Light propagation in 2PN approximation in the monopole and quadrupole field of a body at rest: Boundary value problem

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In a recent investigation, the initial value problem of light propagation in the gravitational field of a body at rest with monopole and quadrupole structure has been determined in the second post-Newtonian (2PN) approximation. In reality, the light source as well as the observer are located at finite distances from the solar system bodies. This fact requires to solve the boundary value problem of light propagation. In this investigation, the solution of the boundary value problem is deduced from the initial value problem of light propagation in 2PN approximation. These results are a basic requirement for subsequent investigations aiming at ultra-highly precise tests of light deflection and time delay in the solar system.

I. INTRODUCTION

The precision in astrometric angular measurements has made a giant step from the milli-arcsecond (mas) level, as achieved by the astrometry mission Hipparcos [1] to the micro-arcsecond (μas) scale of accuracy, as achieved by the astrometry mission Gaia [2]. These astrometry missions of the European Space Agency (ESA) were the first space missions developed for highly precise astrometric measurements of positions, distances, and proper motions of celestial objects. Meanwhile, there are several development proposals, both space-based astrometry missions [3–11] and ground-based facilities [12], which are aiming at the sub-micro-arcsecond (sub-μas) and even at the nano-arcsecond (nas) scale of accuracy. The science cases of sub-μas astrometry are overwhelming, like detection of Earth-like exoplanets, measurements of dark matter distributions, tests of general relativity (GR) in the solar system, progress in determining natural constants, considerable extension of a model-independent cosmic distance ladder, and even detection of gravitational waves by astrometric measurements [13–17].

A fundamental problem in relativistic astrometry concerns the precise determination of the trajectory of a light signal, emitted by some celestial light source and propagating through the curved space-time of the solar system towards the observer. The importance of this fact has also been emphasized by the ESA-Senior-Survey-Committee (SSC) in response of the selection of near-future space missions. In particular, the SSC has recommended a further development of the GR framework to model photon trajectories to the required accuracy of high-precision astrometry of the next generation [18].

There are several difficult and serious issues regarding a general-relativistic modeling of light propagation aiming at the sub-μas level of accuracy. Let us consider three of these aspects:

(a) Solar system bodies can be of complicated shape and inner structure, which can be described by the multipole expansion of the metric tensor of these bodies [19–21]. The light trajectories in the gravitational field of such arbitrarily shaped bodies at rest have been determined in the 1PN and 1.5PN approximation, both for stationary multipoles [22] as well as for time-dependent multipoles [23].

(b) The second problem concerns the motion of solar system bodies along their complicated world lines. Several investigations have demonstrated, that on the μas-level it is sufficient to determine the light trajectories in the gravitational fields of bodies at rest, if one implements their retarded positions in the GR model [24–28]. In order to investigate this problem further, the light trajectories in the gravitational field of slowly moving solar system bodies with full multipole structure have been determined in the 1PN and 1.5PN approximation [29, 30] by means of the approach developed in [22, 23].

(c) Higher orders of the post-Newtonian (PN) expansion of metric tensor of the solar system need to be taken into account, both for defining highly accurate reference systems as well as for highly precise GR modeling of light trajectories in the solar system. Regarding this problem, it can be stated that the metric tensor for solar system bodies at rest is, in principle, well-known up to the second post-Newtonian approximation (2PN) [19–21, 31–33]. However, such an understanding has by far not been achieved for the light propagation in 2PN approximation. In fact, 2PN light trajectories have only been solved for gravitational fields generated by monopoles, that means spherically symmetric bodies at rest [34, 35]. This problem has later been reconsidered under several aspects in a series of subsequent investigations [36–49]. The next term in the multipole decomposition of the metric tensor of solar system bodies is the quadrupole term. Accordingly, the impact of the quadrupole structure of solar system bodies on light trajectories is the most significant effect beyond the monopole. Such a rigorous 2PN solution for light trajectories has been obtained only recently in our investigation [50]. In that investigation it has also been found that the 2PN quadrupole light deflection amounts up to 0.95 μas and 0.29 μas for grazing light rays at the giant planets Jupiter and Saturn, respectively.

Another aspect of astrometric science concerns the progress in ultra-highly precise time-measurements, both by ground-based facilities as well as space-based atomic clocks. In our recent investigation in [51], the impact of the quadrupole structure of solar system bodies on time-delay has been determined in the 2PN approximation, which amounts up to 0.14 and 0.04 pico-seconds (ps) for grazing rays at the giant planets Jupiter and Saturn. These results can be compared with present-days technological achievements in time measurements. For instance, the atomic clocks on-board of the European Galileo navigation system have a standard deviation of $\Delta t/t \sim 10^{-14}$. The todays most precise atomic clock on-board of a satellite is the Deep Space Atomic Clock (DSAC) [52] (launched 2019) by National Aeronautics and Space Administration (NASA). The DSAC mission is testing a new navigation technology for future space explorers. The DSAC has a standard deviation of $\Delta t/t \sim 10^{-15}$ and it is at least 10 times more stable than atomic clocks of the satellites of the Global Positioning System (GPS). In order to illustrate and to compare the standard deviation with the precision in time measurements, one may consider the light travel time between Earth and Mercury, which is less than 10^3 s. Then, these standard deviations of DSAC would correspond to an accuracy of about $\Delta t \sim 1$ ps in measurements of light travel time. Further examples are the Earth-based atomic clocks NIST-F1 and NIST-F2 at the National Institute of Standard and Technology (NIST), having a standard deviation of $\Delta t/t \sim 10^{-16}$ [53], which corresponds to an accuracy of about $\Delta t \sim 0.1$ ps for such a light signal. The highest accuracies for Earth-bound atomic clocks have been achieved with optical atomic clocks, having a standard deviation of only $\Delta t/t \sim 10^{-19}$ [54], which implies an absolute accuracy of $\Delta t \sim 10^{-4}$ ps for light signals, for instance, between Earth and Mercury.

Thus, in view of advancements of the precision in angular observations as well as in time-measurements, it is clear that in near-future the quadrupole effects in 2PN approximation become relevant for high precision tests of relativity in the solar system. As mentioned above, the 2PN solution of the initial value problem for the light trajectory in the quadrupole field of solar system bodies has been achieved in our recent investigation in [50]. The initial value problem (Cauchy problem) is characterized by two given initial values: the initial direction of the light ray and the spatial position of the light source. However, for high precision tests of relativity in the solar system it is necessary to determine the 2PN light ray solution of the boundary value problem. The boundary value problem is characterized by two given boundary values: the spatial position of the light source and the spatial position of the observer. The solution of the boundary value problem is considerably more involved than the initial value problem. Some parts of the boundary value problem have been considered in [51]. A fundamental and important step towards a comprehensive solution of the boundary value problem will be given in this investigation, where the light trajectory is written in a new form, which contains the spatial positions of the light source and of the observer.

The manuscript is organized as follows: The 2PN solution of the initial value problem of light propagation in the monopole and quadrupole field of a body at rest is summarized in Section II. The problem and the procedure of how to rewrite the initial value solution of light propagation into a new form, which is required to solve the boundary value problem, is described in Sections III and IV. The 2PN solution of the initial value problem, which is appropriate to implement the spatial positions of source and observer, is given in Section V. The solution of the boundary value problem of light propagation, that means the coordinate velocity and trajectory of the light signal in terms of the spatial positions of source and observer, is presented in Section VI. A summary and outlook is given in Section VII. The notations, some details of the calculations, the tensorial coefficients as well as the scalar functions are shifted into a set of several appendices.

II. INITIAL VALUE PROBLEM OF LIGHT PROPAGATION IN THE OLD FORM

We consider the gravitational field generated by a massive solar system body. The curved space-time is assumed to be covered by harmonic four-coordinates $\left(x^0, x^1, x^2, x^3\right)$, in line with the resolutions of the International Astronomical Union (IAU) [55]. They are treated like Cartesian coordinates [19, 35]; see also Section III in [32]. The origin of the spatial axes is assumed to be located at the barycenter of this body. In case of weak gravitational fields and slow motions of the bodies, the metric tensor can be series expanded in inverse powers of the speed of light. This post-Newtonian (PN) expansion reads in the 2PN approximation

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h^{(2)}_{\alpha\beta} + h^{(3)}_{\alpha\beta} + h^{(4)}_{\alpha\beta} + \mathcal{O}\left(c^{-5}\right), \qquad (1)$$

where $h^{(n)}_{\alpha\beta} = \mathcal{O}(c^{-n})$. In the general case, the solar system body can be of arbitrary shape and inner structure and can also be in arbitrary rotational motions. In order to describe the gravitational field generated by such bodies, the metric tensor is decomposed in terms of symmetric trace-free (STF) mass-multipoles M_L and spin-multipoles S_L [19–21, 31]. The mass-multipoles describe the shape and inner structure of the body, while the spin-multipoles account for the rotational motions and inner currents of the body. In the stationary case these multipoles are time-independent and then they are given by

$$M_L = \int d^3x \; x_{<L>} \, \frac{T^{00} + T^{kk}}{c^2} \, , \tag{2}$$

$$S_L = \int d^3x \; x_{<L-1} \, \epsilon_{i_l > jk} \, x^j \, \frac{T^{0k}}{c} \, , \tag{3}$$

where $T^{\alpha\beta}$ is the stress-energy tensor of the body and $x_{<L>}$ is the symmetric and trace-free part of x_L with respect to the spatial indices, where $L = i_1 i_2 \dots i_l$ is a multi-index for these spatial indices. A solar system body, described by these multipoles in (2) and (3), can still be of arbitrary shape, inner structure, and can also be in rotational motions, but the body cannot oscillate and the rotational motions and inner currents have to be time-independent. Then, the post-Newtonian expansion of the metric tensor in (1) reads

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h^{(2)}_{\alpha\beta}\left(M_L\right) + h^{(3)}_{\alpha\beta}\left(S_L\right) + h^{(4)}_{\alpha\beta}\left(M_L\right) \tag{4}$$

up to terms of the order $\mathcal{O}\left(c^{-5}\right)$. The metric perturbations in (4) have been deduced in [32] from the metric density achieved in the basic investigations in [19, 20, 33]. In our investigation [50] we have considered the problem of light propagation in 2PN approximation in the gravitational field of a body at rest and have taken into account the mass-monopole, M, and mass-quadrupole terms, M_{ab},

$$M = \int d^3x \, \frac{T^{00} + T^{kk}}{c^2} \, , \tag{5}$$

$$M_{ab} = \int d^3x \, x_{<ab>} \, \frac{T^{00} + T^{kk}}{c^2} \, , \tag{6}$$

where the integrals run over the three-dimensional volume of the body, and $x_{<ab>} = x_a x_b - \frac{1}{3} \left|\boldsymbol{x}\right|^2 \delta_{ab}$. The mass-dipole terms vanish, $M_i = 0$, because the origin of the spatial axes is located the center of mass of the source. Then, the metric (4) simplifies to the form

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h^{(2)}_{\alpha\beta}\left(M, M_{ab}\right) + h^{(4)}_{\alpha\beta}\left(M, M_{ab}\right) \tag{7}$$

up to terms of the order $\mathcal{O}\left(c^{-6}\right)$. The metric perturbations in (7) were explicitly given in [32, 56].

Light trajectories are null-geodesics and they are governed by the geodesic equation, which in terms of coordinate time reads [35, 57, 58]:

$$\frac{\ddot{x}^i\left(t\right)}{c^2} + \Gamma^i_{\mu\nu} \frac{\dot{x}^\mu\left(t\right)}{c} \frac{\dot{x}^\nu\left(t\right)}{c} = \Gamma^0_{\mu\nu} \frac{\dot{x}^\mu\left(t\right)}{c} \frac{\dot{x}^\nu\left(t\right)}{c} \frac{\dot{x}^i\left(t\right)}{c} \, , \tag{8}$$

where $\Gamma^\alpha_{\mu\nu} = g^{\alpha\beta}\left(g_{\beta\mu,\nu} + g_{\beta\nu,\mu} - g_{\mu\nu,\beta}\right)/2$ are the Christoffel symbols, which are functions of the metric tensor $g_{\alpha\beta}$. The geodesic equation is a differential equation of second order, thus a unique solution requires two initial conditions,

$$\boldsymbol{\sigma} = \frac{\dot{\boldsymbol{x}}\left(t\right)}{c}\bigg|_{t=-\infty} \, , \tag{9}$$

$$\boldsymbol{x}_0 = \boldsymbol{x}\left(t\right)\bigg|_{t=t_0} \, , \tag{10}$$

with $\boldsymbol{\sigma}$ and $\boldsymbol{x}_0$ being the unit-direction of the light ray at past infinity and the spatial position of the light source at the moment of emission of the light signal. The first integration of the geodesic equation yields the coordinate velocity of the light signal, and the second integration of the geodesic equation yields the trajectory of the light signal. By inserting the metric tensor (4) into the geodesic equation (8) one arrives at the geodesic equation in 2PN approximation for the light propagation, as given for instance in the Refs. [36, 46, 50].

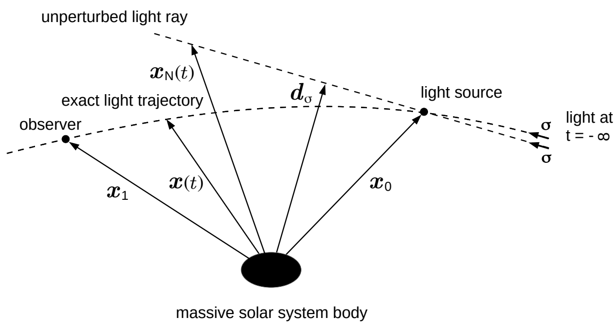

Figure 1: A geometrical representation of the propagation of a light signal through the gravitational field of a massive solar system body at rest. The light signal is emitted by the light source at $\boldsymbol{x}_0$ and propagates along the exact light trajectory $\boldsymbol{x}\left(t\right)$. The three-vector $\boldsymbol{x}_1$ points from the origin of the coordinate system towards the spatial position of the observer. The unit tangent vector along the light trajectory at past infinity is $\boldsymbol{\sigma}$. The unperturbed light ray $\boldsymbol{x}_{\rm N}\left(t\right)$ is given by Eq. (12) and propagates in the direction of $\boldsymbol{\sigma}$ along a straight line through the position of the light source at $\boldsymbol{x}_0$. The impact vector $\boldsymbol{d}_\sigma$ of the unperturbed light ray is given by Eq. (21).

The solutions of the first and second integration of geodesic equation in 2PN approximation have recently been determined for light rays propagating in the gravitational field of a body at rest by means of an iterative approach [50], where the monopole and quadrupole structure of the body have been taken into account. These iterative solutions are given in the following form:

$$\frac{\dot{\boldsymbol{x}}_{\rm N}}{c} = \boldsymbol{\sigma} \, , \tag{11}$$

$$\boldsymbol{x}_{\rm N} = \boldsymbol{x}_0 + c\left(t - t_0\right)\boldsymbol{\sigma} \, , \tag{12}$$

$$\frac{\dot{\boldsymbol{x}}_{\rm 1PN}}{c} = \boldsymbol{\sigma} + \frac{\Delta\dot{\boldsymbol{x}}_{\rm 1PN}\left(\boldsymbol{x}_{\rm N}\right)}{c} \, , \tag{13}$$

$$\boldsymbol{x}_{\rm 1PN} = \boldsymbol{x}_{\rm N} + \Delta\boldsymbol{x}_{\rm 1PN}\left(\boldsymbol{x}_{\rm N}\right) - \Delta\boldsymbol{x}_{\rm 1PN}\left(\boldsymbol{x}_0\right) , \tag{14}$$

$$\frac{\dot{\boldsymbol{x}}_{\rm 2PN}}{c} = \boldsymbol{\sigma} + \frac{\Delta\dot{\boldsymbol{x}}_{\rm 1PN}\left(\boldsymbol{x}_{\rm N}\right)}{c} + \frac{\Delta\dot{\boldsymbol{x}}_{\rm 2PN}\left(\boldsymbol{x}_{\rm N}\right)}{c} \, , \tag{15}$$

$$\begin{aligned} \boldsymbol{x}_{\rm 2PN} = \boldsymbol{x}_{\rm N} &+ \Delta\boldsymbol{x}_{\rm 1PN}\left(\boldsymbol{x}_{\rm N}\right) - \Delta\boldsymbol{x}_{\rm 1PN}\left(\boldsymbol{x}_0\right) \\ &+ \Delta\boldsymbol{x}_{\rm 2PN}\left(\boldsymbol{x}_{\rm N}\right) - \Delta\boldsymbol{x}_{\rm 2PN}\left(\boldsymbol{x}_0\right) , \end{aligned} \tag{16}$$

where the time-arguments have been omitted. The first two equations, (11) and (12), represent the homogeneous solution of geodesic equation (vanishing Christoffel symbols), which yields the unperturbed light ray propagating along a straight line. The 1PN perturbations $\Delta\dot{\boldsymbol{x}}_{\rm 1PN}$ and $\Delta\boldsymbol{x}_{\rm 1PN}$ are terms of the order $\mathcal{O}(c^{-2})$ which have been determined long time ago in [59], while the 2PN perturbations $\Delta\dot{\boldsymbol{x}}_{\rm 2PN}$ and $\Delta\boldsymbol{x}_{\rm 2PN}$ are terms of the order $\mathcal{O}(c^{-4})$, which were the primary results of our recent investigation [50].

The integration of geodesic equation simplifies considerably, if one separates the time-dependent scalar functions from the time-independent tensorial coefficients. This procedure has been applied in [50] and has lead to eight master integrals, which can be solved in their exact form. However, this approach leads to tensorial coefficients, which contain linearly dependent tensors. For the subsequent investigations, it would be more appropriate, to rearrange these tensorial terms into such a form, that the tensorial coefficients contain only tensors which are linearly independent of each other. In this way, we achieve the following expressions for the perturbations: the spatial components of 1PN terms in (13) - (16) are given by

$$\frac{\Delta\dot{x}^i_{\rm 1PN}\left(\boldsymbol{x}_{\rm N}\right)}{c} = \frac{GM}{c^2}\sum_{n=1}^{2} U^i_{(n)}\left(\boldsymbol{x}_{\rm N}\right)\,\dot{F}_{(n)}\left(\boldsymbol{x}_{\rm N}\right) + \frac{GM_{ab}}{c^2}\sum_{n=1}^{8} V^{ab\,i}_{(n)}\left(\boldsymbol{x}_{\rm N}\right)\,\dot{G}_{(n)}\left(\boldsymbol{x}_{\rm N}\right), \tag{17}$$

$$\Delta x^i_{\rm 1PN}\left(\boldsymbol{x}_{\rm N}\right) = \frac{GM}{c^2}\sum_{n=1}^{2} U^i_{(n)}\left(\boldsymbol{x}_{\rm N}\right)\,F_{(n)}\left(\boldsymbol{x}_{\rm N}\right) + \frac{GM_{ab}}{c^2}\sum_{n=1}^{8} V^{ab\,i}_{(n)}\left(\boldsymbol{x}_{\rm N}\right)\,G_{(n)}\left(\boldsymbol{x}_{\rm N}\right), \tag{18}$$

and the spatial components of 2PN terms are given by

$$\frac{\Delta\dot{x}^i_{\rm 2PN}\left(\boldsymbol{x}_{\rm N}\right)}{c} = \frac{GM}{c^2}\,\frac{GM}{c^2}\sum_{n=1}^{2} U^i_{(n)}\left(\boldsymbol{x}_{\rm N}\right)\,\dot{A}_{(n)}\left(\boldsymbol{x}_{\rm N}\right) + \frac{GM}{c^2}\,\frac{GM_{ab}}{c^2}\sum_{n=1}^{8} V^{ab\,i}_{(n)}\left(\boldsymbol{x}_{\rm N}\right)\,\dot{B}_{(n)}\left(\boldsymbol{x}_{\rm N}\right) + \frac{GM_{ab}}{c^2}\,\frac{GM_{cd}}{c^2}\sum_{n=1}^{28} W^{abcd\,i}_{(n)}\left(\boldsymbol{x}_{\rm N}\right)\,\dot{C}_{(n)}\left(\boldsymbol{x}_{\rm N}\right), \tag{19}$$

$$\Delta x^i_{\rm 2PN}\left(\boldsymbol{x}_{\rm N}\right) = \frac{GM}{c^2}\,\frac{GM}{c^2}\sum_{n=1}^{2} U^i_{(n)}\left(\boldsymbol{x}_{\rm N}\right)\,A_{(n)}\left(\boldsymbol{x}_{\rm N}\right) + \frac{GM}{c^2}\,\frac{GM_{ab}}{c^2}\sum_{n=1}^{8} V^{ab\,i}_{(n)}\left(\boldsymbol{x}_{\rm N}\right)\,B_{(n)}\left(\boldsymbol{x}_{\rm N}\right) + \frac{GM_{ab}}{c^2}\,\frac{GM_{cd}}{c^2}\sum_{n=1}^{28} W^{abcd\,i}_{(n)}\left(\boldsymbol{x}_{\rm N}\right)\,C_{(n)}\left(\boldsymbol{x}_{\rm N}\right). \tag{20}$$

The terms $\Delta\boldsymbol{x}_{\rm 1PN}\left(\boldsymbol{x}_0\right)$ and $\Delta\boldsymbol{x}_{\rm 2PN}\left(\boldsymbol{x}_0\right)$ in Eqs. (14) and (16) are obtained from Eqs. (18) and (20) by replacing the arguments $\boldsymbol{x}_{\rm N}$ by $\boldsymbol{x}_0$. The tensorial coefficients, $U^i_{(n)}, V^{ab\,i}_{(n)}, W^{abcd\,i}_{(n)}$, are given in Appendix B. The scalar functions in (17) - (18) are given in Appendix C. The scalar functions in Eqs. (19) - (20) can straightforwardly be deduced from [50], namely by comparison of the tensorial coefficients in Eqs. (83) - (85) and Eqs. (89) - (91) in [50] with Eqs. (19) and (20), respectively. These scalar functions are of simple but extensive algebraic structure. Therefore, in favor of a simpler representation, only the functions $A_{(n)}$ and $B_{(1)}, B_{(8)}$ as well as $C_{(1)}, C_{(28)}$ and their time derivatives are given in Appendix D, while the full set of these functions is represented as supplementary material [60].

The tensorial coefficients contain the impact vector $\boldsymbol{d}_\sigma$ of the unperturbed light ray,

$$\boldsymbol{d}_\sigma = \boldsymbol{\sigma}\times\left(\boldsymbol{x}_{\rm N}\times\boldsymbol{\sigma}\right) = \boldsymbol{\sigma}\times\left(\boldsymbol{x}_0\times\boldsymbol{\sigma}\right), \tag{21}$$

and its absolute value $d_\sigma = |\boldsymbol{d}_\sigma|$ which is called impact parameter d_σ. The impact vector in (21) is time-independent and points from the origin of the coordinate system towards the unperturbed light ray at the moment of its closest encounter; see also Figure 1.

III. STATEMENT OF THE PROBLEM

The solution of the initial value problem, defined by Eqs. (9) and (10), can only be the first step, because in reality the light source and the observer are located at finite distances from the gravitating body. Therefore, real astrometric measurements require the solution of the boundary value problem, that means solving the geodesic equation for light rays in terms of the boundary values,

$$\boldsymbol{x}_0 = \boldsymbol{x}\left(t\right)\Bigg|_{t=t_0}, \tag{22}$$

$$\boldsymbol{x}_1 = \boldsymbol{x}\left(t\right)\Bigg|_{t=t_1}, \tag{23}$$

where $\boldsymbol{x}_0$ is the spatial position of the light source at the moment of emission of the light signal t_0, while $\boldsymbol{x}_1$ is the spatial position of the observer at the moment of reception of the light signal t_1. These equations state, that the spatial position of the exact trajectory of the light signal, $\boldsymbol{x}\left(t\right)$, is in coincidence with the spatial positions of light source and observer at the moment of emission and the moment of reception, respectively; see also Figure 1.

The solution of the boundary value problem can uniquely be deduced from the solution of the initial value problem derived in our investigation [50] and represented by Eqs. (11) - (16). The iterative approach in [50] implies, that coordinate velocity and trajectory of the light signal are given in terms of the unperturbed light ray $\boldsymbol{x}_{\rm N}$. It is essential to realize, that the 1PN perturbations in Eqs. (15) and (16) are terms of the order $\mathcal{O}(c^{-2})$. Therefore, in the arguments of these 1PN perturbations in Eqs. (15) and (16) one cannot replace the unperturbed light ray at the moment of reception by the spatial coordinate of the observer, because such a replacement

$$\boldsymbol{x}_1 = \boldsymbol{x}_{\rm N}\left(t_1\right) + \mathcal{O}\left(c^{-2}\right) \tag{24}$$

would cause terms of the order $\mathcal{O}(c^{-4})$, that means, such a replacement would spoil the 2PN approximation. Therefore, one has to rewrite the 2PN solution in Eqs. (15) and (16) into such a form, where the arguments of these 1PN perturbations are the spatial positions of the 1PN light ray, $\boldsymbol{x}_{\rm 1PN}$, while the arguments of the 2PN perturbations are the spatial position of the unperturbed light ray $\boldsymbol{x}_{\rm N}$. Afterwards, one may replace the arguments of the 1PN terms according to the following relation,

$$\boldsymbol{x}_1 = \boldsymbol{x}_{\rm 1PN}\left(t_1\right) + \mathcal{O}\left(c^{-4}\right). \tag{25}$$

Such a replacement in these 1PN perturbations would cause terms of the order $\mathcal{O}(c^{-6})$ which are neglected, in line with the 2PN approximation. The procedure of how to arrive at this new form of the 2PN solution of the initial value problem is described in the subsequent Section.

IV. DESCRIPTION OF THE PROCEDURE

The procedure, to rewrite the initial value problem into such a new form as described in the previous Section, is relevant for calculations in 2PN approximation and calculations of higher order, but is not necessary in the 1PN or 1.5PN approximation. Furthermore, if one takes into account only the monopole structure of a massive solar system body, then the performance of this technique is more or less straightforward. That might be the reason, that this treatment has almost not been discussed explicitly in the literature. It might well be, that the comments in the text below Eq. (3.2.42) in [35], in the text below Eq. (6.8) in [36], as well as in the text below Eq. (52) in [37] are just three of only a very few explicit hints in the literature about this specific issue. However, if one takes account for the quadrupole structure of a massive solar system body, then the calculations become rather involved. This Section is devoted to represent the method of how to treat that specific problem of 2PN calculations. The procedure is subdivided into three steps.

First step: One changes the arguments of the 1PN perturbations in Eqs. (15) and (16) from the unperturbed light ray, $\boldsymbol{x}_{\rm N}$, to the light ray in 1PN approximation, $\boldsymbol{x}_{\rm 1PN}$. Such a replacement generates additional terms of 2PN order. In order to identify these additional terms, one has to consider the difference between these 1PN perturbations, formally given by,

$$\frac{\delta\dot{\boldsymbol{x}}_{\rm 2PN}\left(\boldsymbol{x}_{\rm N}\right)}{c} = \frac{\Delta\dot{\boldsymbol{x}}_{\rm 1PN}\left(\boldsymbol{x}_{\rm N}\right)}{c} - \frac{\Delta\dot{\boldsymbol{x}}_{\rm 1PN}\left(\boldsymbol{x}_{\rm 1PN}\right)}{c}\,, \tag{26}$$

$$\delta\boldsymbol{x}_{\rm 2PN}\left(\boldsymbol{x}_{\rm N}\right) = \Delta\boldsymbol{x}_{\rm 1PN}\left(\boldsymbol{x}_{\rm N}\right) - \Delta\boldsymbol{x}_{\rm 1PN}\left(\boldsymbol{x}_{\rm 1PN}\right), \tag{27}$$

where label 2PN on the left-hand side in (26) and (27) indicates, that these differences are terms of second post-Newtonian order. The first terms on the right-hand side in (26) and (27) are given by Eqs. (17) and (18), while the second terms on the right-hand side in (26) and (27) read

$$\frac{\Delta\dot{x}^i_{\rm 1PN}\left(\boldsymbol{x}_{\rm 1PN}\right)}{c} = \frac{GM}{c^2}\sum_{n=1}^{2} U^i_{(n)}\left(\boldsymbol{x}_{\rm 1PN}\right)\,\dot{F}_{(n)}\left(\boldsymbol{x}_{\rm 1PN}\right) + \frac{GM_{ab}}{c^2}\sum_{n=1}^{8} V^{ab\,i}_{(n)}\left(\boldsymbol{x}_{\rm 1PN}\right)\,\dot{G}_{(n)}\left(\boldsymbol{x}_{\rm 1PN}\right), \tag{28}$$

$$\Delta x^i_{\rm 1PN}\left(\boldsymbol{x}_{\rm 1PN}\right) = \frac{GM}{c^2}\sum_{n=1}^{2} U^i_{(n)}\left(\boldsymbol{x}_{\rm 1PN}\right)\,F_{(n)}\left(\boldsymbol{x}_{\rm 1PN}\right) + \frac{GM_{ab}}{c^2}\sum_{n=1}^{8} V^{ab\,i}_{(n)}\left(\boldsymbol{x}_{\rm 1PN}\right)\,G_{(n)}\left(\boldsymbol{x}_{\rm 1PN}\right), \tag{29}$$

where the tensorial coefficients are given in Appendix E and the scalar functions are given in Appendix F.

Second step: In order to obtain the terms $\delta\dot{\boldsymbol{x}}_{\rm 2PN}$ in (26) and $\delta\boldsymbol{x}_{\rm 2PN}$ in (27), one has to perform a series expansion of $\Delta\dot{\boldsymbol{x}}_{\rm 1PN}\left(\boldsymbol{x}_{\rm 1PN}\right)$ and $\Delta\boldsymbol{x}_{\rm 1PN}\left(\boldsymbol{x}_{\rm 1PN}\right)$ in (28) and (29), respectively. The tensorial coefficients and the scalar functions in (28) and (29) contain the light ray in 1PN approximation, $\boldsymbol{x}_{\rm 1PN}$, and its absolute value, $x_{\rm 1PN}$. Therefore, for performing that series expansion, one needs the following relations, which are valid up to terms of the order $\mathcal{O}(c^{-4})$,

$$\boldsymbol{x}_{\rm 1PN} = \boldsymbol{x}_{\rm N} + \Delta\boldsymbol{x}_{\rm 1PN}\left(\boldsymbol{x}_{\rm N}\right) - \Delta\boldsymbol{x}_{\rm 1PN}\left(\boldsymbol{x}_0\right)\,, \tag{30}$$

$$\frac{1}{\left(x_{\rm 1PN}\right)^n} = \frac{1}{\left(x_{\rm N}\right)^n} - n\,\frac{\boldsymbol{x}_{\rm N}\cdot\left(\Delta\boldsymbol{x}_{\rm 1PN}\left(\boldsymbol{x}_{\rm N}\right) - \Delta\boldsymbol{x}_{\rm 1PN}\left(\boldsymbol{x}_0\right)\right)}{\left(x_{\rm N}\right)^{n+2}}\,, \tag{31}$$

where n is an arbitrary integer. It is noticed, that relation (31) can also be written in the form

$$\frac{1}{\left(x_{\rm 1PN}\right)^n} = \frac{1}{\left(x_{\rm N}\right)^n} - n\,\frac{\boldsymbol{d}_\sigma\cdot\left(\Delta\boldsymbol{x}_{\rm 1PN}\left(\boldsymbol{x}_{\rm N}\right) - \Delta\boldsymbol{x}_{\rm 1PN}\left(\boldsymbol{x}_0\right)\right)}{\left(x_{\rm N}\right)^{n+2}} - n\left(\boldsymbol{\sigma}\cdot\boldsymbol{x}_{\rm N}\right)\frac{\boldsymbol{\sigma}\cdot\left(\Delta\boldsymbol{x}_{\rm 1PN}\left(\boldsymbol{x}_{\rm N}\right) - \Delta\boldsymbol{x}_{\rm 1PN}\left(\boldsymbol{x}_0\right)\right)}{\left(x_{\rm N}\right)^{n+2}}\,, \tag{32}$$

where the unperturbed light ray (12) has been used in the form $\boldsymbol{x}_{\rm N} = \boldsymbol{d}_\sigma + \left(\boldsymbol{\sigma}\cdot\boldsymbol{x}_{\rm N}\right)\boldsymbol{\sigma}$, which follows from (21).

The 1PN perturbations (28) and (29) do also contain the impact vector with respect to the light ray in 1PN approximation (cf. Eq. (J4) in [50]),

$$\widehat{\boldsymbol{d}_\sigma} = \boldsymbol{\sigma}\times\left(\boldsymbol{x}_{\rm 1PN}\times\boldsymbol{\sigma}\right), \tag{33}$$

as well as its absolute value $\widehat{d_\sigma} = |\widehat{\boldsymbol{d}_\sigma}|$. This impact vector, like the impact vector with respect to the unperturbed light ray (21), is perpendicular to three-vector $\boldsymbol{\sigma}$. By inserting (30) into (33), one finds that these impact vectors in (21) and (33) and their absolute values are related to each other as follows (cf. Eqs. (J5) and (J7) in

[50]),

$$\widehat{\boldsymbol{d}_\sigma} = \boldsymbol{d}_\sigma + \boldsymbol{\sigma} \times \left[\left(\Delta \boldsymbol{x}_{\rm 1PN}\left(\boldsymbol{x}_{\rm N}\right) - \Delta \boldsymbol{x}_{\rm 1PN}\left(\boldsymbol{x}_0\right)\right) \times \boldsymbol{\sigma} \right], \tag{34}$$

$$\frac{1}{(\widehat{d_\sigma})^n} = \frac{1}{(d_\sigma)^n} - \frac{n}{(d_\sigma)^n} \, \frac{\boldsymbol{d}_\sigma \cdot \left(\Delta \boldsymbol{x}_{\rm 1PN}\left(\boldsymbol{x}_{\rm N}\right) - \Delta \boldsymbol{x}_{\rm 1PN}\left(\boldsymbol{x}_0\right)\right)}{(d_\sigma)^2} \,, \tag{35}$$

up to terms of the order $\mathcal{O}(c^{-4})$; n is an arbitrary integer. The relation (35) represents the first term of an infinite series expansion (cf. text below Eq. (J7) in [50]). This series expansion has a convergence radius determined by the condition $n\,|\boldsymbol{d}_\sigma \cdot \Delta \boldsymbol{x}_{\rm 1PN}| \le (d_\sigma)^2$. Using the same arguments as given in text below Eq. (J7) in [50], one finds that this convergence condition is satisfied for any realistic observer, which is naturally assumed to be located in the solar system. Nevertheless, this convergence condition is even satisfied for observers which are located at far distances of more than a few hundred astronomical units from the solar system. The impact vector in (33) is only needed as an intermediate step, because later the replacement in (25) will be performed, where the intermediate impact vector in (33) becomes literally the impact vector of the boundary value problem in Eq. (58), up to terms of the order $\mathcal{O}(c^{-4})$.

Third step: The terms in (26) and (27) are 2PN terms, hence they can be decomposed into terms of the same set of linearly independent tensors like the 2PN terms in (19) and (20), that means

$$\frac{\delta \dot{x}^i_{\rm 2PN}\left(\boldsymbol{x}_{\rm N}\right)}{c} = \frac{GM}{c^2}\,\frac{GM}{c^2} \sum_{n=1}^{2} U^i_{(n)}\left(\boldsymbol{x}_{\rm N}\right)\,\dot{\widetilde{A}}_{(n)}\left(\boldsymbol{x}_{\rm N}\right) + \frac{GM}{c^2}\,\frac{GM_{ab}}{c^2} \sum_{n=1}^{8} V^{ab\,i}_{(n)}\left(\boldsymbol{x}_{\rm N}\right)\,\dot{\widetilde{B}}_{(n)}\left(\boldsymbol{x}_{\rm N}\right) + \frac{GM_{ab}}{c^2}\,\frac{GM_{cd}}{c^2} \sum_{n=1}^{28} W^{abcd\,i}_{(n)}\left(\boldsymbol{x}_{\rm N}\right)\,\dot{\widetilde{C}}_{(n)}\left(\boldsymbol{x}_{\rm N}\right), \tag{36}$$

and

$$\delta x^i_{\rm 2PN}\left(\boldsymbol{x}_{\rm N}\right) = \frac{GM}{c^2}\,\frac{GM}{c^2} \sum_{n=1}^{2} U^i_{(n)}\left(\boldsymbol{x}_{\rm N}\right)\,\widetilde{A}_{(n)}\left(\boldsymbol{x}_{\rm N}\right) + \frac{GM}{c^2}\,\frac{GM_{ab}}{c^2} \sum_{n=1}^{8} V^{ab\,i}_{(n)}\left(\boldsymbol{x}_{\rm N}\right)\,\widetilde{B}_{(n)}\left(\boldsymbol{x}_{\rm N}\right) + \frac{GM_{ab}}{c^2}\,\frac{GM_{cd}}{c^2} \sum_{n=1}^{28} W^{abcd\,i}_{(n)}\left(\boldsymbol{x}_{\rm N}\right)\,\widetilde{C}_{(n)}\left(\boldsymbol{x}_{\rm N}\right), \tag{37}$$

where the tensorial coefficients are given in Appendix B. The calculations in order to get these terms in (36) and (37) are not complicated but lengthy, and the scalar functions $\dot{\widetilde{A}}_{(n)}$, $\dot{\widetilde{B}}_{(n)}$, $\dot{\widetilde{C}}_{(n)}$, as well as $\widetilde{A}_{(n)}$, $\widetilde{B}_{(n)}$, $\widetilde{C}_{(n)}$ are of extensive algebraic structure. In view of this fact and because these functions are considered as an intermediate step, they will not be presented here in their explicit form. Instead, the full set of these functions is represented as supplementary material [60].

As final step, these expressions in (36) and (37) have to be added to the 2PN terms in (19) and (20), respectively, which leads to the following expressions:

$$\frac{\triangle \dot{\boldsymbol{x}}_{\rm 2PN}\left(\boldsymbol{x}_{\rm N}\right)}{c} = \frac{\Delta \dot{\boldsymbol{x}}_{\rm 2PN}\left(\boldsymbol{x}_{\rm N}\right)}{c} + \frac{\delta \dot{\boldsymbol{x}}_{\rm 2PN}\left(\boldsymbol{x}_{\rm N}\right)}{c}\,, \tag{38}$$

$$\triangle \boldsymbol{x}_{\rm 2PN}\left(\boldsymbol{x}_{\rm N}\right) = \Delta \boldsymbol{x}_{\rm 2PN}\left(\boldsymbol{x}_{\rm N}\right) + \delta \boldsymbol{x}_{\rm 2PN}\left(\boldsymbol{x}_{\rm N}\right). \tag{39}$$

The symbol $\triangle$ (Laplace) instead of Δ (Delta) on the left-hand side in (38) and (39) indicates, that these functions have carefully to be distinguished from the 2PN functions in (19) and (20).

V. INITIAL VALUE PROBLEM OF LIGHT PROPAGATION IN THE NEW FORM

By performing this procedure, which has been described in the previous Section, one arrives at a new representation of the same iterative solution as given by Eqs. (11) - (16), but in the following form:

$$\frac{\dot{\boldsymbol{x}}_{\rm N}}{c} = \boldsymbol{\sigma}\,, \tag{40}$$

$$\boldsymbol{x}_{\rm N} = \boldsymbol{x}_0 + c\left(t - t_0\right)\boldsymbol{\sigma}\,, \tag{41}$$

$$\frac{\dot{\boldsymbol{x}}_{\rm 1PN}}{c} = \boldsymbol{\sigma} + \frac{\Delta \dot{\boldsymbol{x}}_{\rm 1PN}\left(\boldsymbol{x}_{\rm N}\right)}{c}\,, \tag{42}$$

$$\boldsymbol{x}_{\rm 1PN} = \boldsymbol{x}_{\rm N} + \Delta \boldsymbol{x}_{\rm 1PN}\left(\boldsymbol{x}_{\rm N}\right) - \Delta \boldsymbol{x}_{\rm 1PN}\left(\boldsymbol{x}_0\right), \tag{43}$$

$$\frac{\dot{\boldsymbol{x}}_{\rm 2PN}}{c} = \boldsymbol{\sigma} + \frac{\Delta \dot{\boldsymbol{x}}_{\rm 1PN}\left(\boldsymbol{x}_{\rm 1PN}\right)}{c} + \frac{\triangle \dot{\boldsymbol{x}}_{\rm 2PN}\left(\boldsymbol{x}_{\rm N}\right)}{c}\,, \tag{44}$$

$$\boldsymbol{x}_{\rm 2PN} = \boldsymbol{x}_{\rm N} + \Delta \boldsymbol{x}_{\rm 1PN}\left(\boldsymbol{x}_{\rm 1PN}\right) - \Delta \boldsymbol{x}_{\rm 1PN}\left(\boldsymbol{x}_0\right) + \triangle \boldsymbol{x}_{\rm 2PN}\left(\boldsymbol{x}_{\rm N}\right) - \triangle \boldsymbol{x}_{\rm 2PN}\left(\boldsymbol{x}_0\right), \tag{45}$$

where the time-arguments have been omitted. The only differences between Eqs. (11) - (16) and Eqs. (40) - (45) are the arguments $\boldsymbol{x}_{\rm 1PN}$ in the 1PN terms in (44) and (45) and the new scalar functions $\triangle \dot{\boldsymbol{x}}_{\rm 2PN}$ and $\triangle \boldsymbol{x}_{\rm 2PN}$ in (44) and (45). It is, however, essential to realize that the iterative solution in Eqs. (40) - (45) is identical to the iterative solution given above by Eqs. (11) - (16), up to terms beyond the 2PN approximation.

These equations (42) - (45) represent a generalization of Eqs. (3.2.35) - (3.2.38) in [35], which are valid for the 2PN light propagation in the monopole field of a body, while the equations presented here are valid for the 2PN light propagation in the monopole and quadrupole field of a body.

The 1PN terms in (42) and (43) are given by Eqs. (17) and (18). The 1PN terms in (44) and (45) are given by Eqs. (28) and (29). The 2PN perturbations in (44) - (45)

are given by

$$\frac{\triangle \dot{x}^i_{\rm 2PN}\left(\boldsymbol{x}_{\rm N}\right)}{c} = \frac{GM}{c^2}\,\frac{GM}{c^2}\sum_{n=1}^{2} U^i_{(n)}\left(\boldsymbol{x}_{\rm N}\right)\,\dot{X}_{(n)}\left(\boldsymbol{x}_{\rm N}\right)$$
$$+\frac{GM}{c^2}\,\frac{GM_{ab}}{c^2}\sum_{n=1}^{8} V^{ab\,i}_{(n)}\left(\boldsymbol{x}_{\rm N}\right)\,\dot{Y}_{(n)}\left(\boldsymbol{x}_{\rm N}\right)$$
$$+\frac{GM_{ab}}{c^2}\,\frac{GM_{cd}}{c^2}\sum_{n=1}^{28} W^{abcd\,i}_{(n)}\left(\boldsymbol{x}_{\rm N}\right)\,\dot{Z}_{(n)}\left(\boldsymbol{x}_{\rm N}\right), \qquad (46)$$

and

$$\triangle x^i_{\rm 2PN}\left(\boldsymbol{x}_{\rm N}\right) = \frac{GM}{c^2}\,\frac{GM}{c^2}\sum_{n=1}^{2} U^i_{(n)}\left(\boldsymbol{x}_{\rm N}\right)\,X_{(n)}\left(\boldsymbol{x}_{\rm N}\right)$$
$$+\frac{GM}{c^2}\,\frac{GM_{ab}}{c^2}\sum_{n=1}^{8} V^{ab\,i}_{(n)}\left(\boldsymbol{x}_{\rm N}\right)\,Y_{(n)}\left(\boldsymbol{x}_{\rm N}\right)$$
$$+\frac{GM_{ab}}{c^2}\,\frac{GM_{cd}}{c^2}\sum_{n=1}^{28} W^{abcd\,i}_{(n)}\left(\boldsymbol{x}_{\rm N}\right)\,Z_{(n)}\left(\boldsymbol{x}_{\rm N}\right). \qquad (47)$$

It is noticed here, that the 2PN monopole terms in (46) and (47) are in agreement with the results obtained in [35–37]. The tensorial coefficients in (46) and (47) are the same as in Eqs. (19) and (20) and they are given in Appendix B. The scalar functions are presented in Appendix J (arguments $\boldsymbol{x}_1$ in Appendix J substituted by the arguments $\boldsymbol{x}_{\rm N}$). The 2PN terms $\Delta \boldsymbol{x}_{\rm 2PN}\left(\boldsymbol{x}_0\right)$ in (45) are obtained from (47) by the replacements $\boldsymbol{x}_{\rm N}$ by $\boldsymbol{x}_0$.

The final step of this procedure concerns the replacement of the arguments in Eqs. (40) - (45) by the spatial positions of source and observer. This issue will be the subject of the subsequent Section.

VI. BOUNDARY VALUE PROBLEM OF LIGHT PROPAGATION

From Eqs. (41), (43), and (45) follows that the position of the light source is given by the position of the light ray at emission time t_0, in any order of the post-Newtonian expansion, that means

$$\boldsymbol{x}_0 = \boldsymbol{x}_{\rm N}\left(t_0\right) = \boldsymbol{x}_{\rm 1PN}\left(t_0\right) = \boldsymbol{x}_{\rm 2PN}\left(t_0\right), \qquad (48)$$

represent exact relations.

On the other side, the spatial position of the observer, $\boldsymbol{x}_1$, coincides with the spatial position of the light signal, propagating along the exact light trajectory, at the moment of reception, $\boldsymbol{x}\left(t_1\right)$, which is determined up to the given order in the post-Newtonian expansion. Therefore, the position of the observer can be obtained by replacing the spatial coordinate of the light signal at the moment of observation by accounting for the correct order, that means according to Eqs. (24) and (25); see also [37, 50, 51]. Only that new representation of the initial value problem, as represented by Eqs. (40) - (45), allows one to insert in these relations (24) and (25). This procedure leads finally to the 1PN and 2PN expressions, which are given in their explicit form in the following two subsections.

A. The 1PN terms

The coordinate velocity and spatial position of the light signal in the 1PN approximation are given by Eqs. (42) and (43). According to relation (24), one may replace the arguments at the moment of reception of the light signal by the spatial position of the observer. In this way, one obtains for the coordinate velocity and spatial position of the light ray in the 1PN approximation:

$$\frac{\dot{\boldsymbol{x}}_{\rm 1PN}\left(t_1\right)}{c} = \boldsymbol{\sigma} + \frac{\Delta\dot{\boldsymbol{x}}_{\rm 1PN}\left(\boldsymbol{x}_1\right)}{c}, \qquad (49)$$

$$\boldsymbol{x}_{\rm 1PN}\left(t_1\right) = \boldsymbol{x}_0 + c\left(t_1 - t_0\right)\boldsymbol{\sigma} + \Delta\boldsymbol{x}_{\rm 1PN}\left(\boldsymbol{x}_1\right) - \Delta\boldsymbol{x}_{\rm 1PN}\left(\boldsymbol{x}_0\right), \qquad (50)$$

where the 1PN perturbation terms in (49) and (50) are given by

$$\frac{\Delta\dot{x}^i_{\rm 1PN}\left(\boldsymbol{x}_1\right)}{c} = \frac{GM}{c^2}\sum_{n=1}^{2} U^i_{(n)}\left(\boldsymbol{x}_1\right)\,\dot{F}_{(n)}\left(\boldsymbol{x}_1\right) + \frac{GM_{ab}}{c^2}\sum_{n=1}^{8} V^{ab\,i}_{(n)}\left(\boldsymbol{x}_1\right)\,\dot{G}_{(n)}\left(\boldsymbol{x}_1\right), \qquad (51)$$

$$\Delta x^i_{\rm 1PN}\left(\boldsymbol{x}_1\right) = \frac{GM}{c^2}\sum_{n=1}^{2} U^i_{(n)}\left(\boldsymbol{x}_1\right)\,F_{(n)}\left(\boldsymbol{x}_1\right) + \frac{GM_{ab}}{c^2}\sum_{n=1}^{8} V^{ab\,i}_{(n)}\left(\boldsymbol{x}_1\right)\,G_{(n)}\left(\boldsymbol{x}_1\right). \qquad (52)$$

The tensorial coefficients and scalar functions are given in Appendices G and I. Clearly, the term $\Delta\boldsymbol{x}_{\rm 1PN}\left(\boldsymbol{x}_0\right)$ in (50) is obtained from (52) by substituting the argument $\boldsymbol{x}_1$ by the argument $\boldsymbol{x}_0$.

B. The 2PN terms

The coordinate velocity and spatial position of the light signal in the 2PN approximation are given by Eqs. (44) and (45). One may replace the arguments at the moment of reception of the light signal by the spatial position of the observer according to relations (24) and (25). In this way, one obtains for the coordinate velocity and spatial position of the light ray in the 2PN approximation:

$$\frac{\dot{\boldsymbol{x}}_{\rm 2PN}\left(t_1\right)}{c} = \boldsymbol{\sigma} + \frac{\Delta\dot{\boldsymbol{x}}_{\rm 1PN}\left(\boldsymbol{x}_1\right)}{c} + \frac{\triangle\dot{\boldsymbol{x}}_{\rm 2PN}\left(\boldsymbol{x}_1\right)}{c}, \qquad (53)$$

$$\boldsymbol{x}_{\rm 2PN}\left(t_1\right) = \boldsymbol{x}_0 + c\left(t_1 - t_0\right)\boldsymbol{\sigma} + \Delta\boldsymbol{x}_{\rm 1PN}\left(\boldsymbol{x}_1\right) - \Delta\boldsymbol{x}_{\rm 1PN}\left(\boldsymbol{x}_0\right) + \triangle\boldsymbol{x}_{\rm 2PN}\left(\boldsymbol{x}_1\right) - \triangle\boldsymbol{x}_{\rm 2PN}\left(\boldsymbol{x}_0\right), \qquad (54)$$

where the 2PN perturbation terms in (53) and (54) are given by

$$\frac{\triangle \dot{x}^i_{\rm 2PN}\left(\boldsymbol{x}_1\right)}{c} = \frac{GM}{c^2}\,\frac{GM}{c^2} \sum_{n=1}^{2} U^i_{(n)}\left(\boldsymbol{x}_1\right)\,\dot{X}_{(n)}\left(\boldsymbol{x}_1\right)$$
$$+\frac{GM}{c^2}\,\frac{GM_{ab}}{c^2} \sum_{n=1}^{8} V^{ab\,i}_{(n)}\left(\boldsymbol{x}_1\right)\,\dot{Y}_{(n)}\left(\boldsymbol{x}_1\right)$$
$$+\frac{GM_{ab}}{c^2}\,\frac{GM_{cd}}{c^2} \sum_{n=1}^{28} W^{abcd\,i}_{(n)}\left(\boldsymbol{x}_1\right)\,\dot{Z}_{(n)}\left(\boldsymbol{x}_1\right), \qquad (55)$$

and

$$\triangle x^i_{\rm 2PN}\left(\boldsymbol{x}_1\right) = \frac{GM}{c^2}\,\frac{GM}{c^2} \sum_{n=1}^{2} U^i_{(n)}\left(\boldsymbol{x}_1\right)\,X_{(n)}\left(\boldsymbol{x}_1\right)$$
$$+\frac{GM}{c^2}\,\frac{GM_{ab}}{c^2} \sum_{n=1}^{8} V^{ab\,i}_{(n)}\left(\boldsymbol{x}_1\right)\,Y_{(n)}\left(\boldsymbol{x}_1\right)$$
$$+\frac{GM_{ab}}{c^2}\,\frac{GM_{cd}}{c^2} \sum_{n=1}^{28} W^{abcd\,i}_{(n)}\left(\boldsymbol{x}_1\right)\,Z_{(n)}\left(\boldsymbol{x}_1\right). \qquad (56)$$

The tensorial coefficients are given in the Appendices G and H, while the scalar functions are presented in Appendix J. Clearly, the term $\triangle \boldsymbol{x}_{\rm 2PN}\left(\boldsymbol{x}_0\right)$ in (54) is obtained from (56) by substituting the argument $\boldsymbol{x}_1$ by the argument $\boldsymbol{x}_0$.

Finally, it should be noticed that the replacements (24) and (48) into the impact vector (21), as well as the replacements in (25) and (48) into the impact vector (33) imply the occurrence of two new impact vectors,

$$\boldsymbol{d}^0_{\sigma} = \boldsymbol{\sigma} \times \left(\boldsymbol{x}_0 \times \boldsymbol{\sigma}\right), \qquad (57)$$
$$\boldsymbol{d}^1_{\sigma} = \boldsymbol{\sigma} \times \left(\boldsymbol{x}_1 \times \boldsymbol{\sigma}\right), \qquad (58)$$

where their absolute values are $d^0_{\sigma} = |\boldsymbol{d}^0_{\sigma}|$ and $d^1_{\sigma} = |\boldsymbol{d}^1_{\sigma}|$. These two impact vectors appear in a natural way if one considers the boundary value problem. It is not surprising, that the impact vector in (57) is actually identical to the impact vector in (21). The reason, that this impact vector appears both in the boundary value problem as well as in the initial value problem, is based in the fact that the initial condition (10) of the initial value problem is identical to the boundary condition (22) of the boundary value problem.

VII. SUMMARY AND OUTLOOK

In our recent investigation [50] the coordinate velocity and the trajectory of a light signal in the gravitational field of a body at rest has been determined in the 2PN approximation, where the monopole and quadrupole terms of the gravitational field have been taken into account. The unique solution of the geodesic equation has been determined in the scheme of the initial value problem. In reality, however, the light source as well as the observer are located at finite distances $\boldsymbol{x}_0$ and $\boldsymbol{x}_1$ from the gravitating solar system body. This fact requires to solve the boundary value of the geodesic equation. The solution of 2PN light propagation in terms of these boundary values $\boldsymbol{x}_0$ and $\boldsymbol{x}_1$ is represented in Section VI. Notably the 2PN terms in Eqs. (53) and (54) with Eqs. (55) - (56), are the primary results of this investigation. These results are a basic requirement for highly precise measurements of light deflection on the sub-μas scale and time delay on sub-pico-second level in the solar system.

The final ambition of the boundary value problem is the determination of three fundamental transformations [37, 61]: $\boldsymbol{k} \rightarrow \boldsymbol{\sigma}$, $\boldsymbol{\sigma} \rightarrow \boldsymbol{n}$, $\boldsymbol{k} \rightarrow \boldsymbol{n}$, where $\boldsymbol{k}$ is the unit direction from the source towards the observer, $\boldsymbol{\sigma}$ is the unit tangent vector of light trajectory at minus infinity, and $\boldsymbol{n}$ is the unit tangent vector of the light ray at the spatial position of the observer. These transformations represent the basis of the Gaia rlativistic model (GREM) [61], which has later been refined by our investigations in [37] and [62]. These transformations would also be implemented in relativistic models for data reduction of possible future space astrometry missions, like the Gaia successor GaiaNIR [3] or Theia [4], planned to be launched in an optimistic case in 2045 as medium-sized mission of ESA [63]. The determination of these transformations will be represented in a subsequent investigation.

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Appendix A: Notation

Throughout the investigation the following notation is in use:

- G is the Newtonian constant of gravitation.
- c is the vacuum speed of light in Minkowskian space-time.
- M is the rest mass of the body.
- M_{ab} is the symmetric trace-free quadrupole moment of the body.
- Lower case Latin indices $i,\, j,\, \ldots$ take values $1, 2, 3$.

- $\dot{f}$ denotes total derivative of f with respect to global coordinate time.
- $\delta_{ij} = \delta^{ij} = \mathrm{diag}\,(+1,+1,+1)$ is Kronecker delta.
- Three-vectors are in boldface: e.g. $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{\sigma}$, $\boldsymbol{x}$.
- Contravariant components of three-vectors: $a^i = \left(a^1, a^2, a^3\right)$.
- Scalar product of three-vectors: $\boldsymbol{a} \cdot \boldsymbol{b} = \delta_{ij}\, a^i\, b^j$.
- Absolute value of three-vector: $|\boldsymbol{a}| = \sqrt{\delta_{ij}\, a^i\, a^j}$.
- Levi-Civita symbol: $\varepsilon_{ijk} = \varepsilon^{ijk}$ with $\varepsilon_{123} = +1$.
- Vector product of two three-vectors: $\left(\boldsymbol{a} \times \boldsymbol{b}\right)^i = \varepsilon_{ijk}\, a^j\, b^k$.
- Lower case Greek indices take values 0,1,2,3.
- $\eta_{\alpha\beta} = \eta^{\alpha\beta} = \mathrm{diag}\,(-1,+1,+1,+1)$ is the metric tensor of flat space-time.
- $g_{\alpha\beta}$ and $g^{\alpha\beta}$ are the covariant and contravariant components of the metric tensor.
- Contravariant components of four-vectors: $a^\mu = \left(a^0, a^1, a^2, a^3\right)$.
- milli-arcsecond (mas): $1\,\mathrm{mas} = \pi/(180 \times 60 \times 60) \times 10^{-3}\,\mathrm{rad}$.
- micro-arcsecond (μas): $1\,\mu\mathrm{as} = \pi/(180 \times 60 \times 60) \times 10^{-6}\,\mathrm{rad}$.
- nano-arcsecond (nas): $1\,\mathrm{nas} = \pi/(180 \times 60 \times 60) \times 10^{-9}\,\mathrm{rad}$.
- pico-second (ps): $1\,\mathrm{ps} = 10^{-12}$ second.
- repeated indices are implicitly summed over (*Einstein's* sum convention).

Appendix B: Tensorial coefficients of 1PN and 2PN solution in (17) - (20)

The tensorial coefficients of the 1PN perturbation terms in (17) and (18) are given by

$$U^{i}_{(1)}\left(\boldsymbol{x}_{\rm N}\right) = \sigma^i\;, \tag{B1}$$
$$U^{i}_{(2)}\left(\boldsymbol{x}_{\rm N}\right) = d^i_\sigma\;, \tag{B2}$$
$$V^{ab\,i}_{(1)}\left(\boldsymbol{x}_{\rm N}\right) = \sigma^a \delta^{bi}\;, \tag{B3}$$
$$V^{ab\,i}_{(2)}\left(\boldsymbol{x}_{\rm N}\right) = d^a_\sigma \delta^{bi}\;, \tag{B4}$$
$$V^{ab\,i}_{(3)}\left(\boldsymbol{x}_{\rm N}\right) = \sigma^a \sigma^b \sigma^i\;, \tag{B5}$$
$$V^{ab\,i}_{(4)}\left(\boldsymbol{x}_{\rm N}\right) = \sigma^a d^b_\sigma \sigma^i\;, \tag{B6}$$
$$V^{ab\,i}_{(5)}\left(\boldsymbol{x}_{\rm N}\right) = d^a_\sigma d^b_\sigma \sigma^i\;, \tag{B7}$$
$$V^{ab\,i}_{(6)}\left(\boldsymbol{x}_{\rm N}\right) = d^a_\sigma d^b_\sigma d^i_\sigma\;, \tag{B8}$$
$$V^{ab\,i}_{(7)}\left(\boldsymbol{x}_{\rm N}\right) = \sigma^a \sigma^b d^i_\sigma\;, \tag{B9}$$
$$V^{ab\,i}_{(8)}\left(\boldsymbol{x}_{\rm N}\right) = \sigma^a d^b_\sigma d^i_\sigma\;. \tag{B10}$$

The coefficients in (B3) - (B10) represent a complete set of linearly-independent tensors with three spatial indices, which can be constructed from two independent three-vectors, σ^a and d^b_σ, and the Kronecker symbol. Note, that a permutation of the indices ($a \leftrightarrow b$) is of no relevance, because of the symmetry of the quadrupole tensor. For instance, there is no need to distinguish between the tensors $\sigma^a \delta^{bi}$ and $\sigma^b \delta^{ai}$, because they yield same result: $\sigma^a \delta^{bi}\, M_{ab} = \sigma^b \delta^{ai}\, M_{ab}$.

The tensorial coefficients of the 2PN perturbation terms in (19) and (20) are given by

$$W^{abcd\,i}_{(1)}\left(\boldsymbol{x}_{\rm N}\right) = \delta^{ac} \sigma^b \delta^{di}\;, \tag{B11}$$
$$W^{abcd\,i}_{(2)}\left(\boldsymbol{x}_{\rm N}\right) = \delta^{ac} d^b_\sigma \delta^{di}\;, \tag{B12}$$
$$W^{abcd\,i}_{(3)}\left(\boldsymbol{x}_{\rm N}\right) = \sigma^a \sigma^b \sigma^c \delta^{di}\;, \tag{B13}$$
$$W^{abcd\,i}_{(4)}\left(\boldsymbol{x}_{\rm N}\right) = \sigma^a \sigma^b d^c_\sigma \delta^{di}\;, \tag{B14}$$
$$W^{abcd\,i}_{(5)}\left(\boldsymbol{x}_{\rm N}\right) = \sigma^a d^b_\sigma \sigma^c \delta^{di}\;, \tag{B15}$$
$$W^{abcd\,i}_{(6)}\left(\boldsymbol{x}_{\rm N}\right) = \sigma^a d^b_\sigma d^c_\sigma \delta^{di}\;, \tag{B16}$$
$$W^{abcd\,i}_{(7)}\left(\boldsymbol{x}_{\rm N}\right) = d^a_\sigma d^b_\sigma \sigma^c \delta^{di}\;, \tag{B17}$$
$$W^{abcd\,i}_{(8)}\left(\boldsymbol{x}_{\rm N}\right) = d^a_\sigma d^b_\sigma d^c_\sigma \delta^{di}\;, \tag{B18}$$
$$W^{abcd\,i}_{(9)}\left(\boldsymbol{x}_{\rm N}\right) = \delta^{ac} \delta^{bd} \sigma^i\;, \tag{B19}$$
$$W^{abcd\,i}_{(10)}\left(\boldsymbol{x}_{\rm N}\right) = \delta^{ac} \sigma^b \sigma^d \sigma^i\;, \tag{B20}$$
$$W^{abcd\,i}_{(11)}\left(\boldsymbol{x}_{\rm N}\right) = \delta^{ac} \sigma^b d^d_\sigma \sigma^i\;, \tag{B21}$$
$$W^{abcd\,i}_{(12)}\left(\boldsymbol{x}_{\rm N}\right) = \delta^{ac} d^b_\sigma d^d_\sigma \sigma^i\;, \tag{B22}$$
$$W^{abcd\,i}_{(13)}\left(\boldsymbol{x}_{\rm N}\right) = \sigma^a \sigma^b \sigma^c \sigma^d \sigma^i\;, \tag{B23}$$
$$W^{abcd\,i}_{(14)}\left(\boldsymbol{x}_{\rm N}\right) = \sigma^a \sigma^b \sigma^c d^d_\sigma \sigma^i\;, \tag{B24}$$
$$W^{abcd\,i}_{(15)}\left(\boldsymbol{x}_{\rm N}\right) = \sigma^a \sigma^b d^c_\sigma d^d_\sigma \sigma^i\;, \tag{B25}$$
$$W^{abcd\,i}_{(16)}\left(\boldsymbol{x}_{\rm N}\right) = \sigma^a d^b_\sigma \sigma^c d^d_\sigma \sigma^i\;, \tag{B26}$$
$$W^{abcd\,i}_{(17)}\left(\boldsymbol{x}_{\rm N}\right) = \sigma^a d^b_\sigma d^c_\sigma d^d_\sigma \sigma^i\;, \tag{B27}$$
$$W^{abcd\,i}_{(18)}\left(\boldsymbol{x}_{\rm N}\right) = d^a_\sigma d^b_\sigma d^c_\sigma d^d_\sigma \sigma^i\;, \tag{B28}$$
$$W^{abcd\,i}_{(19)}\left(\boldsymbol{x}_{\rm N}\right) = \delta^{ac} \delta^{bd} d^i_\sigma\;, \tag{B29}$$
$$W^{abcd\,i}_{(20)}\left(\boldsymbol{x}_{\rm N}\right) = \delta^{ac} \sigma^b \sigma^d d^i_\sigma\;, \tag{B30}$$
$$W^{abcd\,i}_{(21)}\left(\boldsymbol{x}_{\rm N}\right) = \delta^{ac} \sigma^b d^d_\sigma d^i_\sigma\;, \tag{B31}$$
$$W^{abcd\,i}_{(22)}\left(\boldsymbol{x}_{\rm N}\right) = \delta^{ac} d^b_\sigma d^d_\sigma d^i_\sigma\;, \tag{B32}$$
$$W^{abcd\,i}_{(23)}\left(\boldsymbol{x}_{\rm N}\right) = \sigma^a \sigma^b \sigma^c \sigma^d d^i_\sigma\;, \tag{B33}$$
$$W^{abcd\,i}_{(24)}\left(\boldsymbol{x}_{\rm N}\right) = \sigma^a \sigma^b \sigma^c d^d_\sigma d^i_\sigma\;, \tag{B34}$$
$$W^{abcd\,i}_{(25)}\left(\boldsymbol{x}_{\rm N}\right) = \sigma^a \sigma^b d^c_\sigma d^d_\sigma d^i_\sigma\;, \tag{B35}$$
$$W^{abcd\,i}_{(26)}\left(\boldsymbol{x}_{\rm N}\right) = \sigma^a d^b_\sigma \sigma^c d^d_\sigma d^i_\sigma\;, \tag{B36}$$
$$W^{abcd\,i}_{(27)}\left(\boldsymbol{x}_{\rm N}\right) = \sigma^a d^b_\sigma d^c_\sigma d^d_\sigma d^i_\sigma\;, \tag{B37}$$
$$W^{abcd\,i}_{(28)}\left(\boldsymbol{x}_{\rm N}\right) = d^a_\sigma d^b_\sigma d^c_\sigma d^d_\sigma d^i_\sigma\;. \tag{B38}$$

The coefficients in (B11) - (B38) represent a complete set of tensors with five spatial indices, which can be constructed from two independent three-vectors, σ^a and d^b_σ, and the Kronecker symbol, and where the symmetry of the quadrupole tensor is accounted for. That means, the permutations $(a \leftrightarrow b \wedge c \leftrightarrow d)$, $(a \leftrightarrow b \vee c \leftrightarrow d)$, $(a \leftrightarrow c \wedge b \leftrightarrow d)$, $(a \leftrightarrow d \wedge b \leftrightarrow c)$ have no relevance; cf. text below Eq. (B10).

Appendix C: Scalar functions of 1PN and 2PN solution in (17) - (18)

1. Scalar functions of the 1PN Monopole term in (17) and (18)

By comparing with Eqs. (62) and (66) in [50] one obtains the scalar functions of the 1PN monopole terms in Eqs. (17) and (18):

$$\dot{F}_{(1)}\left(\boldsymbol{x}_{\mathrm{N}}\right) = +2\,\dot{\mathcal{W}}_{(3)}\left(\boldsymbol{x}_{\mathrm{N}}\right)\,, \tag{C1}$$

$$\dot{F}_{(2)}\left(\boldsymbol{x}_{\mathrm{N}}\right) = -2\,\dot{\mathcal{X}}_{(3)}\left(\boldsymbol{x}_{\mathrm{N}}\right)\,, \tag{C2}$$

with

$$\dot{\mathcal{W}}_{(3)}\left(\boldsymbol{x}_{\mathrm{N}}\right) = -\frac{1}{x_{\mathrm{N}}}\,, \tag{C3}$$

$$\dot{\mathcal{X}}_{(3)}\left(\boldsymbol{x}_{\mathrm{N}}\right) = +\frac{1}{\left(d_\sigma\right)^2}\left(1+\frac{\boldsymbol{\sigma}\cdot\boldsymbol{x}_{\mathrm{N}}}{x_{\mathrm{N}}}\right), \tag{C4}$$

and

$$F_{(1)}\left(\boldsymbol{x}_{\mathrm{N}}\right) = +2\,\mathcal{W}_{(3)}\left(\boldsymbol{x}_{\mathrm{N}}\right)\,, \tag{C5}$$

$$F_{(2)}\left(\boldsymbol{x}_{\mathrm{N}}\right) = -2\,\mathcal{X}_{(3)}\left(\boldsymbol{x}_{\mathrm{N}}\right)\,, \tag{C6}$$

with

$$\mathcal{W}_{(3)}\left(\boldsymbol{x}_{\mathrm{N}}\right) = +\ln\left(x_{\mathrm{N}}-\boldsymbol{\sigma}\cdot\boldsymbol{x}_{\mathrm{N}}\right), \tag{C7}$$

$$\mathcal{X}_{(3)}\left(\boldsymbol{x}_{\mathrm{N}}\right) = +\frac{1}{\left(d_\sigma\right)^2}\left(x_{\mathrm{N}}+\boldsymbol{\sigma}\cdot\boldsymbol{x}_{\mathrm{N}}\right), \tag{C8}$$

where $\boldsymbol{x}_{\mathrm{N}} = \boldsymbol{x}_{\mathrm{N}}\left(t\right)$.

2. Scalar functions of the 1PN Quadrupole term in (17) and (18)

By comparing with Eqs. (63) and (67) in [50] one obtains the scalar functions of the 1PN quadrupole terms in Eqs. (17) and (18):

$$\dot{G}_{(1)}\left(\boldsymbol{x}_{\mathrm{N}}\right) = +6\,\dot{\mathcal{W}}_{(5)}\left(\boldsymbol{x}_{\mathrm{N}}\right), \tag{C9}$$

$$\dot{G}_{(2)}\left(\boldsymbol{x}_{\mathrm{N}}\right) = +6\,\dot{\mathcal{X}}_{(5)}\left(\boldsymbol{x}_{\mathrm{N}}\right), \tag{C10}$$

$$\dot{G}_{(3)}\left(\boldsymbol{x}_{\mathrm{N}}\right) = +3\,\dot{\mathcal{W}}_{(5)}\left(\boldsymbol{x}_{\mathrm{N}}\right) - 15\left(d_\sigma\right)^2\dot{\mathcal{W}}_{(7)}\left(\boldsymbol{x}_{\mathrm{N}}\right), \tag{C11}$$

$$\dot{G}_{(4)}\left(\boldsymbol{x}_{\mathrm{N}}\right) = +18\,\dot{\mathcal{X}}_{(3)}\left(\boldsymbol{x}_{\mathrm{N}}\right) - 30\left(d_\sigma\right)^2\dot{\mathcal{X}}_{(7)}\left(\boldsymbol{x}_{\mathrm{N}}\right), \tag{C12}$$

$$\dot{G}_{(5)}\left(\boldsymbol{x}_{\mathrm{N}}\right) = +15\,\dot{\mathcal{W}}_{(7)}\left(\boldsymbol{x}_{\mathrm{N}}\right), \tag{C13}$$

$$\dot{G}_{(6)}\left(\boldsymbol{x}_{\mathrm{N}}\right) = -15\,\dot{\mathcal{X}}_{(7)}\left(\boldsymbol{x}_{\mathrm{N}}\right), \tag{C14}$$

$$\dot{G}_{(7)}\left(\boldsymbol{x}_{\mathrm{N}}\right) = -15\,\dot{\mathcal{X}}_{(5)}\left(\boldsymbol{x}_{\mathrm{N}}\right) + 15\left(d_\sigma\right)^2\dot{\mathcal{X}}_{(7)}\left(\boldsymbol{x}_{\mathrm{N}}\right), \tag{C15}$$

$$\dot{G}_{(8)}\left(\boldsymbol{x}_{\mathrm{N}}\right) = -30\,\dot{\mathcal{W}}_{(7)}\left(\boldsymbol{x}_{\mathrm{N}}\right), \tag{C16}$$

with

$$\dot{\mathcal{W}}_{(5)}\left(\boldsymbol{x}_{\mathrm{N}}\right) = -\frac{1}{3}\,\frac{1}{\left(x_{\mathrm{N}}\right)^3}\,, \tag{C17}$$

$$\dot{\mathcal{W}}_{(7)}\left(\boldsymbol{x}_{\mathrm{N}}\right) = -\frac{1}{5}\,\frac{1}{\left(x_{\mathrm{N}}\right)^5}\,, \tag{C18}$$

$$\dot{\mathcal{X}}_{(5)}\left(\boldsymbol{x}_{\mathrm{N}}\right) = +\frac{2}{3}\,\frac{1}{\left(d_\sigma\right)^2} \times\left(\frac{1}{\left(d_\sigma\right)^2}+\frac{1}{\left(d_\sigma\right)^2}\,\frac{\boldsymbol{\sigma}\cdot\boldsymbol{x}_{\mathrm{N}}}{x_{\mathrm{N}}}+\frac{1}{2}\,\frac{\boldsymbol{\sigma}\cdot\boldsymbol{x}_{\mathrm{N}}}{\left(x_{\mathrm{N}}\right)^3}\right), \tag{C19}$$

$$\dot{\mathcal{X}}_{(7)}\left(\boldsymbol{x}_{\mathrm{N}}\right) = +\frac{8}{15}\,\frac{1}{\left(d_\sigma\right)^2}\left(\frac{1}{\left(d_\sigma\right)^4}+\frac{1}{\left(d_\sigma\right)^4}\,\frac{\boldsymbol{\sigma}\cdot\boldsymbol{x}_{\mathrm{N}}}{x_{\mathrm{N}}} +\frac{1}{2}\,\frac{1}{\left(d_\sigma\right)^2}\,\frac{\boldsymbol{\sigma}\cdot\boldsymbol{x}_{\mathrm{N}}}{\left(x_{\mathrm{N}}\right)^3}+\frac{3}{8}\,\frac{\boldsymbol{\sigma}\cdot\boldsymbol{x}_{\mathrm{N}}}{\left(x_{\mathrm{N}}\right)^5}\right), \tag{C20}$$

and

$$G_{(1)}\left(\boldsymbol{x}_{\mathrm{N}}\right) = +6\,\mathcal{W}_{(5)}\left(\boldsymbol{x}_{\mathrm{N}}\right), \tag{C21}$$

$$G_{(2)}\left(\boldsymbol{x}_{\mathrm{N}}\right) = +6\,\mathcal{X}_{(5)}\left(\boldsymbol{x}_{\mathrm{N}}\right), \tag{C22}$$

$$G_{(3)}\left(\boldsymbol{x}_{\mathrm{N}}\right) = +3\,\mathcal{W}_{(5)}\left(\boldsymbol{x}_{\mathrm{N}}\right) - 15\left(d_\sigma\right)^2\mathcal{W}_{(7)}\left(\boldsymbol{x}_{\mathrm{N}}\right), \tag{C23}$$

$$G_{(4)}\left(\boldsymbol{x}_{\mathrm{N}}\right) = +18\,\mathcal{X}_{(3)}\left(\boldsymbol{x}_{\mathrm{N}}\right) - 30\left(d_\sigma\right)^2\mathcal{X}_{(7)}\left(\boldsymbol{x}_{\mathrm{N}}\right), \tag{C24}$$

$$G_{(5)}\left(\boldsymbol{x}_{\mathrm{N}}\right) = +15\,\mathcal{W}_{(7)}\left(\boldsymbol{x}_{\mathrm{N}}\right), \tag{C25}$$

$$G_{(6)}\left(\boldsymbol{x}_{\mathrm{N}}\right) = -15\,\mathcal{X}_{(7)}\left(\boldsymbol{x}_{\mathrm{N}}\right), \tag{C26}$$

$$G_{(7)}\left(\boldsymbol{x}_{\mathrm{N}}\right) = -15\,\mathcal{X}_{(5)}\left(\boldsymbol{x}_{\mathrm{N}}\right) + 15\left(d_\sigma\right)^2\mathcal{X}_{(7)}\left(\boldsymbol{x}_{\mathrm{N}}\right), \tag{C27}$$

$$G_{(8)}\left(\boldsymbol{x}_{\mathrm{N}}\right) = -30\,\mathcal{W}_{(7)}\left(\boldsymbol{x}_{\mathrm{N}}\right), \tag{C28}$$

with

$$\mathcal{W}_{(5)}\left(\boldsymbol{x}_{\mathrm{N}}\right) = -\frac{1}{3}\,\frac{1}{\left(d_\sigma\right)^2}\,\frac{\boldsymbol{\sigma}\cdot\boldsymbol{x}_{\mathrm{N}}}{x_{\mathrm{N}}}\,, \tag{C29}$$

$$\mathcal{W}_{(7)}\left(\boldsymbol{x}_{\mathrm{N}}\right) = -\frac{2}{15}\,\frac{1}{\left(d_\sigma\right)^2}\left(\frac{1}{\left(d_\sigma\right)^2}\,\frac{\boldsymbol{\sigma}\cdot\boldsymbol{x}_{\mathrm{N}}}{x_{\mathrm{N}}}+\frac{1}{2}\,\frac{\boldsymbol{\sigma}\cdot\boldsymbol{x}_{\mathrm{N}}}{\left(x_{\mathrm{N}}\right)^3}\right), \tag{C30}$$

$$\mathcal{X}_{(5)}\left(\boldsymbol{x}_{\mathrm{N}}\right) = +\frac{2}{3}\,\frac{1}{\left(d_\sigma\right)^2}\left(\frac{x_{\mathrm{N}}+\boldsymbol{\sigma}\cdot\boldsymbol{x}_{\mathrm{N}}}{\left(d_\sigma\right)^2}-\frac{1}{2}\,\frac{1}{x_{\mathrm{N}}}\right), \tag{C31}$$

$$\mathcal{X}_{(7)}\left(\boldsymbol{x}_{\mathrm{N}}\right) = +\frac{8}{15}\,\frac{1}{\left(d_\sigma\right)^2} \times\left(\frac{x_{\mathrm{N}}+\boldsymbol{\sigma}\cdot\boldsymbol{x}_{\mathrm{N}}}{\left(d_\sigma\right)^4}-\frac{1}{2}\,\frac{1}{\left(d_\sigma\right)^2}\,\frac{1}{x_{\mathrm{N}}}-\frac{1}{8}\,\frac{1}{\left(x_{\mathrm{N}}\right)^3}\right), \tag{C32}$$

where $\boldsymbol{x}_{\mathrm{N}} = \boldsymbol{x}_{\mathrm{N}}\left(t\right)$.

Appendix D: The scalar functions in (19) - (20)

1. Scalar functions of the 2PN Monopole-Monopole term in (19) and (20)

By comparing with Eqs. (83) and (89) in [50] one obtains the scalar functions for the 2PN monopole-monopole terms in Eqs. (19) and (20):

$$\dot{A}_{(1)}\left(\boldsymbol{x}_{\rm N}\right) = -4\,\dot{\mathcal{W}}_{(4)} - 12\left(x_0 + \boldsymbol{\sigma}\cdot\boldsymbol{x}_0\right)\dot{\mathcal{W}}_{(5)} - 2\left(d_\sigma\right)^2\dot{\mathcal{W}}_{(6)} + 4\,\dot{\mathcal{X}}_{(3)} - 8\,\dot{\mathcal{Z}}_{(3)} + 12\left(d_\sigma\right)^2\dot{\mathcal{Z}}_{(5)}\,, \tag{D1}$$

$$\dot{A}_{(2)}\left(\boldsymbol{x}_{\rm N}\right) = -4\,\dot{\mathcal{W}}_{(3)} - 12\,\dot{\mathcal{W}}_{(5)} - \frac{4}{\left(d_\sigma\right)^2}\dot{\mathcal{X}}_{(2)} - \frac{4}{\left(d_\sigma\right)^2}\left(x_0 + \boldsymbol{\sigma}\cdot\boldsymbol{x}_0\right)\dot{\mathcal{X}}_{(3)} + 6\,\dot{\mathcal{X}}_{(4)} + 12\left(x_0 + \boldsymbol{\sigma}\cdot\boldsymbol{x}_0\right)\dot{\mathcal{X}}_{(5)}$$
$$-2\left(d_\sigma\right)^2\dot{\mathcal{X}}_{(6)} + 12\,\dot{\mathcal{Y}}_{(5)}\,, \tag{D2}$$

and

$$A_{(1)}\left(\boldsymbol{x}_{\rm N}\right) = -4\,\mathcal{W}_{(4)} - 12\left(x_0 + \boldsymbol{\sigma}\cdot\boldsymbol{x}_0\right)\mathcal{W}_{(5)} - 2\left(d_\sigma\right)^2\mathcal{W}_{(6)} + 4\,\mathcal{X}_{(3)} - 8\,\mathcal{Z}_{(3)} + 12\left(d_\sigma\right)^2\mathcal{Z}_{(5)}\,, \tag{D3}$$

$$A_{(2)}\left(\boldsymbol{x}_{\rm N}\right) = -4\,\mathcal{W}_{(3)} - 12\,\mathcal{W}_{(5)} - \frac{4}{\left(d_\sigma\right)^2}\mathcal{X}_{(2)} - \frac{4}{\left(d_\sigma\right)^2}\left(x_0 + \boldsymbol{\sigma}\cdot\boldsymbol{x}_0\right)\mathcal{X}_{(3)} + 6\,\mathcal{X}_{(4)} + 12\left(x_0 + \boldsymbol{\sigma}\cdot\boldsymbol{x}_0\right)\mathcal{X}_{(5)}$$
$$-2\left(d_\sigma\right)^2\mathcal{X}_{(6)} + 12\,\mathcal{Y}_{(5)}\,, \tag{D4}$$

where the scalar functions $\dot{\mathcal{W}}_{(n)}$, $\dot{\mathcal{X}}_{(n)}$, $\dot{\mathcal{Y}}_{(n)}$, $\dot{\mathcal{Z}}_{(n)}$ and $\mathcal{W}_{(n)}$, $\mathcal{X}_{(n)}$, $\mathcal{Y}_{(n)}$, $\mathcal{Z}_{(n)}$ are given in [50]; the argument $\boldsymbol{x}_{\rm N}\left(t\right)$ of these functions has been omitted here.

2. Scalar functions of the 2PN Monopole-Quadrupole term in (19) and (20)

By comparing with Eqs. (84) and (90) in [50] one obtains the scalar functions for the 2PN monopole-quadrupole terms in Eqs. (19) and (20):

$$\dot{B}_{(1)}\left(\boldsymbol{x}_{\rm N}\right) = +\frac{4}{\left(d_\sigma\right)^2}\dot{\mathcal{W}}_{(4)} + 22\,\dot{\mathcal{W}}_{(6)} - 60\left(x_0 + \boldsymbol{\sigma}\cdot\boldsymbol{x}_0\right)\dot{\mathcal{W}}_{(7)} + \frac{21}{2}\left(d_\sigma\right)^2\dot{\mathcal{W}}_{(8)} - \frac{4}{\left(d_\sigma\right)^2}\frac{\boldsymbol{\sigma}\cdot\boldsymbol{x}_0}{x_0}\dot{\mathcal{X}}_{(3)} + 60\,\dot{\mathcal{X}}_{(5)}$$
$$-60\left(d_\sigma\right)^2\dot{\mathcal{X}}_{(7)} - 48\,\dot{\mathcal{Z}}_{(5)} + 60\left(d_\sigma\right)^2\dot{\mathcal{Z}}_{(7)}\,, \tag{D5}$$

$$\vdots \tag{D6}$$

$$\dot{B}_{(8)}\left(\boldsymbol{x}_{\rm N}\right) = -\frac{8}{\left(d_\sigma\right)^4}\dot{\mathcal{W}}_{(4)} - \frac{12}{\left(x_0\right)^3}\dot{\mathcal{W}}_{(5)} + \frac{32}{\left(d_\sigma\right)^2}\dot{\mathcal{W}}_{(6)} - 63\,\dot{\mathcal{W}}_{(8)} + 420\left(x_0 + \boldsymbol{\sigma}\cdot\boldsymbol{x}_0\right)\dot{\mathcal{W}}_{(9)} - 60\left(d_\sigma\right)^2\dot{\mathcal{W}}_{(10)}$$
$$+\frac{8}{\left(d_\sigma\right)^4}\frac{\boldsymbol{\sigma}\cdot\boldsymbol{x}_0}{x_0}\dot{\mathcal{X}}_{(3)} + \frac{4}{\left(d_\sigma\right)^2}\frac{\boldsymbol{\sigma}\cdot\boldsymbol{x}_0}{\left(x_0\right)^3}\dot{\mathcal{X}}_{(3)} + \frac{24}{\left(d_\sigma\right)^2}\dot{\mathcal{X}}_{(5)} - \frac{12}{\left(d_\sigma\right)^2}\frac{\boldsymbol{\sigma}\cdot\boldsymbol{x}_0}{x_0}\dot{\mathcal{X}}_{(5)} - 12\,\frac{\boldsymbol{\sigma}\cdot\boldsymbol{x}_0}{\left(x_0\right)^3}\dot{\mathcal{X}}_{(5)}$$
$$-420\,\dot{\mathcal{X}}_{(7)} + 420\left(d_\sigma\right)^2\dot{\mathcal{X}}_{(9)} + 360\,\dot{\mathcal{Z}}_{(7)} - 420\left(d_\sigma\right)^2\dot{\mathcal{Z}}_{(9)}\,, \tag{D7}$$

and

$$B_{(1)}\left(\boldsymbol{x}_{\rm N}\right) = +\frac{4}{\left(d_\sigma\right)^2}\mathcal{W}_{(4)} + 22\,\mathcal{W}_{(6)} - 60\left(x_0 + \boldsymbol{\sigma}\cdot\boldsymbol{x}_0\right)\mathcal{W}_{(7)} + \frac{21}{2}\left(d_\sigma\right)^2\mathcal{W}_{(8)} - \frac{4}{\left(d_\sigma\right)^2}\frac{\boldsymbol{\sigma}\cdot\boldsymbol{x}_0}{x_0}\mathcal{X}_{(3)} + 60\,\mathcal{X}_{(5)}$$
$$-60\left(d_\sigma\right)^2\mathcal{X}_{(7)} - 48\,\mathcal{Z}_{(5)} + 60\left(d_\sigma\right)^2\mathcal{Z}_{(7)}\,, \tag{D8}$$

$$\vdots \tag{D9}$$

$$B_{(8)}\left(\boldsymbol{x}_{\rm N}\right) = -\frac{8}{\left(d_\sigma\right)^4}\mathcal{W}_{(4)} - \frac{12}{\left(x_0\right)^3}\mathcal{W}_{(5)} + \frac{32}{\left(d_\sigma\right)^2}\mathcal{W}_{(6)} - 63\,\mathcal{W}_{(8)} + 420\left(x_0 + \boldsymbol{\sigma}\cdot\boldsymbol{x}_0\right)\mathcal{W}_{(9)} - 60\left(d_\sigma\right)^2\mathcal{W}_{(10)}$$
$$+\frac{8}{\left(d_\sigma\right)^4}\frac{\boldsymbol{\sigma}\cdot\boldsymbol{x}_0}{x_0}\mathcal{X}_{(3)} + \frac{4}{\left(d_\sigma\right)^2}\frac{\boldsymbol{\sigma}\cdot\boldsymbol{x}_0}{\left(x_0\right)^3}\mathcal{X}_{(3)} + \frac{24}{\left(d_\sigma\right)^2}\mathcal{X}_{(5)} - \frac{12}{\left(d_\sigma\right)^2}\frac{\boldsymbol{\sigma}\cdot\boldsymbol{x}_0}{x_0}\mathcal{X}_{(5)} - 12\,\frac{\boldsymbol{\sigma}\cdot\boldsymbol{x}_0}{\left(x_0\right)^3}\mathcal{X}_{(5)}$$
$$-420\,\mathcal{X}_{(7)} + 420\left(d_\sigma\right)^2\mathcal{X}_{(9)} + 360\,\mathcal{Z}_{(7)} - 420\left(d_\sigma\right)^2\mathcal{Z}_{(9)}\,, \tag{D10}$$

where the scalar functions $\dot{\mathcal{W}}_{(n)}$, $\dot{\mathcal{X}}_{(n)}$, $\dot{\mathcal{Y}}_{(n)}$, $\dot{\mathcal{Z}}_{(n)}$ and $\mathcal{W}_{(n)}$, $\mathcal{X}_{(n)}$, $\mathcal{Y}_{(n)}$, $\mathcal{Z}_{(n)}$ are given in [50]; the argument $\boldsymbol{x}_{\rm N}\left(t\right)$ of these functions has been omitted here.

3. Scalar functions of the 2PN Quadrupole-Quadrupole term in (19) and (20)

By comparing with Eqs. (85) and (91) in [50] one obtains the scalar functions for the 2PN quadrupole-quadrupole terms in (19) and (20):

$$\dot{C}_{(1)}\left(\boldsymbol{x}_{\rm N}\right) = -\frac{12}{\left(d_\sigma\right)^2}\,\dot{\mathcal{W}}_{(6)} + 9\,\dot{\mathcal{W}}_{(8)} - 13\left(d_\sigma\right)^2\dot{\mathcal{W}}_{(10)} + \frac{12}{\left(d_\sigma\right)^2}\,\frac{\boldsymbol{\sigma}\cdot\boldsymbol{x}_0}{x_0}\,\dot{\mathcal{X}}_{(5)}\,, \tag{D11}$$

$$\vdots \tag{D12}$$

$$\dot{C}_{(28)}\left(\boldsymbol{x}_{\rm N}\right) = +\frac{120}{\left(d_\sigma\right)^6}\,\dot{\mathcal{W}}_{(7)} - \frac{420}{\left(d_\sigma\right)^4}\,\dot{\mathcal{W}}_{(9)} + \frac{210}{\left(d_\sigma\right)^4}\,\frac{\boldsymbol{\sigma}\cdot\boldsymbol{x}_0}{x_0}\,\dot{\mathcal{W}}_{(9)} + \frac{105}{\left(d_\sigma\right)^2}\,\frac{\boldsymbol{\sigma}\cdot\boldsymbol{x}_0}{\left(x_0\right)^3}\,\dot{\mathcal{W}}_{(9)} + \frac{120}{\left(d_\sigma\right)^6}\,\dot{\mathcal{X}}_{(6)} + \frac{180}{\left(d_\sigma\right)^4}\,\frac{\dot{\mathcal{X}}_{(7)}}{x_0}$$
$$+\frac{45}{\left(d_\sigma\right)^2}\,\frac{\dot{\mathcal{X}}_{(7)}}{\left(x_0\right)^3} - \frac{360}{\left(d_\sigma\right)^6}\left(x_0 + \boldsymbol{\sigma}\cdot\boldsymbol{x}_0\right)\dot{\mathcal{X}}_{(7)} - \frac{690}{\left(d_\sigma\right)^4}\,\dot{\mathcal{X}}_{(8)} + \frac{300}{\left(d_\sigma\right)^2}\,\dot{\mathcal{X}}_{(10)} + \frac{1005}{2}\,\dot{\mathcal{X}}_{(12)} - \frac{225}{2}\left(d_\sigma\right)^2\dot{\mathcal{X}}_{(14)}\,, \tag{D13}$$

and

$$C_{(1)}\left(\boldsymbol{x}_{\rm N}\right) = -\frac{12}{\left(d_\sigma\right)^2}\,\mathcal{W}_{(6)} + 9\,\mathcal{W}_{(8)} - 13\left(d_\sigma\right)^2\mathcal{W}_{(10)} + \frac{12}{\left(d_\sigma\right)^2}\,\frac{\boldsymbol{\sigma}\cdot\boldsymbol{x}_0}{x_0}\,\mathcal{X}_{(5)}\,, \tag{D14}$$

$$\vdots \tag{D15}$$

$$C_{(28)}\left(\boldsymbol{x}_{\rm N}\right) = +\frac{120}{\left(d_\sigma\right)^6}\,\mathcal{W}_{(7)} - \frac{420}{\left(d_\sigma\right)^4}\,\mathcal{W}_{(9)} + \frac{210}{\left(d_\sigma\right)^4}\,\frac{\boldsymbol{\sigma}\cdot\boldsymbol{x}_0}{x_0}\,\mathcal{W}_{(9)} + \frac{105}{\left(d_\sigma\right)^2}\,\frac{\boldsymbol{\sigma}\cdot\boldsymbol{x}_0}{\left(x_0\right)^3}\,\mathcal{W}_{(9)} + \frac{120}{\left(d_\sigma\right)^6}\,\mathcal{X}_{(6)} + \frac{180}{\left(d_\sigma\right)^4}\,\frac{\mathcal{X}_{(7)}}{x_0}$$
$$+\frac{45}{\left(d_\sigma\right)^2}\,\frac{\mathcal{X}_{(7)}}{\left(x_0\right)^3} - \frac{360}{\left(d_\sigma\right)^6}\left(x_0 + \boldsymbol{\sigma}\cdot\boldsymbol{x}_0\right)\mathcal{X}_{(7)} - \frac{690}{\left(d_\sigma\right)^4}\,\mathcal{X}_{(8)} + \frac{300}{\left(d_\sigma\right)^2}\,\mathcal{X}_{(10)} + \frac{1005}{2}\,\mathcal{X}_{(12)} - \frac{225}{2}\left(d_\sigma\right)^2\mathcal{X}_{(14)}\,, \tag{D16}$$

where the scalar functions $\dot{\mathcal{W}}_{(n)}$, $\dot{\mathcal{X}}_{(n)}$, $\dot{\mathcal{Y}}_{(n)}$, $\dot{\mathcal{Z}}_{(n)}$ and $\mathcal{W}_{(n)}$, $\mathcal{X}_{(n)}$, $\mathcal{Y}_{(n)}$, $\mathcal{Z}_{(n)}$ are given in [50]; the argument $\boldsymbol{x}_{\rm N}\,(t)$ of these functions has been omitted here.

Appendix E: Tensorial coefficients in (28) - (29)

The tensorial coefficients in (28) - (29) are given by

$$U^i_{(1)}\left(\boldsymbol{x}_{\rm 1PN}\right) = \sigma^i\,, \tag{E1}$$
$$U^i_{(2)}\left(\boldsymbol{x}_{\rm 1PN}\right) = \widehat{d_\sigma}^{\,i}\,. \tag{E2}$$
$$V^{ab\,i}_{(1)}\left(\boldsymbol{x}_{\rm 1PN}\right) = \sigma^a\delta^{bi}\,, \tag{E3}$$
$$V^{ab\,i}_{(2)}\left(\boldsymbol{x}_{\rm 1PN}\right) = \widehat{d_\sigma}^{\,a}\delta^{bi}\,, \tag{E4}$$
$$V^{ab\,i}_{(3)}\left(\boldsymbol{x}_{\rm 1PN}\right) = \sigma^a\sigma^b\sigma^i\,, \tag{E5}$$
$$V^{ab\,i}_{(4)}\left(\boldsymbol{x}_{\rm 1PN}\right) = \sigma^a\widehat{d_\sigma}^{\,b}\sigma^i\,, \tag{E6}$$
$$V^{ab\,i}_{(5)}\left(\boldsymbol{x}_{\rm 1PN}\right) = \widehat{d_\sigma}^{\,a}\,\widehat{d_\sigma}^{\,b}\,\sigma^i\,, \tag{E7}$$
$$V^{ab\,i}_{(6)}\left(\boldsymbol{x}_{\rm 1PN}\right) = \widehat{d_\sigma}^{\,a}\,\widehat{d_\sigma}^{\,b}\,\widehat{d_\sigma}^{\,i}\,, \tag{E8}$$
$$V^{ab\,i}_{(7)}\left(\boldsymbol{x}_{\rm 1PN}\right) = \sigma^a\sigma^b\,\widehat{d_\sigma}^{\,i}\,, \tag{E9}$$
$$V^{ab\,i}_{(8)}\left(\boldsymbol{x}_{\rm 1PN}\right) = \sigma^a\,\widehat{d_\sigma}^{\,b}\,\widehat{d_\sigma}^{\,i}\,. \tag{E10}$$

Appendix F: Scalar functions in (28) - (29)

The scalar functions for the monopole term in Eqs. (28) - (29) are given by

$$\dot{F}_{(1)}\left(\boldsymbol{x}_{\rm 1PN}\right) = +2\,\dot{\mathcal{W}}_{(3)}\left(\boldsymbol{x}_{\rm 1PN}\right)\,, \tag{F1}$$
$$\dot{F}_{(2)}\left(\boldsymbol{x}_{\rm 1PN}\right) = -2\,\dot{\mathcal{X}}_{(3)}\left(\boldsymbol{x}_{\rm 1PN}\right)\,, \tag{F2}$$

with

$$\dot{\mathcal{W}}_{(3)}\left(\boldsymbol{x}_{\rm 1PN}\right) = -\frac{1}{x_{\rm 1PN}}\,, \tag{F3}$$
$$\dot{\mathcal{X}}_{(3)}\left(\boldsymbol{x}_{\rm 1PN}\right) = +\frac{1}{(\widehat{d}_\sigma)^2}\left(1 + \frac{\boldsymbol{\sigma}\cdot\boldsymbol{x}_{\rm 1PN}}{x_{\rm 1PN}}\right), \tag{F4}$$

and

$$F_{(1)}\left(\boldsymbol{x}_{\rm 1PN}\right) = +2\,\mathcal{W}_{(3)}\left(\boldsymbol{x}_{\rm 1PN}\right)\,, \tag{F5}$$
$$F_{(2)}\left(\boldsymbol{x}_{\rm 1PN}\right) = -2\,\mathcal{X}_{(3)}\left(\boldsymbol{x}_{\rm 1PN}\right)\,, \tag{F6}$$

with

$$\mathcal{W}_{(3)}\left(\boldsymbol{x}_{\rm 1PN}\right) = +\ln\left(x_{\rm 1PN} - \boldsymbol{\sigma}\cdot\boldsymbol{x}_{\rm 1PN}\right), \tag{F7}$$
$$\mathcal{X}_{(3)}\left(\boldsymbol{x}_{\rm 1PN}\right) = +\frac{1}{(\widehat{d}_\sigma)^2}\left(x_{\rm 1PN} + \boldsymbol{\sigma}\cdot\boldsymbol{x}_{\rm 1PN}\right), \tag{F8}$$

where $\boldsymbol{x}_{1\rm PN} = \boldsymbol{x}_{1\rm PN}\left(t\right)$.

The scalar functions for the quadrupole term in Eqs. (28) - (29) are given by

$$\dot{G}_{(1)}\left(\boldsymbol{x}_{1\rm PN}\right) = +6\,\dot{\mathcal{W}}_{(5)}\left(\boldsymbol{x}_{1\rm PN}\right)\,, \tag{F9}$$

$$\dot{G}_{(2)}\left(\boldsymbol{x}_{1\rm PN}\right) = +6\,\dot{\mathcal{X}}_{(5)}\left(\boldsymbol{x}_{1\rm PN}\right)\,, \tag{F10}$$

$$\dot{G}_{(3)}\left(\boldsymbol{x}_{1\rm PN}\right) = +3\,\dot{\mathcal{W}}_{(5)}\left(\boldsymbol{x}_{1\rm PN}\right) - 15(\widehat{d}_{\sigma})^2\,\dot{\mathcal{W}}_{(7)}\left(\boldsymbol{x}_{1\rm PN}\right)\,, \tag{F11}$$

$$\dot{G}_{(4)}\left(\boldsymbol{x}_{1\rm PN}\right) = +18\,\dot{\mathcal{X}}_{(3)}\left(\boldsymbol{x}_{1\rm PN}\right) - 30(\widehat{d}_{\sigma})^2\,\dot{\mathcal{X}}_{(7)}\left(\boldsymbol{x}_{1\rm PN}\right)\,, \tag{F12}$$

$$\dot{G}_{(5)}\left(\boldsymbol{x}_{1\rm PN}\right) = +15\,\dot{\mathcal{W}}_{(7)}\left(\boldsymbol{x}_{1\rm PN}\right)\,, \tag{F13}$$

$$\dot{G}_{(6)}\left(\boldsymbol{x}_{1\rm PN}\right) = -15\,\dot{\mathcal{X}}_{(7)}\left(\boldsymbol{x}_{1\rm PN}\right)\,, \tag{F14}$$

$$\dot{G}_{(7)}\left(\boldsymbol{x}_{1\rm PN}\right) = -15\,\dot{\mathcal{X}}_{(5)}\left(\boldsymbol{x}_{1\rm PN}\right) + 15(\widehat{d}_{\sigma})^2\,\dot{\mathcal{X}}_{(7)}\left(\boldsymbol{x}_{1\rm PN}\right)\,, \tag{F15}$$

$$\dot{G}_{(8)}\left(\boldsymbol{x}_{1\rm PN}\right) = -30\,\dot{\mathcal{W}}_{(7)}\left(\boldsymbol{x}_{1\rm PN}\right)\,, \tag{F16}$$

with

$$\dot{\mathcal{W}}_{(5)}\left(\boldsymbol{x}_{1\rm PN}\right) = -\frac{1}{3}\,\frac{1}{\left(x_{1\rm PN}\right)^3}\,, \tag{F17}$$

$$\dot{\mathcal{W}}_{(7)}\left(\boldsymbol{x}_{1\rm PN}\right) = -\frac{1}{5}\,\frac{1}{\left(x_{1\rm PN}\right)^5}\,, \tag{F18}$$

$$\dot{\mathcal{X}}_{(5)}\left(\boldsymbol{x}_{1\rm PN}\right) = +\frac{2}{3}\,\frac{1}{(\widehat{d}_{\sigma})^2} \times\left(\frac{1}{(\widehat{d}_{\sigma})^2} + \frac{1}{(\widehat{d}_{\sigma})^2}\,\frac{\boldsymbol{\sigma}\cdot\boldsymbol{x}_{1\rm PN}}{x_{1\rm PN}} + \frac{1}{2}\,\frac{\boldsymbol{\sigma}\cdot\boldsymbol{x}_{1\rm PN}}{\left(x_{1\rm PN}\right)^3}\right), \tag{F19}$$

$$\dot{\mathcal{X}}_{(7)}\left(\boldsymbol{x}_{1\rm PN}\right) = +\frac{8}{15}\,\frac{1}{(\widehat{d}_{\sigma})^2}\left(\frac{1}{(\widehat{d}_{\sigma})^4} + \frac{1}{(\widehat{d}_{\sigma})^4}\,\frac{\boldsymbol{\sigma}\cdot\boldsymbol{x}_{1\rm PN}}{x_{1\rm PN}} + \frac{1}{2}\,\frac{1}{(\widehat{d}_{\sigma})^2}\,\frac{\boldsymbol{\sigma}\cdot\boldsymbol{x}_{1\rm PN}}{\left(x_{1\rm PN}\right)^3} + \frac{3}{8}\,\frac{\boldsymbol{\sigma}\cdot\boldsymbol{x}_{1\rm PN}}{\left(x_{1\rm PN}\right)^5}\right), \tag{F20}$$

and

$$G_{(1)}\left(\boldsymbol{x}_{1\rm PN}\right) = +6\,\mathcal{W}_{(5)}\left(\boldsymbol{x}_{1\rm PN}\right)\,, \tag{F21}$$

$$G_{(2)}\left(\boldsymbol{x}_{1\rm PN}\right) = +6\,\mathcal{X}_{(5)}\left(\boldsymbol{x}_{1\rm PN}\right)\,, \tag{F22}$$

$$G_{(3)}\left(\boldsymbol{x}_{1\rm PN}\right) = +3\,\mathcal{W}_{(5)}\left(\boldsymbol{x}_{1\rm PN}\right) - 15(\widehat{d}_{\sigma})^2\,\mathcal{W}_{(7)}\left(\boldsymbol{x}_{1\rm PN}\right)\,, \tag{F23}$$

$$G_{(4)}\left(\boldsymbol{x}_{1\rm PN}\right) = +18\,\mathcal{X}_{(3)}\left(\boldsymbol{x}_{1\rm PN}\right) - 30(\widehat{d}_{\sigma})^2\,\mathcal{X}_{(7)}\left(\boldsymbol{x}_{1\rm PN}\right)\,, \tag{F24}$$

$$G_{(5)}\left(\boldsymbol{x}_{1\rm PN}\right) = +15\,\mathcal{W}_{(7)}\left(\boldsymbol{x}_{1\rm PN}\right)\,, \tag{F25}$$

$$G_{(6)}\left(\boldsymbol{x}_{1\rm PN}\right) = -15\,\mathcal{X}_{(7)}\left(\boldsymbol{x}_{1\rm PN}\right)\,, \tag{F26}$$

$$G_{(7)}\left(\boldsymbol{x}_{1\rm PN}\right) = -15\,\mathcal{X}_{(5)}\left(\boldsymbol{x}_{1\rm PN}\right) + 15(\widehat{d}_{\sigma})^2\,\mathcal{X}_{(7)}\left(\boldsymbol{x}_{1\rm PN}\right)\,, \tag{F27}$$

$$G_{(8)}\left(\boldsymbol{x}_{1\rm PN}\right) = -30\,\mathcal{W}_{(7)}\left(\boldsymbol{x}_{1\rm PN}\right)\,, \tag{F28}$$

with

$$\mathcal{W}_{(5)}\left(\boldsymbol{x}_{1\rm PN}\right) = -\frac{1}{3}\,\frac{1}{(\widehat{d}_{\sigma})^2}\,\frac{\boldsymbol{\sigma}\cdot\boldsymbol{x}_{1\rm PN}}{x_{1\rm PN}}\,, \tag{F29}$$

$$\mathcal{W}_{(7)}\left(\boldsymbol{x}_{1\rm PN}\right) = -\frac{2}{15}\,\frac{1}{(\widehat{d}_{\sigma})^2}\left(\frac{1}{(\widehat{d}_{\sigma})^2}\frac{\boldsymbol{\sigma}\cdot\boldsymbol{x}_{1\rm PN}}{x_{1\rm PN}} + \frac{1}{2}\,\frac{\boldsymbol{\sigma}\cdot\boldsymbol{x}_{1\rm PN}}{\left(x_{1\rm PN}\right)^3}\right), \tag{F30}$$

$$\mathcal{X}_{(5)}\left(\boldsymbol{x}_{1\rm PN}\right) = +\frac{2}{3}\,\frac{1}{(\widehat{d}_{\sigma})^2} \times\left(\frac{x_{1\rm PN} + \boldsymbol{\sigma}\cdot\boldsymbol{x}_{1\rm PN}}{(\widehat{d}_{\sigma})^2} - \frac{1}{2}\,\frac{1}{x_{1\rm PN}}\right), \tag{F31}$$

$$\mathcal{X}_{(7)}\left(\boldsymbol{x}_{1\rm PN}\right) = +\frac{8}{15}\,\frac{1}{(\widehat{d}_{\sigma})^2} \times\left(\frac{x_{1\rm PN} + \boldsymbol{\sigma}\cdot\boldsymbol{x}_{1\rm PN}}{(\widehat{d}_{\sigma})^4} - \frac{1}{2}\,\frac{1}{(\widehat{d}_{\sigma})^2}\,\frac{1}{x_{1\rm PN}} - \frac{1}{8}\,\frac{1}{\left(x_{1\rm PN}\right)^3}\right), \tag{F32}$$

where $\boldsymbol{x}_{1\rm PN} = \boldsymbol{x}_{1\rm PN}\left(t\right)$.

Appendix G: The tensorial coefficients in (51) and (52)

The tensorial coefficients of the 1PN perturbation terms in (51) and (52) are given by

$$U^{i}_{(1)}\left(\boldsymbol{x}_1\right) = \sigma^i\,, \tag{G1}$$

$$U^{i}_{(2)}\left(\boldsymbol{x}_1\right) = d^{1\,i}_{\sigma}\,, \tag{G2}$$

$$V^{ab\,i}_{(1)}\left(\boldsymbol{x}_1\right) = \sigma^a\delta^{bi}\,, \tag{G3}$$

$$V^{ab\,i}_{(2)}\left(\boldsymbol{x}_1\right) = d^{1\,a}_{\sigma}\delta^{bi}\,, \tag{G4}$$

$$V^{ab\,i}_{(3)}\left(\boldsymbol{x}_1\right) = \sigma^a\sigma^b\sigma^i\,, \tag{G5}$$

$$V^{ab\,i}_{(4)}\left(\boldsymbol{x}_1\right) = \sigma^a d^{1\,b}_{\sigma}\sigma^i\,, \tag{G6}$$

$$V^{ab\,i}_{(5)}\left(\boldsymbol{x}_1\right) = d^{1\,a}_{\sigma} d^{1\,b}_{\sigma}\sigma^i\,, \tag{G7}$$

$$V^{ab\,i}_{(6)}\left(\boldsymbol{x}_1\right) = d^{1\,a}_{\sigma} d^{1\,b}_{\sigma} d^{1\,i}_{\sigma}\,, \tag{G8}$$

$$V^{ab\,i}_{(7)}\left(\boldsymbol{x}_1\right) = \sigma^a\sigma^b d^{1\,i}_{\sigma}\,, \tag{G9}$$

$$V^{ab\,i}_{(8)}\left(\boldsymbol{x}_1\right) = \sigma^a d^{1\,b}_{\sigma} d^{1\,i}_{\sigma}\,, \tag{G10}$$

where the impact vector $\boldsymbol{d}^{1}_{\sigma}$ is defined by Eq. (58).

Appendix H: The tensorial coefficients in (55) and (56)

The tensorial coefficients of the 2PN perturbation terms in (55) and (56) are given by

$$
\begin{aligned}
W_{(1)}^{abcd\,i}\left(\boldsymbol{x}_1\right) &= \delta^{ac}\sigma^b\delta^{di} \,, &\text{(H1)}\\
W_{(2)}^{abcd\,i}\left(\boldsymbol{x}_1\right) &= \delta^{ac}d_\sigma^{1\,b}\delta^{di} \,, &\text{(H2)}\\
W_{(3)}^{abcd\,i}\left(\boldsymbol{x}_1\right) &= \sigma^a\sigma^b\sigma^c\delta^{di} \,, &\text{(H3)}\\
W_{(4)}^{abcd\,i}\left(\boldsymbol{x}_1\right) &= \sigma^a\sigma^b d_\sigma^{1\,c}\delta^{di} \,, &\text{(H4)}\\
W_{(5)}^{abcd\,i}\left(\boldsymbol{x}_1\right) &= \sigma^a d_\sigma^{1\,b}\sigma^c\delta^{di} \,, &\text{(H5)}\\
W_{(6)}^{abcd\,i}\left(\boldsymbol{x}_1\right) &= \sigma^a d_\sigma^{1\,b}d_\sigma^{1\,c}\delta^{di} \,, &\text{(H6)}\\
W_{(7)}^{abcd\,i}\left(\boldsymbol{x}_1\right) &= d_\sigma^{1\,a}d_\sigma^{1\,b}\sigma^c\delta^{di} \,, &\text{(H7)}\\
W_{(8)}^{abcd\,i}\left(\boldsymbol{x}_1\right) &= d_\sigma^{1\,a}d_\sigma^{1\,b}d_\sigma^{1\,c}\delta^{di} \,, &\text{(H8)}\\
W_{(9)}^{abcd\,i}\left(\boldsymbol{x}_1\right) &= \delta^{ac}\delta^{bd}\sigma^i \,, &\text{(H9)}\\
W_{(10)}^{abcd\,i}\left(\boldsymbol{x}_1\right) &= \delta^{ac}\sigma^b\sigma^d\sigma^i \,, &\text{(H10)}\\
W_{(11)}^{abcd\,i}\left(\boldsymbol{x}_1\right) &= \delta^{ac}\sigma^b d_\sigma^{1\,d}\sigma^i \,, &\text{(H11)}\\
W_{(12)}^{abcd\,i}\left(\boldsymbol{x}_1\right) &= \delta^{ac}d_\sigma^{1\,b}d_\sigma^{1\,d}\sigma^i \,, &\text{(H12)}\\
W_{(13)}^{abcd\,i}\left(\boldsymbol{x}_1\right) &= \sigma^a\sigma^b\sigma^c\sigma^d\sigma^i \,, &\text{(H13)}\\
W_{(14)}^{abcd\,i}\left(\boldsymbol{x}_1\right) &= \sigma^a\sigma^b\sigma^c d_\sigma^{1\,d}\sigma^i \,, &\text{(H14)}\\
W_{(15)}^{abcd\,i}\left(\boldsymbol{x}_1\right) &= \sigma^a\sigma^b d_\sigma^{1\,c}d_\sigma^{1\,d}\sigma^i \,, &\text{(H15)}\\
W_{(16)}^{abcd\,i}\left(\boldsymbol{x}_1\right) &= \sigma^a d_\sigma^{1\,b}\sigma^c d_\sigma^{1\,d}\sigma^i \,, &\text{(H16)}\\
W_{(17)}^{abcd\,i}\left(\boldsymbol{x}_1\right) &= \sigma^a d_\sigma^{1\,b}d_\sigma^{1\,c}d_\sigma^{1\,d}\sigma^i \,, &\text{(H17)}\\
W_{(18)}^{abcd\,i}\left(\boldsymbol{x}_1\right) &= d_\sigma^{1\,a}d_\sigma^{1\,b}d_\sigma^{1\,c}d_\sigma^{1\,d}\sigma^i \,, &\text{(H18)}\\
W_{(19)}^{abcd\,i}\left(\boldsymbol{x}_1\right) &= \delta^{ac}\delta^{bd}d_\sigma^{1\,i} \,, &\text{(H19)}\\
W_{(20)}^{abcd\,i}\left(\boldsymbol{x}_1\right) &= \delta^{ac}\sigma^b\sigma^d d_\sigma^{1\,i} \,, &\text{(H20)}\\
W_{(21)}^{abcd\,i}\left(\boldsymbol{x}_1\right) &= \delta^{ac}\sigma^b d_\sigma^{1\,d}d_\sigma^{1\,i} \,, &\text{(H21)}\\
W_{(22)}^{abcd\,i}\left(\boldsymbol{x}_1\right) &= \delta^{ac}d_\sigma^{1\,b}d_\sigma^{1\,d}d_\sigma^{1\,i} \,, &\text{(H22)}\\
W_{(23)}^{abcd\,i}\left(\boldsymbol{x}_1\right) &= \sigma^a\sigma^b\sigma^c\sigma^d d_\sigma^{1\,i} \,, &\text{(H23)}\\
W_{(24)}^{abcd\,i}\left(\boldsymbol{x}_1\right) &= \sigma^a\sigma^b\sigma^c d_\sigma^{1\,d}d_\sigma^{1\,i} \,, &\text{(H24)}\\
W_{(25)}^{abcd\,i}\left(\boldsymbol{x}_1\right) &= \sigma^a\sigma^b d_\sigma^{1\,c}d_\sigma^{1\,d}d_\sigma^{1\,i} \,, &\text{(H25)}\\
W_{(26)}^{abcd\,i}\left(\boldsymbol{x}_1\right) &= \sigma^a d_\sigma^{1\,b}\sigma^c d_\sigma^{1\,d}d_\sigma^{1\,i} \,, &\text{(H26)}\\
W_{(27)}^{abcd\,i}\left(\boldsymbol{x}_1\right) &= \sigma^a d_\sigma^{1\,b}d_\sigma^{1\,c}d_\sigma^{1\,d}d_\sigma^{1\,i} \,, &\text{(H27)}\\
W_{(28)}^{abcd\,i}\left(\boldsymbol{x}_1\right) &= d_\sigma^{1\,a}d_\sigma^{1\,b}d_\sigma^{1\,c}d_\sigma^{1\,d}d_\sigma^{1\,i} \,, &\text{(H28)}
\end{aligned}
$$

where the impact vector $\boldsymbol{d}_\sigma^1$ is defined by Eq. (58).

Appendix I: Scalar functions of 1PN solution in (51) and (52)

In order to simplify the notation, it is useful to introduce the following abbreviations:

$$
\begin{aligned}
a_{(n)} &= \left(x_1 + \boldsymbol{\sigma}\cdot\boldsymbol{x}_1\right)^n \,, &\text{(I1)}\\
b_{(n)} &= \frac{1}{\left(x_1\right)^n} \,, &\text{(I2)}\\
c_{(n)} &= \frac{\boldsymbol{\sigma}\cdot\boldsymbol{x}_1}{\left(x_1\right)^n} \,, &\text{(I3)}\\
d_{(1)} &= \ln\left(x_1 - \boldsymbol{\sigma}\cdot\boldsymbol{x}_1\right) , &\text{(I4)}\\
d_{(2)} &= \arctan\frac{\boldsymbol{\sigma}\cdot\boldsymbol{x}_1}{d_\sigma^1} + \frac{\pi}{2} \,, &\text{(I5)}\\
d_{(3)} &= \arctan\frac{\boldsymbol{\sigma}\cdot\boldsymbol{x}_1}{d_\sigma^1} \,, &\text{(I6)}\\
d_{(4)} &= \frac{\boldsymbol{\sigma}\cdot\boldsymbol{x}_1}{d_\sigma^1}\left(\arctan\frac{\boldsymbol{\sigma}\cdot\boldsymbol{x}_1}{d_\sigma^1} + \frac{\pi}{2}\right) , &\text{(I7)}
\end{aligned}
$$

where $n = 1, 2, 3, \ldots$ in Eqs. (I1) - (I3) is a natural number.

1. Scalar functions of the 1PN Monopole term in (51) and (52)

The scalar functions of the 1PN monopole term in Eqs. (51) and (52) are given by

$$
\begin{aligned}
\dot{F}_{(1)}\left(\boldsymbol{x}_1\right) &= -2\, b_{(1)} \,, &\text{(I8)}\\
\dot{F}_{(2)}\left(\boldsymbol{x}_1\right) &= -\frac{2}{(d_\sigma^1)^2}\left(1 + c_{(1)}\right) , &\text{(I9)}\\
F_{(1)}\left(\boldsymbol{x}_1\right) &= +2\, d_{(1)} \,, &\text{(I10)}\\
F_{(2)}\left(\boldsymbol{x}_1\right) &= -\frac{2}{(d_\sigma^1)^2}\, a_{(1)} \,. &\text{(I11)}
\end{aligned}
$$

2. Scalar functions of the 1PN Quadrupole term in (51) and (52)

The scalar functions of the 1PN quadrupole term in Eqs. (51) and (52) are given by

$$\dot{G}_{(1)}\left(\boldsymbol{x}_1\right) = -2\, b_{(3)}\,, \tag{I12}$$

$$\dot{G}_{(2)}\left(\boldsymbol{x}_1\right) = +\frac{4}{(d^1_\sigma)^4}\left(1 + c_{(1)}\right) + \frac{2}{(d^1_\sigma)^2}\, c_{(3)}\,, \tag{I13}$$

$$\dot{G}_{(3)}\left(\boldsymbol{x}_1\right) = -b_{(3)} + 3\,(d^1_\sigma)^2\, b_{(5)}\,, \tag{I14}$$

$$\dot{G}_{(4)}\left(\boldsymbol{x}_1\right) = -\frac{4}{(d^1_\sigma)^4}\left(1 + c_{(1)}\right) - \frac{2}{(d^1_\sigma)^2}\, c_{(3)} - 6\, c_{(5)}\,, \tag{I15}$$

$$\dot{G}_{(5)}\left(\boldsymbol{x}_1\right) = -3\, b_{(5)}\,, \tag{I16}$$

$$\dot{G}_{(6)}\left(\boldsymbol{x}_1\right) = -\frac{8}{(d^1_\sigma)^6}\left(1 + c_{(1)}\right) - \frac{4}{(d^1_\sigma)^4}\, c_{(3)} - \frac{3}{(d^1_\sigma)^2}\, c_{(5)}\,, \tag{I17}$$

$$\dot{G}_{(7)}\left(\boldsymbol{x}_1\right) = -\frac{2}{(d^1_\sigma)^4}\left(1 + c_{(1)}\right) - \frac{1}{(d^1_\sigma)^2}\, c_{(3)} + 3\, c_{(5)}\,, \tag{I18}$$

$$\dot{G}_{(8)}\left(\boldsymbol{x}_1\right) = +6\, b_{(5)}\,, \tag{I19}$$

and

$$G_{(1)}\left(\boldsymbol{x}_1\right) = -\frac{2}{(d^1_\sigma)^2}\, c_{(1)}\,, \tag{I20}$$

$$G_{(2)}\left(\boldsymbol{x}_1\right) = +\frac{4}{(d^1_\sigma)^4}\, a_{(1)} - \frac{2}{(d^1_\sigma)^2}\, b_{(1)}\,, \tag{I21}$$

$$G_{(3)}\left(\boldsymbol{x}_1\right) = +\frac{c_{(1)}}{(d^1_\sigma)^2} + c_{(3)}\,, \tag{I22}$$

$$G_{(4)}\left(\boldsymbol{x}_1\right) = -\frac{4}{(d^1_\sigma)^4}\, a_{(1)} + \frac{2}{(d^1_\sigma)^2}\, b_{(1)} + 2\, b_{(3)}\,, \tag{I23}$$

$$G_{(5)}\left(\boldsymbol{x}_1\right) = -\frac{2}{(d^1_\sigma)^4}\, c_{(1)} - \frac{c_{(3)}}{(d^1_\sigma)^2}\,, \tag{I24}$$

$$G_{(6)}\left(\boldsymbol{x}_1\right) = -\frac{8}{(d^1_\sigma)^6}\, a_{(1)} + \frac{4}{(d^1_\sigma)^4}\, b_{(1)} + \frac{b_{(3)}}{(d^1_\sigma)^2}\,, \tag{I25}$$

$$G_{(7)}\left(\boldsymbol{x}_1\right) = -\frac{2}{(d^1_\sigma)^4}\, a_{(1)} + \frac{b_{(1)}}{(d^1_\sigma)^2} - b_{(3)}\,, \tag{I26}$$

$$G_{(8)}\left(\boldsymbol{x}_1\right) = +\frac{4}{(d^1_\sigma)^4}\, c_{(1)} + \frac{2}{(d^1_\sigma)^2}\, c_{(3)}\,. \tag{I27}$$

Appendix J: The scalar functions of the 2PN solution in (55) and (56)

The scalar functions of the monopole-monopole term in (55) and (56) are given by

$$\dot{X}_{(1)}\left(\boldsymbol{x}_1\right) = -\frac{4}{\left(d^1_\sigma\right)^2}\left(1 + c_{(1)}\right) + 4\, b_{(2)} + \frac{b_{(4)}}{2}\left(d^1_\sigma\right)^2\,, \tag{J1}$$

$$\dot{X}_{(2)}\left(\boldsymbol{x}_1\right) = +\frac{8}{\left(d^1_\sigma\right)^4}\, a_{(1)} + \frac{4}{\left(d^1_\sigma\right)^2}\, b_{(1)} + \frac{17}{4}\,\frac{c_{(2)}}{\left(d^1_\sigma\right)^2} - \frac{c_{(4)}}{2} - \frac{15}{4}\,\frac{d_{(2)}}{\left(d^1_\sigma\right)^3}\,, \tag{J2}$$

and

$$X_{(1)}\left(\boldsymbol{x}_1\right) = +\frac{4}{\left(d^1_\sigma\right)^2}\, a_{(1)} + \frac{c_{(2)}}{4} - \frac{15}{4}\,\frac{d_{(3)}}{d^1_\sigma}\,, \tag{J3}$$

$$X_{(2)}\left(\boldsymbol{x}_1\right) = +\frac{4}{\left(d^1_\sigma\right)^4}\, a_{(2)} + \frac{b_{(2)}}{4} - \frac{15}{4}\,\frac{d_{(4)}}{\left(d^1_\sigma\right)^2}\,. \tag{J4}$$

The scalar functions of the monopole-quadrupole term in (55) and (56) are given by

$$\dot{Y}_{(1)}\left(\boldsymbol{x}_1\right) = + \frac{4}{\left(d^1_{\sigma}\right)^4}\left(1 + c_{(1)}\right) + \frac{2}{\left(d^1_{\sigma}\right)^2}\, b_{(2)} + \frac{7}{2}\, b_{(4)} - \frac{7}{4}\left(d^1_{\sigma}\right)^2 b_{(6)} + \frac{4}{\left(d^1_{\sigma}\right)^2}\, c_{(3)}\,, \tag{J5}$$

$$\dot{Y}_{(2)}\left(\boldsymbol{x}_1\right) = - \frac{32}{\left(d^1_{\sigma}\right)^6}\, a_{(1)} - \frac{8}{\left(d^1_{\sigma}\right)^4}\, b_{(1)} + \frac{4}{\left(d^1_{\sigma}\right)^2}\, b_{(3)} - \frac{303}{32}\,\frac{c_{(2)}}{\left(d^1_{\sigma}\right)^4} - \frac{37}{16}\,\frac{c_{(4)}}{\left(d^1_{\sigma}\right)^2} + \frac{7}{4}\, c_{(6)} + \frac{465}{32}\,\frac{d_{(2)}}{\left(d^1_{\sigma}\right)^5}\,, \tag{J6}$$

$$\dot{Y}_{(3)}\left(\boldsymbol{x}_1\right) = - \frac{12}{\left(d^1_{\sigma}\right)^4}\left(1 + c_{(1)}\right) + \frac{21}{2}\, b_{(4)} - 7\left(d^1_{\sigma}\right)^2 b_{(6)} - \frac{15}{4}\left(d^1_{\sigma}\right)^4 b_{(8)} - \frac{6}{\left(d^1_{\sigma}\right)^2}\, c_{(3)} + 6\, c_{(5)}\,, \tag{J7}$$

$$\dot{Y}_{(4)}\left(\boldsymbol{x}_1\right) = + \frac{32}{\left(d^1_{\sigma}\right)^6}\, a_{(1)} + \frac{8}{\left(d^1_{\sigma}\right)^4}\, b_{(1)} - \frac{8}{\left(d^1_{\sigma}\right)^2}\, b_{(3)} + 12\, b_{(5)} + \frac{303}{32}\,\frac{c_{(2)}}{\left(d^1_{\sigma}\right)^4} - \frac{27}{16}\,\frac{c_{(4)}}{\left(d^1_{\sigma}\right)^2} + \frac{81}{4}\, c_{(6)} + \frac{15}{2}\left(d^1_{\sigma}\right)^2 c_{(8)} - \frac{465}{32}\,\frac{d_{(2)}}{\left(d^1_{\sigma}\right)^5}\,, \tag{J8}$$

$$\dot{Y}_{(5)}\left(\boldsymbol{x}_1\right) = - \frac{16}{\left(d^1_{\sigma}\right)^6}\left(1 + c_{(1)}\right) + \frac{4}{\left(d^1_{\sigma}\right)^4}\, b_{(2)} - \frac{2}{\left(d^1_{\sigma}\right)^2}\, b_{(4)} + \frac{19}{2}\, b_{(6)} + \frac{15}{4}\left(d^1_{\sigma}\right)^2 b_{(8)} - \frac{4}{\left(d^1_{\sigma}\right)^4}\, c_{(3)} - \frac{6}{\left(d^1_{\sigma}\right)^2}\, c_{(5)}\,, \tag{J9}$$

$$\dot{Y}_{(6)}\left(\boldsymbol{x}_1\right) = + \frac{96}{\left(d^1_{\sigma}\right)^8}\, a_{(1)} - \frac{8}{\left(d^1_{\sigma}\right)^4}\, b_{(3)} + \frac{6}{\left(d^1_{\sigma}\right)^2}\, b_{(5)} + \frac{747}{64}\,\frac{c_{(2)}}{\left(d^1_{\sigma}\right)^6} + \frac{121}{32}\,\frac{c_{(4)}}{\left(d^1_{\sigma}\right)^4} + \frac{101}{8}\,\frac{c_{(6)}}{\left(d^1_{\sigma}\right)^2} - \frac{15}{4}\, c_{(8)} - \frac{2325}{64}\,\frac{d_{(2)}}{\left(d^1_{\sigma}\right)^7}\,, \tag{J10}$$

$$\dot{Y}_{(7)}\left(\boldsymbol{x}_1\right) = + \frac{32}{\left(d^1_{\sigma}\right)^6}\, a_{(1)} - \frac{4}{\left(d^1_{\sigma}\right)^4}\, b_{(1)} + \frac{10}{\left(d^1_{\sigma}\right)^2}\, b_{(3)} - 6\, b_{(5)} - \frac{87}{64}\,\frac{c_{(2)}}{\left(d^1_{\sigma}\right)^4} + \frac{355}{32}\,\frac{c_{(4)}}{\left(d^1_{\sigma}\right)^2} - \frac{121}{8}\, c_{(6)} + \frac{15}{4}\left(d^1_{\sigma}\right)^2 c_{(8)} - \frac{855}{64}\,\frac{d_{(2)}}{\left(d^1_{\sigma}\right)^5}\,, \tag{J11}$$

$$\dot{Y}_{(8)}\left(\boldsymbol{x}_1\right) = - \frac{16}{\left(d^1_{\sigma}\right)^6}\left(1 + c_{(1)}\right) - \frac{4}{\left(d^1_{\sigma}\right)^4}\, b_{(2)} + \frac{20}{\left(d^1_{\sigma}\right)^2}\, b_{(4)} - \frac{63}{2}\, b_{(6)} + \frac{15}{2}\left(d^1_{\sigma}\right)^2 b_{(8)} - \frac{12}{\left(d^1_{\sigma}\right)^4}\, c_{(3)} + \frac{12}{\left(d^1_{\sigma}\right)^2}\, c_{(5)}\,, \tag{J12}$$

and

$$Y_{(1)}\left(\boldsymbol{x}_1\right) = +12\,\frac{a_{(1)}}{\left(d^1_{\sigma}\right)^4} - 4\,\frac{b_{(1)}}{\left(d^1_{\sigma}\right)^2} - \frac{93}{32}\,\frac{c_{(2)}}{\left(d^1_{\sigma}\right)^2} - \frac{7}{16}\, c_{(4)} - \frac{285}{32}\,\frac{d_{(3)}}{\left(d^1_{\sigma}\right)^3}\,, \tag{J13}$$

$$Y_{(2)}\left(\boldsymbol{x}_1\right) = -16\,\frac{a_{(2)}}{\left(d^1_{\sigma}\right)^6} - \frac{91}{32}\,\frac{b_{(2)}}{\left(d^1_{\sigma}\right)^2} - \frac{7}{16}\, b_{(4)} + 4\,\frac{c_{(1)}}{\left(d^1_{\sigma}\right)^4} + \frac{465}{32}\,\frac{d_{(4)}}{\left(d^1_{\sigma}\right)^4}\,, \tag{J14}$$

$$Y_{(3)}\left(\boldsymbol{x}_1\right) = -8\,\frac{a_{(1)}}{\left(d^1_{\sigma}\right)^4} + 2\,\frac{b_{(1)}}{\left(d^1_{\sigma}\right)^2} + 2\, b_{(3)} + \frac{29}{64}\,\frac{c_{(2)}}{\left(d^1_{\sigma}\right)^2} + \frac{111}{32}\, c_{(4)} - \frac{5}{8}\left(d^1_{\sigma}\right)^2 c_{(6)} + \frac{285}{64}\,\frac{d_{(3)}}{\left(d^1_{\sigma}\right)^3}\,, \tag{J15}$$

$$Y_{(4)}\left(\boldsymbol{x}_1\right) = +16\,\frac{a_{(2)}}{\left(d^1_{\sigma}\right)^6} + \frac{27}{32}\,\frac{b_{(2)}}{\left(d^1_{\sigma}\right)^2} + \frac{111}{16}\, b_{(4)} - \frac{5}{4}\left(d^1_{\sigma}\right)^2 b_{(6)} - 8\,\frac{c_{(1)}}{\left(d^1_{\sigma}\right)^4} - 4\,\frac{c_{(3)}}{\left(d^1_{\sigma}\right)^2} - \frac{465}{32}\,\frac{d_{(4)}}{\left(d^1_{\sigma}\right)^4}\,, \tag{J16}$$

$$Y_{(5)}\left(\boldsymbol{x}_1\right) = +8\,\frac{a_{(1)}}{\left(d^1_{\sigma}\right)^6} - 4\,\frac{b_{(1)}}{\left(d^1_{\sigma}\right)^4} - 2\,\frac{b_{(3)}}{\left(d^1_{\sigma}\right)^2} - \frac{209}{64}\,\frac{c_{(2)}}{\left(d^1_{\sigma}\right)^4} - \frac{91}{32}\,\frac{c_{(4)}}{\left(d^1_{\sigma}\right)^2} + \frac{5}{8}\, c_{(6)} - \frac{465}{64}\,\frac{d_{(3)}}{\left(d^1_{\sigma}\right)^5}\,, \tag{J17}$$

$$Y_{(6)}\left(\boldsymbol{x}_1\right) = +48\,\frac{a_{(2)}}{\left(d^1_{\sigma}\right)^8} + \frac{263}{64}\,\frac{b_{(2)}}{\left(d^1_{\sigma}\right)^4} + \frac{91}{32}\,\frac{b_{(4)}}{\left(d^1_{\sigma}\right)^2} + \frac{5}{8}\, b_{(6)} - 16\,\frac{c_{(1)}}{\left(d^1_{\sigma}\right)^6} - 4\,\frac{c_{(3)}}{\left(d^1_{\sigma}\right)^4} - \frac{2325}{64}\,\frac{d_{(4)}}{\left(d^1_{\sigma}\right)^6}\,, \tag{J18}$$

$$Y_{(7)}\left(\boldsymbol{x}_1\right) = +16\,\frac{a_{(2)}}{\left(d^1_{\sigma}\right)^6} + \frac{285}{64}\,\frac{b_{(2)}}{\left(d^1_{\sigma}\right)^2} - \frac{71}{32}\, b_{(4)} - \frac{5}{8}\left(d^1_{\sigma}\right)^2 b_{(6)} + 4\,\frac{c_{(3)}}{\left(d^1_{\sigma}\right)^2} - \frac{855}{64}\,\frac{d_{(4)}}{\left(d^1_{\sigma}\right)^4}\,, \tag{J19}$$

$$Y_{(8)}\left(\boldsymbol{x}_1\right) = -32\,\frac{a_{(1)}}{\left(d^1_{\sigma}\right)^6} + 12\,\frac{b_{(1)}}{\left(d^1_{\sigma}\right)^4} + 8\,\frac{b_{(3)}}{\left(d^1_{\sigma}\right)^2} + \frac{81}{32}\,\frac{c_{(2)}}{\left(d^1_{\sigma}\right)^4} + \frac{91}{16}\,\frac{c_{(4)}}{\left(d^1_{\sigma}\right)^2} + \frac{5}{4}\, c_{(6)} + \frac{465}{32}\,\frac{d_{(3)}}{\left(d^1_{\sigma}\right)^5}\,. \tag{J20}$$

The scalar functions of the quadrupole-quadrupole term in (55) and (56) are given by

$$\dot{Z}_{(1)}\left(\boldsymbol{x}_1\right) = +\frac{8}{\left(d_\sigma^1\right)^6}\left(1+c_{(1)}\right) - \frac{4}{\left(d_\sigma^1\right)^4}\,b_{(2)} - \frac{b_{(4)}}{\left(d_\sigma^1\right)^2} - \frac{3}{2}\,b_{(6)} + \frac{13}{8}\left(d_\sigma^1\right)^2 b_{(8)}\,, \tag{J21}$$

$$\dot{Z}_{(2)}\left(\boldsymbol{x}_1\right) = -\frac{32}{\left(d_\sigma^1\right)^8}\,a_{(1)} + \frac{16}{\left(d_\sigma^1\right)^6}\,b_{(1)} + \frac{985}{128}\,\frac{c_{(2)}}{\left(d_\sigma^1\right)^6} + \frac{217}{192}\,\frac{c_{(4)}}{\left(d_\sigma^1\right)^4} + \frac{5}{48}\,\frac{c_{(6)}}{\left(d_\sigma^1\right)^2} - \frac{13}{8}\,c_{(8)} + \frac{985}{128}\,\frac{d_{(2)}}{\left(d_\sigma^1\right)^7}\,, \tag{J22}$$

$$\dot{Z}_{(3)}\left(\boldsymbol{x}_1\right) = -\frac{4}{\left(d_\sigma^1\right)^6}\left(1+c_{(1)}\right) + \frac{6}{\left(d_\sigma^1\right)^4}\,b_{(2)} - \frac{3}{2}\,\frac{b_{(4)}}{\left(d_\sigma^1\right)^2} + \frac{37}{4}\,b_{(6)} - \frac{189}{16}\left(d_\sigma^1\right)^2 b_{(8)} + \frac{9}{4}\left(d_\sigma^1\right)^4 b_{(10)} + \frac{4}{\left(d_\sigma^1\right)^4}\,c_{(3)}\,, \tag{J23}$$

$$\dot{Z}_{(4)}\left(\boldsymbol{x}_1\right) = -\frac{32}{\left(d_\sigma^1\right)^8}\,a_{(1)} + \frac{16}{\left(d_\sigma^1\right)^6}\,b_{(1)} - \frac{28}{\left(d_\sigma^1\right)^4}\,b_{(3)} + \frac{24}{\left(d_\sigma^1\right)^2}\,b_{(5)} + \frac{5515}{512}\,\frac{c_{(2)}}{\left(d_\sigma^1\right)^6} - \frac{19061}{768}\,\frac{c_{(4)}}{\left(d_\sigma^1\right)^4} + \frac{2255}{192}\,\frac{c_{(6)}}{\left(d_\sigma^1\right)^2}$$
$$+\frac{329}{32}\,c_{(8)} - \frac{9}{4}\left(d_\sigma^1\right)^2 c_{(10)} + \frac{5515}{512}\,\frac{d_{(2)}}{\left(d_\sigma^1\right)^7}\,, \tag{J24}$$

$$\dot{Z}_{(5)}\left(\boldsymbol{x}_1\right) = +\frac{32}{\left(d_\sigma^1\right)^8}\,a_{(1)} - \frac{16}{\left(d_\sigma^1\right)^6}\,b_{(1)} - \frac{2285}{256}\,\frac{c_{(2)}}{\left(d_\sigma^1\right)^6} - \frac{749}{384}\,\frac{c_{(4)}}{\left(d_\sigma^1\right)^4} - \frac{73}{96}\,\frac{c_{(6)}}{\left(d_\sigma^1\right)^2} + \frac{305}{16}\,c_{(8)} - \frac{9}{2}\left(d_\sigma^1\right)^2 c_{(10)}$$
$$-\frac{2285}{256}\,\frac{d_{(2)}}{\left(d_\sigma^1\right)^7}\,, \tag{J25}$$

$$\dot{Z}_{(6)}\left(\boldsymbol{x}_1\right) = +\frac{16}{\left(d_\sigma^1\right)^6}\,b_{(2)} - \frac{56}{\left(d_\sigma^1\right)^4}\,b_{(4)} + \frac{22}{\left(d_\sigma^1\right)^2}\,b_{(6)} + \frac{41}{2}\,b_{(8)} - \frac{9}{2}\left(d_\sigma^1\right)^2 b_{(10)} + \frac{16}{\left(d_\sigma^1\right)^6}\,c_{(3)} - \frac{48}{\left(d_\sigma^1\right)^4}\,c_{(5)}\,, \tag{J26}$$

$$\dot{Z}_{(7)}\left(\boldsymbol{x}_1\right) = +\frac{8}{\left(d_\sigma^1\right)^6}\,b_{(2)} - \frac{4}{\left(d_\sigma^1\right)^4}\,b_{(4)} - \frac{1}{\left(d_\sigma^1\right)^2}\,b_{(6)} + \frac{19}{2}\,b_{(8)} - \frac{9}{4}\left(d_\sigma^1\right)^2 b_{(10)} + \frac{8}{\left(d_\sigma^1\right)^6}\,c_{(3)}\,, \tag{J27}$$

$$\dot{Z}_{(8)}\left(\boldsymbol{x}_1\right) = +\frac{16}{\left(d_\sigma^1\right)^6}\,b_{(3)} - \frac{24}{\left(d_\sigma^1\right)^4}\,b_{(5)} - \frac{2205}{512}\,\frac{c_{(2)}}{\left(d_\sigma^1\right)^8} + \frac{3361}{256}\,\frac{c_{(4)}}{\left(d_\sigma^1\right)^6} - \frac{1171}{64}\,\frac{c_{(6)}}{\left(d_\sigma^1\right)^4} - \frac{255}{32}\,\frac{c_{(8)}}{\left(d_\sigma^1\right)^2} + \frac{9}{4}\,c_{(10)}$$
$$-\frac{2205}{512}\,\frac{d_{(2)}}{\left(d_\sigma^1\right)^9}\,, \tag{J28}$$

$$\dot{Z}_{(9)}\left(\boldsymbol{x}_1\right) = -\frac{b_{(6)}}{2} + \frac{5}{8}\left(d_\sigma^1\right)^2 b_{(8)}\,, \tag{J29}$$

$$\dot{Z}_{(10)}\left(\boldsymbol{x}_1\right) = -\frac{8}{\left(d_\sigma^1\right)^6}\left(1+c_{(1)}\right) + \frac{4}{\left(d_\sigma^1\right)^4}\,b_{(2)} + \frac{b_{(4)}}{\left(d_\sigma^1\right)^2} + \frac{5}{2}\,b_{(6)} - \frac{95}{8}\left(d_\sigma^1\right)^2 b_{(8)} + \frac{15}{2}\left(d_\sigma^1\right)^4 b_{(10)}\,, \tag{J30}$$

$$\dot{Z}_{(11)}\left(\boldsymbol{x}_1\right) = +\frac{32}{\left(d_\sigma^1\right)^8}\,a_{(1)} - \frac{16}{\left(d_\sigma^1\right)^6}\,b_{(1)} - \frac{8}{\left(d_\sigma^1\right)^4}\,b_{(3)} - \frac{985}{128}\,\frac{c_{(2)}}{\left(d_\sigma^1\right)^6} + \frac{1319}{192}\,\frac{c_{(4)}}{\left(d_\sigma^1\right)^4} + \frac{187}{48}\,\frac{c_{(6)}}{\left(d_\sigma^1\right)^2} + \frac{85}{8}\,c_{(8)}$$
$$-15\left(d_\sigma^1\right)^2 c_{(10)} - \frac{985}{128}\,\frac{d_{(2)}}{\left(d_\sigma^1\right)^7}\,, \tag{J31}$$

$$\dot{Z}_{(12)}\left(\boldsymbol{x}_1\right) = -\frac{16}{\left(d_\sigma^1\right)^8}\left(1+c_{(1)}\right) + \frac{6}{\left(d_\sigma^1\right)^4}\,b_{(4)} + \frac{2}{\left(d_\sigma^1\right)^2}\,b_{(6)} + \frac{25}{4}\,b_{(8)} - \frac{15}{2}\left(d_\sigma^1\right)^2 b_{(10)} - \frac{8}{\left(d_\sigma^1\right)^6}\,c_{(3)}\,, \tag{J32}$$

$$\dot{Z}_{(13)}\left(\boldsymbol{x}_1\right) = -\frac{6}{\left(d_\sigma^1\right)^4}\,b_{(2)} - \frac{9}{\left(d_\sigma^1\right)^2}\,b_{(4)} + \frac{33}{2}\,b_{(6)} + \frac{39}{16}\left(d_\sigma^1\right)^2 b_{(8)} - \frac{27}{2}\left(d_\sigma^1\right)^4 b_{(10)} + \frac{75}{8}\left(d_\sigma^1\right)^6 b_{(12)}$$
$$-\frac{6}{\left(d_\sigma^1\right)^4}\,c_{(3)} + \frac{6}{\left(d_\sigma^1\right)^2}\,c_{(5)}\,, \tag{J33}$$

$$\dot{Z}_{(14)}\left(\boldsymbol{x}_1\right) = +\frac{16}{\left(d_\sigma^1\right)^4}\,b_{(3)} - \frac{12}{\left(d_\sigma^1\right)^2}\,b_{(5)} - \frac{945}{512}\,\frac{c_{(2)}}{\left(d_\sigma^1\right)^6} + \frac{3781}{256}\,\frac{c_{(4)}}{\left(d_\sigma^1\right)^4} - \frac{319}{64}\,\frac{c_{(6)}}{\left(d_\sigma^1\right)^2} - \frac{411}{32}\,c_{(8)} + \frac{81}{4}\left(d_\sigma^1\right)^2 c_{(10)}$$
$$-\frac{75}{2}\left(d_\sigma^1\right)^4 c_{(12)} - \frac{945}{512}\,\frac{d_{(2)}}{\left(d_\sigma^1\right)^7}\,, \tag{J34}$$

$$\dot{Z}_{(15)}\left(\boldsymbol{x}_1\right) = -\frac{16}{\left(d_\sigma^1\right)^8}\left(1+c_{(1)}\right) - \frac{8}{\left(d_\sigma^1\right)^6}\,b_{(2)} - \frac{16}{\left(d_\sigma^1\right)^4}\,b_{(4)} - 4\,b_{(8)} + 12\left(d_\sigma^1\right)^2 b_{(10)} - \frac{75}{4}\left(d_\sigma^1\right)^4 b_{(12)} - \frac{16}{\left(d_\sigma^1\right)^6}\,c_{(3)}$$
$$+\frac{6}{\left(d_\sigma^1\right)^4}\,c_{(5)}\,, \tag{J35}$$

$$\dot{Z}_{(16)}\left(\boldsymbol{x}_1\right) = +\frac{16}{\left(d_\sigma^1\right)^8}\left(1+c_{(1)}\right) - \frac{16}{\left(d_\sigma^1\right)^6}\,b_{(2)} + \frac{50}{\left(d_\sigma^1\right)^4}\,b_{(4)} - \frac{24}{\left(d_\sigma^1\right)^2}\,b_{(6)} - \frac{35}{4}\,b_{(8)} + 24\left(d_\sigma^1\right)^2 b_{(10)} - \frac{75}{2}\left(d_\sigma^1\right)^4 b_{(12)}$$
$$-\frac{8}{\left(d_\sigma^1\right)^6}\,c_{(3)} + \frac{48}{\left(d_\sigma^1\right)^4}\,c_{(5)}\,, \tag{J36}$$

$$\dot{Z}_{(17)}\left(\boldsymbol{x}_1\right) = -\frac{32}{\left(d_\sigma^1\right)^6} b_{(3)} + \frac{48}{\left(d_\sigma^1\right)^4} b_{(5)} + \frac{2205}{512} \frac{c_{(2)}}{\left(d_\sigma^1\right)^8} - \frac{7457}{256} \frac{c_{(4)}}{\left(d_\sigma^1\right)^6} + \frac{2195}{64} \frac{c_{(6)}}{\left(d_\sigma^1\right)^4} + \frac{447}{32} \frac{c_{(8)}}{\left(d_\sigma^1\right)^2} + \frac{39}{4} c_{(10)} + \frac{75}{2} \left(d_\sigma^1\right)^2 c_{(12)} + \frac{2205}{512} \frac{d_{(2)}}{\left(d_\sigma^1\right)^9}, \tag{J37}$$

$$\dot{Z}_{(18)}\left(\boldsymbol{x}_1\right) = -\frac{12}{\left(d_\sigma^1\right)^6} b_{(4)} + \frac{6}{\left(d_\sigma^1\right)^4} b_{(6)} + \frac{15}{2} \frac{b_{(8)}}{\left(d_\sigma^1\right)^2} + \frac{3}{2} b_{(10)} + \frac{75}{8} \left(d_\sigma^1\right)^2 b_{(12)} - \frac{12}{\left(d_\sigma^1\right)^6} c_{(5)}, \tag{J38}$$

$$\dot{Z}_{(19)}\left(\boldsymbol{x}_1\right) = +\frac{5}{128} \frac{c_{(2)}}{\left(d_\sigma^1\right)^6} + \frac{5}{192} \frac{c_{(4)}}{\left(d_\sigma^1\right)^4} + \frac{1}{48} \frac{c_{(6)}}{\left(d_\sigma^1\right)^2} - \frac{5}{8} c_{(8)} + \frac{5}{128} \frac{d_{(2)}}{\left(d_\sigma^1\right)^7}, \tag{J39}$$

$$\dot{Z}_{(20)}\left(\boldsymbol{x}_1\right) = +\frac{925}{256} \frac{c_{(2)}}{\left(d_\sigma^1\right)^6} + \frac{925}{384} \frac{c_{(4)}}{\left(d_\sigma^1\right)^4} + \frac{185}{96} \frac{c_{(6)}}{\left(d_\sigma^1\right)^2} + \frac{95}{16} c_{(8)} - \frac{15}{2} \left(d_\sigma^1\right)^2 c_{(10)} + \frac{925}{256} \frac{d_{(2)}}{\left(d_\sigma^1\right)^7}, \tag{J40}$$

$$\dot{Z}_{(21)}\left(\boldsymbol{x}_1\right) = -\frac{48}{\left(d_\sigma^1\right)^8} \left(1 + c_{(1)}\right) + \frac{16}{\left(d_\sigma^1\right)^6} b_{(2)} + \frac{10}{\left(d_\sigma^1\right)^4} b_{(4)} + \frac{4}{\left(d_\sigma^1\right)^2} b_{(6)} + \frac{55}{4} b_{(8)} - 15 \left(d_\sigma^1\right)^2 b_{(10)} - \frac{8}{\left(d_\sigma^1\right)^6} c_{(3)}, \tag{J41}$$

$$\dot{Z}_{(22)}\left(\boldsymbol{x}_1\right) = +\frac{128}{\left(d_\sigma^1\right)^{10}} a_{(1)} - \frac{64}{\left(d_\sigma^1\right)^8} b_{(1)} - \frac{32}{\left(d_\sigma^1\right)^6} b_{(3)} + \frac{24}{\left(d_\sigma^1\right)^4} b_{(5)} - \frac{6895}{256} \frac{c_{(2)}}{\left(d_\sigma^1\right)^8} - \frac{13039}{384} \frac{c_{(4)}}{\left(d_\sigma^1\right)^6} - \frac{227}{96} \frac{c_{(6)}}{\left(d_\sigma^1\right)^4} - \frac{5}{16} \frac{c_{(8)}}{\left(d_\sigma^1\right)^2} + \frac{15}{2} c_{(10)} - \frac{6895}{256} \frac{d_{(2)}}{\left(d_\sigma^1\right)^9}, \tag{J42}$$

$$\dot{Z}_{(23)}\left(\boldsymbol{x}_1\right) = +\frac{24}{\left(d_\sigma^1\right)^8} a_{(1)} - \frac{12}{\left(d_\sigma^1\right)^6} b_{(1)} + \frac{14}{\left(d_\sigma^1\right)^4} b_{(3)} - \frac{6}{\left(d_\sigma^1\right)^2} b_{(5)} - \frac{25875}{2048} \frac{c_{(2)}}{\left(d_\sigma^1\right)^6} + \frac{8783}{1024} \frac{c_{(4)}}{\left(d_\sigma^1\right)^4} - \frac{701}{256} \frac{c_{(6)}}{\left(d_\sigma^1\right)^2} - \frac{2577}{128} c_{(8)} + \frac{399}{16} \left(d_\sigma^1\right)^2 c_{(10)} - \frac{75}{8} \left(d_\sigma^1\right)^4 c_{(12)} - \frac{25875}{2048} \frac{d_{(2)}}{\left(d_\sigma^1\right)^7}, \tag{J43}$$

$$\dot{Z}_{(24)}\left(\boldsymbol{x}_1\right) = +\frac{24}{\left(d_\sigma^1\right)^8} \left(1 + c_{(1)}\right) - \frac{24}{\left(d_\sigma^1\right)^6} b_{(2)} + \frac{15}{\left(d_\sigma^1\right)^4} b_{(4)} - \frac{6}{\left(d_\sigma^1\right)^2} b_{(6)} - \frac{609}{8} b_{(8)} + \frac{207}{2} \left(d_\sigma^1\right)^2 b_{(10)} - \frac{75}{2} \left(d_\sigma^1\right)^4 b_{(12)} - \frac{12}{\left(d_\sigma^1\right)^6} c_{(3)} + \frac{12}{\left(d_\sigma^1\right)^4} c_{(5)}, \tag{J44}$$

$$\dot{Z}_{(25)}\left(\boldsymbol{x}_1\right) = +\frac{128}{\left(d_\sigma^1\right)^{10}} a_{(1)} - \frac{64}{\left(d_\sigma^1\right)^8} b_{(1)} + \frac{48}{\left(d_\sigma^1\right)^6} b_{(3)} - \frac{30}{\left(d_\sigma^1\right)^4} b_{(5)} - \frac{19405}{1024} \frac{c_{(2)}}{\left(d_\sigma^1\right)^8} + \frac{78899}{1536} \frac{c_{(4)}}{\left(d_\sigma^1\right)^6} - \frac{41}{384} \frac{c_{(6)}}{\left(d_\sigma^1\right)^4} + \frac{49}{64} \frac{c_{(8)}}{\left(d_\sigma^1\right)^2} - \frac{279}{8} c_{(10)} + \frac{75}{4} \left(d_\sigma^1\right)^2 c_{(12)} - \frac{19405}{1024} \frac{d_{(2)}}{\left(d_\sigma^1\right)^9}, \tag{J45}$$

$$\dot{Z}_{(26)}\left(\boldsymbol{x}_1\right) = -\frac{128}{\left(d_\sigma^1\right)^{10}} a_{(1)} + \frac{64}{\left(d_\sigma^1\right)^8} b_{(1)} + \frac{8}{\left(d_\sigma^1\right)^6} b_{(3)} + \frac{6395}{512} \frac{c_{(2)}}{\left(d_\sigma^1\right)^8} + \frac{251}{768} \frac{c_{(4)}}{\left(d_\sigma^1\right)^6} - \frac{1025}{192} \frac{c_{(6)}}{\left(d_\sigma^1\right)^4} + \frac{73}{32} \frac{c_{(8)}}{\left(d_\sigma^1\right)^2} - \frac{279}{4} c_{(10)} + \frac{75}{2} \left(d_\sigma^1\right)^2 c_{(12)} + \frac{6395}{512} \frac{d_{(2)}}{\left(d_\sigma^1\right)^9}, \tag{J46}$$

$$\dot{Z}_{(27)}\left(\boldsymbol{x}_1\right) = -\frac{16}{\left(d_\sigma^1\right)^{10}} \left(1 + c_{(1)}\right) - \frac{48}{\left(d_\sigma^1\right)^8} b_{(2)} + \frac{96}{\left(d_\sigma^1\right)^6} b_{(4)} - \frac{30}{\left(d_\sigma^1\right)^4} b_{(6)} - \frac{6}{\left(d_\sigma^1\right)^2} b_{(8)} - \frac{147}{2} b_{(10)} + \frac{75}{2} \left(d_\sigma^1\right)^2 b_{(12)} - \frac{56}{\left(d_\sigma^1\right)^8} c_{(3)} + \frac{72}{\left(d_\sigma^1\right)^6} c_{(5)}, \tag{J47}$$

$$\dot{Z}_{(28)}\left(\boldsymbol{x}_1\right) = -\frac{24}{\left(d_\sigma^1\right)^8} b_{(3)} + \frac{36}{\left(d_\sigma^1\right)^6} b_{(5)} + \frac{19845}{2048} \frac{c_{(2)}}{\left(d_\sigma^1\right)^{10}} - \frac{17961}{1024} \frac{c_{(4)}}{\left(d_\sigma^1\right)^8} + \frac{7467}{256} \frac{c_{(6)}}{\left(d_\sigma^1\right)^6} + \frac{1719}{128} \frac{c_{(8)}}{\left(d_\sigma^1\right)^4} + \frac{159}{16} \frac{c_{(10)}}{\left(d_\sigma^1\right)^2} - \frac{75}{8} c_{(12)} + \frac{19845}{2048} \frac{d_{(2)}}{\left(d_\sigma^1\right)^{11}}, \tag{J48}$$

and

$$Z_{(1)}\left(\boldsymbol{x}_1\right) = +8\,\frac{a_{(1)}}{\left(d_\sigma^1\right)^6} - 8\,\frac{b_{(1)}}{\left(d_\sigma^1\right)^4} - \frac{327}{128}\,\frac{c_{(2)}}{\left(d_\sigma^1\right)^4} - \frac{7}{192}\,\frac{c_{(4)}}{\left(d_\sigma^1\right)^2} + \frac{13}{48}\,c_{(6)} + \frac{185}{128}\,\frac{d_{(3)}}{\left(d_\sigma^1\right)^5}\,, \tag{J49}$$

$$Z_{(2)}\left(\boldsymbol{x}_1\right) = -16\,\frac{a_{(2)}}{\left(d_\sigma^1\right)^8} - \frac{985}{384}\,\frac{b_{(2)}}{\left(d_\sigma^1\right)^4} - \frac{5}{192}\,\frac{b_{(4)}}{\left(d_\sigma^1\right)^2} + \frac{13}{48}\,b_{(6)} + 8\,\frac{c_{(1)}}{\left(d_\sigma^1\right)^6} + \frac{985}{128}\,\frac{d_{(4)}}{\left(d_\sigma^1\right)^6}\,, \tag{J50}$$

$$Z_{(3)}\left(\boldsymbol{x}_1\right) = +4\,\frac{a_{(1)}}{\left(d_\sigma^1\right)^6} + 4\,\frac{b_{(1)}}{\left(d_\sigma^1\right)^4} - \frac{2103}{512}\,\frac{c_{(2)}}{\left(d_\sigma^1\right)^4} + \frac{451}{256}\,\frac{c_{(4)}}{\left(d_\sigma^1\right)^2} + \frac{23}{64}\,c_{(6)} + \frac{9}{32}\left(d_\sigma^1\right)^2 c_{(8)} - \frac{5175}{512}\,\frac{d_{(3)}}{\left(d_\sigma^1\right)^5}\,, \tag{J51}$$

$$\begin{aligned} Z_{(4)}\left(\boldsymbol{x}_1\right) = &-16\,\frac{a_{(2)}}{\left(d_\sigma^1\right)^8} - \frac{27019}{1536}\,\frac{b_{(2)}}{\left(d_\sigma^1\right)^4} + \frac{1585}{768}\,\frac{b_{(4)}}{\left(d_\sigma^1\right)^2} + \frac{55}{192}\,b_{(6)} + \frac{9}{32}\left(d_\sigma^1\right)^2 b_{(8)} + 20\,\frac{c_{(1)}}{\left(d_\sigma^1\right)^6} - 8\,\frac{c_{(3)}}{\left(d_\sigma^1\right)^4} \\ &+ \frac{5515}{512}\,\frac{d_{(4)}}{\left(d_\sigma^1\right)^6}\,, \end{aligned} \tag{J52}$$

$$Z_{(5)}\left(\boldsymbol{x}_1\right) = +16\,\frac{a_{(2)}}{\left(d_\sigma^1\right)^8} - \frac{3859}{768}\,\frac{b_{(2)}}{\left(d_\sigma^1\right)^4} + \frac{1609}{384}\,\frac{b_{(4)}}{\left(d_\sigma^1\right)^2} + \frac{79}{96}\,b_{(6)} + \frac{9}{16}\left(d_\sigma^1\right)^2 b_{(8)} - 8\,\frac{c_{(1)}}{\left(d_\sigma^1\right)^6} - \frac{2285}{256}\,\frac{d_{(4)}}{\left(d_\sigma^1\right)^6}\,, \tag{J53}$$

$$\begin{aligned} Z_{(6)}\left(\boldsymbol{x}_1\right) = &-16\,\frac{a_{(1)}}{\left(d_\sigma^1\right)^8} + 24\,\frac{b_{(1)}}{\left(d_\sigma^1\right)^6} - 16\,\frac{b_{(3)}}{\left(d_\sigma^1\right)^4} + \frac{6381}{256}\,\frac{c_{(2)}}{\left(d_\sigma^1\right)^6} - \frac{2323}{384}\,\frac{c_{(4)}}{\left(d_\sigma^1\right)^4} - \frac{119}{96}\,\frac{c_{(6)}}{\left(d_\sigma^1\right)^2} - \frac{9}{16}\,c_{(8)} \\ &+ \frac{2285}{256}\,\frac{d_{(3)}}{\left(d_\sigma^1\right)^7}\,, \end{aligned} \tag{J54}$$

$$Z_{(7)}\left(\boldsymbol{x}_1\right) = +16\,\frac{a_{(1)}}{\left(d_\sigma^1\right)^8} - \frac{1419}{512}\,\frac{c_{(2)}}{\left(d_\sigma^1\right)^6} - \frac{2443}{768}\,\frac{c_{(4)}}{\left(d_\sigma^1\right)^4} - \frac{143}{192}\,\frac{c_{(6)}}{\left(d_\sigma^1\right)^2} - \frac{9}{32}\,c_{(8)} - \frac{5515}{512}\,\frac{d_{(3)}}{\left(d_\sigma^1\right)^7}\,, \tag{J55}$$

$$Z_{(8)}\left(\boldsymbol{x}_1\right) = +\frac{4831}{512}\,\frac{b_{(2)}}{\left(d_\sigma^1\right)^6} - \frac{877}{256}\,\frac{b_{(4)}}{\left(d_\sigma^1\right)^4} - \frac{43}{64}\,\frac{b_{(6)}}{\left(d_\sigma^1\right)^2} - \frac{9}{32}\,b_{(8)} + 8\,\frac{c_{(3)}}{\left(d_\sigma^1\right)^6} - \frac{2205}{512}\,\frac{d_{(4)}}{\left(d_\sigma^1\right)^8}\,, \tag{J56}$$

$$Z_{(9)}\left(\boldsymbol{x}_1\right) = +\frac{1}{128}\,\frac{c_{(2)}}{\left(d_\sigma^1\right)^4} + \frac{1}{192}\,\frac{c_{(4)}}{\left(d_\sigma^1\right)^2} + \frac{5}{48}\,c_{(6)} + \frac{1}{128}\,\frac{d_{(3)}}{\left(d_\sigma^1\right)^5}\,, \tag{J57}$$

$$Z_{(10)}\left(\boldsymbol{x}_1\right) = -8\,\frac{a_{(1)}}{\left(d_\sigma^1\right)^6} + 8\,\frac{b_{(1)}}{\left(d_\sigma^1\right)^4} + \frac{839}{256}\,\frac{c_{(2)}}{\left(d_\sigma^1\right)^4} + \frac{199}{384}\,\frac{c_{(4)}}{\left(d_\sigma^1\right)^2} - \frac{85}{96}\,c_{(6)} + \frac{15}{16}\left(d_\sigma^1\right)^2 c_{(8)} - \frac{185}{256}\,\frac{d_{(3)}}{\left(d_\sigma^1\right)^5}\,, \tag{J58}$$

$$Z_{(11)}\left(\boldsymbol{x}_1\right) = +16\,\frac{a_{(2)}}{\left(d_\sigma^1\right)^8} + \frac{2521}{384}\,\frac{b_{(2)}}{\left(d_\sigma^1\right)^4} + \frac{197}{192}\,\frac{b_{(4)}}{\left(d_\sigma^1\right)^2} - \frac{85}{48}\,b_{(6)} + \frac{15}{8}\left(d_\sigma^1\right)^2 b_{(8)} - 8\,\frac{c_{(1)}}{\left(d_\sigma^1\right)^6} - \frac{985}{128}\,\frac{d_{(4)}}{\left(d_\sigma^1\right)^6}\,, \tag{J59}$$

$$Z_{(12)}\left(\boldsymbol{x}_1\right) = -\frac{985}{256}\,\frac{c_{(2)}}{\left(d_\sigma^1\right)^6} - \frac{217}{384}\,\frac{c_{(4)}}{\left(d_\sigma^1\right)^4} - \frac{5}{96}\,\frac{c_{(6)}}{\left(d_\sigma^1\right)^2} - \frac{15}{16}\,c_{(8)} - \frac{985}{256}\,\frac{d_{(3)}}{\left(d_\sigma^1\right)^7}\,, \tag{J60}$$

$$\begin{aligned} Z_{(13)}\left(\boldsymbol{x}_1\right) = &-4\,\frac{a_{(1)}}{\left(d_\sigma^1\right)^6} - 4\,\frac{b_{(1)}}{\left(d_\sigma^1\right)^4} + \frac{3237}{2048}\,\frac{c_{(2)}}{\left(d_\sigma^1\right)^4} - \frac{969}{1024}\,\frac{c_{(4)}}{\left(d_\sigma^1\right)^2} + \frac{395}{256}\,c_{(6)} - \frac{369}{128}\left(d_\sigma^1\right)^2 c_{(8)} + \frac{15}{16}\left(d_\sigma^1\right)^4 c_{(10)} \\ &+ \frac{15525}{2048}\,\frac{d_{(3)}}{\left(d_\sigma^1\right)^5}\,, \end{aligned} \tag{J61}$$

$$\begin{aligned} Z_{(14)}\left(\boldsymbol{x}_1\right) = &+\frac{8507}{512}\,\frac{b_{(2)}}{\left(d_\sigma^1\right)^4} - \frac{1217}{256}\,\frac{b_{(4)}}{\left(d_\sigma^1\right)^2} + \frac{393}{64}\,b_{(6)} - \frac{369}{32}\left(d_\sigma^1\right)^2 b_{(8)} + \frac{15}{4}\left(d_\sigma^1\right)^4 b_{(10)} - 12\,\frac{c_{(1)}}{\left(d_\sigma^1\right)^6} + 8\,\frac{c_{(3)}}{\left(d_\sigma^1\right)^4} \\ &- \frac{945}{512}\,\frac{d_{(4)}}{\left(d_\sigma^1\right)^6}\,, \end{aligned} \tag{J62}$$

$$Z_{(15)}\left(\boldsymbol{x}_1\right) = -16\,\frac{a_{(1)}}{\left(d_\sigma^1\right)^8} - \frac{2677}{1024}\,\frac{c_{(2)}}{\left(d_\sigma^1\right)^6} + \frac{5515}{1536}\,\frac{c_{(4)}}{\left(d_\sigma^1\right)^4} + \frac{335}{384}\,\frac{c_{(6)}}{\left(d_\sigma^1\right)^2} + \frac{249}{64}\,c_{(8)} - \frac{15}{8}\left(d_\sigma^1\right)^2 c_{(10)} + \frac{5515}{1024}\,\frac{d_{(3)}}{\left(d_\sigma^1\right)^7}\,, \tag{J63}$$

$$\begin{aligned} Z_{(16)}\left(\boldsymbol{x}_1\right) = &+16\,\frac{a_{(1)}}{\left(d_\sigma^1\right)^8} - 24\,\frac{b_{(1)}}{\left(d_\sigma^1\right)^6} + 16\,\frac{b_{(3)}}{\left(d_\sigma^1\right)^4} - \frac{10477}{512}\,\frac{c_{(2)}}{\left(d_\sigma^1\right)^6} + \frac{5395}{768}\,\frac{c_{(4)}}{\left(d_\sigma^1\right)^4} + \frac{311}{192}\,\frac{c_{(6)}}{\left(d_\sigma^1\right)^2} + \frac{249}{32}\,c_{(8)} - \frac{15}{4}\left(d_\sigma^1\right)^2 c_{(10)} \\ &- \frac{2285}{512}\,\frac{d_{(3)}}{\left(d_\sigma^1\right)^7}\,, \end{aligned} \tag{J64}$$

$$Z_{(17)}\left(\boldsymbol{x}_1\right)=-\frac{8927}{512}\,\frac{b_{(2)}}{\left(d^1_\sigma\right)^6}+\frac{1901}{256}\,\frac{b_{(4)}}{\left(d^1_\sigma\right)^4}+\frac{107}{64}\,\frac{b_{(6)}}{\left(d^1_\sigma\right)^2}+\frac{249}{32}\,b_{(8)}-\frac{15}{4}\left(d^1_\sigma\right)^2 b_{(10)}-8\,\frac{c_{(3)}}{\left(d^1_\sigma\right)^6}+\frac{2205}{512}\,\frac{d_{(4)}}{\left(d^1_\sigma\right)^8}\,, \tag{J65}$$

$$Z_{(18)}\left(\boldsymbol{x}_1\right)=+\frac{2205}{2048}\,\frac{c_{(2)}}{\left(d^1_\sigma\right)^8}-\frac{3361}{1024}\,\frac{c_{(4)}}{\left(d^1_\sigma\right)^6}-\frac{365}{256}\,\frac{c_{(6)}}{\left(d^1_\sigma\right)^4}-\frac{129}{128}\,\frac{c_{(8)}}{\left(d^1_\sigma\right)^2}+\frac{15}{16}\,c_{(10)}+\frac{2205}{2048}\,\frac{d_{(3)}}{\left(d^1_\sigma\right)^9}\,, \tag{J66}$$

$$Z_{(19)}\left(\boldsymbol{x}_1\right)=-\frac{5}{384}\,\frac{b_{(2)}}{\left(d^1_\sigma\right)^4}-\frac{1}{192}\,\frac{b_{(4)}}{\left(d^1_\sigma\right)^2}+\frac{5}{48}\,b_{(6)}+\frac{5}{128}\,\frac{d_{(4)}}{\left(d^1_\sigma\right)^6}\,, \tag{J67}$$

$$Z_{(20)}\left(\boldsymbol{x}_1\right)=-\frac{3997}{768}\,\frac{b_{(2)}}{\left(d^1_\sigma\right)^4}-\frac{569}{384}\,\frac{b_{(4)}}{\left(d^1_\sigma\right)^2}-\frac{95}{96}\,b_{(6)}+\frac{15}{16}\left(d^1_\sigma\right)^2 b_{(8)}+\frac{925}{256}\,\frac{d_{(4)}}{\left(d^1_\sigma\right)^6}\,, \tag{J68}$$

$$Z_{(21)}\left(\boldsymbol{x}_1\right)=-64\,\frac{a_{(1)}}{\left(d^1_\sigma\right)^8}+48\,\frac{b_{(1)}}{\left(d^1_\sigma\right)^6}+\frac{3033}{128}\,\frac{c_{(2)}}{\left(d^1_\sigma\right)^6}+\frac{601}{192}\,\frac{c_{(4)}}{\left(d^1_\sigma\right)^4}+\frac{5}{48}\,\frac{c_{(6)}}{\left(d^1_\sigma\right)^2}-\frac{15}{8}c_{(8)}+\frac{985}{128}\,\frac{d_{(3)}}{\left(d^1_\sigma\right)^7}\,, \tag{J69}$$

$$Z_{(22)}\left(\boldsymbol{x}_1\right)=+64\,\frac{a_{(2)}}{\left(d^1_\sigma\right)^{10}}+\frac{13039}{768}\,\frac{b_{(2)}}{\left(d^1_\sigma\right)^6}+\frac{611}{384}\,\frac{b_{(4)}}{\left(d^1_\sigma\right)^4}+\frac{5}{96}\,\frac{b_{(6)}}{\left(d^1_\sigma\right)^2}-\frac{15}{16}\,b_{(8)}-\frac{48}{\left(d^1_\sigma\right)^8}\,c_{(1)}-\frac{6895}{256}\,\frac{d_{(4)}}{\left(d^1_\sigma\right)^8}\,, \tag{J70}$$

$$\begin{aligned}Z_{(23)}\left(\boldsymbol{x}_1\right)=&+12\frac{a_{(2)}}{\left(d^1_\sigma\right)^8}+\frac{31153}{2048}\frac{b_{(2)}}{\left(d^1_\sigma\right)^4}-\frac{2371}{1024}\frac{b_{(4)}}{\left(d^1_\sigma\right)^2}-\frac{37}{256}b_{(6)}-\frac{111}{128}\left(d^1_\sigma\right)^2 b_{(8)}+\frac{15}{16}\left(d^1_\sigma\right)^4 b_{(10)}-4\frac{c_{(1)}}{\left(d^1_\sigma\right)^6}+8\frac{c_{(3)}}{\left(d^1_\sigma\right)^4}\\&-\frac{25875}{2048}\frac{d_{(4)}}{\left(d^1_\sigma\right)^6}\,,\end{aligned} \tag{J71}$$

$$\begin{aligned}Z_{(24)}\left(\boldsymbol{x}_1\right)=&+24\,\frac{a_{(1)}}{\left(d^1_\sigma\right)^8}-36\,\frac{b_{(1)}}{\left(d^1_\sigma\right)^6}+16\,\frac{b_{(3)}}{\left(d^1_\sigma\right)^4}-\frac{11343}{512}\,\frac{c_{(2)}}{\left(d^1_\sigma\right)^6}+\frac{59}{256}\,\frac{c_{(4)}}{\left(d^1_\sigma\right)^4}-\frac{65}{64}\,\frac{c_{(6)}}{\left(d^1_\sigma\right)^2}-\frac{9}{32}\,c_{(8)}-\frac{15}{4}\left(d^1_\sigma\right)^2 c_{(10)}\\&+\frac{945}{512}\,\frac{d_{(3)}}{\left(d^1_\sigma\right)^7}\,,\end{aligned} \tag{J72}$$

$$\begin{aligned}Z_{(25)}\left(\boldsymbol{x}_1\right)=&+64\frac{a_{(2)}}{\left(d^1_\sigma\right)^{10}}+\frac{93133}{3072}\frac{b_{(2)}}{\left(d^1_\sigma\right)^6}-\frac{11479}{1536}\frac{b_{(4)}}{\left(d^1_\sigma\right)^4}-\frac{433}{384}\frac{b_{(6)}}{\left(d^1_\sigma\right)^2}-\frac{9}{64}b_{(8)}-\frac{15}{8}\left(d^1_\sigma\right)^2 b_{(10)}-48\frac{c_{(1)}}{\left(d^1_\sigma\right)^8}+16\frac{c_{(3)}}{\left(d^1_\sigma\right)^6}\\&-\frac{19405}{1024}\frac{d_{(4)}}{\left(d^1_\sigma\right)^8}\,,\end{aligned} \tag{J73}$$

$$\begin{aligned}Z_{(26)}\left(\boldsymbol{x}_1\right)=&-64\,\frac{a_{(2)}}{\left(d^1_\sigma\right)^{10}}+\frac{5893}{1536}\,\frac{b_{(2)}}{\left(d^1_\sigma\right)^6}-\frac{5887}{768}\,\frac{b_{(4)}}{\left(d^1_\sigma\right)^4}-\frac{457}{192}\,\frac{b_{(6)}}{\left(d^1_\sigma\right)^2}-\frac{9}{32}\,b_{(8)}-\frac{15}{4}\left(d^1_\sigma\right)^2 b_{(10)}+48\,\frac{c_{(1)}}{\left(d^1_\sigma\right)^8}\\&+\frac{6395}{512}\,\frac{d_{(4)}}{\left(d^1_\sigma\right)^8}\,,\end{aligned} \tag{J74}$$

$$\begin{aligned}Z_{(27)}\left(\boldsymbol{x}_1\right)=&-48\,\frac{b_{(1)}}{\left(d^1_\sigma\right)^8}+\frac{48}{\left(d^1_\sigma\right)^6}\,b_{(3)}-\frac{26781}{512}\,\frac{c_{(2)}}{\left(d^1_\sigma\right)^8}+\frac{7457}{256}\,\frac{c_{(4)}}{\left(d^1_\sigma\right)^6}+\frac{493}{64}\,\frac{c_{(6)}}{\left(d^1_\sigma\right)^4}+\frac{129}{32}\,\frac{c_{(8)}}{\left(d^1_\sigma\right)^2}+\frac{15}{4}\,c_{(10)}\\&-\frac{2205}{512}\,\frac{d_{(3)}}{\left(d^1_\sigma\right)^9}\,,\end{aligned} \tag{J75}$$

$$Z_{(28)}\left(\boldsymbol{x}_1\right)=-\frac{55767}{2048}\,\frac{b_{(2)}}{\left(d^1_\sigma\right)^8}+\frac{10965}{1024}\,\frac{b_{(4)}}{\left(d^1_\sigma\right)^6}+\frac{579}{256}\,\frac{b_{(6)}}{\left(d^1_\sigma\right)^4}+\frac{129}{128}\,\frac{b_{(8)}}{\left(d^1_\sigma\right)^2}+\frac{15}{16}\,b_{(10)}-24\,\frac{c_{(3)}}{\left(d^1_\sigma\right)^8}+\frac{19845}{2048}\,\frac{d_{(4)}}{\left(d^1_\sigma\right)^{10}}\,. \tag{J76}$$

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