Beyond GREM
Relativity for higher accuracy

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1. Introduction

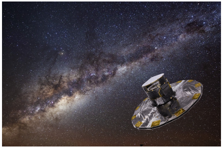
1.1 The new era of space astrometry missions by ESA

Hipparcos
Launched: 1989

milli-arcsecond
astrometry (VIS)

Gaia
launched: 2013

micro-arcseond
astrometry (VIS)



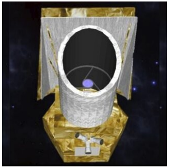
Several space astrometry missions proposed to ESA

GaiaNIR
ideal case: 2045

micro-arcsecond
astrometry (IR)

Theia
ideal case: 2045

sub-micro-arcsecond
astrometry (VIS)



1.2 Science cases of sub-micro-arcsecond astrometry

Science themes of sub-μas-astrometry are overwhelming

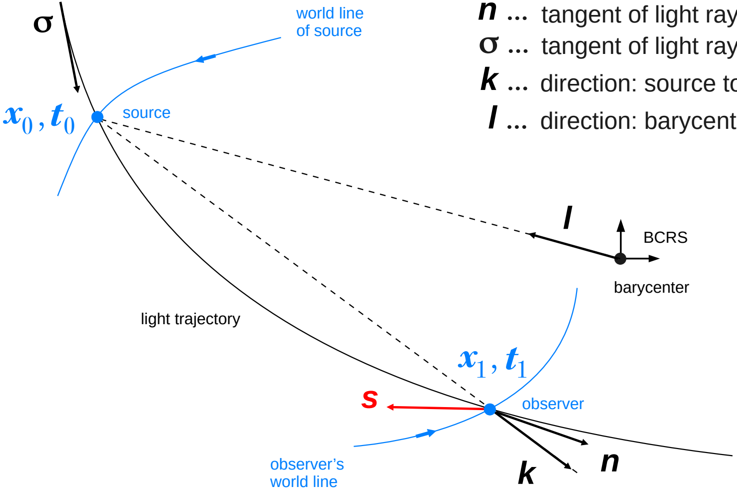
- discovery of Earth-like exoplanets
- detection of gravitational waves
- measurements of dark matter distributions
- new precise test of general relativity
- extension of model-independent distance ladder
- and many many more ...

- Comment:

  theoretical light propagation model should be 10-times more accurate than end-of-mission astrometric accuracy

1.3 The General Relativistic Model (GREM)

s ... observed direction at t_0
n ... tangent of light ray at t_0
σ ... tangent of light ray at $-\infty$
k ... direction: source to observer
l ... direction: barycenter to source



GREM: transformations among these vectors

$$\boldsymbol{s} \rightarrow \boldsymbol{n} \rightarrow \boldsymbol{\sigma} \rightarrow \boldsymbol{k} \rightarrow \boldsymbol{l}$$

- Development of GREM:

  S.A. Klioner, AJ **125** (2003) 1580.

  S.A. Klioner, PRD **69** (2004) 124001.

- Refinements of GREM:

  S.A. Klioner, S. Zschocke, CQG **27** (2010) 075015.

  S. Zschocke, S.A. Klioner, CQG **28** (2011) 015009.

Final aim of GREM: position of the source $\boldsymbol{x}_0$

Three important effects

1. Aberration described by: $\boldsymbol{s} \rightarrow \boldsymbol{n}$
   is related to the motion of observer

2. Parallax described by: $\boldsymbol{k} \rightarrow \boldsymbol{l}$
   is related to distance of observer and barycenter

3. Light deflection described by: $\boldsymbol{n} \rightarrow \boldsymbol{\sigma} \rightarrow \boldsymbol{k}$
   they are independent of an observer

Talk is focused on effect of light deflection

Light deflection described by sequence: $\boldsymbol{n} \rightarrow \boldsymbol{\sigma} \rightarrow \boldsymbol{k}$

Tangent vectors determined by light trajectory: $\dot{\boldsymbol{x}}(t)$, $\boldsymbol{x}(t)$

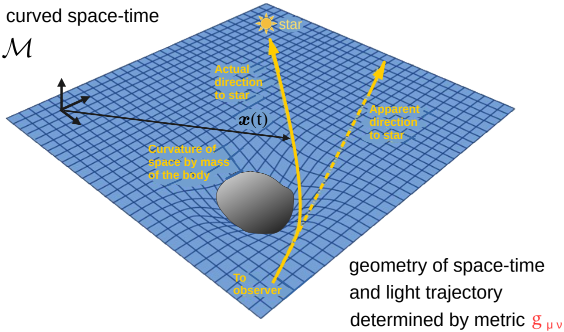
$$\boldsymbol{\sigma} = \lim_{t \rightarrow -\infty} \frac{\dot{\boldsymbol{x}}(t)}{c} \qquad (1)$$

$$\boldsymbol{k} = \frac{\boldsymbol{x}(t_1) - \boldsymbol{x}(t_0)}{|\boldsymbol{x}(t_1) - \boldsymbol{x}(t_0)|} \qquad (2)$$

$$\boldsymbol{n} = \frac{\dot{\boldsymbol{x}}(t_1)}{|\dot{\boldsymbol{x}}(t_1)|} \qquad (3)$$

2. Theory of light propagation

- space-time is curved due to mass and energy of matter

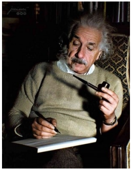


- astrometry needs to determine light trajectories $\boldsymbol{x}(t)$

- light trajectories are governed by geodesic equation

$$\frac{\ddot{x}^i}{c^2} + \Gamma^i_{\mu\nu}\frac{\dot{x}^\mu}{c}\frac{\dot{x}^\nu}{c} - \Gamma^0_{\mu\nu}\frac{\dot{x}^\mu}{c}\frac{\dot{x}^\nu}{c}\frac{\dot{x}^i}{c} = 0 \quad \Longrightarrow \quad \boldsymbol{x}(t) \qquad (4)$$

- historically: geodesic equation for light (Einstein, 1914)



KÖNIGLICH PREUSSISCHEN

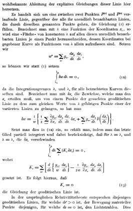
AKADEMIE DER WISSENSCHAFTEN.

Gesammtsitzung vom 19. November.

Mitth. aus der Sitzung der phys.-math. Classe vom 29. October.

Die formale Grundlage der allgemeinen Relativitätstheorie.

Von A. Einstein.



wohlbekannte Ableitung der expliziten Gleichungen dieser Linie hier hersetzen.

Es handelt sich um eine zwischen zwei Punkten $P^{(1)}$ und $P^{(2)}$ verlaufende Linie, gegenüber der alle ihr unendlich benachbarten Linien, die durch dieselben genannten Punkte gehen, die Gleichung (1) erfüllen. Bezeichnet man mit λ eine Funktion der Koordinaten x_ν, so wird eine »Fläche« von konstanten λ auf allen diesen unendlich benachbarten Linien je einen Punkt herausschneiden, dessen Koordinaten bei gegebener Kurve als Funktionen von λ allein aufzufassen sind. Setzen wir

$$w^2 = \sum_{\mu\nu} g_{\mu\nu}\frac{dx_\mu}{d\lambda}\frac{dx_\nu}{d\lambda},$$

so können wir statt (1) setzen

$$\int_{\lambda_1}^{\lambda_2} \delta w \, d\lambda = 0, \qquad (1a)$$

da die Integrationsgrenzen λ_1 und λ_2 für alle betrachteten Kurven dieselben sind. Bezeichnet man mit δx_ν die Zuwächse, welche man den x_ν erteilen muß, um von einem Punkte der gesuchten geodätischen Linie zu dem zum gleichen Werte von λ gehörigen Punkte einer der variierten Linien zu gelangen, so hat man

$$\delta w = \frac{1}{w}\left\{\frac{1}{2}\sum_{\mu\nu\sigma}\frac{\partial g_{\mu\nu}}{\partial x_\sigma}\frac{dx_\mu}{d\lambda}\frac{dx_\nu}{d\lambda}\delta x_\sigma + \sum_{\mu\nu} g_{\mu\nu}\frac{dx_\mu}{d\lambda}\delta\left(\frac{dx_\nu}{d\lambda}\right)\right\}.$$

Setzt man dies in (1a) ein, so erhält man, indem man das letzte Glied partiell integriert und dabei berücksichtigt, daß für $\lambda = \lambda_1$ und $\lambda = \lambda_2$ die δx_ν verschwinden

$$\int_{\lambda_1}^{\lambda_2} d\lambda \sum_\sigma (K_\sigma \delta x_\sigma) = 0,$$

wobei

$$K_\sigma = \sum_{\mu\nu}\left[\frac{d}{d\lambda}\left\{\frac{g_{\mu\sigma}}{w}\frac{dx_\mu}{d\lambda}\right\} - \frac{1}{2w}\frac{\partial g_{\mu\nu}}{\partial x_\sigma}\frac{dx_\mu}{d\lambda}\frac{dx_\nu}{d\lambda}\right]$$

gesetzt ist. Es folgt hieraus, daß

$$K_\sigma = 0 \qquad (23)$$

die Gleichung der geodätischen Linie ist.

In der ursprünglichen Relativitätstheorie entsprechen diejenigen geodätischen Linien, für welche $ds^2 > 0$ ist, der Bewegung materieller Punkte diejenigen, für welche $ds = 0$ ist, den Lichtstrahlen. Dies

Albert Einstein:

Die Grundlage der allgemeinen Relativitätstheorie

Preussische Akademie der Wissenschaften **2** (1914) 1030.

- Christoffel symbols:

$$\boxed{\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2}\, g^{\alpha\beta} \left(\frac{\partial g_{\beta\mu}}{\partial x^{\nu}} + \frac{\partial g_{\beta\nu}}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\beta}} \right)} \qquad (5)$$

- metric tensor $g_{\mu\nu}$ from field equations:

$$\boxed{R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu}\, R = \kappa\, T_{\mu\nu}} \qquad (6)$$

$R_{\mu\nu}$ ... Ricci tensor

$T_{\mu\nu}$ ... energy-momentum tensor of matter

$(\sqrt{-g}\, g^{\mu\nu})_{,\,\nu} = 0$ ... harmonic coordinates

Post-Newtonian (PN) approximation scheme

weak-field slow-motion approximation

Post-Newtonian expansion of metric tensor in 1.5PN

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}^{1\mathrm{PN}} + h_{\mu\nu}^{1.5\mathrm{PN}} + \mathcal{O}\left(c^{-4}\right) \qquad (7)$$

First integration of geodesic equation in 1.5PN (velocity)

$$\frac{\dot{\boldsymbol{x}}\left(t\right)}{c} = \boldsymbol{\sigma} + \frac{\Delta \dot{\boldsymbol{x}}_{1\mathrm{PN}}}{c} + \frac{\Delta \dot{\boldsymbol{x}}_{1.5\mathrm{PN}}}{c} + \mathcal{O}\left(c^{-4}\right) \qquad (8)$$

Second integration of geodesic equation in 1.5PN (trajectory)

$$\boldsymbol{x}\left(t\right) = \boldsymbol{x}_{\mathrm{N}} + \Delta \boldsymbol{x}_{1\mathrm{PN}} + \Delta \boldsymbol{x}_{1.5\mathrm{PN}} + \mathcal{O}\left(c^{-4}\right) \qquad (9)$$

- Transformation $\boldsymbol{k} \rightarrow \boldsymbol{\sigma}$ is given by:

$$\boxed{\boldsymbol{\sigma} = \boldsymbol{k} - \frac{1}{R}\,\boldsymbol{k} \times \left(\frac{\Delta \boldsymbol{x}_{\mathrm{PN}}\left(t_1, t_0\right)}{c} \times \boldsymbol{k} \right)} \qquad (10)$$

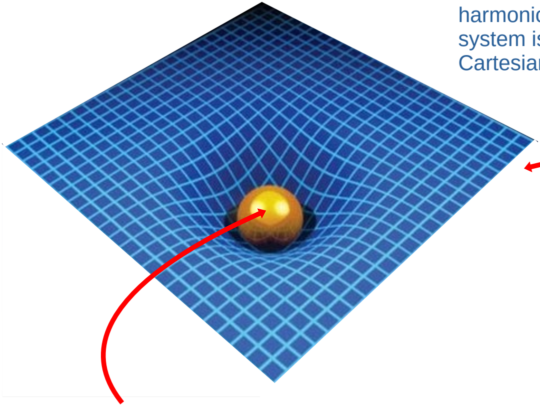
- Transformation $\boldsymbol{\sigma} \rightarrow \boldsymbol{n}$ is given by:

$$\boxed{\boldsymbol{n} = \boldsymbol{\sigma} + \boldsymbol{\sigma} \times \left(\frac{\Delta \dot{\boldsymbol{x}}_{\mathrm{PN}}\left(t_1\right)}{c} \times \boldsymbol{\sigma} \right)} \qquad (11)$$

- Transformation $\boldsymbol{k} \rightarrow \boldsymbol{n}$ is given by:

$$\boxed{\boldsymbol{n} = \boldsymbol{k} + \boldsymbol{k} \times \left(\frac{\Delta \dot{\boldsymbol{x}}_{\mathrm{PN}}}{c} \times \boldsymbol{k} \right) - \frac{1}{R}\,\boldsymbol{k} \times \left(\frac{\Delta \boldsymbol{x}_{\mathrm{PN}}}{c} \times \boldsymbol{k} \right)} \qquad (12)$$

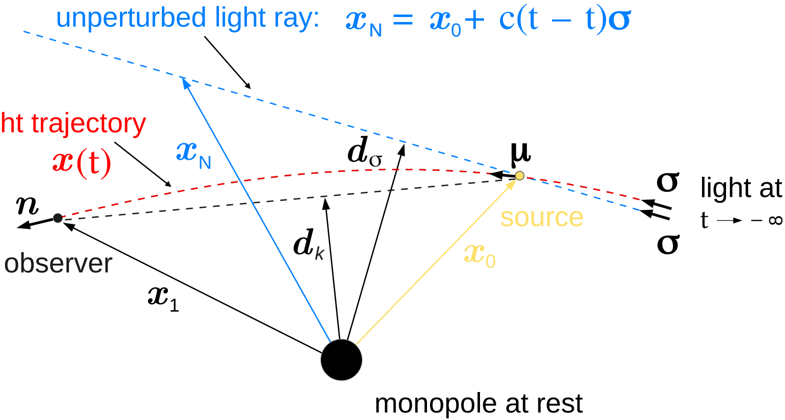
3. Light deflection in the solar system



harmonic coordinate system is asymptotically Cartesian

body's center-of-mass is at rest with respect to the harmonic coordinate system

3.1 Light propagation in field of monopole at rest



- geodesic equation (4) in 1PN for monopole

$$\ddot{\boldsymbol{x}} = -\frac{2\,G\,M\,\boldsymbol{x}}{r^3} + 4\,G\,M\,\frac{\boldsymbol{\sigma}\cdot\boldsymbol{x}}{r^3}\,\boldsymbol{\sigma} \tag{13}$$

- first integration of geodesic equation (13) in 1PN

$$\frac{\Delta\dot{\boldsymbol{x}}^{M}_{\rm 1PN}}{c} = -\frac{2GM}{c^2}\,\frac{\boldsymbol{\sigma}}{x_{\rm N}} - \frac{2GM}{c^2}\,\frac{\boldsymbol{d}_\sigma}{\left(d_\sigma\right)^2}\left(\frac{\boldsymbol{\sigma}\cdot\boldsymbol{x}_{\rm N}}{x_{\rm N}} + 1\right)$$

- second integration of geodesic equation (13) in 1PN

$$\Delta\boldsymbol{x}^{M}_{\rm 1PN} = -\frac{2GM}{c^2}\,\boldsymbol{\sigma}\,\ln\frac{x_{\rm N} + \boldsymbol{\sigma}\cdot\boldsymbol{x}_{\rm N}}{x_0 + \boldsymbol{\sigma}\cdot\boldsymbol{x}_0} - \frac{2GM}{c^2}\,\frac{\boldsymbol{d}_\sigma}{\left(d_\sigma\right)^2}\left(x_{\rm N} + \boldsymbol{\sigma}\cdot\boldsymbol{x}_{\rm N} - x_0 - \boldsymbol{\sigma}\cdot\boldsymbol{x}_0\right)$$

- vector $\boldsymbol{n}$ follows from Eq. (12):

$$\boldsymbol{n}_{1\mathrm{PN}}^{M} = \boldsymbol{k} - 2\,\frac{GM}{c^2}\,\frac{\boldsymbol{d}_k}{\left(d_k\right)^2}\,\frac{x_0\,x_1 - \boldsymbol{x}_0\cdot\boldsymbol{x}_1}{x_1\,R} \tag{14}$$

- light deflection is the angle between $\boldsymbol{n}$ and $\boldsymbol{k}$

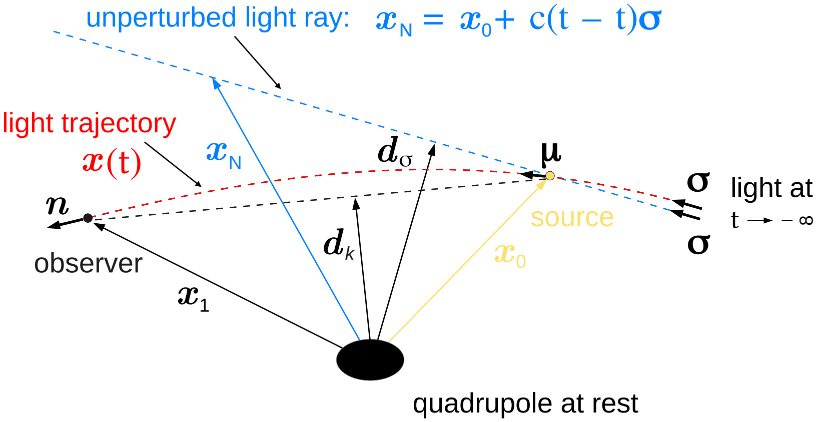
$$\delta\left(\boldsymbol{k},\boldsymbol{n}_{1\mathrm{PN}}^{M}\right) = \arcsin\left|\boldsymbol{k}\times\boldsymbol{n}_{1\mathrm{PN}}^{M}\right| \tag{15}$$

- magnitude of light deflection is criterion for GREM
- for monopole one obtains for the upper limit:

$$\left|\delta\left(\boldsymbol{k},\boldsymbol{n}_{1\mathrm{PN}}^{M}\right)\right| \le \frac{4\,GM}{c^2}\,\frac{1}{d_k} \tag{16}$$

(Sun: 1.75 arcsec predicted by Einstein in 1915)

3.2 Light propagation in field of quadrupole at rest



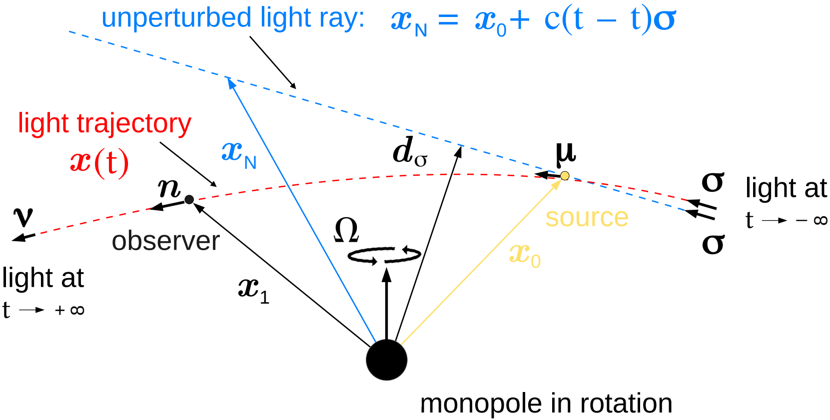
- light trajectory $\Delta\dot{\boldsymbol{x}}^{Q}_{\rm 1PN}$ and $\Delta\boldsymbol{x}^{Q}_{\rm 1PN}$ determined by
  S. Klioner, Sov. Astronomy **35** (1991) 523
  these are pretty much involved expressions
- upper limit of light deflection caused by quadrupole:
  S. Zschocke, S. Klioner: CQG **28** (2011) 015009

$$\left|\delta\left(\boldsymbol{k},\boldsymbol{n}^{Q}_{\rm 1PN}\right)\right| \le \frac{4\,GM}{c^2}\,\frac{|J_2|}{d_k}\left(\frac{P}{d_k}\right)^2 \qquad (17)$$

P ... equatorial radius of body

J_2 ... second zonal harmonic coefficient of body

3.3 Light propagation in field of spin-dipole at rest



- light trajectory $\Delta \dot{\boldsymbol{x}}^{S}_{\rm 1PN}$ and $\Delta \boldsymbol{x}^{S}_{\rm 1PN}$ determined by
  S. Klioner, Sov. Astronomy **35** (1991) 523

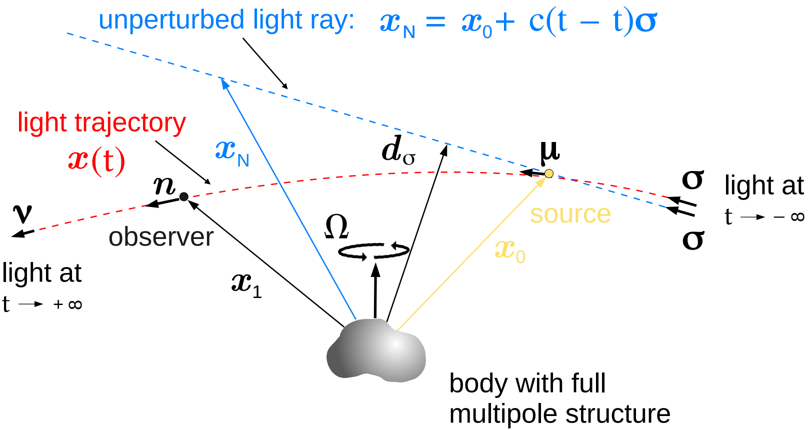
- upper limit of angle $\delta\left(\boldsymbol{k}, \boldsymbol{n}^{S}_{\rm 1.5PN}\right)$ has not been calculated
- upper limit of total light deflection caused by spin dipole
  S. Klioner, Sov. Astronomy **35** (1991) 523
  S. Kopeikin, B. Mashhoon, PRD **65** (2002) 064025
  S. Zschocke, PRD **107** (2023) 124055

$$\boxed{\left|\delta\left(\boldsymbol{\sigma}, \boldsymbol{\nu}^{S}_{\rm 1.5PN}\right)\right| \le \frac{4\,GM}{c^3}\,\Omega\,\kappa^2} \qquad (18)$$

Ω ... angular velocity of body
κ^2 ... dimensionless moment of inertia

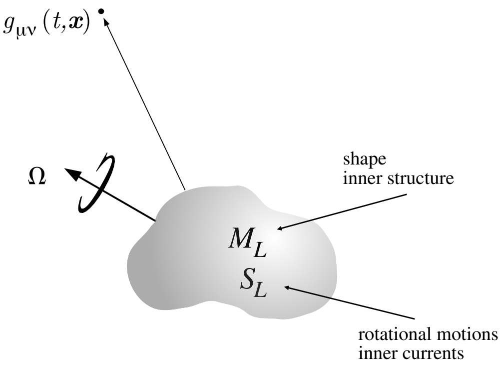
3.4 Light propagation in field of multipoles at rest

- solar system bodies can be of arbitrary shape and structure



Metric of such a body given in terms of multipoles

- $\hat{M}_L$ ... shape and inner structure
- $\hat{S}_L$ ... rotation and inner currents



Metric in 1.5PN approximation: $g_{\mu\nu} = \eta_{\mu\nu} + h^{1\mathrm{PN}}_{\mu\nu} + h^{1.5\mathrm{PN}}_{\mu\nu}$

K. Thorne, Rev.Mod.Phys. **52** (1980) 299

L. Blanchet, T. Damour, Phil.Trans.R.Soc.L. A **320** (1986) 30

T. Damour, B. Iyer, PRD **43** (1991) 3259

$$h^{1\mathrm{PN}}_{00}(t,\boldsymbol{x}) = \frac{2G}{c^2} \sum_{l=0}^{\infty} \frac{(-1)^l}{l!}\, \partial_L \frac{\hat{M}_L}{r}$$

$$h^{1.5\mathrm{PN}}_{0i}(t,\boldsymbol{x}) = \frac{4G}{c^3} \sum_{l=1}^{\infty} \frac{(-1)^l\, l}{(l+1)!}\, \epsilon_{iab}\, \partial_{aL-1} \frac{\hat{S}_{bL-1}}{r}$$

$$h^{1\mathrm{PN}}_{ij}(t,\boldsymbol{x}) = \frac{2G}{c^2}\, \delta_{ij} \sum_{l=0}^{\infty} \frac{(-1)^l}{l!}\, \partial_L \frac{\hat{M}_L}{r}$$

where $\partial_L = \dfrac{\partial}{\partial x^1} \dots \dfrac{\partial}{\partial x^l}$ and $r = |\boldsymbol{x}|$

- first integration of geodesic equation (4)

$$\frac{\dot{\boldsymbol{x}}(t)}{c} = \boldsymbol{\sigma} + \sum_{l=0}^{\infty} \frac{\Delta \dot{\boldsymbol{x}}_{1\mathrm{PN}}^{M_L}}{c} + \sum_{l=1}^{\infty} \frac{\Delta \dot{\boldsymbol{x}}_{1.5\mathrm{PN}}^{S_L}}{c} + \mathcal{O}\left(c^{-4}\right) \qquad (19)$$

- second integration of geodesic equation (4)

$$\boldsymbol{x}(t) = \boldsymbol{x}_{\mathrm{N}} + \sum_{l=0}^{\infty} \Delta \boldsymbol{x}_{1\mathrm{PN}}^{M_L} + \sum_{l=1}^{\infty} \Delta \boldsymbol{x}_{1.5\mathrm{PN}}^{S_L} + \mathcal{O}\left(c^{-4}\right) \qquad (20)$$

- important progress in the theory of light propagation:
  S. Kopeikin, J. Math. Phys. **38** (1997) 2587

Solutions $\dot{\boldsymbol{x}}(t)$, $\boldsymbol{x}(t)$ highly involved. The question arises:
Which terms in (19) and (20) relevant for goal accuracy?

Total light deflection

From Eq. (11) one obtains the total light deflection

$$\boldsymbol{\nu} = \boldsymbol{\sigma} + \boldsymbol{\sigma} \times \left(\lim_{t_1 \to +\infty} \frac{\Delta \dot{\boldsymbol{x}}_{\rm PN}\,(t_1)}{c} \times \boldsymbol{\sigma} \right) \qquad (21)$$

- allows one to find relevant terms for given goal accuracy

$$\delta\left(\boldsymbol{\sigma}, \boldsymbol{\nu}_{\rm 1PN}^{\hat{M}_L}\right) = -\frac{4\,GM}{c^2\,d_\sigma}\,\frac{(-1)^l}{(l-1)!}\,\hat{M}_L\,\hat{\partial}_L\,\ln|\boldsymbol{\xi}|$$

$$\delta\left(\boldsymbol{\sigma}, \boldsymbol{\nu}_{\rm 1.5PN}^{S_L}\right) = -\frac{8G}{c^3}\,\frac{1}{|\boldsymbol{\xi}|}\epsilon_{abc}\,\sigma^c\,\frac{(-1)^l\,l^2}{(l+1)}\hat{S}_{bL-1}\hat{\partial}_{aL-1}\ln|\boldsymbol{\xi}|$$

where $\hat{\partial}_L = {\rm STF}_{i_1 \dots i_l}\,\dfrac{\partial}{\partial \xi^1} \dots \dfrac{\partial}{\partial \xi^l}$ and $\boldsymbol{\xi} = \boldsymbol{d}_\sigma$

What is the magnitude of these terms?

- Time-Transfer Function (TTF) approach to get $\Delta\boldsymbol{x}^{M_L}_{1\mathrm{PN}}$, $\Delta\boldsymbol{x}^{S_L}_{1.5\mathrm{PN}}$
- represents another method in the theory of light propagation
- light travel time between two events: $(t_A, \boldsymbol{x}_A)$ and $(t_B, \boldsymbol{x}_B)$

$$t_A - t_B \;=\; \underbrace{\mathcal{T}\left(\boldsymbol{x}_A, \boldsymbol{x}_B\right)}_{\mathrm{time-transfer-function}} \tag{22}$$

- applied to the case of axisymmetric bodies:

  C. Le Poncine-Lafitte & P. Teyssandier, PRD **77** (2008) 044029

- total light deflection determined for axisymmetric bodies

$$\left|\delta\left(\boldsymbol{\sigma}, \boldsymbol{\nu}^{M_L}_{1\mathrm{PN}}\right)\right| \leq \frac{4\,GM}{c^2}\,\frac{|J_l|}{d_\sigma}\left(\frac{P}{d_\sigma}\right)^l \tag{23}$$

- proof for $l = 2, 3, 4$, but validity for $l \geq 4$ was only conjectured
- no spin-multipoles

- total light deflection for axisymmetric bodies in rotation
- by using the approach of S. Kopeikin:

  Total light deflection is related to Chebyshev polynomials

  S. Zschocke: Physical Review D **107** (2023) 124055

$$\left|\delta\left(\boldsymbol{\sigma},\boldsymbol{\nu}_{1\rm PN}^{M_L}\right)\right| \le \frac{4\,GM}{c^2}\,\frac{|J_l|}{d_\sigma}\left(\frac{P}{d_\sigma}\right)^l$$

$$\left|\delta\left(\boldsymbol{\sigma},\boldsymbol{\nu}_{1.5\rm PN}^{S_L}\right)\right| \le \frac{8\,GM}{c^3}\,\Omega\,\frac{l^2}{l+4}\,|J_{l-1}|\left(\frac{P}{d_\sigma}\right)^{l+1}$$

- rigorous proof for all values of $l \ge 0$ in $\delta(\boldsymbol{\sigma},\boldsymbol{\nu}_{1\rm PN}^{M_L})$
- rigorous proof for all values of $l \ge 1$ in $\delta(\boldsymbol{\sigma},\boldsymbol{\nu}_{1.5\rm PN}^{S_L})$

3.5 Numerical values of total light deflection $[\mu\mathrm{as}]$

Light deflection	Sun	Jupiter	Saturn
$\lvert\delta(\boldsymbol{\sigma},\boldsymbol{\nu}^{M_0}_{1\mathrm{PN}})\rvert$	1.75×10^6	16.3×10^3	5.8×10^3
$\lvert\delta(\boldsymbol{\sigma},\boldsymbol{\nu}^{M_2}_{1\mathrm{PN}})\rvert$	0.455	239	94
$\lvert\delta(\boldsymbol{\sigma},\boldsymbol{\nu}^{M_4}_{1\mathrm{PN}})\rvert$	0.008	9.6	5.41
$\lvert\delta(\boldsymbol{\sigma},\boldsymbol{\nu}^{M_6}_{1\mathrm{PN}})\rvert$	$-$	0.55	0.50
$\lvert\delta(\boldsymbol{\sigma},\boldsymbol{\nu}^{M_8}_{1\mathrm{PN}})\rvert$	$-$	0.04	0.06
$\lvert\delta(\boldsymbol{\sigma},\boldsymbol{\nu}^{M_{10}}_{1\mathrm{PN}})\rvert$	$-$	0.003	0.01
$\lvert\delta(\boldsymbol{\sigma},\boldsymbol{\nu}^{S_1}_{1.5\mathrm{PN}})\rvert$	0.7	0.17	0.04
$\lvert\delta(\boldsymbol{\sigma},\boldsymbol{\nu}^{S_3}_{1.5\mathrm{PN}})\rvert$	$-$	0.026	0.008

- S. Zschocke: Physical Review D **107** (2023) 124055
  J_n of Sun: I.W. Roxburgh, A&A **377** (2001) 688.

Some comments about 2PN calculations

- 2PN calculations only for monopole and quadrupole
- 2PN metric density: L. Blanchet, T. Damour (1986).
- 2PN metric: S. Zschocke, PRD **100** (2019) 084005

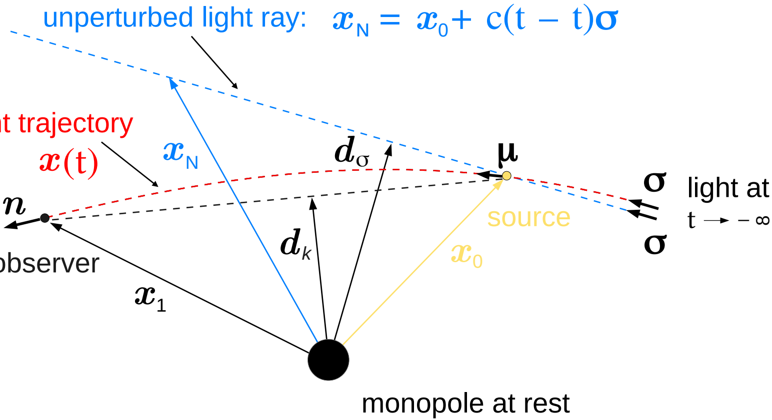
$$h_{00}^{(2\mathrm{PM})} = -\frac{2}{c^4}\left(\sum_{l=0}^{\infty}\frac{(-1)^l}{l!}\,\partial_L\,\frac{\hat{M}_L}{r}\right)^2$$

$$h_{0i}^{(2\mathrm{PM})} = 0$$

$$h_{ij}^{(2\mathrm{PM})} = +\frac{2}{c^4}\,\delta_{ij}\left(\sum_{l=0}^{\infty}\frac{(-1)^l}{l!}\,\partial_L\,\frac{\hat{M}_L}{r}\right)^2 - \frac{4}{c^4}\,\Delta^{-1}\left(\frac{\partial}{\partial x_i}\sum_{l=0}^{\infty}\frac{(-1)^l}{l!}\,\partial_L\,\frac{\hat{M}_L}{r}\right)\left(\frac{\partial}{\partial x_j}\sum_{l=0}^{\infty}\frac{(-1)^l}{l!}\,\partial_L\,\frac{\hat{M}_L}{r}\right)$$

- Hadamard regularization necessary for the last term

3.6 2PN light propagation in field of monopole at rest



- first integration of geodesic equation in 2PN

$$\frac{\dot{\boldsymbol{x}}\,(t)}{c} = \boldsymbol{\sigma} + \frac{\Delta\dot{\boldsymbol{x}}_{\rm 1PN}^{M}}{c} + \frac{\Delta\dot{\boldsymbol{x}}_{\rm 2PN}^{M\times M}}{c}$$

- second integration of geodesic equation in 2PN

$$\boldsymbol{x}\,(t) = \boldsymbol{x}_{\rm N} + \Delta\boldsymbol{x}_{\rm 1PN}^{M} + \Delta\boldsymbol{x}_{\rm 2PN}^{M\times M}$$

2PN monopole solution determined at the first time by:

V. A. Brumberg: *Post-post Newtonian propagation of light in the Schwarzschild field*, Kinematica i physika nebeshykh tel **3** (1987) 8

- 2PN monopole solution investigated in several articles
  boundary value problem, PPN, transformations, etc ...
- recalling: upper limit of 1PN terms

$$\left|\delta\left(\boldsymbol{k},\boldsymbol{n}^{M}_{\rm 1PN}\right)\right| \le \frac{4\,G\,M}{c^2}\,\frac{1}{d_k}$$

- upper limit of 2PN terms $\Longrightarrow$ discovery of enhanced terms
  $\Longrightarrow$ 2PN enhanced terms terms implemented in GREM

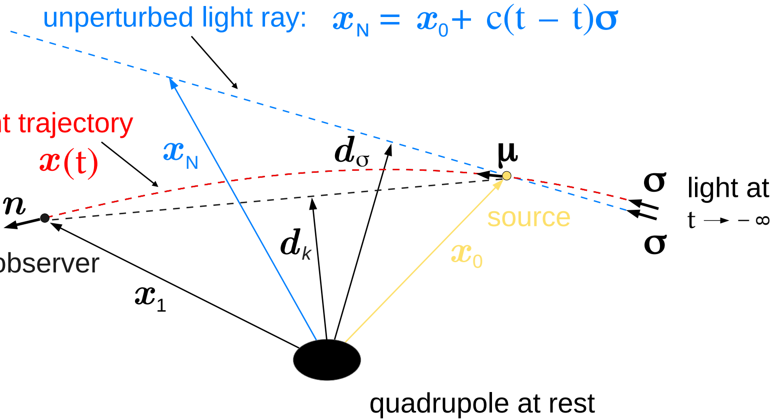
$$\left|\delta\left(\boldsymbol{k},\boldsymbol{n}^{M\times M}_{\rm 2PN}\right)\right| \le \frac{16\,G^2\,M^2}{c^4}\,\frac{1}{\left(d_k\right)^2}\,\frac{x_1}{d_k}$$

S.A. Klioner, S. Zschocke, CQG **27** (2010) 075015

N. Ashby, B. Bertotti, CQG **27** (2010) 145013

P. Teyssandier, C. Poncine-Lafitte, CQG **29** (2012) 245010

3.7 2PN light propagation in field of quadrupole at rest



- first integration of geodesic equation in 2PN

$$\frac{\dot{\boldsymbol{x}}\,(t)}{c} = \boldsymbol{\sigma} + \frac{\Delta\dot{\boldsymbol{x}}^{M}_{\rm 1PN}}{c} + \frac{\Delta\dot{\boldsymbol{x}}^{M\times M}_{\rm 2PN}}{c} + \frac{\Delta\dot{\boldsymbol{x}}^{M\times Q}_{\rm 2PN}}{c} + \frac{\Delta\dot{\boldsymbol{x}}^{Q\times Q}_{\rm 2PN}}{c}$$

- second integration of geodesic equation in 2PN

$$\boldsymbol{x}\,(t) = \boldsymbol{x}_{\rm N} + \Delta\boldsymbol{x}^{M}_{\rm 1PN} + \Delta\boldsymbol{x}^{M\times M}_{\rm 2PN} + \Delta\boldsymbol{x}^{M\times Q}_{\rm 2PN} + \Delta\boldsymbol{x}^{Q\times Q}_{\rm 2PN}$$

- first determined by:

  initial value: S. Zschocke: PRD **105** (2022) 024040

  boundary value: S. Zschocke: PRD **111** (2025) 104069

- upper limit of 2PN terms $\Longrightarrow$ discovery of enhanced terms

$$\left|\delta\left(\boldsymbol{k},\boldsymbol{n}_{\rm 2PN}^{M\times M}\right)\right| \leq \frac{16\,G^2\,M^2}{c^4}\,\frac{1}{\left(d_k\right)^2}\,\frac{x_1}{d_k}$$

$$\left|\delta\left(\boldsymbol{k},\boldsymbol{n}_{\rm 2PN}^{M\times Q}\right)\right| \leq \frac{64\,G^2\,M^2}{c^4}\,\frac{|J_2|}{\left(d_k\right)^2}\left(\frac{P}{d_k}\right)^2\frac{x_1}{d_k}$$

$$\left|\delta\left(\boldsymbol{k},\boldsymbol{n}_{\rm 2PN}^{Q\times Q}\right)\right| \leq \frac{48\,G^2\,M^2}{c^4}\,\frac{|J_2|^2}{\left(d_k\right)^2}\left(\frac{P}{d_k}\right)^4\frac{x_1}{d_k}$$

3.8 Numerical values of 2PN light deflection $[\mu\mathrm{as}]$

Object	$\left\lvert\delta\left(\boldsymbol{k},\boldsymbol{n}_{2\mathrm{PN}}^{M\times M}\right)\right\rvert$	$\left\lvert\delta\left(\boldsymbol{k},\boldsymbol{n}_{2\mathrm{PN}}^{M\times Q}\right)\right\rvert$	$\left\lvert\delta\left(\boldsymbol{k},\boldsymbol{n}_{2\mathrm{PN}}^{Q\times Q}\right)\right\rvert$
Jupiter	16.11	0.95	0.010
Saturn	4.42	0.29	0.003
Uranus	2.58	0.04	0.001
Neptune	5.83	0.08	0.002

3.9 Light propagation in 3PN approximation

- 3PN calculations are extremely rare:

- S. Zschocke: *A generalized lens equation for light deflection in weak gravitational fields*, CQG **28** (2011) 125016
- B. Linet, P. Teyssandier: *New method for determining the light travel time in static spherically symmetric spacetimes. Calculation of terms of order G^3*, CQG **30** (2013) 175008

$$\boxed{\left|\delta\left(\boldsymbol{k},\boldsymbol{n}^{M}_{\rm 3PN}\right)\right| = \frac{128\,G^3 M^3}{c^6}\,\frac{1}{\left(d_k\right)^3}\,\left(\frac{x_1}{d_k}\right)^2} \qquad (24)$$

- this result has been found independently by:

  Eq. (27) in S. Zschocke, CQG **28** (2011) 125016

  Eq. (93) in B. Linet, P. Teyssandier, CQG **30** (2013) 175008
- numerical value: $12\,\mu{\rm as}$ in case of grazing light ray at Sun

4. Summary and Conclusions

What is known thus far?

- 1PN and 1.5PN solutions for full set of multipoles
- 2PN solutions are known for monopole and quadrupole
- 3PN upper limits are known for monopole

What is necessary for GREM on sub-$\mu\mathrm{as}$ level?

- mass-multipoles up to $l = 10$
- spin-multipoles up to $l = 3$
- 2PN solution for monopole and quadrupole
- most probably 2PN solution for octupole