_k}\right)^l, \quad (206)$$

$$|\delta(\boldsymbol{\sigma}, \Delta\boldsymbol{\nu}_{S_L})| \leq \frac{8GM}{c^3} \Omega \frac{l^2}{l+4} |J_{l-1}| \left(\frac{P}{d_k}\right)^{l+1}, \quad (207)$$

$$|\delta(\boldsymbol{\sigma}, \Delta\boldsymbol{\nu}_{S_1})| \leq \frac{4GM}{c^3} \Omega \kappa^2 \left(\frac{P}{d_k}\right)^2, \quad (208)$$

where (206) is valid for  $l \geq 0$  and (207) for  $l > 1$ , while (208) is the upper limit of the spin-dipole  $l = 1$ . These upper limits are valid for astrometric configurations with source and observer at infinite spatial distance from the gravitating Solar System body. They coincide with the upper limits of light deflection in case of finite spatial distances, which are represented by Eqs. (201) - (202) and (203). Thus, we have demonstrated that the upper limits of the total light deflection do also represent an upper limit for finite distances between source and observer from the Solar System body. This fact confirms a corresponding statement made in the introductory section of our previous work [30]. The only restriction is that the observer has to be located somewhere nearby the Earth, for instance at Lagrange point  $L_2$ , like in case of the Gaia mission [1, 2] or in case of the missions *GaiaNIR* [3] and *Theia* [4] which have been proposed to ESA.

Table VI: The magnitude of the 2PN terms of Eqs. (211) and (212) and of 3PN terms of Eq. (215) to the angle of light deflection in case of a grazing ray at massive Solar System bodies. All values are given in micro-arcseconds ( $\mu\text{s}$ ). A blank entry means less than 1 nas. It is noticed here that in case of a Sun shield with an aspect angle of 45 degree (like in the Gaia mission) the 2PN terms of the light rays in the gravitational field of the Sun contribute less than 1 nas [40].

body	$ \delta(\mathbf{k}, \Delta\mathbf{n}_{2\text{PN}}^{\text{M}\times\text{M}}) $	$ \delta(\mathbf{k}, \Delta\mathbf{n}_{2\text{PN}}^{\text{M}\times\text{Q}}) $	$ \delta(\mathbf{k}, \Delta\mathbf{n}_{3\text{PN}}^{\text{M}\times\text{M}\times\text{M}}) $
Sun	3136.1	–	11.2
Jupiter	16.11	0.95	0.014
Saturn	4.42	0.29	0.004
Uranus	2.58	0.04	0.004
Neptune	5.83	0.08	0.023

## XII. IMPACT OF 2PN AND 3PN TERMS

On the sub-micro-arcsecond level of precision also post-post-Newtonian (2PN) terms become relevant and it might be worth to consider here briefly the order of magnitude of the dominant 2PN contributions. The expansion of the metric in (3) in the 2PN approximation becomes

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}^{(2)} + h_{\alpha\beta}^{(3)} + h_{\alpha\beta}^{(4)} + \mathcal{O}(c^{-5}), \quad (209)$$

where  $h_{\alpha\beta}^{(4)}$  are the 2PN perturbations of the metric tensor. This expansion implies a corresponding expansion of the light trajectory and its unit tangent vector (22),

$$\mathbf{n} = \mathbf{k} + \Delta\mathbf{n}_{1\text{PN}} + \Delta\mathbf{n}_{1.5\text{PN}} + \Delta\mathbf{n}_{2\text{PN}} + \mathcal{O}(c^{-5}). \quad (210)$$

The dominant contributions of the 2PN terms are the mass-monopole and mass-quadrupole terms. The 2PN mass-monopole term,  $\Delta\mathbf{n}_{2\text{PN}}^{\text{M}\times\text{M}}$ , is given by Eq. (87) in our work [40] and the 2PN mass-quadrupole term,  $\Delta\mathbf{n}_{2\text{PN}}^{\text{M}\times\text{Q}}$ , has been given by Eq. (30) in our investigation [44]. The effect of light deflection is defined by Eq. (180). The upper limits of these 2PN terms have been determined in our investigations [40, 44]. For grazing rays they are given by

$$\delta(\mathbf{k}, \Delta\mathbf{n}_{2\text{PN}}^{\text{M}\times\text{M}}) \leq 16 \frac{G^2 M^2}{c^4} \frac{1}{(d_k)^2} \frac{r_1}{d_k}, \quad (211)$$

$$\delta(\mathbf{k}, \Delta\mathbf{n}_{2\text{PN}}^{\text{M}\times\text{Q}}) \leq 64 \frac{G^2 M^2}{c^4} \frac{1}{(d_k)^2} |J_2| \frac{r_1}{d_k}. \quad (212)$$

The large factors  $r_1/d_k \gg 1$  are the so-called *enhancing factors* which have been found in case of 2PN monopole by several independent investigations [40, 45, 46]. Later this *enhancing factors* have also been found in case of 2PN quadrupole in our work [44].

The numerical magnitude of these terms is represented here again by Table VI. These values in Table VI show that the mass-monopole and mass-quadrupole terms

need to be enclosed in the theory of light propagation for astrometry on the sub-micro-arcsecond level. For the moment being it remains an open question of whether the mass-octupole term in the 2PN approximation needs also to be taken into account.

A final word should also be done about the 3PN approximation. Higher multipoles in the 3PN approximation are certainly negligible and have, thus far, never been calculated. So we consider the gravitational field of a non-rotating spherically symmetric body at rest (i.e. there are no 1.5PN, 2.5PN, 3.5PN terms in the metric), where the expansion of the metric in (3) in the 3PN approximation becomes

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}^{(2)} + h_{\alpha\beta}^{(4)} + h_{\alpha\beta}^{(6)} + \mathcal{O}(c^{-8}), \quad (213)$$

where  $h_{\alpha\beta}^{(6)}$  are the 3PN perturbations of the metric tensor. This expansion implies a corresponding expansion of the light trajectory and its unit tangent vector (22),

$$\mathbf{n} = \mathbf{k} + \Delta\mathbf{n}_{1\text{PN}}^{\text{M}} + \Delta\mathbf{n}_{2\text{PN}}^{\text{M}\times\text{M}} + \Delta\mathbf{n}_{3\text{PN}}^{\text{M}\times\text{M}\times\text{M}} + \mathcal{O}(c^{-8}). \quad (214)$$

The magnitude of the angle of light deflection caused by 3PN mass-monopole terms of the unit tangent vector has been determined by means of a lens equation which is valid for finite distances [47], where the following formula (cf. Eq. (27) *ibid.*) has been derived,

$$\delta(\mathbf{k}, \Delta\mathbf{n}_{3\text{PN}}^{\text{M}\times\text{M}\times\text{M}}) = 128 \frac{G^3 M^3}{c^6} \frac{1}{(d_k)^3} \left( \frac{r_1}{d_k} \right)^2. \quad (215)$$

Like in case of 2PN monopole and 2PN quadrupole, the *enhancing factor*  $r_1/d_k \gg 1$  appears also in case of 3PN monopole in (215). The same result (215) has also been obtained by the approach of Time-Transfer-Function [48] (cf. Eqs. (21) and (22) *ibid.*). Numerical values of these 3PN terms in (215) in case of grazing rays are presented in Table VI.

## XIII. SUMMARY AND OUTLOOK

Astrometric measurements have recently achieved a level of a few micro-arcseconds in angular measurements of celestial objects by the ESA astrometry mission *Gaia* [1, 2]. One of the major goals in astrometry is to improve the accuracy of angular resolution, because it is directly related to the precision in determining the spatial positions of celestial objects. In fact, future space-astrometry missions, like *GaiaNIR* [3] and *Theia* [4], have been proposed to ESA which are aiming at the sub-micro-arcsecond level of precision. As written already in the introductory Section, the term *sub-micro-arcsecond level* refers to an astrometric precision of angular measurements of 0.1 micro-arcseconds. Such an accuracy necessitates a relativistic model of light propagation which is about 10 times better than the end-of-mission accuracy. Accordingly, the given threshold of our model is

0.01 micro-arcseconds. Such an advancement in astrometric measurements needs a corresponding progress in the theory of light propagation in the curved space-time of the Solar System.

The determination of the spatial positions of celestial light sources is based on the observed direction of light signals, which are emitted by these sources. In other words, the determination of the spatial positions of celestial objects is based on measurements of the unit tangent vector of these light trajectories when they receive the observer. As outlined in the introductory Section, numerical integrations of the geodesic equation of light trajectories are not workable with regard to the huge amount of astrometric observations in astrometry missions like *Gaia* [1, 2] or *GaiaNIR* [3]. Instead of that, analytical solutions of the light trajectories are required for the treatment of such a huge amount of astrometric measurements. However, the analytical expressions of the unit tangent vector are rather cumbersome. In view of that involved structure of the tangent vector, it becomes clear how important it is to simplify the analytical solution of the tangent vector by neglecting all those terms which contribute less than  $0.01 \mu\text{as}$  to the angle of light deflection. Such a considerably simplified expression of the unit tangent vector has been obtained in this investigation. The results of this investigation are summarized as follows:

1. The unit tangent vector of the light ray in the field of an arbitrary body has been derived in the 1.5PN approximation, given by Eq. (47) with the mass-multipole terms (66) - (69) and the spin-multipole terms (70) - (76). These expressions become awful as soon as one performs the partial derivatives.
2. A simplified unit tangent vector of the light ray in the field of an arbitrary body has been obtained by Eq. (135) with the mass-multipole terms of Eqs. (136) - (139) and spin-multipole terms of Eqs. (140) - (143), where all partial derivatives have been performed already. In the simplified tangent vector (135) all those terms are neglected, which in total contribute less than a few nano-arcseconds to the angle of light deflection in all astrometric configurations. In particular:
  - (a) The total sum of all neglected terms is less than 10 nano-arcseconds for all Solar system bodies and all configurations, except for grazing rays at Jupiter and Saturn.
  - (b) In case of grazing rays at Jupiter, the total sum of all neglected terms is at most 36.2 nano-arcseconds, as given by Eq. (144).
  - (c) In case of grazing rays at Saturn, the total sum of all neglected terms is at most 14.9 nano-arcseconds, as given by Eq. (145).
  - (d) If one implements the mass-quadrupole term, which is exact in the 1PN approximation, then

one arrives at a simplified unit tangent vector where only terms are neglected which contribute in total less than 10 nas for all possible astrometric configurations, including grazing rays at the Sun and at giant planets.

3. The simplified unit tangent vector of the light ray has been applied for the gravitational field of an axi-symmetric body. It has been demonstrated that in case of an axi-symmetric body the unit tangent vector can be expressed in terms of Chebyshev polynomials, given by Eq. (160) with the mass-multipole terms of Eqs. (161) - (165) and spin-multipole terms of Eqs. (170) - (174). The mass-monopole and spin-dipole terms are given by Eqs. (162) and (171).
4. Upper limits of the effect of light deflection have been determined for an axi-symmetric body for the full set of mass-multipoles and spin-multipoles, given by Eqs. (201) - (203).
5. These these upper limits for finite distances of source and observer, given by Eqs. (201) - (203), coincide with the upper limits for infinite distances of source and observer, given by Eqs. (206) - (208). Thus it has been demonstrated that the total light deflection, where light source and observer are located at infinite spatial distances from the body, represents an upper limit for bending of light.
6. It has been shown by numerical values presented of Tables IV and V that astrometric measurements on the sub-micro-arcsecond level of accuracy necessitates to account for mass-multipoles of the order  $0 \leq l \leq 8$  and for spin-multipoles of the order  $0 \leq l \leq 3$  in the simplified unit tangent vector.
7. In Section XII the impact of those 2PN and 3PN terms is discussed, which are relevant on the sub-micro-arcsecond level.

The fifth point was already asserted in the introductory section in our recent investigation [30]. This conclusion has been shown here for the case of an observer which is located somewhere nearby the Earth, that means sufficiently far away from the massive Solar System bodies.

The sub-micro-arcsecond scale of accuracy will certainly be achieved in near future. For instance, the astrometry missions *GaiaNIR* [3] and *Theia* [4], recently proposed to ESA, are the most promising candidates to get realized within the next few decades [49]. Both these missions are aiming at ultra highly precise angular measurements on the sub-micro-arcsecond level. In particular, the mission *GaiaNIR* aims for three-dimensional mapping of about 50 billion ( $50 \times 10^9$ ) stars in the near-infrared band. Such a huge amount of stellar objects implies the need for highly effective methods and algorithms of observational data reduction. The simplified expression for the unit tangent vector of light rays, which has

been obtained in our investigation, aims at such a highly effective performance of data reduction and can directly be implemented in relativistic models of future space astrometry missions like *GaiaNIR* [3] or *Theia* [4]. The implementation of the simplified unit tangent vector of light rays into relativistic models for possible future astrometry missions would proceed in the same way, as it has been described in Section III of our recent work [44].

#### XIV. ACKNOWLEDGMENT

This work was funded by the German Research Foundation (Deutsche Forschungsgemeinschaft DFG) under grant number 447922800. Sincere gratitude is expressed to Sergei A. Klioner, Michael H. Soffel, Ralf Schützhold, William G. Unruh, Anke Theuser, Lutz Graefe, Alexey Butkevich, Günter Plunien, Jürgen Schreiber, Burkhard Kämpfer, and Laszlo P. Csernai for kind support and inspiring discussions about astrometry and general theory of relativity.

#### Appendix A: Notation

The following notation is in use:

- Newtonian constant of gravitation:  $G$ .
- vacuum speed of light in flat space-time:  $c$ .
- Newtonian mass of the body:  $M$ .
- Equatorial radius of the body:  $P$ .
- Angular velocity of the body:  $\Omega$ .
- Zonal harmonic coefficients of the body:  $J_l$ .
- $\eta_{\alpha\beta} = \text{diag}(-1, +1, +1, +1)$  is the metric tensor of flat space-time.
- $g^{\alpha\beta}$  and  $g_{\alpha\beta}$  are the contravariant and covariant components of the metric tensor with signature  $(-, +, +, +)$ .
- 1 mas (milli – arcsecond)  $\simeq 4.85 \times 10^{-9}$  rad.
- 1  $\mu\text{as}$  (micro – arcsecond)  $\simeq 4.85 \times 10^{-12}$  rad.
- 1 nas (nano – arcsecond)  $\simeq 4.85 \times 10^{-15}$  rad.
- 1 pas (pico – arcsecond)  $\simeq 4.85 \times 10^{-18}$  rad.
- $n! = n(n-1)(n-2) \cdots 2 \cdot 1$  is the factorial; by definition:  $0! = 1$ .
- $n!! = n(n-2)(n-4) \cdots (2 \text{ or } 1)$  is the double factorial; by definition:  $0!! = 1$  and  $(-1)!! = 1$ .
- $(2n)!! = 2^n n!$
- $(2n-1)!! = \frac{(2n)!}{2^n n!}$

- $(2n+1)!! = \frac{(2n+1)!}{2^n n!}$
- Lower case Greek indices take values 0,1,2,3.
- The contravariant components of four-vectors:  $a^\mu = (a^0, a^1, a^2, a^3)$ .
- Lower case Latin indices take values 1,2,3.
- The three-dimensional coordinate quantities (three-vectors) referred to the spatial axes of the reference system are in boldface:  $\mathbf{a}$ .
- The contravariant components of three-vectors:  $a^i = (a^1, a^2, a^3)$ .
- The absolute value of a three-vector:  $a = |\mathbf{a}| = \sqrt{a^1 a^1 + a^2 a^2 + a^3 a^3}$ .
- The scalar product of two three-vectors:  $\mathbf{a} \cdot \mathbf{b} = \delta_{ij} a^i b^j = a^i b^i$  with Kronecker delta  $\delta_{ij}$ .
- The vector product of two three-vectors reads  $(\mathbf{a} \times \mathbf{b})^i = \varepsilon_{ijk} a^j b^k$  with Levi-Civita symbol  $\varepsilon_{ijk}$ .
- The angle between two three-vectors  $\mathbf{a}$  and  $\mathbf{b}$  is designated as  $\delta(\mathbf{a}, \mathbf{b})$ .
- These angles can uniquely be computed by

$$\delta(\mathbf{a}, \mathbf{b}) = \arccos \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}| |\mathbf{b}|}. \quad (\text{A1})$$

#### Appendix B: STF tensors

Here we will present some standard notations about symmetric trace-free (STF) tensors, which are necessary for our considerations, while further STF relations can be found in [7, 11–13, 50].

- $L = i_1 i_2 \dots i_l$  is a Cartesian multi-index of a given tensor  $T$ , that means  $T_L \equiv T_{i_1 i_2 \dots i_l}$ .
- two identical multi-indices imply summation:

$$A_L B_L \equiv \sum_{i_1 \dots i_l} A_{i_1 \dots i_l} B_{i_1 \dots i_l}. \quad (\text{B1})$$

- The symmetric part of a Cartesian tensor  $T_L$  is (cf. Eq. (2.1) in [11]):

$$T_{(L)} = T_{(i_1 \dots i_l)} = \frac{1}{l!} \sum_{\sigma} T_{i_{\sigma(1)} \dots i_{\sigma(l)}} \quad (\text{B2})$$

where  $\sigma$  is running over *all permutations* of  $(1, 2, \dots, l)$ .

For instance: let  $T_{i_1 i_2 i_3}$  be a Cartesian tensor which is already symmetric in all of its Cartesian indices. Then the symmetric part reads:

$$T_{(i_1 i_2 i_3)} = T_{i_1 i_2 i_3}. \quad (\text{B3})$$

For instance: let  $T_{i_1 i_2 i_3}$  be a Cartesian tensor which is not symmetric in any of its Cartesian indices. Then the symmetric part reads:

$$T_{(i_1 i_2 i_3)} = \frac{1}{3!} \times (T_{i_1 i_2 i_3} + T_{i_1 i_3 i_2} + T_{i_2 i_1 i_3} + T_{i_2 i_3 i_1} + T_{i_3 i_1 i_2} + T_{i_3 i_2 i_1}). \quad (\text{B4})$$

- The un-normalized symmetric part of a Cartesian tensor  $T_L$  is (cf. text above Eq. (A19) in [12]):

$$T_{\{L\}} = T_{\{i_1 \dots i_l\}} = \sum_{\sigma \in S} T_{i_{\sigma(1)} \dots i_{\sigma(l)}} \quad (\text{B5})$$

where  $S$  is the *smallest set of permutations* of  $(1, 2, \dots, l)$  which makes  $T_{i_1 \dots i_l}$  fully symmetric in its indices. Let us give some examples:

$$\delta_{\{i_1 i_2\}} = \delta_{i_1 i_2}, \quad (\text{B6})$$

$$\delta_{\{i_1 i_2 n_{i_3}\}} = \delta_{i_1 i_2} n_{i_3} + \delta_{i_1 i_3} n_{i_2} + \delta_{i_2 i_3} n_{i_1}, \quad (\text{B7})$$

$$\delta_{\{i_1 i_2 \delta_{i_3 i_4}\}} = \delta_{i_1 i_2} \delta_{i_3 i_4} + \delta_{i_1 i_3} \delta_{i_2 i_4} + \delta_{i_1 i_4} \delta_{i_2 i_3}. \quad (\text{B8})$$

In case  $T_{i_1 i_2 i_3}$  is a Cartesian tensor which is not symmetric in any of its indices, then the un-normalized symmetric part reads:

$$T_{\{i_1 i_2 i_3\}} = 3! T_{(i_1 i_2 i_3)}. \quad (\text{B9})$$

- The symmetric trace-free part of a Cartesian tensor  $T_L$  (notation:  $\hat{T}_L \equiv \text{STF}_L T_L = T_{\langle i_1 \dots i_l \rangle}$ ) is (cf. Eq. (2.2) in [11]):

$$\hat{T}_L = \sum_{k=0}^{\lfloor l/2 \rfloor} a_{lk} \delta_{(i_1 i_2 \dots i_{2k} \dots i_{2k-1} i_{2k} S_{i_{2k+1} \dots i_l} a_1 a_1 \dots a_k a_k} \quad (\text{B10})$$

and  $S_L \equiv T_{(L)}$  abbreviates the symmetric part of tensor  $T_L$ . The coefficient in (B10) is given by

$$a_{lk} = (-1)^k \frac{l!}{(l-2k)!} \frac{(2l-2k-1)!!}{(2l-1)!! (2k)!!}. \quad (\text{B11})$$

For instance:

$$\begin{aligned} \hat{T}_{i_1 i_2 i_3} &\equiv T_{\langle i_1 i_2 i_3 \rangle} \\ &= T_{(i_1 i_2 i_3)} \\ &\quad - \frac{1}{5} (\delta_{i_1 i_2} T_{(i_3 k k)} + \delta_{i_2 i_3} T_{(i_1 k k)} + \delta_{i_3 i_1} T_{(i_2 k k)}). \end{aligned} \quad (\text{B12})$$

Three comments are in order about STF. First of all, the Kronecker delta has no symmetric trace-free part,

$$\text{STF}_{ab} \delta^{ab} = 0. \quad (\text{B13})$$

Second, the symmetric trace-free part of any tensor which contains Kronecker delta is zero, if the Kronecker delta has not any summation (dummy) index, for instance,

$$\text{STF}_{abc} \delta^{ab} d_k^c = 0, \quad (\text{B14})$$

$$\text{STF}_{abc} \delta^{ab} k^c = 0. \quad (\text{B15})$$

And third, the following relation is very useful [7, 11–13]

$$\text{STF}_{i_1 \dots i_l} A_{i_1 \dots i_l} \text{STF}_{i_1 \dots i_l} B_{i_1 \dots i_l} = A_{i_1 \dots i_l} \text{STF}_{i_1 \dots i_l} B_{i_1 \dots i_l}, \quad (\text{B16})$$

which allows for simplifying expressions at several occasions. In particular, we need the following Cartesian STF tensor,

$$\hat{n}_L = \frac{x_{\langle i_1}}{r} \dots \frac{x_{i_l \rangle}}{r}, \quad (\text{B17})$$

where  $x_i$  are the spatial coordinates of some arbitrary field point and  $r = |\mathbf{x}|$ ; we note that  $x_i = x^i$  and  $\hat{n}_L = \hat{n}^L$ . A very useful relation for later purposes is the expansion of the Cartesian STF tensor  $\hat{n}_L$  in terms of Cartesian tensor  $n_L$  (cf. Eq. (A20a) in [12]),

$$\hat{n}_L = \sum_{k=0}^{\lfloor l/2 \rfloor} (-1)^k \frac{(2l-2k-1)!!}{(2l-1)!!} \delta_{\{i_1 i_2 \dots i_{2k-1} i_{2k} n_{i_{2k+1} \dots i_l\}}. \quad (\text{B18})$$

### Appendix C: Proof of relations (93) - (94)

In this Appendix we will show the validity of relations (93) - (94) in case of spin-dipole ( $l = 1$ ), which is the most dominant term, and we will determine the coefficients  $C_1^{S_1}$  and  $B_1^{S_1}$ , which were given in the text below Eq. (94). For that we consider Eqs. (74) - (76), which in case of spin-dipole read

$$\Delta n_{S_1}^{A_1} = + \frac{2G\hat{S}_b}{c^3} \hat{\epsilon}_{iab} \hat{\partial}_a^{\tau_1} \frac{1}{r_1}, \quad (\text{C1})$$

$$\Delta n_{S_1}^{S_1} = + \frac{2G\hat{S}_b}{c^3} \hat{\epsilon}_{iab} \frac{1}{R} \hat{\partial}_a^{\tau_1} \ln(r_1 - c\tau_1), \quad (\text{C2})$$

$$\Delta n_{S_1}^{\delta_1} = - \frac{2G\hat{S}_b}{c^3} \hat{\epsilon}_{iab} \frac{1}{R} \hat{\partial}_a^{\tau_0} \ln(r_0 - c\tau_0), \quad (\text{C3})$$

where  $\hat{\epsilon}_{iab} = \epsilon_{iab} - k_i \epsilon_{jab} k^j$ . The differential operators (61) and (62) for  $l = 1$  read

$$\hat{\partial}_a^{\tau_0} = P_a^{j_1} \frac{\partial}{\partial \xi^{j_1}} + k_a \frac{\partial}{\partial c\tau_0}, \quad (\text{C4})$$

$$\hat{\partial}_a^{\tau_1} = P_a^{j_1} \frac{\partial}{\partial \xi^{j_1}} + k_a \frac{\partial}{\partial c\tau_1}, \quad (\text{C5})$$

and the spin-dipole (150) is a three-vector,

$$\mathbf{S} = -M \Omega (P)^2 J_0 \kappa^2 \mathbf{e}_3, \quad (\text{C6})$$

where  $\mathbf{e}_3$  is the principal axis of the body, which equals with the rotational axis of the axi-symmetric body.

In order to determine the contribution of Eqs. (C1) - (C3) to the angle of light deflection we make use of relation (184). Then one finds that the second term of  $\hat{\epsilon}_{iab}$  and the first term on the right-hand side of (C4) and (C5) do not contribute to the angle of light deflection. Thus, by performing the derivatives, we get for the individual contributions to the angle of light deflection

$$\delta(\mathbf{k}, \Delta \mathbf{n}_{S_1}^4) = -\frac{2GM}{c^3} \Omega(P)^2 \kappa^2 J_0(\mathbf{k} \times \mathbf{e}_3) \cdot \frac{\hat{\xi}}{\hat{\xi}} \frac{c\tau_1}{(r_1)^3}, \quad (\text{C7})$$

$$\delta(\mathbf{k}, \Delta \mathbf{n}_{S_1}^5) = -\frac{2GM}{c^3} \Omega(P)^2 \kappa^2 J_0(\mathbf{k} \times \mathbf{e}_3) \cdot \frac{\hat{\xi}}{\hat{\xi}} \frac{1}{R} \frac{1}{r_1}, \quad (\text{C8})$$

$$\delta(\mathbf{k}, \Delta \mathbf{n}_{S_1}^6) = +\frac{2GM}{c^3} \Omega(P)^2 \kappa^2 J_0(\mathbf{k} \times \mathbf{e}_3) \cdot \frac{\hat{\xi}}{\hat{\xi}} \frac{1}{R} \frac{1}{r_0}. \quad (\text{C9})$$

In order to determine the absolute value of these equations we use

$$\left| (\mathbf{k} \times \mathbf{e}_3) \cdot \frac{\hat{\xi}}{\hat{\xi}} \right| \leq 1, \quad (\text{C10})$$

$$\left| \frac{c\tau_1}{r_1} \right| \leq 1 \quad (\text{C11})$$

$$\frac{1}{R} \left( \frac{1}{r_1} - \frac{1}{r_0} \right) \leq \frac{1}{r_1} \frac{1}{\hat{\xi}}, \quad (\text{C12})$$

and obtain

$$|\delta(\mathbf{k}, \Delta \mathbf{n}_{S_1}^4)| \leq 2 \kappa^2 \frac{GM}{c^3} \Omega |J_0| \left( \frac{P}{r_1} \right)^2, \quad (\text{C13})$$

$$|\delta(\mathbf{k}, \Delta \mathbf{n}_{S_1}^{5+6})| \leq 2 \kappa^2 \frac{GM}{c^3} \Omega |J_0| \left( \frac{P}{r_1} \right) \left( \frac{P}{d_k} \right), \quad (\text{C14})$$

where (C14) is the absolute value of the sum of (C8) and (C9), and  $\hat{\xi}$  has been replaced by  $d_k$ . These relations show the validity of Eqs. (93) and (94) in case of spin-dipole. The coefficients  $C_1^{S_1} = 2 \kappa^2$  and  $B_1^{S_1} = 2 \kappa^2$ , can be read of from (C13) and (C14) and they agree with the coefficients asserted in the text below Eq. (94). We note that the distance  $R$  between the source and the observer is arbitrarily and can also be zero. That is why one has to consider the sum of the terms (C8) and (C9), because each individual term would be infinite if  $R$  tends to zero.

#### Appendix D: Chebyshev polynomials

In this Section we will briefly review the Chebyshev polynomials [51–53]. There are Chebyshev polynomials of first and second kind,  $T_l(x)$  and  $U_l(x)$ , which form a sequence of orthogonal polynomials. The power representation of Chebyshev polynomials of first kind reads

(cf. Eqs. (13.67) and (13.88a) in [51])

$$T_0(x) = 1, \quad (\text{D1})$$

$$T_l(x) = \frac{l}{2} \sum_{n=0}^{[l/2]} \frac{(-1)^n}{n!} \frac{(l-n-1)!}{(l-2n)!} (2x)^{l-2n}, \quad (\text{D2})$$

where  $l \geq 1$ . The power representation of Chebyshev polynomials of second kind reads (cf. Eq. (13.88b) in [51])

$$U_l(x) = \sum_{n=0}^{[l/2]} \frac{(-1)^n}{n!} \frac{(l-n)!}{(l-2n)!} (2x)^{l-2n}, \quad (\text{D3})$$

where  $l \geq 0$ . The argument of the Chebyshev polynomials is a real number of the closed interval  $x \in [-1, +1]$ . The Chebyshev polynomials of first and second kind are related by

$$T_l(x) = U_l(x) - x U_{l-1}(x), \quad (\text{D4})$$

$$U_l(x) = \frac{x T_{l+1}(x) - T_{l+2}(x)}{1-x^2}. \quad (\text{D5})$$

The first derivative of Chebyshev polynomials of first kind (D2) is related to the Chebyshev polynomials of second kind as follows (see also p. 794 in [51])

$$\frac{dT_l(x)}{dx} = l U_{l-1}(x). \quad (\text{D6})$$

The first derivative of Chebyshev polynomials of second kind (D3) is related to the Chebyshev polynomials of first kind as follows

$$\frac{dU_l(x)}{dx} = \frac{x U_l(x) - (l+1) T_{l+1}(x)}{1-x^2}. \quad (\text{D7})$$

The Chebyshev polynomials of first kind (D2) can be written in terms of trigonometric functions (cf. Eq. (13.83a) in [51])

$$T_l(x) = \cos(l \arccos x) \quad (\text{D8})$$

for  $l \geq 0$ . From (D8) follows

$$T_l(x) \leq 1. \quad (\text{D9})$$

The Chebyshev polynomials of second kind (D3) can be written in terms of trigonometric functions (cf. Eqs. (13.83b) and (13.85a) in [51])

$$U_{l-1}(x) = \frac{1}{\sqrt{1-x^2}} \sin(l \arccos x). \quad (\text{D10})$$

From (D8) follows

$$U_{l-1}(x) \leq l. \quad (\text{D11})$$

### Appendix E: Gegenbauer polynomials

The Gegenbauer polynomials, also known as ultraspherical polynomials, can be represented by hypergeometric functions [51–53]. According to these representations, they can be written in the form

$$C_l^{(\alpha)}(x) = \sum_{n=0}^{\lfloor l/2 \rfloor} (-1)^n \frac{\Gamma(l + \alpha - n)}{\Gamma(\alpha) n! (l - 2n)!} (2x)^{l-2n} \quad (\text{E1})$$

where  $l$  is the degree of the polynomial,  $\alpha$  is the Gegenbauer index of the polynomial,  $x$  is here a real number, and  $\Gamma(z)$  is the Gamma function

$$\Gamma(w) = \int_0^\infty t^{w-1} e^{-t} dt, \quad (\text{E2})$$

where  $w \geq 0$  is a positive real number. Let  $n$  be an integer. Then, the Gamma function is related to the factorials,

$$\Gamma(n) = (n-1)! \quad \text{i.e.} \quad \Gamma(n+1) = n!. \quad (\text{E3})$$

Let  $n + 1/2$  be a half-integer. Then, the Gamma function is related to the factorials by

$$\Gamma\left(n + \frac{1}{2}\right) = \sqrt{\pi} \frac{(2n)!}{2^{2n} n!}. \quad (\text{E4})$$

By using this relation one may deduce from (E1) the following form of Gegenbauer polynomials,

$$C_l^{(q+\frac{1}{2})}(x) = \frac{q!}{(2q)!} \sum_{n=0}^{\lfloor l/2 \rfloor} \frac{(-1)^n (2l + 2q - 2n)!}{2^l (l + q - n)!} \frac{(x)^{l-2n}}{n! (l - 2n)!} \quad (\text{E5})$$

where  $q \geq 0$  is a positive integer. These Gegenbauer polynomials satisfy the arithmetic operation

$$C_l^{(q+\frac{1}{2})}(-x) = (-1)^l C_l^{(q+\frac{1}{2})}(x). \quad (\text{E6})$$

We also note the value of Gegenbauer polynomials when the argument  $x = 1$  [51–53], which is frequently used as normalization constant,

$$C_l^{(q+\frac{1}{2})}(1) = \binom{l+2q}{l}. \quad (\text{E7})$$

In case of  $q = 0$  the Gegenbauer polynomials (E5) become Legendre polynomials [51–53]

$$P_l(x) = \sum_{n=0}^{\lfloor l/2 \rfloor} \frac{(-1)^n (2l - 2n)!}{2^l (l - n)!} \frac{x^{l-2n}}{n! (l - 2n)!}. \quad (\text{E8})$$

### Appendix F: Some relations

First of all, we notice that

$$\frac{\mathbf{k} \cdot \mathbf{x}_0}{r_0} = \frac{r_1^2 - r_0^2 - R^2}{2Rr_0}, \quad (\text{F1})$$

$$\frac{\mathbf{k} \cdot \mathbf{x}_1}{r_1} = \frac{r_1^2 - r_0^2 + R^2}{2Rr_1}.$$

By introducing the variable

$$z = \frac{r_0}{r_1}, \quad (\text{F2})$$

with  $z \in [0, \infty]$  we may rewrite these relations into the form

$$\frac{\mathbf{k} \cdot \mathbf{x}_0}{r_0} = \frac{1 - z^2 - \widehat{R}^2}{2z\widehat{R}}, \quad (\text{F3})$$

$$\frac{\mathbf{k} \cdot \mathbf{x}_1}{r_1} = \frac{1 - z^2 + \widehat{R}^2}{2\widehat{R}}, \quad (\text{F4})$$

where

$$\widehat{R} = \sqrt{1 + z^2 - 2z \cos \delta(\mathbf{x}_1, \mathbf{x}_0)} \quad (\text{F5})$$

is a dimensionless term which we call *reduced distance* between source and observer, in order to distinguish it from the spatial distance  $R$  between source and observer as defined by Eq. (50). They are related to each other by  $R = r_1 \widehat{R}$ .

As usual, for the differential operators we are using the following notations,

$$\partial_L = \frac{\partial}{\partial \xi^{i_1}} \dots \frac{\partial}{\partial \xi^{i_l}}, \quad (\text{F6})$$

$$\tilde{\partial}_L = P_{j_1}^{(i_1)} \dots P_{j_l}^{(i_l)} \frac{\partial}{\partial \xi^{j_1}} \dots \frac{\partial}{\partial \xi^{j_l}}, \quad (\text{F7})$$

$$\widehat{\partial}_L = P_{j_1}^{<i_1} \dots P_{j_l}^{>i_l} \frac{\partial}{\partial \xi^{j_1}} \dots \frac{\partial}{\partial \xi^{j_l}}, \quad (\text{F8})$$

and the abbreviations

$$\xi_L = \xi_{i_1} \dots \xi_{i_l}, \quad (\text{F9})$$

$$\widehat{\xi}_L = P_{i_1}^{j_1} \dots P_{i_l}^{j_l} \xi_{j_1} \dots \xi_{j_l}. \quad (\text{F10})$$

We recall here that three-vector  $\boldsymbol{\xi}$  has three independent spatial components, while three-vector  $\widehat{\boldsymbol{\xi}}$  is perpendicular to three-vector  $\mathbf{k}$  and because of this restriction has only two independent spatial components. Therefore, three-vector  $\widehat{\boldsymbol{\xi}}$  equals impact vector  $\mathbf{d}_k$ .

First, we consider the spatial derivatives acting on several functions, where the differential operator has been defined by Eq. (63). In particular, we need the following relations:

$$\partial_L \frac{1}{\xi} = (-1)^l (2l - 1)!! \text{STF}_{i_1 \dots i_l} \frac{\xi_L}{(\xi)^{2l+1}}, \quad (\text{F11})$$

$$\widehat{\partial}_L \frac{1}{\widehat{\xi}} = (-1)^l (2l-1)!! \text{STF}_{i_1 \dots i_l} \frac{\widehat{\xi}_L}{(\widehat{\xi})^{2l+1}}, \quad (\text{F12})$$

$$\begin{aligned} \widehat{\partial}_L \frac{\widehat{\xi}^i}{(\widehat{\xi})^2} &= P^{ij} \frac{\partial}{\partial \xi^j} (-1)^{l+1} \text{STF}_{i_1 \dots i_l} \sum_{n=0}^{[l/2]} G_n^l \\ &\times \frac{P_{i_1 i_2} \dots P_{i_{2n-1} i_{2n}} \widehat{\xi}_{i_{2n+1}} \dots \widehat{\xi}_{i_l}}{(\widehat{\xi})^{2l-2n}}, \end{aligned} \quad (\text{F13})$$

$$\begin{aligned} \widetilde{\partial}_L \frac{1}{r} &= (-1)^l \sum_{n=0}^{[l/2]} K_n^l \\ &\times \frac{P_{(i_1 i_2} \dots P_{i_{2n-1} i_{2n}} \widehat{\xi}_{i_{2n+1}} \dots \widehat{\xi}_{i_l})}}{(r)^{2l-2n+1}}, \end{aligned} \quad (\text{F14})$$

$$\begin{aligned} \widehat{\partial}_L \frac{1}{r} &= (-1)^l \text{STF}_{i_1 \dots i_l} \sum_{n=0}^{[l/2]} K_n^l \\ &\times \frac{P_{i_1 i_2} \dots P_{i_{2n-1} i_{2n}} \widehat{\xi}_{i_{2n+1}} \dots \widehat{\xi}_{i_l}}{(r)^{2l-2n+1}}, \end{aligned} \quad (\text{F15})$$

$$\begin{aligned} \widehat{\partial}_L r &= (-1)^{l+1} \text{STF}_{i_1 \dots i_l} \sum_{n=0}^{[l/2]} G_n^l W_n^l \\ &\times \frac{P_{i_1 i_2} \dots P_{i_{2n-1} i_{2n}} \widehat{\xi}_{i_{2n+1}} \dots \widehat{\xi}_{i_l}}{(r)^{2l-2n-1}}, \end{aligned} \quad (\text{F16})$$

$$\begin{aligned} \widehat{\partial}_L \frac{\widehat{\xi}^i}{(\widehat{\xi})^2} \frac{c\tau}{r} &= P^{ij} \frac{\partial}{\partial \xi^j} (-1)^{l+1} \text{STF}_{i_1 \dots i_l} \sum_{n=0}^{[l/2]} G_n^l F_n^l(c\tau, r) \\ &\times \frac{P_{i_1 i_2} \dots P_{i_{2n-1} i_{2n}} \widehat{\xi}_{i_{2n+1}} \dots \widehat{\xi}_{i_l}}{(\widehat{\xi})^{2l-2n}}, \end{aligned} \quad (\text{F17})$$

where

$$[l/2] = l/2 \quad \text{for even values of } l, \quad (\text{F18})$$

$$[l/2] = (l-1)/2 \quad \text{for odd values of } l, \quad (\text{F19})$$

is the largest natural number less than or equal to  $l/2$ . These relations are valid for  $n \geq 1$ . The coefficients  $G_n^l$  and  $W_n^l$  are given by Eqs. (121) and (122), while the coefficient  $K_n^l$  reads

$$K_n^l = (-1)^n \frac{l!}{2^l} \frac{(2l-2n)!}{(l-n)! (l-2n)! n!}. \quad (\text{F20})$$

These coefficients are identical with the coefficients of the Legendre polynomials (E8) times the faculty of the multipole index  $l$ . The scalar function in (F17) is defined by

$$F_n^l(c\tau, r) = \frac{c\tau}{r} \sum_{k=0}^{l-n-1} \frac{1}{2^{2k}} \binom{2k}{k} \left( \frac{\widehat{\xi}}{r} \right)^{2k}, \quad (\text{F21})$$

where we note that  $\widehat{\xi} = \sqrt{(r)^2 - (c\tau)^2}$ . This polynomial can also be written in the form

$$F_n^l(c\tau, r) = \frac{c\tau}{r} + \widetilde{F}_n^l(c\tau, r), \quad (\text{F22})$$

where the scalar function  $\widetilde{F}_n^l(c\tau, r)$  is defined by

$$\widetilde{F}_n^l(c\tau, r) = \frac{c\tau}{r} \sum_{k=1}^{l-n-1} \frac{1}{2^{2k}} \binom{2k}{k} \left( \frac{\widehat{\xi}}{r} \right)^{2k}. \quad (\text{F23})$$

There are congruences of these polynomials [54, 55] and that one may express these polynomials (F21) into ordinary hypergeometric functions [56, 57]. The coefficients  $\binom{2k}{k}$  are the central binomial coefficients. By means of the generating function of these coefficients,

$$\frac{1}{\sqrt{1-x^2}} = \sum_{k=0}^{\infty} \frac{1}{2^{2k}} \binom{2k}{k} (x)^{2k}, \quad (\text{F24})$$

one may show that  $|F_n^l(c\tau, r)| \leq 1$ .

Relations like (F11) - (F16) were given in several articles, e.g. [12, 21, 30, 58]. A proof of relation (F13) has been shown in [30], while the proof of relation (F17) is very similar to the approach in [30].

Second, we consider scalar derivatives. Especially, we need the following relations

$$\left( \frac{\partial}{\partial c\tau_1} \right)^{p-1} \frac{1}{(r_1)^m} = \sum_{k=1}^{\{p/2\}} L_k^{p,m} \frac{(c\tau_1)^{p-2k+1}}{(r_1)^{m+2p-2k}}, \quad (\text{F25})$$

$$\left( \frac{\partial}{\partial c\tau_1} \right)^{p-2} \frac{1}{(r_1)^m} = \sum_{k=1}^{\{p/2\}} Q_k^{p,m} \frac{(c\tau_1)^{p-2k}}{(r_1)^{m+2p-2k-2}}, \quad (\text{F26})$$

where (F25) and (F26) are valid for  $p \geq 1$  and  $p \geq 2$ , respectively. The upper limit of the sum in (F25) means

$$\{l/2\} = l/2 \quad \text{for even values of } l, \quad (\text{F27})$$

$$\{l/2\} = (l+1)/2 \quad \text{for odd values of } l, \quad (\text{F28})$$

which is the smallest natural number larger than or equal to  $l/2$ , while the upper limit of the sum in (F26) has been explained by Eqs. (F18) and (F19). The coefficients in (F26) are given by

$$\begin{aligned} L_k^{p,m} &= (-1)^{p+k} \binom{p-1}{p-2k+1} (2k-3)!! \\ &\times \frac{(m+2p-2k-2)!!}{(m-2)!!}, \end{aligned} \quad (\text{F29})$$

and the coefficients in (F25) are given by

$$\begin{aligned} Q_k^{p,m} &= (-1)^{p+k+1} \binom{p-2}{p-2k} (2k-3)!! \\ &\times \frac{(m+2p-2k-4)!!}{(m-2)!!}. \end{aligned} \quad (\text{F30})$$

### Appendix G: Proof of relations (114) and (117)

In this Appendix we will show the validity of relations (114) and (117); the terms  $c\tau_1$  and  $r_1$  are replaced by  $c\tau$  and  $r$ . By inserting the relations

$$\frac{\hat{\xi}}{r - c\tau} \frac{1}{r} = \frac{\partial}{\partial \hat{\xi}} \ln(r - c\tau), \quad (\text{G1})$$

$$\frac{\hat{\xi}}{r - c\tau} = \frac{\partial}{\partial \hat{\xi}} [(c\tau) \ln(r - c\tau) + r], \quad (\text{G2})$$

into the left-hand side of Eqs. (114) and (117), we get the following two functions

$$\hat{\partial}_L U_L = \text{STF}_{i_1 \dots i_l} P_{i_1}^{j_1} \dots P_{i_l}^{j_l} \frac{\partial}{\partial \xi^{j_1}} \dots \frac{\partial}{\partial \xi^{j_l}} \ln(r - c\tau), \quad (\text{G3})$$

$$\hat{\partial}_L V_L = (c\tau) \hat{\partial}_L U_L + \hat{\partial}_L r, \quad (\text{G4})$$

where we have inserted the differential operator of Eq. (63). Using

$$\frac{\partial}{\partial \xi^{j_1}} \ln(r - c\tau) = \frac{\xi^{j_1}}{r - c\tau} \frac{1}{r}, \quad (\text{G5})$$

which is just relation (G1) without the projector, one obtains for expression (G3)

$$\hat{\partial}_L U_L = \text{STF}_{i_1 \dots i_l} P_{i_2}^{j_2} \dots P_{i_l}^{j_l} \frac{\partial}{\partial \xi^{j_2}} \dots \frac{\partial}{\partial \xi^{j_l}} \frac{\hat{\xi}^{j_1}}{(\hat{\xi})^2} \left(1 + \frac{c\tau}{r}\right), \quad (\text{G6})$$

where  $(\hat{\xi})^2 = r^2 - (c\tau)^2$  has been inserted. By inserting relations (F13) and (F17) into (G6) one recovers Eq. (114). Similarly, by inserting (F13) and (F17) and relation (F16) into (G4) one recovers Eq. (117).

### Appendix H: The scalar function $S_n^{l,p}$

If one determines the upper limits of neglected terms, one encounters in the subsequent appendices the following dimensionless scalar function,

$$S_{p,n}^l = k_{i_1} \dots k_{i_{p+2n}} \frac{\hat{\xi}_{i_{p+2n+1}} \dots \hat{\xi}_{i_l}}{(\hat{\xi})^{l-p-2n}} \delta_{<i_1}^3 \dots \delta_{i_l}^3. \quad (\text{H1})$$

The tensor  $k_{i_1} \dots k_{i_{p+2n}} \hat{\xi}_{i_{p+2n+1}} \dots \hat{\xi}_{i_l}$  originates from the action of the differential operators (64) and (65) on some scalar functions, while the STF tensor  $\delta_{<i_1}^3 \dots \delta_{i_l}^3$  originates from the mass-multipoles (149) and spin-multipoles (151) of an axi-symmetric body. The scalar function (H1) carries the natural number  $l$  which is the multipole-order. In addition, the scalar function (H1) carries the natural numbers  $p$  and  $n$ , where  $p$  originates from these differential operators, while index  $n$  originates from relations like (F13) - (F17).

In this Appendix we will determine the upper limit of the absolute value of (H1),

$$|S_{p,n}^l| = \left| k_{i_1} \dots k_{i_{p+2n}} \frac{\hat{\xi}_{i_{p+2n+1}} \dots \hat{\xi}_{i_l}}{(\hat{\xi})^{l-p-2n}} \delta_{<i_1}^3 \dots \delta_{i_l}^3 \right|. \quad (\text{H2})$$

Clearly, an upper limit of (H2) is given by

$$|S_{p,n}^l| \leq 1 \quad (\text{H3})$$

for all possible values of  $l, p, n$ . As we will see below, such an upper limit is far too inaccurate for a correct determination of the upper limits of the neglected terms of the unit tangent vector. That is why we have to consider this expression (H1) more carefully. By implementing the STF tensor (154) into (H1) one obtains

$$S_{p,n}^l = \sum_{m=0}^{\lfloor \frac{l-p-2n}{2} \rfloor} M_m^{l,p,n} \left( \frac{\hat{\xi} \cdot \mathbf{e}_3}{\hat{\xi}} \right)^{l-p-2n-2m} \times \sum_{s=m}^{\lfloor \frac{p+2n+2m}{2} \rfloor} H_s^l N_{m,s}^{p,n} (\mathbf{k} \cdot \mathbf{e}_3)^{p+2n+2m-2s}, \quad (\text{H4})$$

with the coefficients

$$M_m^{l,p,n} = \frac{(l-p-2n)!}{(l-p-2n-2m)! (2m)!}, \quad (\text{H5})$$

$$N_{m,s}^{p,n} = \frac{(p+2n)!}{(p+2n+2m-2s)! (2s-2m)!}. \quad (\text{H6})$$

As an explicit example we consider the case of mass-quadrupole  $l=2$ . From (H4) we get

$$S_{p=0,n=0}^{l=2} = \left( \frac{\hat{\xi} \cdot \mathbf{e}_3}{\hat{\xi}} \right)^2 - \frac{1}{3}, \quad (\text{H7})$$

$$S_{p=0,n=1}^{l=2} = (\mathbf{k} \cdot \mathbf{e}_3)^2 - \frac{1}{3}, \quad (\text{H8})$$

$$S_{p=1,n=0}^{l=2} = \left( \frac{\hat{\xi} \cdot \mathbf{e}_3}{\hat{\xi}} \right) (\mathbf{k} \cdot \mathbf{e}_3). \quad (\text{H9})$$

Furthermore, in two specific cases the function  $S_{p,n}^l$  in (H4) becomes a Legendre polynomial (E8),

$$S_{p=l,n=0}^l = \frac{l!}{(2l-1)!} P_l(\mathbf{e}_3 \cdot \mathbf{k}), \quad (\text{H10})$$

$$S_{p=0,n=0}^l = \frac{l!}{(2l-1)!} P_l \left( \frac{\mathbf{e}_3 \cdot \hat{\xi}}{\hat{\xi}} \right). \quad (\text{H11})$$

The sum over variable  $s$  in (H4) leads to Gegenbauer polynomials,

$$\begin{aligned} & \sum_{s=m}^{\lfloor \frac{p+2n+2m}{2} \rfloor} H_s^l N_{m,s}^{p,n} (\mathbf{k} \cdot \mathbf{e}_3)^{p+2n+2m-2s} \\ &= (-1)^m 2^{p+2n+m-1} \frac{(l-1)! (2l-p-2n-2m)!}{(2l-1)! (l-p-2n-m)!} \\ & \times \frac{C_{p+2n}^{l-p-2n-m+1/2}(\mathbf{k} \cdot \mathbf{e}_3)}{C_{p+2n}^{l-p-2n-m+1/2}(1)}, \end{aligned} \quad (\text{H12})$$

where the Gegenbauer polynomials are given by Eq. (E5). The arguments of these polynomials are within the closed interval  $\mathbf{k} \cdot \mathbf{e}_3 \in [-1, +1]$ . Within this interval, the maximal and minimal value of the Gegenbauer polynomials is reached at the boundary values  $\mathbf{k} \cdot \mathbf{e}_3 = \pm 1$ . Thus, using the maximal and minimal values of Gegenbauer polynomials as given by Eq. (E7), one may show that the upper limit of (H4) is given by

$$|S_{p,n}^l| \leq \left| \sum_{m=0}^{\lfloor \frac{l-p-2n}{2} \rfloor} M_m^{l,p,n} \sum_{s=m}^{\lfloor \frac{p+2n+2m}{2} \rfloor} H_s^l N_{m,s}^{p,n} \right|. \quad (\text{H13})$$

A Fortran 90 code has been employed [59] as independent check that (H13) represents the upper limit of (H4). In addition, the relations (H10) and (H11) have been used to check this numerical Fortran 90 code. In view of its importance, the computer algebra systems *Mathematica* [60] and *Maple* [61] have also been used to confirm relation (H13). By using (H12) one finds that the sum over variable  $s$  in (H13) reads

$$\begin{aligned} \sum_{s=m}^{\lfloor \frac{p+2n+2m}{2} \rfloor} H_s^l N_{m,s}^{p,n} &= (-1)^m \frac{(l-1)!}{(2l-1)!} \\ &\times 2^{p+2n+m-1} \frac{(2l-p-2n-2m)!}{(l-p-2n-m)!}. \end{aligned} \quad (\text{H14})$$

Thus, by inserting (H14) into (H13) we obtain the upper limit

$$\begin{aligned} |S_{p,n}^l| &\leq 2^{p+2n-1} \frac{(l-1)!}{(2l-1)!} (l-p-2n)! \\ &\times \sum_{m=0}^{\lfloor \frac{l-p-2n}{2} \rfloor} \frac{(-1)^m}{(l-p-2n-2m)!} \frac{(2l-p-2n-2m)!}{(l-p-2n-m)! m!} \end{aligned} \quad (\text{H15})$$

where  $(2m)!! = 2^m m!$  has been used. This upper limit depends on the chosen values of  $l, p, n$  and is much more fine-tuned than the estimate as given above by Eq. (H3) which is too inaccurate. As explicit example we consider the absolute values of (H7) - (H9), which read

$$|S_{p=0,n=0}^{l=2}| \leq \frac{2}{3}, \quad (\text{H16})$$

$$|S_{p=0,n=1}^{l=2}| \leq \frac{2}{3}, \quad (\text{H17})$$

$$|S_{p=1,n=0}^{l=2}| \leq 1. \quad (\text{H18})$$

The application of relation (H15) yields the same upper limits (H16) - (H18). However, an additional comment needs to be done about the absolute value in (H18). The impact vector  $\hat{\boldsymbol{\xi}}$  and the unit-vector of the unperturbed light ray  $\mathbf{k}$  are perpendicular to each other:  $\hat{\boldsymbol{\xi}} \cdot \mathbf{k} = 0$ . Therefore, the possible values of their components are not independent of each other, but they are restricted by this constraint. This fine-tuning fact has no impact of

the upper limits of (H16) and (H17), but it has an impact on the upper limit of (H18). Namely, if one takes into account that  $\hat{\boldsymbol{\xi}}$  and  $\mathbf{k}$  are perpendicular to each other, then it actually follows from (H9) that

$$|S_{p=1,n=0}^{l=2}| \leq \frac{1}{2}. \quad (\text{H19})$$

That relation is needed below and has, for instance, also been used in [43] (cf. Eq. (64) *ibid.*). The upper limit in (H19) is by a factor 2 smaller than the upper limit in (H18). This example shows that relation (H15) is a correct upper limit. But if one takes account of the fact that  $\hat{\boldsymbol{\xi}}$  and  $\mathbf{k}$  are perpendicular to each other, then the values for  $|S_{p,n}^l|$  are even smaller than (H15). However, this fine-tuning fact, that  $\hat{\boldsymbol{\xi}}$  and  $\mathbf{k}$  are perpendicular to each other, turns out to be of no relevance for the determination of the upper limit of the neglected terms of higher multipole order  $l > 2$ , because these terms are below the given threshold of 10 nas anyway. But for the quadrupole  $l = 2$  we will take the upper limit of (H19) instead of (H18), because the neglected terms of quadrupole are very nearby this threshold.

## Appendix I: Proof of relation (90)

The perturbation of the unit tangent vector in the left-hand side of relation (90) is given by Eq. (87) and reads

$$\Delta \mathbf{n}_{M_L}^{1,p \neq 0} = -\frac{2G\hat{M}_L}{c^2} \frac{(-1)^l}{l!} \hat{\partial}_{\tau_1, p \neq 0} \frac{\hat{\boldsymbol{\xi}}}{r_1 - c\tau_1} \frac{1}{r_1}, \quad (\text{I1})$$

with  $l \geq 2$  and where the differential operator is given by Eq. (65). We will determine the impact of (I1) on light deflection, that means we consider the contribution of this term to the angle of light deflection. Using relation (188) as well as the relations

$$\frac{\partial}{\partial c\tau_1} \frac{\hat{\boldsymbol{\xi}}}{r_1 - c\tau_1} \frac{1}{r_1} = +\frac{\hat{\boldsymbol{\xi}}}{(r_1)^3}, \quad (\text{I2})$$

$$\frac{\partial}{\partial \hat{\boldsymbol{\xi}}} \frac{1}{r_1} = -\frac{\hat{\boldsymbol{\xi}}}{(r_1)^3}, \quad (\text{I3})$$

one obtains for the angle between  $\mathbf{k}$  and  $\Delta \mathbf{n}_{M_L}^{1,p \neq 0}$  the expression

$$\begin{aligned} \delta \left( \mathbf{k}, \Delta \mathbf{n}_{M_L}^{1,p \neq 0} \right) &= -\frac{2G\hat{M}_L}{c^2} \frac{(-1)^l}{l!} \frac{\partial}{\partial \hat{\boldsymbol{\xi}}} \sum_{p=1}^l \binom{l}{p} \left( \frac{\partial}{\partial c\tau_1} \right)^{p-1} \\ &\times \text{STF}_{i_1 \dots i_l} k_{i_1} \dots k_{i_p} P_{i_{p+1}}^{j_{p+1}} \dots P_{i_l}^{j_l} \frac{\partial}{\partial \xi^{j_{p+1}}} \dots \frac{\partial}{\partial \xi^{j_l}} \frac{1}{r_1}. \end{aligned} \quad (\text{I4})$$

By means of relation (F14) one obtains

$$\begin{aligned} \delta(\mathbf{k}, \Delta \mathbf{n}_{M_L}^{1,p \neq 0}) &= -\frac{2GM_L}{c^2} \frac{1}{l!} \frac{\partial}{\partial \hat{\xi}} \sum_{p=1}^l \binom{l}{p} \left( \frac{\partial}{\partial c\tau_1} \right)^{p-1} \\ &\times \frac{(-1)^p}{(r_1)^{l-p+1}} \sum_{n=0}^{[(l-p)/2]} (-1)^n K_n^{l-p} \left( \frac{\hat{\xi}}{r_1} \right)^{l-p-2n} S_{p,n}^{i_1 \dots i_l} \end{aligned} \quad (\text{I5})$$

with the dimensionless tensorial function

$$S_{p,n}^{i_1 \dots i_l} = \text{STF}_{i_1 \dots i_l} k_{i_1} \dots k_{i_{p+2n}} \frac{\hat{\xi}_{i_{p+2n+1}} \dots \hat{\xi}_{i_l}}{(\hat{\xi})^{l-p-2n}}. \quad (\text{I6})$$

Here it has been taken into account that a contraction of the mass-multipoles with a Kronecker symbol vanishes, hence the projectors can be simplified by means of relation (123). By implementing the mass-multipoles (149) of an axi-symmetric body one gets

$$\begin{aligned} \delta(\mathbf{k}, \Delta \mathbf{n}_{M_L}^{1,p \neq 0}) &= +\frac{2GM}{c^2} \frac{(P)^l}{l!} J_l \frac{\partial}{\partial \hat{\xi}} \sum_{p=1}^l \binom{l}{p} \left( \frac{\partial}{\partial c\tau_1} \right)^{p-1} \\ &\times \frac{(-1)^p}{(r_1)^{l-p+1}} \sum_{n=0}^{[(l-p)/2]} (-1)^n K_n^{l-p} \left( \frac{\hat{\xi}}{r_1} \right)^{l-p-2n} S_{p,n}^l \end{aligned} \quad (\text{I7})$$

with the dimensionless scalar function

$$S_{p,n}^l = k_{i_1} \dots k_{i_{p+2n}} \frac{\hat{\xi}_{i_{p+2n+1}} \dots \hat{\xi}_{i_l}}{(\hat{\xi})^{l-p-2n}} \delta_{<i_1}^3 \dots \delta_{i_l}^3. \quad (\text{I8})$$

In the step from Eq. (I6) to Eq. (I8) the STF operation has been omitted in view of relation (B16). The function of Eq. (I8) is considered in Appendix H. Here it is only noticed that  $S_{p,n}^l$  does not depend on variable  $c\tau_1$ , hence the derivatives with respect to variable  $c\tau_1$  of Eq. (I7) do not act on  $S_{p,n}^l$ . Furthermore,  $S_{p,n}^l$  depends on the unit direction of three-vector  $\hat{\xi}$ , but not on its absolute value  $\hat{\xi}$ . In other words, according to relation (190), the derivative of  $S_{p,n}^l$  with respect to  $\hat{\xi}$  of Eq. (I7) vanishes too. Thus, by performing the partial derivative with respect to  $\hat{\xi}$  and afterwards the partial derivatives with respect to  $c\tau_1$  by means of relation (F25), one obtains

$$\delta(\mathbf{k}, \Delta \mathbf{n}_{M_L}^{1,p \neq 0}) = \delta_1 + \delta_2, \quad (\text{I9})$$

with the dimensionless terms

$$\begin{aligned} \delta_1 &= +\frac{2GM}{c^2 r_1} \left( \frac{P}{r_1} \right)^l \frac{J_l}{l!} \sum_{p=1}^l (-1)^p \binom{l}{p} \\ &\times \sum_{n=0}^{[(l-p)/2]} (-1)^n K_n^{l-p} S_{p,n}^l (l-p-2n) \\ &\times \sum_{k=1}^{\{p/2\}} L_k^{p,m_1} \left( \frac{\hat{\xi}}{r_1} \right)^{l-p-2n-1} \left( \frac{c\tau_1}{r_1} \right)^{p-2k+1} \end{aligned} \quad (\text{I10})$$

and

$$\begin{aligned} \delta_2 &= -\frac{2GM}{c^2 r_1} \left( \frac{P}{r_1} \right)^l \frac{J_l}{l!} \sum_{p=1}^l (-1)^p \binom{l}{p} \\ &\times \sum_{n=0}^{[(l-p)/2]} (-1)^n K_n^{l-p} S_{p,n}^l (2l-2p-2n+1) \\ &\times \sum_{k=1}^{\{p/2\}} L_k^{p,m_2} \left( \frac{\hat{\xi}}{r_1} \right)^{l-p-2n+1} \left( \frac{c\tau_1}{r_1} \right)^{p-2k+1}. \end{aligned} \quad (\text{I11})$$

The coefficients  $L_k^{p,m}$  are given by Eq. (F29), where  $m$  is replaced by  $m_1 = 2l-2p-2n+1$  and  $m_2 = 2l-2p-2n+3$ , respectively. The summation over  $k$  of Eqs. (I10) and (I11) leads to Gegenbauer polynomials (E1), that means

$$\sum_{k=1}^{\{p/2\}} L_k^{p,m_1} \left( \frac{\tau_1}{r_1} \right)^{p-2k+1} = (-1)^p (p-1)! C_{p-1}^{m_1/2} \left( \frac{\tau_1}{r_1} \right), \quad (\text{I12})$$

$$\sum_{k=1}^{\{p/2\}} L_k^{p,m_2} \left( \frac{\tau_1}{r_1} \right)^{p-2k+1} = (-1)^p (p-1)! C_{p-1}^{m_2/2} \left( \frac{\tau_1}{r_1} \right). \quad (\text{I13})$$

Inserting (I12) and (I13) into (I10) and (I11) yields for the absolute value of (I9)

$$\left| \delta(\mathbf{k}, \Delta \mathbf{n}_{M_L}^{1,p \neq 0}) \right| = C_1^{M_L} \frac{GM}{c^2 r_1} |J_l| \left( \frac{P}{r_1} \right)^l, \quad (\text{I14})$$

which shows the validity of relation (90). The coefficients are given by

$$\begin{aligned} C_1^{M_L} &= 2 \sum_{p=1}^l \binom{l}{p} \sum_{n=0}^{[(l-p)/2]} (-1)^n K_n^{l-p} |S_{p,n}^l| (p-1)! \\ &\times \left| (l-p-2n) \left( \frac{\hat{\xi}}{r_1} \right)^{l-p-2n-1} C_{p-1}^{l-p-n+1/2} \left( \frac{\tau_1}{r_1} \right) \right. \\ &\left. - (2l-2p-2n+1) \left( \frac{\hat{\xi}}{r_1} \right)^{l-p-2n+1} C_{p-1}^{l-p-n+3/2} \left( \frac{\tau_1}{r_1} \right) \right|. \end{aligned} \quad (\text{I15})$$

The coefficients  $K_n^{l-p}$  are given by Eq. (F20) where  $l$  is replaced by  $l-p$ . It is noticed that this coefficient contains a factor  $(-1)^n$  so that (I15) represents a double sum over positive terms. The insertion of (H15) into (I15) results into three sums over the variables  $p, n, m$ . In order to determine the term of Eq. (I15) a Fortran 90 code has been employed [59] and the relations

$$\frac{\hat{\xi}}{r_1} = \sin \delta(\mathbf{k}, \mathbf{x}_1), \quad (\text{I16})$$

$$\frac{c\tau_1}{r_1} = \cos \delta(\mathbf{k}, \mathbf{x}_1), \quad (\text{I17})$$

Table VII: The maximal values of the coefficients  $C_1^{M_L}$  of Eq. (I15). The values are simplified to one digit after the decimal point. These coefficients are the prefactors of relation (90).

$l$	$C_1^{M_L}$
2	2.0
4	13.1
6	77.3
8	431.5
10	2366.8

have been inserted with the angle  $\delta(\mathbf{k}, \mathbf{x}_1) \in [0, \pi]$ . The maximal values of these coefficients (I15) are given in Table VII. By inserting these numerical results for the coefficients into (I14) one finds that this term contributes less  $10^{-7}$  nas.

By inserting these numerical results for the coefficients into (I14) one finds

$$\left| \delta(\mathbf{k}, \Delta \mathbf{n}_{M_L}^{1,p \neq 0}) \right| \ll 1 \text{ nas} \quad (\text{I18})$$

to any multipole order and any solar system body and can safely be neglected. For instance, for Jupiter we get  $2 \times 10^{-7}$  nas for  $l = 2$ .

### Appendix J: Proof of relation (91)

The term on the left-hand side of relation (91) is given by the sum of Eqs. (88) and (89)

$$\Delta \mathbf{n}_{M_L}^{2+3,p \neq 0} = \Delta \mathbf{n}_{M_L}^{2,p \neq 0} + \Delta \mathbf{n}_{M_L}^{3,p \neq 0}, \quad (\text{J1})$$

and reads

$$\Delta \mathbf{n}_{M_L}^{2+3,p \neq 0} = + \frac{2G\hat{M}_L}{c^2} \frac{(-1)^l}{l!} \frac{1}{R} \times \left( \hat{\partial}_L^{\tau_1, p \neq 0} \frac{\hat{\xi}}{r_1 - c\tau_1} - \hat{\partial}_L^{\tau_0, p \neq 0} \frac{\hat{\xi}}{r_0 - c\tau_0} \right), \quad (\text{J2})$$

with  $l \geq 2$  and where the differential operators are given by Eqs. (64) and (65). In this Appendix, we will determine the impact of (J1) on the angle of light deflection. By means of the relation

$$\frac{\partial}{\partial c\tau} \frac{\hat{\xi}}{r - c\tau} = \frac{\hat{\xi}}{r - c\tau} \frac{1}{r}, \quad (\text{J3})$$

and afterwards making use of (G1) and (188), one obtains for the angle between  $\mathbf{k}$  and  $\Delta \mathbf{n}_{M_L}^{2+3,p \neq 0}$  the following ex-

pression

$$\begin{aligned} \delta(\mathbf{k}, \Delta \mathbf{n}_{M_L}^{2+3,p \neq 0}) &= - \frac{2G\hat{M}_L}{c^2} \frac{1}{R} \frac{(-1)^l}{l!} \frac{\partial}{\partial \hat{\xi}} \sum_{p=1}^l \binom{l}{p} \\ &\times \text{STF}_{i_1 \dots i_l} k_{i_1} \dots k_{i_p} P_{i_{p+1}}^{j_{p+1}} \dots P_{i_l}^{j_l} \frac{\partial}{\partial \xi^{j_{p+1}}} \dots \frac{\partial}{\partial \xi^{j_l}} \\ &\times \left[ \left( \frac{\partial}{\partial c\tau_1} \right)^{p-1} \ln(r_1 - c\tau_1) - \left( \frac{\partial}{\partial c\tau_0} \right)^{p-1} \ln(r_0 - c\tau_0) \right]. \end{aligned} \quad (\text{J4})$$

We separate (J4) into two terms,

$$\delta(\mathbf{k}, \Delta \mathbf{n}_{M_L}^{2+3,p \neq 0}) = \delta(\mathbf{k}, \Delta \mathbf{n}_{M_L}^{2+3,p=1}) + \delta(\mathbf{k}, \Delta \mathbf{n}_{M_L}^{2+3,p \geq 2}), \quad (\text{J5})$$

where the term with  $p = 1$  reads

$$\begin{aligned} \delta(\mathbf{k}, \Delta \mathbf{n}_{M_L}^{2+3,p=1}) &= - \frac{2G\hat{M}_L}{c^2} \frac{1}{R} \frac{(-1)^l}{(l-1)!} \frac{\partial}{\partial \hat{\xi}} \\ &\times \text{STF}_{i_1 \dots i_l} k_{i_1} P_{i_2}^{j_2} \dots P_{i_l}^{j_l} \frac{\partial}{\partial \xi^{j_2}} \dots \frac{\partial}{\partial \xi^{j_l}} \ln \frac{r_1 - c\tau_1}{r_0 - c\tau_0}, \end{aligned} \quad (\text{J6})$$

while the term with  $p \geq 2$  reads

$$\begin{aligned} \delta(\mathbf{k}, \Delta \mathbf{n}_{M_L}^{2+3,p \geq 2}) &= + \frac{2G\hat{M}_L}{c^2} \frac{1}{R} \frac{(-1)^l}{l!} \frac{\partial}{\partial \hat{\xi}} \sum_{p=2}^l \binom{l}{p} \\ &\times \text{STF}_{i_1 \dots i_l} k_{i_1} \dots k_{i_p} P_{i_{p+1}}^{j_{p+1}} \dots P_{i_l}^{j_l} \frac{\partial}{\partial \xi^{j_{p+1}}} \dots \frac{\partial}{\partial \xi^{j_l}} \\ &\times \left[ \left( \frac{\partial}{\partial c\tau_1} \right)^{p-2} \frac{1}{r_1} - \left( \frac{\partial}{\partial c\tau_0} \right)^{p-2} \frac{1}{r_0} \right]. \end{aligned} \quad (\text{J7})$$

In (J7) the relation

$$\frac{\partial}{\partial c\tau} \ln(r - c\tau) = - \frac{1}{r} \quad (\text{J8})$$

has been used.

First of all, we consider (J6) which we denote as  $\delta_3$ . Using relations (F13) and (F17) we get

$$\begin{aligned} &P_{j_2}^{(i_2)} \dots P_{j_l}^{(i_l)} \frac{\partial}{\partial \xi^{j_2}} \dots \frac{\partial}{\partial \xi^{j_l}} \ln \frac{r_1 - c\tau_1}{r_0 - c\tau_0} \\ &= (-1)^l \sum_{n=0}^{\lfloor \frac{l-1}{2} \rfloor} G_n^{l-1} \frac{P_{i_2 i_3} \dots P_{i_{2n} i_{2n+1}} \hat{\xi}_{i_{2n+2}} \dots \hat{\xi}_{i_l}}{(\hat{\xi})^{2l-2n-2}} \\ &\times (F_n^{l-1}(c\tau_1, r_1) - F_n^{l-1}(c\tau_0, r_0)). \end{aligned} \quad (\text{J9})$$

By inserting (J9) into (J6) follows

$$\begin{aligned} \delta_3 &= - \frac{2G\hat{M}_L}{c^2} \frac{1}{R} \frac{1}{(l-1)!} \frac{\partial}{\partial \hat{\xi}} \frac{1}{(\hat{\xi})^{l-1}} \sum_{n=0}^{\lfloor \frac{l-1}{2} \rfloor} (-1)^n G_n^{l-1} \\ &\times S_{p=1, n}^{i_1 \dots i_l} (F_n^{l-1}(c\tau_1, r_1) - F_n^{l-1}(c\tau_0, r_0)), \end{aligned} \quad (\text{J10})$$

where the dimensionless scalar function  $F_n^{l-1}(c\tau, r)$  is given by Eq. (F21) and the dimensionless tensorial function reads

$$S_{p=1,n}^{i_1 \dots i_l} = \text{STF}_{i_1 \dots i_l} k_{i_1} \dots k_{i_{2n+1}} \frac{\hat{\xi}_{i_{2n+2}} \dots \hat{\xi}_{i_l}}{(\hat{\xi})^{l-2n-1}}, \quad (\text{J11})$$

which is the special case  $p = 1$  of (I6). Because a contraction of the STF mass multipoles  $M_L$  with a Kronecker symbol vanishes, we have simplified the projectors in (J9) by means of (123) with  $p = 1$  in order to get (J10) with (J11). By implementing these mass-multipoles (149) of an axi-symmetric body one gets

$$\begin{aligned} \delta_3 = & + \frac{2GM}{c^2} \frac{(P)^l}{R} J_l \frac{1}{(l-1)!} \frac{\partial}{\partial \hat{\xi}} \frac{1}{(\hat{\xi})^{l-1}} \\ & \times \sum_{n=0}^{\lfloor \frac{l-1}{2} \rfloor} (-1)^n G_n^{l-1} S_{p=1,n}^l (F_n^{l-1}(c\tau_1, r_1) - F_n^{l-1}(c\tau_0, r_0)), \end{aligned} \quad (\text{J12})$$

with the dimensionless scalar function

$$S_{p=1,n}^l = k_{i_1} \dots k_{i_{2n+1}} \frac{\hat{\xi}_{i_{2n+2}} \dots \hat{\xi}_{i_l}}{(\hat{\xi})^{l-2n-1}} \delta_{<i_1}^3 \dots \delta_{i_l}^3, \quad (\text{J13})$$

which is the special case  $p = 1$  of (H1). In the step from Eq. (J11) to Eq. (J13) the STF operation has been omitted due to relation (B16). The function of Eq. (J13) is considered in Appendix H. As it was mentioned in the text below Eq. (I8), the derivative of (J13) with respect to  $\hat{\xi}$  vanishes. Thus, the partial derivative in (J12) with respect to  $\hat{\xi}$  yields

$$\delta_3 = - \frac{2GM}{c^2 r_1} \left( \frac{P}{\hat{\xi}} \right)^l J_l \sum_{n=0}^{\lfloor \frac{l-1}{2} \rfloor} (-1)^n G_n^{l-1} S_{p=1,n}^l T(z, \cos \psi) \quad (\text{J14})$$

where  $\psi = \delta(\mathbf{x}_1, \mathbf{x}_0)$  is the angle between  $\mathbf{x}_0$  and  $\mathbf{x}_1$ . The function  $T$  in (J14) reads

$$\begin{aligned} T(z, \cos \psi) = & \frac{1}{\hat{R}} \left[ \frac{1}{(l-2)!} (F_n^{l-1}(c\tau_1, r_1) - F_n^{l-1}(c\tau_0, r_0)) \right. \\ & \left. - \frac{1}{(l-1)!} (E_n^{l-1}(c\tau_1, r_1) - E_n^{l-1}(c\tau_0, r_0)) \right], \end{aligned} \quad (\text{J15})$$

where  $F_n^{l-1}(c\tau, r)$  is given by Eq. (F21), while  $E_n^{l-1}(c\tau, r)$  is  $\hat{\xi}$  times the derivative of  $F_n^{l-1}(c\tau, r)$  with respect to  $\hat{\xi}$ ,

$$\begin{aligned} E_n^{l-1}(c\tau, r) = & -F_n^{l-1}(c\tau, r) \\ & + \left( \frac{c\tau}{r} \right)^3 \sum_{k=0}^{l-n-2} \frac{2k+1}{2^{2k}} \binom{2k}{k} \left( \frac{\hat{\xi}}{r} \right)^{2k}, \end{aligned} \quad (\text{J16})$$

and  $\hat{R}$  is the reduced distance (F5). The function (J15) can be written in the more explicit form

$$T(z, \cos \psi) = \frac{1}{(l-1)!} (T_1 - T_2), \quad (\text{J17})$$

with

$$T_1 = \sum_{k=0}^{l-n-2} \frac{l}{2^{2k}} \binom{2k}{k} \left( \frac{\hat{\xi}}{r_0} \right)^{2k} \frac{1}{\hat{R}} \left[ \frac{\mathbf{k} \cdot \mathbf{x}_1}{r_1} z^{2k} - \frac{\mathbf{k} \cdot \mathbf{x}_0}{r_0} \right], \quad (\text{J18})$$

$$\begin{aligned} T_2 = & \sum_{k=0}^{l-n-2} \frac{2k+1}{2^{2k}} \binom{2k}{k} \left( \frac{\hat{\xi}}{r_0} \right)^{2k} \\ & \times \frac{1}{\hat{R}} \left[ \left( \frac{\mathbf{k} \cdot \mathbf{x}_1}{r_1} \right)^3 z^{2k} - \left( \frac{\mathbf{k} \cdot \mathbf{x}_0}{r_0} \right)^3 \right], \end{aligned} \quad (\text{J19})$$

where  $c\tau_0$  and  $c\tau_1$  have been replaced by  $\mathbf{k} \cdot \mathbf{x}_0$  and  $\mathbf{k} \cdot \mathbf{x}_1$ , respectively. In view of  $\hat{\xi}/r_0 = \sin \delta(\mathbf{k}, \mathbf{x}_0)$  and by means of relations (F3) and (F4), it becomes apparent that the function (J17) depends on two variables only:  $z$  and  $\psi$ . For  $l = 2$  one obtains  $|T(z, \cos \psi)| \leq 2.1$ , while for higher multipole orders  $l > 2$  we have used the computer algebra system *Maple* [61] and have found that the absolute value of (J17) is maximized when  $z = 1$ ,

$$|T(z, \cos \psi)| \leq |T(z = 1, \cos \psi)| \quad \text{for } l > 2. \quad (\text{J20})$$

By inserting the limits

$$\lim_{z \rightarrow 1} \frac{1}{\hat{R}} \left[ \frac{\mathbf{k} \cdot \mathbf{x}_1}{r_1} z^{2k} - \frac{\mathbf{k} \cdot \mathbf{x}_0}{r_0} \right] = 1, \quad (\text{J21})$$

$$\lim_{z \rightarrow 1} \frac{1}{\hat{R}} \left[ \left( \frac{\mathbf{k} \cdot \mathbf{x}_1}{r_1} \right)^3 z^{2k} - \left( \frac{\mathbf{k} \cdot \mathbf{x}_0}{r_0} \right)^3 \right] = \frac{1}{2} (1 - \cos \psi), \quad (\text{J22})$$

into (J18) and (J19) one arrives at the function

$$T(z = 1, \cos \psi) = \frac{1}{(l-1)!} (L_1 - L_2), \quad (\text{J23})$$

which depends on one variable,  $\psi$ , and where

$$L_1 = \sum_{k=0}^{l-n-2} \frac{l}{2^{2k}} \binom{2k}{k} \left( \frac{\hat{\xi}}{r_0} \right)^{2k}, \quad (\text{J24})$$

$$L_2 = \sum_{k=0}^{l-n-2} \frac{2k+1}{2^{2k}} \binom{2k}{k} \left( \frac{\hat{\xi}}{r_0} \right)^{2k} \frac{1}{2} (1 - \cos \psi). \quad (\text{J25})$$

Both these polynomials are positive,  $L_1 > 0$  and  $L_2 \geq 0$ , hence

$$|L_1 - L_2| \leq \begin{cases} L_1 & \text{if } L_1 \geq L_2 \\ L_2 & \text{if } L_2 \geq L_1 \end{cases}. \quad (\text{J26})$$

The largest values of  $L_1$  and  $L_2$  are achieved for  $\hat{\xi}/r_0 = 1$ . Hence, we get

$$L_1 \leq \sum_{k=0}^{l-n-2} \frac{l}{2^{2k}} \binom{2k}{k}, \quad (\text{J27})$$

$$L_2 \leq \sum_{k=0}^{l-n-2} \frac{2k+1}{2^{2k}} \binom{2k}{k}. \quad (\text{J28})$$

Table VIII: The coefficients  $B_{1,A}^{M_L}$  of Eq. (J34). The values are simplified to one digit after the decimal point. The relation (H19) needs to be used, in order to get the coefficient  $B_{1,A}^{M_L}$  for  $l = 2$ .

$l$	$B_{1,A}^{M_L}$
2	2.1
4	18.3
6	85.6
8	387.4
10	1711.5

These terms can be evaluated by using the relations [56]

$$\sum_{k=0}^m \frac{1}{2^{2k}} \binom{2k}{k} = \frac{2m+1}{2^{2m}} \binom{2m}{m}, \quad (\text{J29})$$

$$\sum_{k=0}^m \frac{k}{2^{2k}} \binom{2k}{k} = \frac{m}{3} \frac{2m+1}{2^{2m}} \binom{2m}{m}, \quad (\text{J30})$$

from which one concludes that  $L_1 > L_2$  because of the factor  $l$  in (J27). Thus, in view of (J26), we have only to consider the term

$$L_1 \leq l \frac{2l-2n-3}{2^{2l-2n-4}} \binom{2l-2n-4}{l-n-2} \quad (\text{J31})$$

and obtain the following upper limit

$$|T(z, \cos \psi)| \leq \frac{l}{(l-1)!} \frac{2l-2n-3}{2^{2l-2n-4}} \binom{2l-2n-4}{l-n-2}. \quad (\text{J32})$$

By inserting (J32) into (J12) we get for the absolute value of  $\delta_3$  the following upper limit

$$|\delta_3| \leq B_{1,A}^{M_L} \frac{GM}{c^2 r_1} \left(\frac{P}{\hat{\xi}}\right)^l |J_l|, \quad (\text{J33})$$

where

$$B_{1,A}^{M_L} = \frac{2l}{(l-1)!} \sum_{n=0}^{\lfloor \frac{l-1}{2} \rfloor} (-1)^n G_n^{l-1} |S_{p=1,n}^l| \times \frac{2l-2n-3}{2^{2l-2n-4}} \binom{2l-2n-4}{l-n-2}. \quad (\text{J34})$$

The coefficients  $G_n^{l-1}$  are given by Eq. (121) and for  $|S_{p=1,n}^l|$  we take relation (H15), except for the case of  $l = 2$ , where we use relation (H19). In order to determine the sum over variable  $n$  of Eq. (J34) a Fortran 90 code has been employed [59]. The values of these coefficients (J34) are given in Table VIII.

Now we consider (J7). The mathematical structure of the individual terms with parameters  $(c\tau_0, r_0)$  and  $(c\tau_1, r_1)$  of Eq. (J7) is almost identical to Eq. (I4). In fact, we may

use the very same steps as performed from Eq. (I4) to Eq. (I9) and use relation (F26). Then we get

$$\delta(\mathbf{k}, \Delta \mathbf{n}_{M_L}^{2+3, p \geq 2}) = \delta_4 + \delta_5 \quad (\text{J35})$$

with the dimensionless terms

$$\begin{aligned} \delta_4 = & -\frac{2GM}{c^2 r_1} \left(\frac{P}{\hat{\xi}}\right)^l \frac{J_l}{l!} \sum_{p=2}^l (-1)^p \binom{l}{p} \\ & \times \sum_{n=0}^{\lfloor (l-p)/2 \rfloor} (-1)^n K_n^{l-p} S_{p,n}^l (l-p-2n) \\ & \times \sum_{k=1}^{\lfloor p/2 \rfloor} Q_k^{p, m_1} T_4 \end{aligned} \quad (\text{J36})$$

and

$$\begin{aligned} \delta_5 = & +\frac{2GM}{c^2 r_1} \left(\frac{P}{\hat{\xi}}\right)^l \frac{J_l}{l!} \sum_{p=2}^l (-1)^p \binom{l}{p} \\ & \times \sum_{n=0}^{\lfloor (l-p)/2 \rfloor} (-1)^n K_n^{l-p} S_{p,n}^l (2l-2p-2n+1) \\ & \times \sum_{k=1}^{\lfloor p/2 \rfloor} Q_k^{p, m_2} T_5. \end{aligned} \quad (\text{J37})$$

The coefficients  $Q_k^{p,m}$  are given by Eq. (F30), where  $m$  is replaced by  $m_1 = 2l-2p-2n+1$  and  $m_2 = 2l-2p-2n+3$ , respectively, and the dimensionless functions read

$$T_4 = \frac{(\hat{\xi})^{2l-p-2n-1}}{\hat{R}} \left[ \frac{(\mathbf{k} \cdot \mathbf{x}_1)^{p-2k}}{(r_1)^{2l-2n-2k-1}} - \frac{(\mathbf{k} \cdot \mathbf{x}_0)^{p-2k}}{(r_0)^{2l-2n-2k-1}} \right], \quad (\text{J38})$$

$$T_5 = \frac{(\hat{\xi})^{2l-p-2n+1}}{\hat{R}} \left[ \frac{(\mathbf{k} \cdot \mathbf{x}_1)^{p-2k}}{(r_1)^{2l-2n-2k+1}} - \frac{(\mathbf{k} \cdot \mathbf{x}_0)^{p-2k}}{(r_0)^{2l-2n-2k+1}} \right], \quad (\text{J39})$$

where the terms  $c\tau_0$  and  $c\tau_1$  have been replaced by  $\mathbf{k} \cdot \mathbf{x}_0$  and  $\mathbf{k} \cdot \mathbf{x}_1$ , respectively. In view of relations (F3) and (F4), it becomes apparent that these functions (J38) and (J39) depend on two variables only:  $z$  and  $\psi$ . Using the computer algebra system *Maple* [61] one finds that their maximal absolute values are given by  $|T_4| \leq 1.0$  and  $|T_5| \leq 1.1$  for  $l = 2$ , while for higher multipole order we get

$$|T_4| \leq 2.0 \quad \text{for } l > 2, \quad (\text{J40})$$

$$|T_5| \leq 2.0 \quad \text{for } l > 2. \quad (\text{J41})$$

These upper limits are valid for any set of parameters  $p, k, n$  which is within the region of possible values. The region of possible values is given by the upper limits of the sum of these parameters.

Table IX: The coefficients  $B_{1,B}^{M_L}$  and  $B_{1,C}^{M_L}$  of Eqs. (J44) and (J45). The values are simplified to one digit after the decimal point.

$l$	$B_{1,B}^{M_L}$	$B_{1,C}^{M_L}$
2	0.0	0.7
4	1.5	3.3
6	3.9	10.3
8	15.4	39.2
10	54.5	142.3

Thus we get for the upper limits of (J36) and (J37)

$$|\delta_4| \leq B_{1,B}^{M_L} \frac{GM}{c^2 r_1} \left(\frac{P}{\hat{\xi}}\right)^l |J_l|, \quad (\text{J42})$$

$$|\delta_5| \leq B_{1,C}^{M_L} \frac{GM}{c^2 r_1} \left(\frac{P}{\hat{\xi}}\right)^l |J_l|, \quad (\text{J43})$$

with

$$B_{1,B}^{M_L} = \frac{2}{l!} |T_4| \sum_{p=2}^l (-1)^p \binom{l}{p} \sum_{n=0}^{[(l-p)/2]} (-1)^n K_n^{l-p} |S_{p,n}^l| \times (l-p-2n) \sum_{k=1}^{[p/2]} Q_k^{p,2l-2p-2n+1}, \quad (\text{J44})$$

$$B_{1,C}^{M_L} = \frac{2}{l!} |T_5| \sum_{p=2}^l (-1)^p \binom{l}{p} \sum_{n=0}^{[(l-p)/2]} (-1)^n K_n^{l-p} |S_{p,n}^l| \times (2l-2p-2n+1) \sum_{k=1}^{[p/2]} Q_k^{p,2l-2p-2n+3}. \quad (\text{J45})$$

These terms can be simplified by means of

$$\sum_{k=1}^{[p/2]} Q_k^{p,2l-2p-2n+1} = (-1)^p \frac{(2l-p-2n-1)!}{(2l-2p-2n+1)!},$$

$$\sum_{k=1}^{[p/2]} Q_k^{p,2l-2p-2n+3} = (-1)^p \frac{(2l-p-2n)!}{(2l-2p-2n+2)!}. \quad (\text{J46})$$

In order to perform the summation over variables  $p, n$  of Eqs. (J44) and (J45) a Fortran 90 code has been employed [59]. The values of these coefficients of Eqs. (J44) and (J45) are given in Table IX.

According to Eq. (J5) we may add the terms (J33) as well as (J42) and (J43) together and obtain

$$\left| \delta(\mathbf{k}, \Delta \mathbf{n}_{M_L}^{2+3, p \neq 0}) \right| \leq B_1^{M_L} \frac{GM}{c^2 r_1} \left(\frac{P}{d_k}\right)^l |J_l|, \quad (\text{J47})$$

where  $\hat{\xi}$  has been replaced by the impact parameter  $d_k$  and the coefficients of Eq. (91)

$$B_1^{M_L} = B_{1,A}^{M_L} + B_{1,B}^{M_L} + B_{1,C}^{M_L}. \quad (\text{J48})$$

Table X: The maximal values of the coefficients  $B_1^{M_L}$  of Eq. (J48). The values are simplified to one digit after the decimal point. These coefficients are the prefactors of relation (91).

$l$	$B_1^{M_L}$
2	2.8
4	23.1
6	99.8
8	442.0
10	1908.3

which are summarized in Table X.

Eq. (J47) agrees with Eq. (91). By inserting the values of Table VIII and Table IX into (J47) one obtains the contribution of this term to the angle of light deflection. Numerical values for grazing light ray at Jupiter are presented in Table I.

### Appendix K: Proof of relation (125)

In this Appendix we will show that the maximal contribution of the term (124) to the angle of light deflection is given by Eq. (125). In view of relation (183) and by means of (188) we find that the contribution of (124) to the angle of light deflection is given by

$$\delta(\mathbf{k}, \mathbf{U}_2^{M_L}) = + \frac{2GM_L}{c^2} \frac{1}{l!} \frac{\partial}{\partial \hat{\xi}} \sum_{n=0}^{[l/2]} \hat{F}_n^l(c\tau, r_1) \times G_n^l P_{i_1 i_2} \dots P_{i_{2n-1} i_{2n}} \frac{\hat{\xi}_{i_{2n+1}} \dots \hat{\xi}_{i_l}}{(\hat{\xi})^{2l-2n}}. \quad (\text{K1})$$

Using the mass-multipoles of an axi-symmetric body as given by Eq. (149) and taking account of relation (123) we obtain

$$\delta(\mathbf{k}, \mathbf{U}_2^{M_L}) = - \frac{2GM}{c^2} \frac{J_l}{l!} \sum_{n=0}^{[l/2]} (-1)^n G_n^l S_{p=0,n}^l \times \frac{\partial}{\partial \hat{\xi}} \left(\frac{P}{\hat{\xi}}\right)^l \hat{F}_n^l(c\tau_1, r_1) \quad (\text{K2})$$

where  $S_{p=0,n}^l$  is the special case  $p=0$  of the scalar function of Eq. (18). In Eq. (K2) it has been taken into account that the derivative of this scalar function (18) with respect to  $\hat{\xi}$  vanishes. By performing the derivative in (K2) we get

$$\delta(\mathbf{k}, \mathbf{U}_2^{M_L}) = \frac{2GM}{c^2 \hat{\xi}} \frac{J_l}{l!} \left(\frac{P}{\hat{\xi}}\right)^l \sum_{n=0}^{[l/2]} (-1)^n G_n^l S_{p=0,n}^l \times \left[ l \hat{F}_n^l(c\tau_1, r_1) - \hat{E}_n^l(c\tau_1, r_1) \right], \quad (\text{K3})$$

where  $\hat{F}_n^l(c\tau_1, r_1)$  is given by Eq. (F23) and  $\hat{E}_n^l(c\tau_1, r_1)$  reads

$$\begin{aligned} \hat{E}_n^l(c\tau_1, r_1) &= -\hat{F}_n^l(c\tau_1, r_1) \\ &+ \left(\frac{c\tau_1}{r_1}\right)^3 \sum_{k=1}^{l-n-1} \frac{2k+1}{2^{2k}} \binom{2k}{k} \left(\frac{\hat{\xi}}{r_1}\right)^{2k}, \end{aligned} \quad (\text{K4})$$

which resembles a similar structure encountered above by Eq. (J16). Using  $0 \leq \hat{\xi}/r_1 \leq 1$  and  $(-1)^n G_n^l > 0$  we find for the upper limit of the absolute value of (K3)

$$\begin{aligned} |\delta(\mathbf{k}, \mathbf{U}_2^{ML})| &\leq \frac{2GM}{c^2 r_1} \left(\frac{P}{r_1}\right) \frac{|J_l|}{l!} \left(\frac{P}{\hat{\xi}}\right)^{l-1} \sum_{n=0}^{\lfloor l/2 \rfloor} (-1)^n G_n^l \\ &\times |S_{p=0,n}^l| \left( l \left| \hat{F}_n^l(c\tau_1, r_1) \right| + \left| \hat{E}_n^l(c\tau_1, r_1) \right| \right). \end{aligned} \quad (\text{K5})$$

Furthermore, since  $|c\tau_1/r_1| \leq 1$  we conclude from (F23) and (K4) that

$$\left| \hat{F}_n^l(c\tau_1, r_1) \right| \leq \sum_{k=1}^{l-n-1} \frac{1}{2^{2k}} \binom{2k}{k}, \quad (\text{K6})$$

$$\left| \hat{E}_n^l(c\tau_1, r_1) \right| \leq \left| \hat{F}_n^l(c\tau_1, r_1) \right| + \sum_{k=1}^{l-n-1} \frac{2k+1}{2^{2k}} \binom{2k}{k}. \quad (\text{K7})$$

Both these polynomials can be evaluated by means relations (J29) and (J30). One finds that

$$(l+1) \left| \hat{F}_n^l(c\tau_1, r_1) \right| \geq \sum_{k=1}^{l-n-1} \frac{2k+1}{2^{2k}} \binom{2k}{k}. \quad (\text{K8})$$

Therefore,

$$\begin{aligned} |\delta(\mathbf{k}, \mathbf{U}_2^{ML})| &\leq \frac{4GM}{c^2 r_1} \left(\frac{P}{r_1}\right) \frac{|J_l|}{l!} \left(\frac{P}{\hat{\xi}}\right)^{l-1} \sum_{n=0}^{\lfloor l/2 \rfloor} (-1)^n G_n^l \\ &\times |S_{p=0,n}^l| (l+1) \left| \hat{F}_n^l(c\tau_1, r_1) \right|. \end{aligned} \quad (\text{K9})$$

For (K6) we get from (J29)

$$\sum_{k=0}^{l-n-1} \frac{1}{2^{2k}} \binom{2k}{k} = \frac{2l-2n-1}{2^{2l-2n-2}} \binom{2l-2n-2}{l-n-1} - 1. \quad (\text{K10})$$

Hence

$$\hat{F}_n^l(c\tau_1, r_1) \leq \frac{2l-2n-1}{2^{2l-2n-2}} \binom{2l-2n-2}{l-n-1} \quad (\text{K11})$$

because the first term on the right-hand side of (K10) is larger than 1. Inserting (K11) into (K9) yields

$$|\delta(\mathbf{k}, \mathbf{U}_2^{ML})| \leq C_2^{ML} \frac{GM}{c^2 r_1} \left(\frac{P}{r_1}\right) |J_l| \left(\frac{P}{\hat{\xi}}\right)^{l-1} \quad (\text{K12})$$

Table XI: The values of the coefficients  $C_2^{ML}$  of Eq. (K13). The numerical values were rounded to integers.

$l$	$C_2^{ML}$
2	32
4	154
6	720
8	3224
10	14156

with the coefficient

$$\begin{aligned} C_2^{ML} &= 4 \frac{l+1}{l!} \sum_{n=0}^{\lfloor l/2 \rfloor} (-1)^n G_n^l |S_{p=0,n}^l| \\ &\times \frac{2l-2n-1}{2^{2l-2n-2}} \binom{2l-2n-2}{l-n-1}. \end{aligned} \quad (\text{K13})$$

In order to perform the summation over variable  $n$  of Eq. (K13) a Fortran 90 code has been employed [59]. Numerical values of these coefficients are presented in Table XI.

By inserting these numerical results for the coefficients into (K12) one finds

$$|\delta(\mathbf{k}, \mathbf{U}_2^{ML})| \ll 1 \text{ nas} \quad (\text{K14})$$

to any multipole order and any solar system body and can safely be neglected. For instance, for Jupiter we get  $3 \times 10^{-6}$  nas for  $l=2$ .

## Appendix L: Proof of relation (131)

In this Appendix we will show that the maximal contribution of the term (130) to the angle of light deflection is given by Eq. (131). In view of relation (183) and by means of (188) we find that the contribution of (130) to the angle of light deflection is given by

$$\delta(\mathbf{k}, \mathbf{V}_2^{ML}) = -\frac{2GM_L}{c^2} \frac{1}{l!} \frac{1}{R} (V_L^2(c\tau_1, r_1) - V_L^2(c\tau_0, r_0)) \quad (\text{L1})$$

with

$$\begin{aligned} V_L^2(c\tau_1, r_1) &= \frac{\partial}{\partial \hat{\xi}} \sum_{n=0}^{\lfloor l/2 \rfloor} G_n^l P_{i_1 i_2} \dots P_{i_{2n-1} i_{2n}} \\ &\times \left[ c\tau_1 \hat{F}_n^l(c\tau_1, r_1) + r_1 W_n^l \left(\frac{\hat{\xi}}{r_1}\right)^{2l-2n} \right] \frac{\hat{\xi}_{i_{2n+1}} \dots \hat{\xi}_{i_l}}{\left(\hat{\xi}\right)^{2l-2n}}. \end{aligned} \quad (\text{L2})$$

The expression  $V_L^2(c\tau_0, r_0)$  of Eq. (L1) is obtained from (L2) by replacing  $c\tau_1$  by  $c\tau_0$  and  $r_1$  by  $r_0$ . The function

$\hat{F}_n^l(c\tau_1, r_1)$  is given by Eq. (F23) and the coefficients  $G_n^l$  and  $W_n^l$  are given by Eqs. (121) and (122). In order to determine the magnitude of (L1) we insert the mass-multipole (149) of an axi-symmetric body and get

$$\delta(\mathbf{k}, \mathbf{V}_2^{ML}) = + \frac{2GM}{c^2} \frac{J_l}{l!} \frac{1}{R} (V_L^2(c\tau_1, r_1) - V_L^2(c\tau_0, r_0)) \quad (\text{L3})$$

with

$$V_L^2(c\tau_1, r_1) = \sum_{n=0}^{\lfloor l/2 \rfloor} (-1)^n G_n^l S_{p=0,n}^l \times \frac{\partial}{\partial \hat{\xi}} \left( \frac{P}{\hat{\xi}} \right)^l \left[ c\tau_1 \hat{F}_n^l(c\tau_1, r_1) + r_1 W_n^l \left( \frac{\hat{\xi}}{r_1} \right)^{2l-2n} \right], \quad (\text{L4})$$

where we have used relation (123) and the dimensionless scalar function  $S_{p,n}^l$  is given by Eq. (H1). Furthermore, it has been taken into account that the derivative of this scalar function with respect to  $\hat{\xi}$  vanishes. The difference between (L2) and (L4) is the STF tensor of Eq. (154) and a term  $(P)^l$ , which is the radius of the body to power  $l$ , but we shall avoid a new notation and keep the notation  $V_L^2(c\tau_1, r_1)$  of Eq. (L4) as is. By performing the derivative in (L4) and inserting into (L3) we get

$$\delta(\mathbf{k}, \mathbf{V}_2^{ML}) = B_2^{ML} \frac{GM}{c^2 r_1} J_l \left( \frac{P}{\hat{\xi}} \right)^l, \quad (\text{L5})$$

which coincides with the asserted relation (131). The dimensionless coefficients are given by

$$B_2^{ML} = B_2^{ML}(c\tau_1, r_1) - B_2^{ML}(c\tau_0, r_0), \quad (\text{L6})$$

with

$$B_2^{ML}(c\tau_1, r_1) = - \frac{2}{l!} \frac{1}{\hat{R}} \sum_{n=0}^{\lfloor l/2 \rfloor} (-1)^n G_n^l S_{p=0,n}^l \times \left[ l f_1(\hat{\xi}, r_1) - (l-1) f_2(\hat{\xi}, r_1) - f_3(\hat{\xi}, r_1) - g_1(\hat{\xi}, r_1) + 2g_2(\hat{\xi}, r_1) - g_3(\hat{\xi}, r_1) + l W_n^l \left( \frac{\hat{\xi}}{r_1} \right)^{2l-2n-1} - W_n^l \left( \frac{\hat{\xi}}{r_1} \right)^{2l-2n+1} - (2l-2n) W_n^l \left( \frac{\hat{\xi}}{r_1} \right)^{2l-2n+1} + (2l-2n) W_n^l \left( \frac{\hat{\xi}}{r_1} \right)^{2l-2n+3} \right], \quad (\text{L7})$$

where  $\hat{R}$  is the reduced distance (F5) between source and observer. The dimensionless function  $S_{p=0,n}^l$  is a special case of Eq. (H4). The coefficients  $G_n^l$  and  $W_n^l$  are defined by Eqs. (121) and (122). The term  $B_2^{ML}(c\tau_0, r_0)$  in (L6)

is obtained from (L7) by the replacements  $c\tau_1$  by  $c\tau_0$  and  $r_1$  by  $r_0$ . In order to obtain (L7) from (L4) we have used the relations

$$(c\tau_1)^2 = (r_1)^2 - (\hat{\xi})^2, \quad (\text{L8})$$

$$(c\tau_0)^2 = (r_0)^2 - (\hat{\xi})^2. \quad (\text{L9})$$

The functions in (L7) are given by

$$f_a(\hat{\xi}, r_1) = \sum_{k=1}^{l-n-1} \frac{1}{2^{2k}} \binom{2k}{k} \left( \frac{\hat{\xi}}{r_1} \right)^{2k+2a-3}, \quad (\text{L10})$$

$$g_a(\hat{\xi}, r_1) = \sum_{k=1}^{l-n-1} \frac{2k}{2^{2k}} \binom{2k}{k} \left( \frac{\hat{\xi}}{r_1} \right)^{2k+2a-3}. \quad (\text{L11})$$

According to Eq. (L7) the coefficients  $B_2^{ML}$  in (L6) contain only terms of the structure

$$\frac{1}{\hat{R}} \left[ \left( \frac{\hat{\xi}}{r_1} \right)^b - \left( \frac{\hat{\xi}}{r_0} \right)^b \right] = \frac{r_0 - r_1}{R} \sum_{j=0}^{b-1} \left( \frac{\hat{\xi}}{r_1} \right)^j \left( \frac{\hat{\xi}}{r_0} \right)^{b-j} \quad (\text{L12})$$

where  $b \geq 1$  is a natural number and on the right-hand side of (L12) we have used the third binomial theorem. The distance  $R$  between source and observer of Eq. (50) and the dimensionless reduced distance  $\hat{R}$  of Eq. (F5) are related to each other by  $R = r_1 \hat{R}$ . We may write the identity (L12) in the following form

$$\frac{1}{\hat{R}} \left[ \left( \frac{\hat{\xi}}{r_1} \right)^b - \left( \frac{\hat{\xi}}{r_0} \right)^b \right] = \frac{r_0 - r_1}{R} \left( \frac{\hat{\xi}}{r_0} \right)^b + \mathcal{O} \left( \frac{\hat{\xi}}{r_1} \right). \quad (\text{L13})$$

The terms of the order  $\mathcal{O}(\hat{\xi}/r_1)$  in (L13) turn out to be terms of the order  $\mathcal{O}\left(\frac{GM}{c^2 r_1} \frac{J_l P}{r_1}\right)$  in the final expression (L5). These expressions are at least by a factor  $10^{-4}$  smaller than the first term on the right-hand side of (L13) and contributes less than  $10^{-3}$  nas to the angle of light deflection and can safely be neglected. Then, by inserting (L7) into the coefficients (L6) and using relation (L13) one obtains for these coefficients

$$B_2^{ML} = + \frac{2}{l!} \frac{r_1 - r_0}{R} \sum_{n=0}^{\lfloor l/2 \rfloor} (-1)^n G_n^l S_{p=0,n}^l D_n^l \quad (\text{L14})$$

with the function

$$D_n^l = l f_1(\hat{\xi}, r_0) - (l-1) f_2(\hat{\xi}, r_0) - f_3(\hat{\xi}, r_0) - g_1(\hat{\xi}, r_0) + 2g_2(\hat{\xi}, r_0) - g_3(\hat{\xi}, r_0) + l W_n^l \left( \frac{\hat{\xi}}{r_0} \right)^{2l-2n-1} - W_n^l \left( \frac{\hat{\xi}}{r_0} \right)^{2l-2n+1} - (2l-2n) W_n^l \left( \frac{\hat{\xi}}{r_0} \right)^{2l-2n+1} + (2l-2n) W_n^l \left( \frac{\hat{\xi}}{r_0} \right)^{2l-2n+3}, \quad (\text{L15})$$

Table XII: The values of the coefficients  $B_2^{ML}$  of Eq. (L25). For the case  $l = 2$  the upper limit is given by Eq. (L23). The values are simplified to one digit after the decimal point.

$l$	$B_2^{ML}$
2	0.7
4	4.7
6	19.2
8	77.9
10	315.5

where

$$f_a(\hat{\xi}, r_0) = \sum_{k=1}^{l-n-1} \frac{1}{2^{2k}} \binom{2k}{k} \left( \frac{\hat{\xi}}{r_0} \right)^{2k+2a-3}, \quad (\text{L16})$$

$$g_a(\hat{\xi}, r_0) = \sum_{k=1}^{l-n-1} \frac{2k}{2^{2k}} \binom{2k}{k} \left( \frac{\hat{\xi}}{r_0} \right)^{2k+2a-3}. \quad (\text{L17})$$

For the ratio of impact parameter,  $\hat{\xi}$ , and distance between body and source,  $r_0$ , we introduce the variable

$$y = \frac{\hat{\xi}}{r_0} \quad (\text{L18})$$

and notice that this real-valued variable is in the closed interval  $x \in [0, 1]$ .

The case of mass-quadrupole  $l = 2$  needs special care. So, before we proceed further, we will consider the function  $D_n^l$  in (L15) explicitly for  $l = 2$ . We get

$$D_{n=0}^{l=2} = \frac{5}{2} y^3 - 4 y^5 + 2 y^7, \quad (\text{L19})$$

$$D_{n=1}^{l=2} = 2 y - 3 y^3 + 2 y^5. \quad (\text{L20})$$

Inserting (L19) and (L20) into (L14) we get for the coefficients  $B_2^{ML}$

$$B_{l=2} = \frac{r_1 - r_0}{R} \left( (e_3^x)^2 - \frac{1}{3} \right) (5 y^3 - 8 y^5 + 4 y^7) + \frac{r_1 - r_0}{R} \left( (e_3^y)^2 - \frac{1}{3} \right) (2 y - 3 y^3 + 2 y^5), \quad (\text{L21})$$

where we have used (H7) and (H8) as well as the coefficients (121) have been inserted, that is  $G_{n=0}^{l=2} = 2$  and  $G_{n=1}^{l=2} = -1$ . In addition, the coordinate system has been rotated such that the  $x^1$ -axis is aligned with  $\hat{\xi}$  and the  $x^2$ -axis is aligned with  $\mathbf{k}$ , hence  $\hat{\xi} \cdot \mathbf{e}_3 / \hat{\xi} = e_3^x$  and  $\mathbf{k} \cdot \mathbf{e}_3 = e_3^y$ .

This orientation of the coordinate system is always possible, because the vectors  $\hat{\xi}$  and  $\mathbf{k}$  are perpendicular to each other. The components of the unit-vector  $\mathbf{e}_3$  are not independent of each other, but restricted by the condition:  $(e_3^x)^2 + (e_3^y)^2 + (e_3^z)^2 = 1$ . Taking into account this fact and using the inequality

$$\left| \frac{r_1 - r_0}{R} \right| \leq 1 \quad (\text{L22})$$

one finds that the upper limit of  $B_{l=2}$  is given by

$$|B_{l=2}| \leq \frac{2}{3}. \quad (\text{L23})$$

A Fortran 90 code has been employed [59], in order to show that the function (L15) is larger or equal to zero,  $D_n^l \geq 0$ . The maximal value of function (L15) is given by

$$D_n^l \leq c_l (l-1) W_n^l. \quad (\text{L24})$$

The coefficients  $c_l$  have been determined numerically. The results are:  $c_4 = 1.1$ ,  $c_6 = 1.3$ ,  $c_8 = 1.5$ ,  $c_{10} = 1.6$ . These maximal values (L24) have been checked for all possible values of  $l, n$  with  $4 \leq l \leq 10$ . Inserting (L24) into (L14) and taking the absolute value yields

$$\left| B_2^{ML} \right| \leq \frac{2}{l!} c_l (l-1) \sum_{n=0}^{\lfloor l/2 \rfloor} (-1)^n G_n^l |S_{p=0,n}^l| W_n^l. \quad (\text{L25})$$

Furthermore, the inequality (L22) has been used. The numerical values of the upper limits of Eq. (L25) are presented in Table XII and the contribution of the term (L5) to the angle of light deflection is presented in Table II. It is mentioned that relation (L25) yields  $B_{l=2} = 2.7$  instead of the more sophisticated estimation in (L23). The reason is that relation (L25) we have not taken into account that  $\hat{\xi}$  and  $\mathbf{k}$  are perpendicular to each other. Furthermore, the upper limit presented in (L24) slightly overestimates the upper limit of  $D_n^l$ . Both these effects together lead to the fact that relation (L25) overestimates the upper limits of  $B_n^l$  by approximately a factor of 2. Nevertheless, the tiny values presented in Table II show that relation (L25) is sufficient to demonstrate that the contribution of  $\mathbf{V}_L^2$  of Eq. (119) is negligible for astrometry on the sub-micro-arcsecond level.

## References

[1] M.A.C. Perryman, K.S. de Boer, G. Gilmore, et al., *GAIA: composition, formation and evolution of the*

*galaxy*, *Astronomy and Astrophysics* **369** (2001) 339.  
[2] T. Prusti, J.H. J. de Bruijne, A.G.A. Brown, et al., *The*

- Gaia mission*, *Astronomy and Astrophysics* **595** (2016) A1.
- [3] D. Hobbs, A. Brown, A. Mora, et al. (2017) *GaiaNIR - Combining optical and Near-Infra-Red (NIR) capabilities with Time-Delay-Integration (TDI) sensors for a future Gaia-like mission*, arXiv:astro-ph/1609.07325.
- [4] Theia Collaboration (2017) *Theia: Faint objects in motion or the new astrometry frontier*, arXiv:astro-ph/1707.01348.
- [5] A. Einstein, *Näherungsweise Integration der Feldgleichungen der Gravitation*, Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften **Part 1** (1916) 688.
- [6] C.W. Misner, K.S. Thorne, J.A. Wheeler, *Gravitation*, Palgrave Macmillan (1973).
- [7] E. Poisson, C.M. Will, *Gravity - Newtonian, Post-Newtonian, Relativistic*, Cambridge University Press, 2014.
- [8] S. Carroll, *Spacetime and Geometry - An Introduction to General Relativity*, Pearson New International, Edinburgh Gate, 2013.
- [9] S. Kopeikin, M. Efroimsky, G. Kaplan, *Relativistic Celestial Mechanics of the Solar System*, Wiley-VCH, Singapore (2012).
- [10] A.N. Petrov, S.M. Kopeikin, R.R. Lompay, B. Tekin, *Metric Theories of Gravity: Perturbations and Conservation Laws*, De Gruyter Studies in Mathematical Physics **38** (2017), Boston, USA.
- [11] K.S. Thorne, *Multipole expansions of gravitational radiation*, *Review of Modern Physics* **52** (1980) 299.
- [12] L. Blanchet, T. Damour, *Radiative gravitational fields in general relativity: I. General structure of the field outside the source*, *Philosophical Transactions of the Royal Society of London A* **320** (1986) 379.
- [13] T. Damour, B.R. Iyer, *Multipole analysis for electromagnetism and linearized gravity with irreducible Cartesian tensors*, *Physical Review D* **43** (1991) 3259.
- [14] L. Blanchet, *Second-post-Newtonian generation of gravitational radiation*, *Physical Review D* **51** (1995) 2559.
- [15] S. Zschocke, *A detailed proof of the fundamental theorem of STF multipole expansion in linearized gravity*, *International Journal of Modern Physics D* **23** (2014) 1450003.
- [16] The post-Newtonian expansion is an expansion in inverse powers of the speed of light up to the order  $\mathcal{O}(c^{-8})$ ; see Ref. [12].
- [17] L. Blanchet, G. Faye, B.R. Iyer, and S. Sinha, *The third post-Newtonian gravitational wave polarizations and associated spherical harmonic modes for inspiralling compact binaries in quasi-circular orbits*, *Classical and Quantum Gravity* **25** (2008) 165003.
- [18] S. Zschocke, *Post-linear metric of a compact source of matter*, *Physical Review D* **100** (2019) 084005.
- [19] V.A. Brumberg, *Essential Relativistic Celestial Mechanics*, (1991) Bristol: Adam Hilder.
- [20] S.M. Kopeikin, G. Schäfer, C.R. Gwinn, T.M. Eubanks, *Astrometric and timing effects of gravitational waves from localized sources*, *Physical Review D* **59** (1999) 084023.
- [21] S.M. Kopeikin, *Propagation of light in the stationary field of multipole gravitational lens*, *Journal of Mathematical Physics* **38** (1997) 2587.
- [22] S.A. Klioner, *Influence of the quadrupole field and rotation of objects on light propagation*, *Soviet Astronomy* **35** (1991) 523.
- [23] S.A. Klioner, S.M. Kopeikin, *Microarcsecond Astrometry in Space: Relativistic Effects and Reduction of Observations*, *Astronomical Journal* **104** (1992) 897.
- [24] S. Zschocke, *Light propagation in the gravitational field of  $N$  arbitrarily moving bodies in 1PN approximation for high-precision astrometry*, *Physical Review D* **92** (2015) 063015.
- [25] S. Zschocke, *Light propagation in the gravitational field of  $N$  arbitrarily moving bodies in the 1.5PN approximation for high-precision astrometry*, *Physical Review D* **93** (2016) 103010.
- [26] S.A. Klioner, *Practical Relativistic Model of Microarcsecond Astrometry in Space*, *The Astronomical Journal* **125** (2003) 1580.
- [27] M. Soffel, W.B. Han, *Applied General Relativity*, Springer, 2019, Cham, Switzerland.
- [28] The differential operator of Eq. (37) is just the STF version of Eq. (101) in [24]. It also coincides with the summand  $q = 0$  of Eq. (94) in [25]. The same differential operator is given by Eq. (11.2.23) in [27]. Let us also notice that the differential operator of Eq. (37) resembles Eq. (24) in [21], where the only difference is just the projector as described in the main text between Eqs. (27) and (37).
- [29] This value of 25.3 nas for the neglected mass-quadrupole terms slightly improves a previous estimation of 32.6 nas for the neglected mass-quadrupole terms for grazing rays at Jupiter in our previous investigations [42, 43]. In particular, by Eq. (15) in [42] and by Eq. (80) in [43] a factor 4.5 in front of the dominant neglected quadrupole terms has been found. Here we get  $B_1^{ML} = 2.8$  for the coefficient of Eq. (91) and  $B_2^{ML} = 0.7$  for the coefficient of Eq. (131) for mass-quadrupole, which in total gives a factor 3.5 in front of the dominant neglected quadrupole terms, which is a bit smaller than the factor 4.5 found in [42, 43].
- [30] S. Zschocke, *Total light deflection in the gravitational field of an axi-symmetric body at rest with full mass and spin multipole structure*, *Physical Review D* **107** (2023) 124055.
- [31] S. Zschocke, *Time delay in the quadrupole field of a body at rest in the 2PN approximation*, *Physical Review D* **106** (2022) 104052.
- [32] H. Essen, *The physics of rotational flattening and the point core model*, *International Journal of Geosciences* **5** (2014) 555.
- [33] P. Teyssandier, *Rotating nonspherical masses and their effects on the precession of a gyroscope*, *Physical Review D* **16** (1977) 946.
- [34] I.W. Roxburgh, *Gravitational multipole moments of the Sun determined from helioseismic estimates of the internal structure and rotation*, *Astronomy & Astrophysics* **377** (2001) 688.
- [35] I. de Pater, J.J. Lissauer, *Planetary Science*, 2.Ed., Cambridge University Press, 2015.
- [36] W. B. Hubbard, B. Militzer, *A preliminary Jupiter model*, *The Astronomical Journal* **820** (2016) 80.
- [37] J.D. Anderson, G. Schubert, *Saturn's gravitational field, internal rotation, and interior structure*, *Science* **317** (2007) 1384.
- [38] R. Helled, J.D. Anderson, M. Podolak, G. Schubert, *Interior models of Uranus and Neptune* *The Astronomical Journal* **726** (2011) 1.
- [39] B.A. Neuenschwander, R. Helled, *Empirical structure*

- models for Uranus and Neptune*, Monthly Notices of the Royal Astronomical Society **512** (2022) 3124.
- [40] S.A. Klioner, S. Zschocke, *Numerical versus analytical accuracy of the formulas for light propagation*, Classical and Quantum Gravity **27** (2010) 075015.
- [41] S. Kopeikin, B. Mashhoon, *Gravitomagnetic effects in the propagation of electromagnetic waves in variable gravitational fields of arbitrary-moving and spinning bodies*, Physical Review D **65** (2002) 064025.
- [42] S. Zschocke, S.A. Klioner, *On the efficient computation of the quadrupole light deflection*, Classical and Quantum Gravity **28** (2011) 015009.
- [43] S. Zschocke, S.A. Klioner, (2010) *Efficient computation of the quadrupole light deflection*, arXiv:astro-ph/0907.4318.
- [44] S. Zschocke, *Light propagation in the 2PN approximation in the monopole and quadrupole field of a body at rest: The basic transformations*, Physical Review D **112** (2026) 064087.
- [45] N. Ashby, B. Bertotti, *Accurate light-time correction due to a gravitating mass*, Classical Quantum Gravity **27** (2010) 145013.
- [46] P. Teyssandier, *Direction of light propagation to order  $G^2$  in static spherically symmetric spacetimes: A new derivation* Classical Quantum Gravity **29** (2012) 245010.
- [47] S. Zschocke, *A generalized lens equation for light deflection in weak gravitational fields*, Classical Quantum Gravity **28** (2011) 125016.
- [48] B. Linet, P. Teyssandier, (2013) *Enhanced term of order  $G^3$  in the light travel time: discussion for some solar system experiments*, arXiv:gr-qc/1312.3510
- [49] E. Høg, *A review of 70 years with astrometry*, Astrophysics and Space Science **369** (2024) 23.
- [50] T. Hartmann, M.H. Soffel, T. Kioustelidis, *On the use of STF-tensors in celestial mechanics*, Celestial Mechanics and Dynamical Astronomy **60** (1994) 139.
- [51] G.B. Arfken, H.J. Weber, *Mathematical methods for physicists*, London, Academic Press, 4th Edition, 1995.
- [52] I.S. Gradshteyn, I.W. Ryzhik, *Tables of series, products, and integrals*, 1th Edition, Harri Deutsch, Thun, Frankfurt/M. 1981.
- [53] M. Abramowitz, I.A. Stegun, *Handbook of mathematical functions*, 9th Edition, Dover Publications, Inc., New York, 1970.
- [54] S. Mattarei, R. Tauraso, *From generating series to polynomial congruences*, Journal of Number Theory **182** (2018) 179.
- [55] R. Tauraso, *Some congruences for central binomial sums involving Fibonacci and Lucas numbers*, Journal of Integer Sequences **19** (2016) Article 16.5.4.
- [56] K. Adegoke, R. Frontczak, (2025) *Three combinatorial sums involving central binomial coefficients*, arXiv:math.NT/2505.09636.
- [57] G. Gasper, M. Rahman, *Basic Hypergeometric Series*, Encyclopedia of Mathematics and its Applications **96**, Cambridge University Press (2004).
- [58] S.A. Klioner, M. Soffel, *Relativistic celestial mechanics with PPN parameters*, Physical Review D **62** (2000) 024019.
- [59] M. Metcalf, J. Reid, M. Cohen, R. Bader, *Modern Fortran Explained: Incorporating Fortran 6th ed.*, (2023), Oxford University Press, UK.
- [60] M Trott, *The Mathematica GuideBook for Symbolics*, Springer, 2006.
- [61] F. Garvan, *The MAPLE Book*, Chapman & Hall (2002), London, UK.