

# Theory of light propagation in the Solar System

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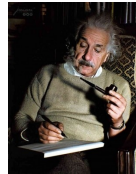
# 1. History of light deflection



Isaac Newton  
(1704)

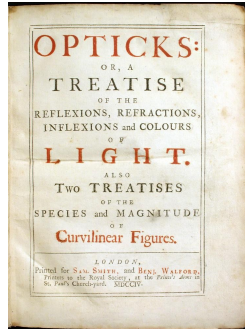
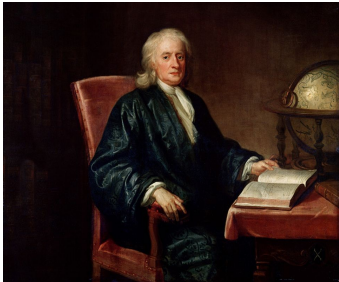


Johann Georg von Soldner  
(1801)



Albert Einstein  
(1907)

# 1.1 A comment of Isaac Newton

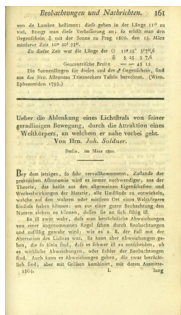
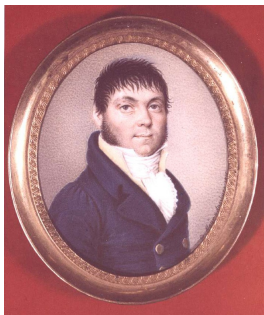


- Isaac Newton in "Opticks" (1704):

"Do not bodies act upon light at a distance, and by their action bend its rays? And is not this action strongest at the minimal distance?"

# 1.2 Does Newtonian theory predict light deflection?

- Johann G. von Soldner (1801): light deflection in flat space



birthplace  
Georgenhof  
(in Bavaria)



Monument in Munich



Street in Fürth

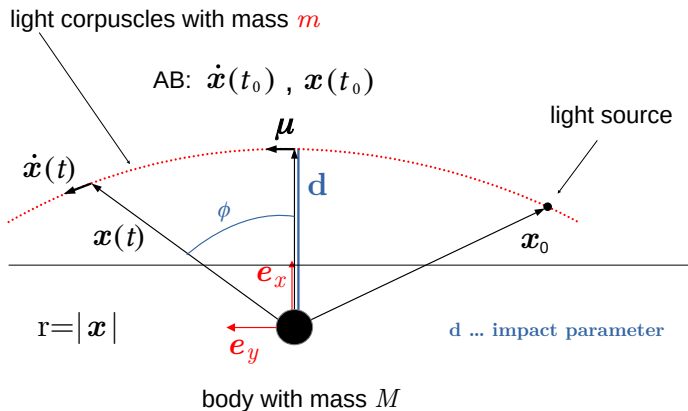
*Über die Ablenkung eines Lichtstrahls*

Astronomisches Jahrbuch **1801**, pp. 161 - 172, Berlin.

- highly remarkable: J.G. v. Soldner's motivation was astrometry

- v. Soldner: light composed of corpuscles with mass  $m \neq 0$
- light corpuscles propagate in field of body with mass  $M$

$$m \ddot{\mathbf{x}} = -m \frac{GM \mathbf{x}}{r^3} \quad (1)$$



- dividing both sides by  $m$  yields differential equation (ODE)

$$\ddot{\mathbf{x}} = -\frac{GM\mathbf{x}}{r^3} \quad (2)$$

- exact solution of ODE (can be treated like Kepler problem)

first integration: coordinate velocity of light corpuscles

$$\dot{\mathbf{x}}(t) = -\sqrt{\frac{GM}{d(1+e)}} \left( \mathbf{e}_x \sin \phi(t) - \mathbf{e}_y (\cos \phi(t) + e) \right) \quad (3)$$

second integration: trajectory of light corpuscles

$$\mathbf{x}(t) = \frac{p}{1+e \cos \phi(t)} \left( \mathbf{e}_x \cos \phi(t) + \mathbf{e}_y \sin \phi(t) \right) \quad (4)$$

where the angle  $\phi$

$$\dot{\phi}(t) = \sqrt{\frac{GM}{d(1+e)}} \frac{(1+e \cos \phi(t))^2}{d(1+e)}$$

parameter  $p$  and numerical eccentricity  $e$

$$p = d(1 + e)$$
$$e = \frac{d}{GM/c^2} - 1$$

- angle  $\alpha$  of total light deflection is defined by

$$\frac{\alpha}{2} = \arcsin \left| \frac{\dot{\mathbf{x}}(t \rightarrow \infty)}{|\dot{\mathbf{x}}(t \rightarrow \infty)|} \times \frac{\dot{\mathbf{x}}(t \rightarrow 0)}{|\dot{\mathbf{x}}(t \rightarrow 0)|} \right| \quad (5)$$

$$\text{light deflection : } \alpha = \frac{2GM}{c^2 d} \quad (6)$$

- $\alpha = 0.84 \text{ arcsec}$  at Sun, which is **one-half** of GR value
- result in Eq. (6) has also been found by Henry Cavendish in 1784, but he never has published his investigation
- $c$  enters equation via initial condition  $\dot{\mathbf{x}}(t \rightarrow 0) = c \boldsymbol{\mu}$



- v. Soldner did not know that (rest) mass of light is zero:  $m = 0$

$$\underbrace{m \ddot{\mathbf{x}} = -m \frac{GM \mathbf{x}}{r^3}}_{\text{o.k. but useless}} \implies \underbrace{\ddot{\mathbf{x}} = -\frac{GM \mathbf{x}}{r^3}}_{\text{nonsense}} \quad (7)$$

- SRT (Einstein, 1905) states that rest mass of light is zero:  $m = 0$
- Clearly, dividing by  $m = 0$  is meaningless. Let's consider an example:

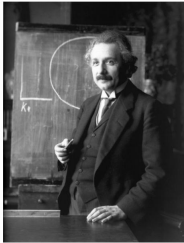
$$0 \cdot \text{donkey} = 0 \cdot \text{kangaroo} \quad \text{o.k. but useless}$$

$$\implies \text{donkey} = \text{kangaroo} \quad \text{nonsense}$$

- $\implies$  Eq. (7) has no justification for massless particles  $m = 0$
- but Eq. (7) not total nonsense for massless particles (cf. Eq. (21))

# 1.3 Light deflection by equivalence principle

- Albert Einstein (1907, 1911): equivalence principle



## Über das Relativitätsprinzip und die aus demselben gezogenen Folgerungen.

Von A. Einstein.

Die Newtonschen Bewegungsgleichungen behalten ihre Form, wenn man auf ein System, relativ zu dem ursprünglich besetzten in gleichförmiger Translationsbewegung begriffenen Koordinatensystem, beliebigfort sich das Mitschieben

## Über den Einfluss der Schwerkraft auf die Ausbreitung des Lichtes.

Von A. Einstein.<sup>1)</sup>

Die Frage, ob die Ausbreitung des Lichtes durch die Schwere beeinflusst wird, habe ich schon im Jahre vier Jahren vorläufigen Abklärung zu beantworten gesucht<sup>2)</sup>. Ich konnte auf dem Thema wieder zurück, weil mich meine damalige Darstellung des Gegenstandes nicht befriedigte, noch mehr aber, weil ich nun nachträglich erkannte, daß eine der wichtigsten Konsequenzen jener Betrachtung der experimentellen Prüfung zugänglich ist. Es ergibt sich nämlich, daß Lichtstrahlen, die in der Nähe der Sonne verlaufen, durch die Gravitationsfeld derselben nach der vorerwähnten Theorie eine Ablenkung erfahren, so daß eine scheinbare Vergrößerung des Winkels zwischen einem Stern an der Sonne beobachteten Fixsternen von dieser im Betrage von fast einer Bogensekunde entsteht.

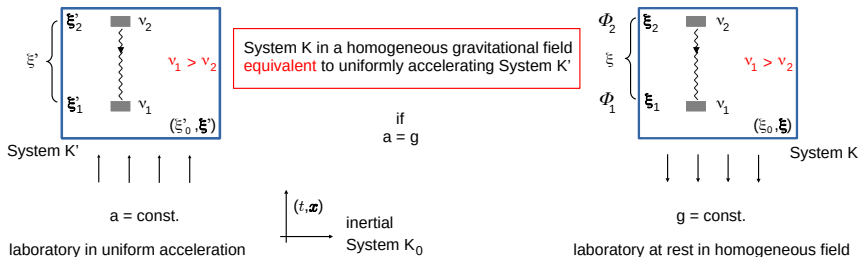
Es haben sich bei der Durchführung der Überlegungen noch noch weitere Resultate ergeben, die sich auf die Gravitation beziehen. Da eine der Grundlagen der ganzen Betrachtung nämlich unbenutzlich wird, sollen im folgenden nur einige ganz elementare Überlegungen gegeben werden, aus denen man sich sagen über die Voraussetzungen und den Gehalt der Theorie orientieren kann. Die hier abgeleiteten Resultate sind, auch wenn die theoretische Grundlage nicht, nur in erster Näherung gültig.

### § 1. Hypothese über die physikalische Natur des Gravitationsfeldes.

In einem homogenen Schwerefeld (Schwerkraftschiebung  $g$ ) behält sich ein ruhendes Koordinatensystem  $K$ , das so orientiert ist, daß die Kraftlinien des Schwerefeldes in Richtung der positiven  $z$ -Achse verlaufen. In einem von Gravitationsfeldern freien Raum befindet sich ein ruhendes Koordinatensystem  $K'$ , das in Richtung seiner positiven  $z$ -Achse eine gleichförmig beschleunigte Bewegung (Beschleunigung  $g$ ) ausführt. Um die Betrachtung nicht unnötig zu komplizieren, sollen wir dabei von der Relativitätstheorie

1. *Relativitätsprinzip und die aus demselben gezogenen Folgerungen*  
Jahrbuch der Radioaktivität **4** (1907) 411 – 462.
2. *Über den Einfluss der Schwerkraft auf die Ausbreitung des Lichtes*  
Annalen der Physik **35** (1911) 898 - 908.

- equivalence principle: completely new approach to determine light deflection in gravitational fields



$$\Phi = \Phi_2 - \Phi_1 = g \xi$$

- Einstein considers a light signal in System K' from  $\xi_2$  to  $\xi_1$
- Doppler effect implies:  $\nu_1 = \nu_2 (1 + v/c)$  (where  $v = at, t = \xi/c$ )

$$\nu_1 = \nu_2 \left( 1 + \frac{a \xi}{c^2} \right) \quad \text{i.e.} \quad \nu_1 > \nu_2 \quad (8)$$

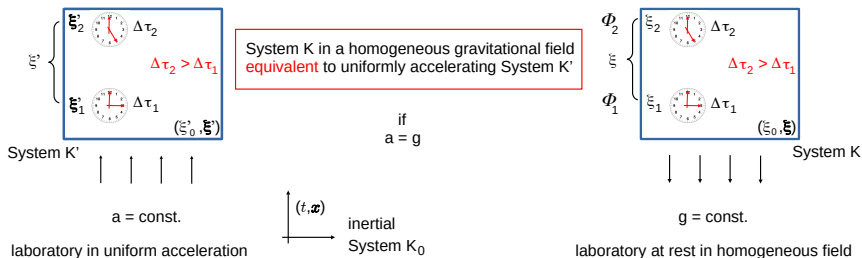
- from equivalence principle follows in  $K$  ( $\Phi = \Phi_2 - \Phi_1$  hence  $\Phi > 0$ ):

$$\nu_1 = \nu_2 \left( 1 + \frac{\Phi}{c^2} \right) \quad \text{i.e.} \quad \nu_1 > \nu_2 \quad (9)$$

(i) number  $N$  of periods is the same at  $\xi_2$  and  $\xi_1$

(ii) frequencies:  $\nu_2 = N/\Delta t_2$  and  $\nu_1 = N/\Delta t_1$

- Einstein (1907,1911): time depends on grav. potential:



$$\Delta\tau_2 = \Delta\tau_1 \left( 1 + \frac{\Phi}{c^2} \right) \quad \text{i.e.} \quad \Delta\tau_2 > \Delta\tau_1 \quad (10)$$

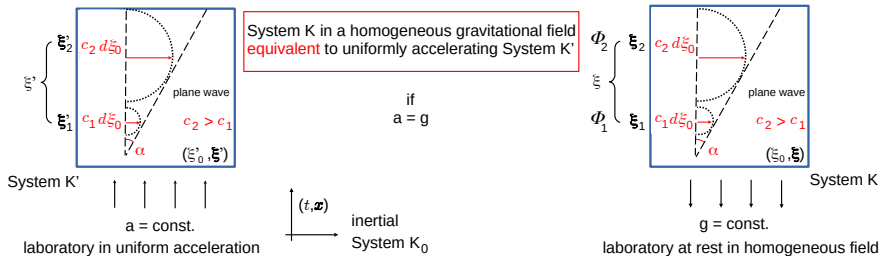
- from Eq. (10) follows line element (Rindler metric)

$$ds^2 = - (1 + \Phi/c^2)^2 c^2 d\xi_0^2 + d\xi^2 = 0$$

$\implies$  speed of light depends on grav. potential:

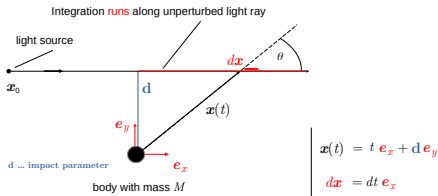
$$c_2 = c_1 \left( 1 + \frac{\Phi}{c^2} \right) \quad \text{i.e.} \quad c_2 > c_1 \quad (11)$$

- from Huygens principle Einstein concludes light deflection:



- for  $\Phi = -GM/r$  the light deflection is given by:

$$\frac{\alpha}{2} = \frac{1}{c^2} \int_{t=0}^{t=\infty} \frac{\partial \Phi}{\partial \mathbf{r}} \cdot d\mathbf{x} \quad \Longrightarrow \quad \frac{\alpha}{2} = \frac{GM}{c^2} \int_{t=0}^{t=\infty} \frac{\mathbf{x} \cdot d\mathbf{x}}{r^3} \quad (12)$$



$$\text{light deflection : } \alpha = \frac{2GM}{c^2 d} \quad (13)$$

- $\alpha = 0.83$  arcsec at Sun, which is one-half of GR value
- recall:  $\alpha = 0.84$  arcsec at Sun by J.G. v. Soldner in 1801
- $c$  enters equation via Doppler and equivalence principle

## 1.4 The correct value of light deflection

- Einstein 1911: time is curved but space is still flat
- Exact field equations 1915: time and space are curved  
⇒ correct light deflection

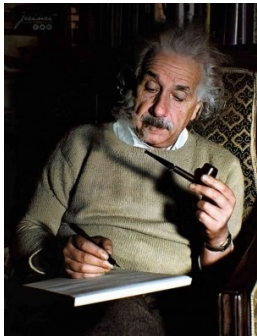
$$c_2 = c_1 \left( 1 + \frac{2\Phi}{c^2} \right) \quad \text{i.e.} \quad c_2 > c_1 \quad (14)$$

$$\text{light deflection : } \alpha = \frac{4GM}{c^2 d} \quad (15)$$

$\alpha = 1.7$  arcsec at Sun, which is the exact GR value

- historically: this correct result was given in November 1915 in:  
A. Einstein: *Erklärung der Perihelbewegung des Merkur aus der allgemeinen Relativitätstheorie*  
Preussische Akademie der Wissenschaften **3** (1915) 831–839.

## 2. Field equations of general theory of relativity



Albert Einstein  
(1915)



# • Albert Einstein (1915): general theory of relativity (GR)



Die im nachfolgenden dargelegte Theorie bildet die denkbar weitgehendste Verallgemeinerung der bereits allgemein als „Relativitätstheorie“ bezeichneten Theorie; die letztere nennt sich im folgenden zur Unterscheidung von der ersten „spezielle Relativitätstheorie“ und setzt die sich bekannt voraus. Die Verallgemeinerung der Relativitätstheorie wurde sehr erleichtert durch die Geometrie, welche der speziellen Relativitätstheorie durch Minkowski gegeben wurde, wiewohl Mathematiker meist die formale Gleichwertigkeit der überlieferten Koordinaten und der Zeithorizonte klar erkannt und für den Aufbau der Theorie nutzbar machten. Die für die allgemeine Relativitätstheorie nötigen mathematischen Hilfsmittel lagen fertig bereit in dem „abstrakten Differentialkalkül“, welches auf den Forschungen von GAUSS, RIEMANN und CHRISTOFFEL über nichteuklidische Mannigfaltigkeiten ruht und von RICCI und LEVI-CIVITA in die System gebracht und bereits auf Probleme der theoretischen Physik angewandt wurde. Ich habe im Abschnitt E der vorliegenden Abhandlung alle für sie nötigen, bei dem Physiker nicht als bekannt voraussetzbaren mathematischen Hilfsmittel in möglichst einfacher und durchsichtiger Weise angeschlossen, so daß ein Studium mathematischer Literatur für das Verständnis der vorliegenden Abhandlung nicht erforderlich ist. Hofflich sei es dieser Stelle dankbar seinen Freunden, des Mathematikers GROSSMANN, gedacht, der mir durch seine Hilfe nicht nur das Studium der einschlägigen mathematischen Literatur ersparte, sondern mich auch beim Schreiben nach den Feldgleichungen der Gravitation unterstützte.

## 1. Die Feldgleichungen der Gravitation

Preussische Akademie der Wissenschaften **4** (1915) 844–847.

## 2. Die Grundlage der allgemeinen Relativitätstheorie

Annalen der Physik **49** (1916) 769 - 822.

- field equations of gravity ( $\kappa = 8\pi G/c^4$ )

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa T_{\mu\nu} \quad \Longrightarrow \quad g_{\mu\nu} \quad (16)$$

energy-momentum tensor:  $T_{\mu\nu}$

Ricci tensor:

$$R_{\mu\nu} = \Gamma_{\mu\nu, \rho}^{\rho} - \Gamma_{\mu\rho, \nu}^{\rho} + \Gamma_{\sigma\rho}^{\rho} \Gamma_{\mu\nu}^{\sigma} - \Gamma_{\sigma\nu}^{\rho} \Gamma_{\mu\rho}^{\sigma}$$

Christoffel symbols:

$$\Gamma_{\mu\nu}^{\alpha} = g^{\alpha\beta} (g_{\beta\mu, \nu} + g_{\beta\nu, \mu} - g_{\mu\nu, \beta}) / 2$$

- metric  $g_{\mu\nu}$  of physical manifold  $\mathcal{M}$  determined by (16)
- physical space-time is a pair  $(\mathcal{M}, g_{\mu\nu})$

- weak gravitational fields  $\implies$  metric is decomposed

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad \text{with} \quad |h_{\mu\nu}| \ll 1$$

- where  $\eta_{\mu\nu} = \text{diag}(-1, +1, +1, +1)$
- background space-time is a pair  $(\mathcal{M}_0, \eta_{\mu\nu})$
- fields  $h_{\mu\nu}$  propagate in flat background manifold  $\mathcal{M}_0$
- reduced field equations (harmonic gauge:  $\bar{h}^{\mu\nu}_{,\nu} = 0$ )

$$\square \bar{h}^{\mu\nu} = -2\kappa (\tau^{\mu\nu} + t^{\mu\nu}) \quad (17)$$

- $h_{\mu\nu}$  determined by reduced field equations of gravity (17)
- Eq. (17) wave-equation in flat space-time:  $\square = \eta^{\mu\nu} \partial_\mu \partial_\nu$
- $\tau^{\mu\nu} = -g T^{\mu\nu}$  where  $g$  is the determinant  $g = \det(g_{\mu\nu})$
- $t^{\mu\nu} = -g \tau_{LL}^{\mu\nu} + (\bar{h}^{\mu\alpha}_{,\beta} \bar{h}^{\nu\beta}_{,\alpha} - \bar{h}^{\mu\nu}_{,\alpha\beta} \bar{h}^{\alpha\beta}) / 2\kappa$
- metric density:  $\bar{h}^{\mu\nu} = \eta^{\mu\nu} - \sqrt{-g} g^{\mu\nu}$

## 2.1 Post-Minkowskian expansion

- expansion of  $g_{\mu\nu}$  w.r.t.  $G^n$  (weak-field)

- 1PM expansion of metric tensor

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}^{(1\text{PM})} + \mathcal{O}(G^2)$$

- 2PM expansion of metric tensor

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}^{(1\text{PM})} + h_{\mu\nu}^{(2\text{PM})} + \mathcal{O}(G^3)$$

- 3PM expansion of metric tensor

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}^{(1\text{PM})} + h_{\mu\nu}^{(2\text{PM})} + h_{\mu\nu}^{(3\text{PM})} + \mathcal{O}(G^4)$$

## 2.2 Post-Newtonian expansion

- expansion  $h_{\mu\nu}$  w.r.t.  $c^{-n}$  (weak-field slow-motion)

- 1PN expansion of metric tensor

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}^{(2)} + \mathcal{O}(c^{-3})$$

- 1.5PN expansion of metric tensor

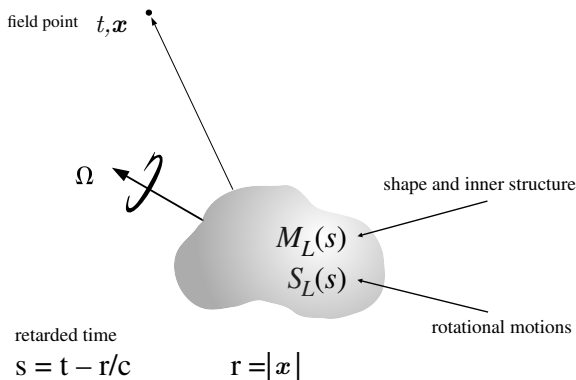
$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}^{(2)} + h_{\mu\nu}^{(3)} + \mathcal{O}(c^{-4})$$

- 2PN expansion of metric tensor

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}^{(2)} + h_{\mu\nu}^{(3)} + h_{\mu\nu}^{(4)} + \mathcal{O}(c^{-5})$$

## 2.3 Example: body at rest

- body's center-of-mass at rest w.r.t. harmonic coordinates
- time-dependent multipoles  $\hat{M}_L, \hat{S}_L$
- $\hat{M}_L$  : body of arbitrary shape, structure, and oscillations
- $\hat{S}_L$  : body in arbitrary rotation and inner currents



## 2.3.1 Metric of a body at rest in 1PM approximation

$$h_{00}^{(1\text{PM})}(t, \mathbf{x}) = + \frac{2}{c^2} \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \hat{\partial}_L \frac{\hat{M}_L(s)}{r},$$

$$h_{0i}^{(1\text{PM})}(t, \mathbf{x}) = + \frac{4}{c^3} \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \hat{\partial}_{L-1} \frac{\hat{M}_{iL-1}(s)}{r} \\ + \frac{4}{c^3} \sum_{l=1}^{\infty} \frac{(-1)^l l}{(l+1)!} \epsilon_{iab} \hat{\partial}_{aL-1} \frac{\hat{S}_{bL-1}(s)}{r},$$

$$h_{ij}^{(1\text{PM})}(t, \mathbf{x}) = + \frac{2}{c^2} \delta_{ij} \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \hat{\partial}_L \frac{\hat{M}_L(s)}{r} \\ + \frac{4}{c^4} \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \hat{\partial}_{L-2} \frac{\hat{M}_{ijL-2}(s)}{r} \\ + \frac{8}{c^4} \sum_{l=1}^{\infty} \frac{(-1)^l l}{(l+1)!} \hat{\partial}_{aL-2} \frac{\epsilon_{ab(i} \hat{S}_{j)bL-2}(s)}{r}.$$

## 2.3.2 Metric of a body at rest in 1.5PN approximation

$$h_{00}^{(2)}(t, \mathbf{x}) = +\frac{2G}{c^2} \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \hat{\partial}_L \frac{\hat{M}_L}{r},$$

$$h_{0i}^{(3)}(t, \mathbf{x}) = +\frac{4}{c^3} \sum_{l=1}^{\infty} \frac{(-1)^l l}{(l+1)!} \epsilon_{iab} \hat{\partial}_{aL-1} \frac{\hat{S}_{bL-1}}{r}$$

$$+\frac{4G}{c^3} \sum_{l=1}^{\infty} \frac{(-1)^l}{l!} \hat{\partial}_{L-1} \frac{\hat{M}_{iL-1}}{r},$$

$$h_{ij}^{(2)}(t, \mathbf{x}) = +\frac{2G}{c^2} \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \hat{\partial}_L \frac{\hat{M}_L}{r} \delta_{ij}.$$

- with the STF differential operator

$$\hat{\partial}_L = \text{STF}_{i_1 \dots i_l} \frac{\partial}{\partial x^{i_1}} \cdots \frac{\partial}{\partial x^{i_l}}$$



## 2.3.3 The mass-multipoles and spin-multipoles

- mass multipoles

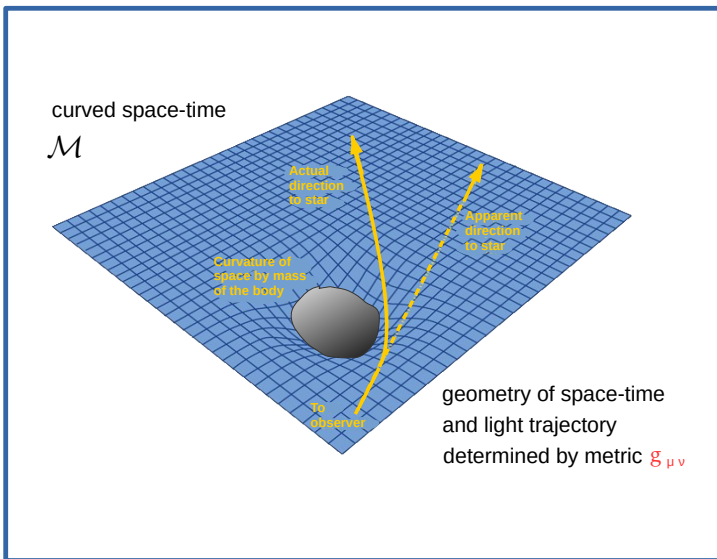
$$\hat{M}_L = \int d^3x \hat{x}_L \Sigma + \mathcal{O}(c^{-2})$$

- spin multipoles

$$\hat{S}_L = \int d^3x \epsilon_{j k < i_l} \hat{x}_{L-1 > x^j \Sigma^k + \mathcal{O}(c^{-2})$$

- where  $\Sigma = (T^{00} + T^{kk})/c^2$  and  $\Sigma^k = T^{0k}/c$   
with  $T^{\alpha\beta}$  is the stress-energy tensor of body

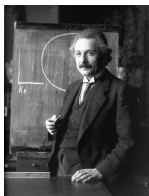
### 3. Theory of light propagation



- astrometry needs to determine light trajectory  $\mathbf{x}(t)$
- geodesic equation

$$\frac{\ddot{x}^i}{c^2} + \Gamma_{\mu\nu}^i \frac{\dot{x}^\mu}{c} \frac{\dot{x}^\nu}{c} - \Gamma_{\mu\nu}^0 \frac{\dot{x}^\mu}{c} \frac{\dot{x}^\nu}{c} \frac{\dot{x}^i}{c} = 0 \quad \implies \quad \mathbf{x}(t) \quad (18)$$

- historically: geodesic equation for light (Einstein, 1914)



KÖNIGLICH PREUSSISCHEN  
 AKADEMIE DER WISSENSCHAFTEN.  
 Zusammenstellung vom 19. November.  
 Mittheilung aus der Sitzung der phys.-math. Classe vom 23. October.  
 Die formale Grundlage der allgemeinen  
 Relativitätstheorie.  
 Von A. EINSTEIN.

weltbekannte Ableitung der euklidischen Geometrie einer Linie von  
 besetzen.  
 Es handelt sich um eine ebene von Punkten  $P^0$  und  $P^1$  ver-  
 laufende Linie, gegeben für alle die unendlich benachbarten Linienelemente durch die beiden benachbarten Punkte  $dx^0$  und  $dx^1$  (Abbildung (1) im  
 Bild). Umfasst man die Linie zwischen den Punkten  $P^0$  und  $P^1$  und  
 wird sich "Hinter" von Kurven  $\alpha$  auf allen diese unendlich benach-  
 barten Linienelemente  $\alpha$  von Punkt  $P^0$  nach dem Punkte  $P^1$  hin  
 gegeben durch die Punkte  $\alpha$  von  $\alpha$  abwärts sind. Dann  
 ist  

$$\alpha^i = \sum_{\mu=0}^1 \frac{\partial x^i}{\partial x^\mu} dx^\mu$$
 in einem  $\alpha$  mit  $\alpha^i = 0$   

$$\int_{\alpha} dx^i = 0$$
 (10)  
 da die Integrationskurve  $\alpha$ , und  $\alpha$  für alle benachbarten Kurven  
 verläuft sind. Bezeichnet man mit  $\alpha$  die Punkte, welche aus den  
 $\alpha$  verlaufen sind, um von einem Punkte  $P^0$  zu einem gegebenen  
 Punkte  $P^1$  zu gelangen, so ist  

$$\alpha^i = \int_{P^0}^{P^1} \frac{\partial x^i}{\partial x^\mu} dx^\mu = \sum_{\mu=0}^1 \frac{\partial x^i}{\partial x^\mu} (x^\mu)$$
 Man kann dies in (10) die  $\alpha$  nicht sein, wenn man die Linie  
 gleich parallel längere und nicht berücksichtigen, daß für  $\alpha = 1$ , und  
 $\alpha = 0$ , die veränderliche  

$$\int_{\alpha} dx^i = \sum_{\mu=0}^1 \frac{\partial x^i}{\partial x^\mu} dx^\mu = 0$$
  
 wobei  

$$\alpha^i = \sum_{\mu=0}^1 \frac{\partial x^i}{\partial x^\mu} \frac{dx^\mu}{dx^0} = \sum_{\mu=0}^1 \frac{\partial x^i}{\partial x^\mu} \frac{dx^\mu}{dx^0}$$
 gesetzt ist. Es folgt hieraus, daß  

$$\alpha^i = 0$$
 (11)  
 die Ableitung der geodesischen Linie ist.  
 In der ursprünglichen Relativitätstheorie verlaufen die geodesischen  
 geodesischen Linien, die welche  $\alpha^i = 0$  ist, die Bewegung materieller  
 Punkte abgeben, die welche  $\alpha^i = 0$  ist, die Lichtstrahlen sind.

A. Einstein:

Die formale Grundlage der allgemeinen Relativitätstheorie

Preussische Akademie der Wissenschaften 2 (1914) 1030.

## 3.1 Post-Minkowskian expansion of light trajectory

- 1PM expansion of light trajectory

$$\mathbf{x}(t) = \mathbf{x}_N + \Delta\mathbf{x}_{1\text{PM}} + \mathcal{O}(G^2)$$

- 2PM expansion of light trajectory

$$\mathbf{x}(t) = \mathbf{x}_N + \Delta\mathbf{x}_{1\text{PM}} + \Delta\mathbf{x}_{2\text{PM}} + \mathcal{O}(G^3)$$

- 3PM expansion of light trajectory

$$\mathbf{x}(t) = \mathbf{x}_N + \Delta\mathbf{x}_{1\text{PM}} + \Delta\mathbf{x}_{2\text{PM}} + \Delta\mathbf{x}_{3\text{PM}} + \mathcal{O}(G^4)$$

where  $\mathbf{x}_N$  is the unperturbed light ray:

$$\mathbf{x}_N = \mathbf{x}_0 + c(t - t_0) \boldsymbol{\sigma}$$

(19)

## 3.2 Post-Newtonian expansion of light trajectory

- 1PN expansion of light trajectory

$$\mathbf{x}(t) = \mathbf{x}_N + \Delta\mathbf{x}_{1\text{PN}} + \mathcal{O}(c^{-3})$$

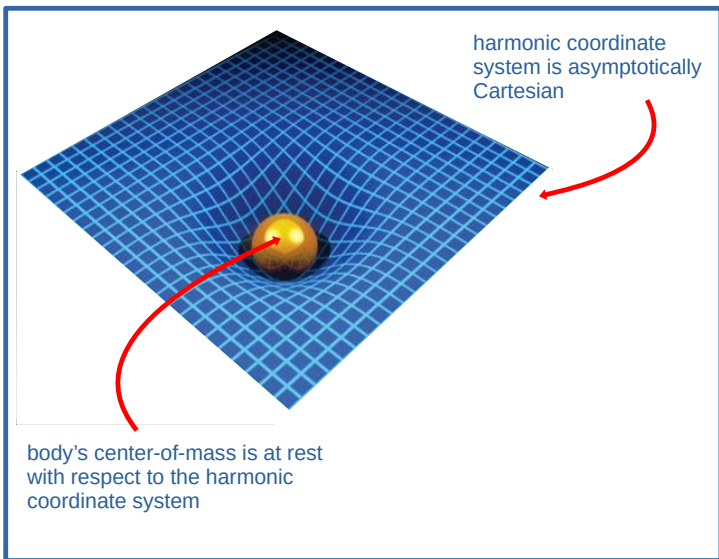
- 1.5PN expansion of light trajectory

$$\mathbf{x}(t) = \mathbf{x}_N + \Delta\mathbf{x}_{1\text{PN}} + \Delta\mathbf{x}_{1.5\text{PN}} + \mathcal{O}(c^{-4})$$

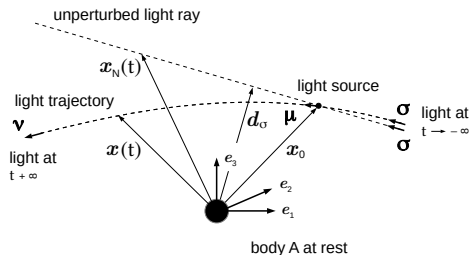
- 2PN expansion of light trajectory

$$\mathbf{x}(t) = \mathbf{x}_N + \Delta\mathbf{x}_{1\text{PN}} + \Delta\mathbf{x}_{1.5\text{PN}} + \Delta\mathbf{x}_{2\text{PN}} + \mathcal{O}(c^{-5})$$

## 4. Theory of light propagation: bodies at rest



## 4.1 Light propagation in field of monopole at rest



$$\delta(\boldsymbol{\sigma}, \boldsymbol{\nu}_{1PN}^M) = \frac{4GM}{c^2} \frac{1}{d_\sigma} \implies \alpha = 1.75 \text{ arcsec at Sun} \quad (20)$$

- historically: this correct result was given in November 1915 in:  
A. Einstein: *Erklärung der Perihelbewegung des Merkur*  
Preussische Akademie der Wissenschaften **3** (1915) 831–839.

- geodesic equation (14) in 1PN for monopole

$$\ddot{\mathbf{x}} = \underbrace{-\frac{2GM\mathbf{x}}{r^3}}_{2 \times \text{Soldner, cf. Eq. (7)}} + 4GM \frac{\boldsymbol{\sigma} \cdot \mathbf{x}}{r^3} \boldsymbol{\sigma} \quad (21)$$

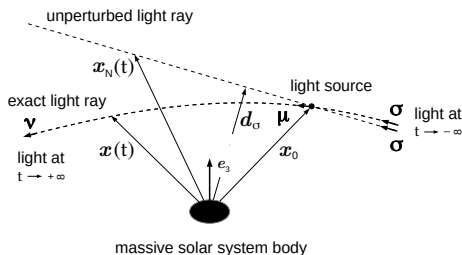
- first integration of geodesic equation (14) in 1PN

$$\frac{\Delta \dot{\mathbf{x}}_{1\text{PN}}^M}{c} = -\frac{2GM}{c^2} \frac{\boldsymbol{\sigma}}{x_N} - \frac{2GM}{c^2} \frac{d_\sigma}{(d_\sigma)^2} \left( \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_N}{x_N} + 1 \right)$$

- V. A. Brumberg: *Essential Relativistic Celestial Mechanics* 1991, Adam Hilger, Bristol
- note:  $|\dot{\mathbf{x}}| = c \left( 1 - \frac{2GM}{c^2 x_N} \right)$  i.e.:  $|\dot{\mathbf{x}}| = c \left( 1 + \frac{2\Phi}{c^2} \right)$
- in agreement with Eq. (14)



## 4.2 Light propagation in field of quadrupole at rest

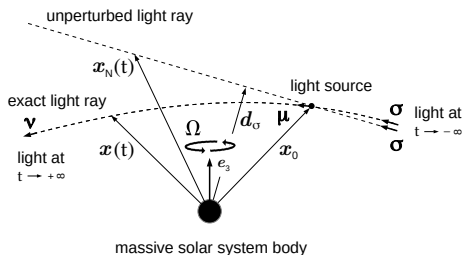


- total light deflection caused by quadrupole:

$$\left| \delta \left( \sigma, \nu_{\text{IPN}}^Q \right) \right| \leq \frac{4GM}{c^2} \frac{|J_2|}{d_\sigma} \left( \frac{P}{d_\sigma} \right)^2 \quad (22)$$

- S. Klioner: Sov. Astronomy **35** (1991) 523

## 4.3 Light propagation in field of spin-dipole at rest



- total light deflection caused by spin-dipole:

$$\left| \delta \left( \boldsymbol{\sigma}, \boldsymbol{\nu}_{1.5\text{PN}}^S \right) \right| \leq \frac{4G}{c^3 (d_\sigma)^2} |\mathbf{S}| \quad (23)$$

- S. Klioner: Sov. Astronomy **35** (1991) 523

## 4.4 Light propagation in field of multipoles at rest

- advanced integration methods to get  $\dot{\mathbf{x}}(t)$ ,  $\mathbf{x}(t)$  by S. Kopeikin  
S. Kopeikin: Journal of Mathematical Physics **38** (1997) 2587
- an important progress in the theory of light propagation:

$$\frac{\dot{\mathbf{x}}(t)}{c} = \boldsymbol{\sigma} + \sum_{l=0}^{\infty} \frac{\Delta \dot{\mathbf{x}}_{1\text{PN}}^{M_L}}{c} + \sum_{l=1}^{\infty} \frac{\Delta \dot{\mathbf{x}}_{1.5\text{PN}}^{S_L}}{c} + \mathcal{O}(c^{-4})$$
$$\mathbf{x}(t) = \mathbf{x}_N + \sum_{l=0}^{\infty} \Delta \mathbf{x}_{1\text{PN}}^{M_L} + \sum_{l=1}^{\infty} \Delta \mathbf{x}_{1.5\text{PN}}^{S_L} + \mathcal{O}(c^{-4})$$

- total light deflection caused by mass- and spin-multipoles:

$$\delta(\boldsymbol{\sigma}, \boldsymbol{\nu}_{1\text{PN}}^{M_L}) = -\frac{4GM}{c^2 d_\sigma} \frac{(-1)^l}{(l-1)!} M_L \hat{\partial}_L \ln |\boldsymbol{\xi}|$$
$$\delta(\boldsymbol{\sigma}, \boldsymbol{\nu}_{1.5\text{PN}}^{S_L}) = -\frac{8G}{c^3} \frac{1}{|\boldsymbol{\xi}|} \epsilon_{abc} \sigma^c \frac{(-1)^l l^2}{(l+1)!} \hat{S}_{bL-1} \hat{\partial}_{aL-1} \ln |\boldsymbol{\xi}|$$

- Time-Transfer Function (TTF) approach to get  $\dot{\mathbf{x}}(t)$ ,  $\mathbf{x}(t)$  applied by B. Linet, C. Le Poncine-Lafitte & P. Teysandier  
Physical Review D **66** (2002) 024045  
Physical Review D **77** (2008) 044029
- represents another method in the theory of light propagation
- light travel time between two events:  $(t_A, \mathbf{x}_A)$  and  $(t_B, \mathbf{x}_B)$

$$t_A - t_B = \underbrace{\mathcal{T}(\mathbf{x}_A, \mathbf{x}_B)}_{\text{time-transfer-function}} \quad (24)$$

- total light deflection determined for axisymmetric bodies

$$\left| \delta(\boldsymbol{\sigma}, \boldsymbol{\nu}_{\text{IPN}}^{ML}) \right| \leq \frac{4GM}{c^2} \frac{|J_l|}{d_\sigma} \left( \frac{P}{d_\sigma} \right)^l \quad (25)$$

- shown only for  $l = 2, 3, 4$  but validity conjectured for  $l \geq 4$

- total light deflection for axisymmetric bodies in rotation
- by using the approach of S. Kopeikin

S. Zschocke: Physical Review D **107** (2023) 124055

$$\left| \delta \left( \boldsymbol{\sigma}, \boldsymbol{\nu}_{1\text{PN}}^{M_L} \right) \right| \leq \frac{4GM}{c^2} \frac{|J_l|}{d_\sigma} \left( \frac{P}{d_\sigma} \right)^l$$

$$\left| \delta \left( \boldsymbol{\sigma}, \boldsymbol{\nu}_{1.5\text{PN}}^{S_L} \right) \right| \leq \frac{8GM}{c^3} \Omega \frac{l^2}{l+4} |J_{l-1}| \left( \frac{P}{d_\sigma} \right)^{l+1}$$

- shown for any values of  $l \geq 0$  and  $l \geq 1$  respectively
- total light deflection is related to Chebyshev polynomials

## 4.5 Numerical values of total light deflection [ $\mu\text{as}$ ]

Light deflection	Sun	Jupiter	Saturn
$ \delta(\sigma, \nu_{1\text{PN}}^{M_0}) $	$1.75 \times 10^6$	$16.3 \times 10^3$	$5.8 \times 10^3$
$ \delta(\sigma, \nu_{1\text{PN}}^{M_2}) $	0.35	239	94
$ \delta(\sigma, \nu_{1\text{PN}}^{M_4}) $	1.72	9.6	5.41
$ \delta(\sigma, \nu_{1\text{PN}}^{M_6}) $	0.07	0.55	0.50
$ \delta(\sigma, \nu_{1\text{PN}}^{M_8}) $	0.007	0.04	0.06
$ \delta(\sigma, \nu_{1\text{PN}}^{M_{10}}) $	—	0.003	0.01
$ \delta(\sigma, \nu_{1.5\text{PN}}^{S_1}) $	0.7	0.17	0.04
$ \delta(\sigma, \nu_{1.5\text{PN}}^{S_3}) $	—	0.026	0.008
$ \delta(\sigma, \nu_{1.5\text{PN}}^{S_5}) $	—	0.001	—

- S. Zschocke: Physical Review D **107** (2023) 124055

## 4.6 2PN light propagation in field of monopole at rest

- 2PN calculations are very rare, because they are by far much more complicated than 1PN or 1.5PN calculations
- 2PN expansion of light velocity for monopole

$$\frac{\dot{\boldsymbol{x}}(t)}{c} = \boldsymbol{\sigma} + \frac{\Delta \dot{\boldsymbol{x}}_{1\text{PN}}^M}{c} + \frac{\Delta \dot{\boldsymbol{x}}_{2\text{PN}}^{M \times M}}{c} + \mathcal{O}(c^{-6})$$

- 2PN expansion of light trajectory for monopole

$$\boldsymbol{x}(t) = \boldsymbol{x}_N + \Delta \boldsymbol{x}_{1\text{PN}}^M + \Delta \boldsymbol{x}_{2\text{PN}}^{M \times M} + \mathcal{O}(c^{-6})$$

- V. A. Brumberg: *Post-post Newtonian propagation of light in the Schwarzschild field*, Kinematica i physika nebesnykh tel **3** (1987) 8





- unit tangent vector at observer's position

$$\mathbf{n} = \frac{\dot{\mathbf{x}}(t_1)}{|\dot{\mathbf{x}}(t_1)|} = \mathbf{n}_{1\text{PN}}^M + \mathbf{n}_{2\text{PN}}^{M \times M}$$

- upper limit of 1PN terms

$$|\delta(\boldsymbol{\sigma}, \mathbf{n}_{1\text{PN}}^M)| \leq \frac{4GM}{c^2} \frac{1}{d_\sigma}$$

- upper limit of 2PN terms  $\implies$  discovery of enhanced terms

$$|\delta(\boldsymbol{\sigma}, \mathbf{n}_{2\text{PN}}^{M \times M})| \leq \frac{16G^2 M^2}{c^4} \frac{1}{(d_\sigma)^2} \frac{x_1}{d_\sigma}$$

- S.A. Klioner, S. Zschocke, CQG **27** (2010) 075015
- N. Ashby, B. Bertotti, CQG **27** (2010) 145013
- P. Teysandier, C. Poncine-Lafitte, CQG **29** (2012) 245010
- the term  $x_1/d_\sigma$  implies no divergence:  
cf. Appendix J in S. Zschocke, PRD **105** (2022) 024040

## 4.7 2PN light propagation in field of quadrupole at rest

- 2PN light velocity for monopole and quadrupole

$$\frac{\dot{\mathbf{x}}(t)}{c} = \boldsymbol{\sigma} + \frac{\Delta\dot{\mathbf{x}}_{1\text{PN}}^M}{c} + \frac{\Delta\dot{\mathbf{x}}_{2\text{PN}}^{M\times M}}{c} + \frac{\Delta\dot{\mathbf{x}}_{2\text{PN}}^{M\times Q}}{c} + \frac{\Delta\dot{\mathbf{x}}_{2\text{PN}}^{Q\times Q}}{c}$$

- 2PN light trajectory for monopole and quadrupole

$$\mathbf{x}(t) = \mathbf{x}_N + \Delta\mathbf{x}_{1\text{PN}}^M + \Delta\mathbf{x}_{2\text{PN}}^{M\times M} + \Delta\mathbf{x}_{2\text{PN}}^{M\times Q} + \Delta\mathbf{x}_{2\text{PN}}^{Q\times Q}$$

- S. Zschocke: Physical Review D **105** (2022) 024040
- S. Zschocke: Physical Review D **106** (2022) 104052
- unit tangent vector at observer's position

$$\mathbf{n} = \frac{\dot{\mathbf{x}}(t_1)}{|\dot{\mathbf{x}}(t_1)|} = \mathbf{n}_{1\text{PN}}^M + \mathbf{n}_{1\text{PN}}^Q + \mathbf{n}_{2\text{PN}}^{M\times M} + \mathbf{n}_{2\text{PN}}^{M\times Q} + \mathbf{n}_{2\text{PN}}^{Q\times Q}$$

- upper limit of 1PN terms

$$\left| \delta \left( \boldsymbol{\sigma}, \mathbf{n}_{1\text{PN}}^M \right) \right| = \frac{4GM}{c^2} \frac{1}{d_\sigma}$$

$$\left| \delta \left( \boldsymbol{\sigma}, \mathbf{n}_{1\text{PN}}^Q \right) \right| \leq \frac{4GM}{c^2} \frac{|J_2|}{d_\sigma} \left( \frac{P}{d_\sigma} \right)^2$$

- upper limit of 2PN terms  $\implies$  discovery of enhanced terms

$$\left| \delta \left( \boldsymbol{\sigma}, \mathbf{n}_{2\text{PN}}^{M \times M} \right) \right| \leq \frac{16 G^2 M^2}{c^4} \frac{1}{(d_\sigma)^2} \frac{x_1}{d_\sigma}$$

$$\left| \delta \left( \boldsymbol{\sigma}, \mathbf{n}_{2\text{PN}}^{M \times Q} \right) \right| \leq \frac{64 G^2 M^2}{c^4} \frac{|J_2|}{(d_\sigma)^2} \left( \frac{P}{d_\sigma} \right)^2 \frac{x_1}{d_\sigma}$$

$$\left| \delta \left( \boldsymbol{\sigma}, \mathbf{n}_{2\text{PN}}^{Q \times Q} \right) \right| \leq \frac{48 G^2 M^2}{c^4} \frac{|J_2|^2}{(d_\sigma)^2} \left( \frac{P}{d_\sigma} \right)^4 \frac{x_1}{d_\sigma}$$

## 4.8 Numerical values of 2PN light deflection [ $\mu\text{as}$ ]

Object	$ \delta(\boldsymbol{\sigma}, \mathbf{n}_{2\text{PN}}^{M \times M}) $	$ \delta(\boldsymbol{\sigma}, \mathbf{n}_{2\text{PN}}^{M \times Q}) $	$ \delta(\boldsymbol{\sigma}, \mathbf{n}_{2\text{PN}}^{Q \times Q}) $
Jupiter	16.11	0.95	0.010
Saturn	4.42	0.29	0.003
Uranus	2.58	0.04	0.001
Neptune	5.83	0.08	0.002

- S. Zschocke: Light propagation in 2PN approximation in the monopole and quadrupole field of a body at rest: Initial value problem  
Physical Review D **105** (2022) 024040

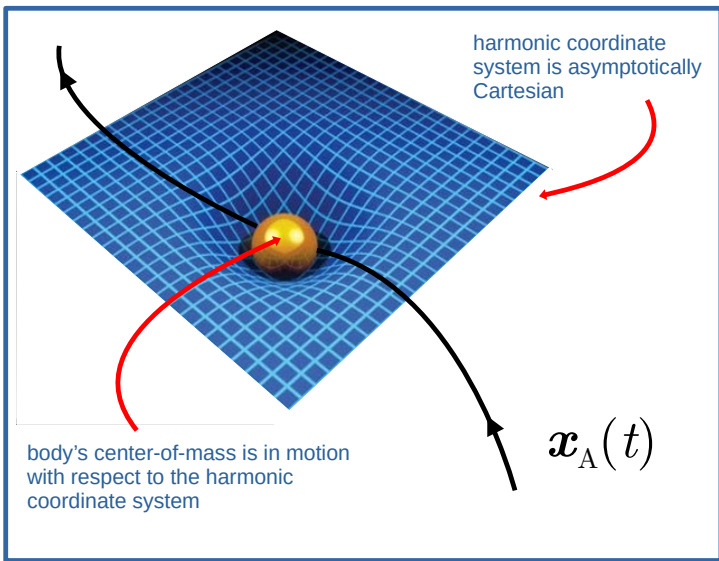
## 4.9 Light propagation in 3PN approximation

- 3PN calculations are extremely rare:
- S. Zschocke: *A generalized lens equation for light deflection in weak gravitational fields*, CQG **28** (2011) 125016
- B. Linet, P. Teyssandier: *New method for determining the light travel time in static spherically symmetric spacetimes. Calculation of terms of order  $G^3$* , CQG **30** (2013) 175008

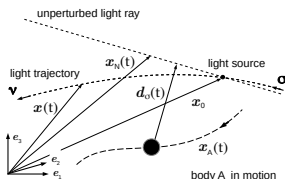
$$\left| \delta \left( \boldsymbol{\sigma}, \mathbf{n}_{3\text{PN}}^M \right) \right| = \frac{128 G^3 M^3}{c^3} \frac{1}{(d_\sigma)^3} \left( \frac{x_1}{d_\sigma} \right)^2 \quad (26)$$

- this result has been found independently by:
  - Eq. (27) in S. Zschocke, CQG **28** (2011) 125016
  - Eq. (93) in B. Linet, P. Teyssandier, CQG **30** (2013) 175008
- numerical value:  $12 \mu\text{as}$  in case of grazing light ray at Sun

## 5. Theory of light propagation: bodies in motion



## 5.1 Light propagation in field of monopole in motion



- total light deflection caused by moving monopole

$$\delta(\boldsymbol{\sigma}, \boldsymbol{\nu}_{1\text{PM}}^M) = \frac{4GM}{c^2} \frac{1}{d_\sigma(s_1)} \left( 1 + \frac{v_A(s_1)}{c} \right) \quad (27)$$

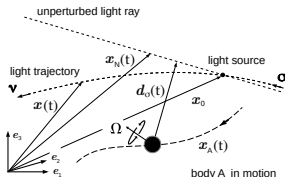
- where  $s_1$  is the retarded time, defined implicitly:

$$s_1 = t_1 - \frac{|\mathbf{x}(t_1) - \mathbf{x}_A(s_1)|}{c} \quad (28)$$

- **monopole in uniform motion in 1PN**  
S.A. Klioner, Comm. of Inst. Appl. Astron. **6**, 1989
- **arbitrarily slowly moving monopoles in 1PN**  
S.A. Klioner, Sov. Astron. **35** (1991) 523
- **arbitrarily moving monopoles in 1PM**  
S.M. Kopeikin, G. Schäfer: Physical Review D **60** (1999) 124002
- **uniformly moving monopoles in 1PM**  
S.A. Klioner, Astronomy & Astrophysics **404** (2003) 783
- **arbitrarily slowly moving monopoles in 1PN**  
S. Zschocke, Physical Review D **92** (2015) 063015
- **arbitrarily slowly moving monopoles in 1.5PN**  
S. Zschocke, Physical Review D **93** (2016) 103010
- **arbitrarily slowly moving monopoles in 2PN**  
S. Zschocke, Classical Quantum Gravity **35** (2018) 055013  
S. Zschocke, Classical Quantum Gravity **36** (2019) 015007



## 5.2 Light propagation in field of spin-dipole in motion

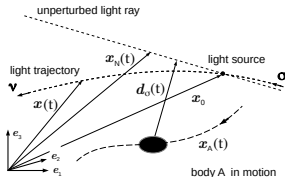


- total light deflection caused by moving spin-dipole

$$\left| \delta \left( \boldsymbol{\sigma}, \boldsymbol{\nu}_{1.5\text{PN}}^S \right) \right| \leq \frac{4G}{c^3 (d_\sigma(s_1))^2} |\mathbf{S}(s_1)| \quad (29)$$

- arbitrarily slowly moving spin-dipoles in 1.5PN  
S.A. Klioner, Sov. Astron. **35** (1991) 523
- arbitrarily moving spin-dipoles in 1PM  
S.M. Kopeikin, B. Mashhoon: Physical Review D **65** (2002) 064025
- arbitrarily slowly moving spin-dipoles in 1.5PN  
S. Zschocke, Physical Review D **93** (2016) 103010

## 5.3 Light propagation in field of quadrupole in motion

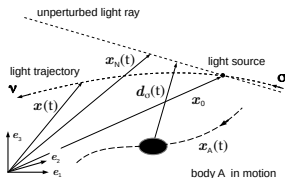


- total light deflection caused by moving quadrupole

$$\left| \delta \left( \boldsymbol{\sigma}, \boldsymbol{\nu}_{1\text{PN}}^Q \right) \right| \leq \frac{4GM}{c^2} \frac{|J_2|}{d_\sigma(s_1)} \left( \frac{P}{d_\sigma(s_1)} \right)^2 \quad (30)$$

- arbitrarily slowly moving quadrupoles in 1PN  
S.A. Klioner, Sov. Astron. **35** (1991) 523
- arbitrarily moving quadrupoles in 1PM  
S.M. Kopeikin, V.V. Makarov: Physical Review D **75** (2007) 062002
- arbitrarily slowly moving quadrupoles in 1PN  
S. Zschocke, Physical Review D **92** (2015) 063015

## 5.4 Light propagation in field of multipoles in motion



- total light deflection caused by moving multipoles:

$$\left| \delta \left( \boldsymbol{\sigma}, \boldsymbol{\nu}_{\text{1PN}}^Q \right) \right| \leq \frac{4GM}{c^2} \frac{|J_l|}{d_\sigma(s_1)} \left( \frac{P}{d_\sigma(s_1)} \right)^l \quad (31)$$

this result is only a conjecture, but has not been proven thus far

- arbitrarily slowly moving multipoles in 1PN and 1.5PN:

S. Zschocke, Physical Review D **92** (2015) 063015

S. Zschocke, Physical Review D **93** (2016) 103010

## 6. Summary

- no light deflection in flat space-time
- solutions for light ray in field of **bodies at rest**
  - (a) 1PN and 1.5PN with full set of multipoles
- solutions for light ray in field of **bodies in motion**
  - (b) monopoles in 1PM
  - (c) spin-dipoles in 1PM
  - (d) full set of multipoles in 1PN and 1.5PN  
but many things are still unclear
- only a very few 2PN solutions are known:
  - (e) monopole **at rest**
  - (f) quadrupole **at rest**
  - (g) monopole in **slow motion**
- very rare results in 3PN approximation