Multipolar Post-Minkowskian Formalism

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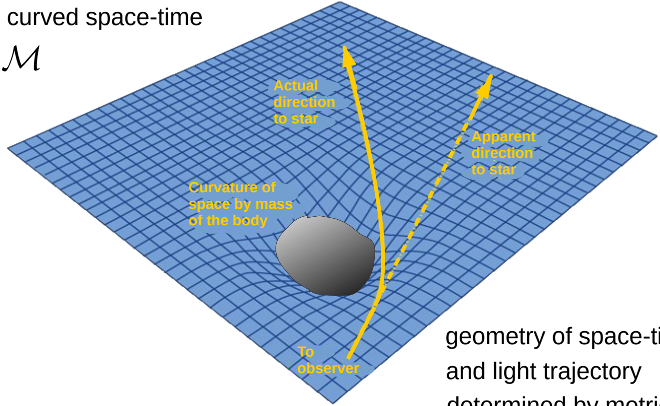
1. Introduction

1.1 Light trajectory through the solar system

- astrometry needs to determine light trajectory $\boldsymbol{x}(t)$

curved space-time

$\mathcal{M}$



geometry of space-time and light trajectory determined by metric $g_{\mu\nu}$

1.2 The geodesic equation

- light trajectory $\boldsymbol{x}(t)$ determined by geodesic equation

$$\boxed{\frac{\ddot{x}^i(t)}{c^2} + \Gamma^i_{\mu\nu}\frac{\dot{x}^\mu(t)}{c}\frac{\dot{x}^\nu(t)}{c} - \Gamma^0_{\mu\nu}\frac{\dot{x}^\mu(t)}{c}\frac{\dot{x}^\nu(t)}{c}\frac{\dot{x}^i(t)}{c} = 0} \qquad (1)$$

- Christoffel symbols are functions of metric tensor

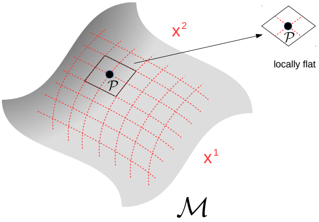
$$\boxed{\Gamma^\alpha_{\mu\nu} = \frac{1}{2}\, g^{\alpha\beta}\left(\frac{\partial g_{\beta\mu}}{\partial x^\nu} + \frac{\partial g_{\beta\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\beta}\right)} \qquad (2)$$

- sum convention: $A^\mu B_\mu = \sum\limits_{\mu=0}^{3} A^\mu B_\mu$ and $A^i B_i = \sum\limits_{i=1}^{3} A^i B_i$

- Talk is concerned with metric $g_{\mu\nu}$ of solar system bodies

1.3 The space-time as semi-Riemannian manifold

a) set of points $\mathcal{P} \in \mathcal{M}$ (Hausdorff space)
b) each point $\mathcal{P} \in \mathcal{M}$ is mapped by coordinates $x^\mu(\mathcal{P})$
c) locally at any $\mathcal{P} \in \mathcal{M}$ flat Minkowskian space-time



$\mathcal{P} \in \mathcal{M}$ possible space-time event iff gauge is fixed , i.e.:

- space-time described by manifold and metric $(\mathcal{M}, \mathbf{g})$
- $(\mathcal{M}, \mathbf{g})$ not unique: two pairs $(\mathcal{M}, \mathbf{g_1})$ and $(\mathcal{M}, \mathbf{g_2})$ are isometric if $\Phi^* \mathbf{g_1} = \mathbf{g_2}$ where $\Phi \in \mathrm{diff}\,(\mathcal{M})$ is an element of all diffeomorphisms $\mathrm{diff}\,(\mathcal{M})$ on $\mathcal{M}$ (equivalence class)
- space-time described by one member of equivalence class
- in practice: gauge is fixed by four coordinate conditions

1.3.1 Classical differential geometry

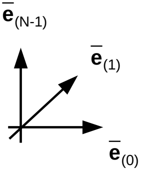
- C.F.Gauß, B. Riemann, E.B. Christoffel, G. Ricci, H. Weyl
  T. Levi-Civita, A. Einstein, M. Grossmann, D. Hilbert
- illustrative approach for (local) basis vectors:
  tangent vectors along coordinate lines $\overline{\mathbf{b}}_{(\mu)} \in \mathrm{T}_{\mathcal{P}}\mathcal{M}$
  and their dual vectors $\overline{\mathbf{b}}^{(\mu)} \in \mathrm{T}^*_{\mathcal{P}}\mathcal{M}$

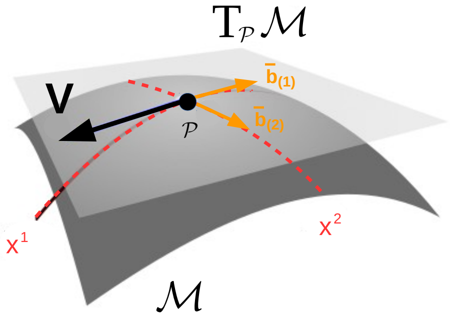
1.3.2 Subsequent developments in differential geometry

- E. Cartan, F. Hausdorff, J.A. Schouten, C. Chevalley,
  J.L. Koszul, N. Nomizu
- abstract approach for (local) basis vectors:
  partial derivatives (of some scalar function) $\overline{\boldsymbol{\partial}}_{(\mu)} \in \mathrm{T}_{\mathcal{P}}\mathcal{M}$
  and their dual vectors (one-forms) $\overline{\mathbf{d}}x^{(\mu)} \in \mathrm{T}^*_{\mathcal{P}}\mathcal{M}$

educational representation of both approaches in Ref.[1]

1.3.3 Tangent space of semi-Riemannian manifold





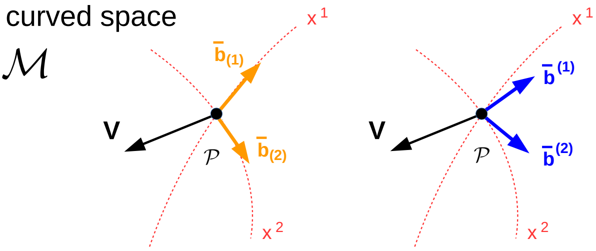
- $\mathrm{T}_{\mathcal{P}}\mathcal{M}$ is Minkowskian space: $\mathrm{n} = \dim \mathrm{T}_{\mathcal{P}}\mathcal{M} = \dim \mathcal{M}$
- $\mathcal{M}$ and $\mathrm{T}_{\mathcal{P}}\mathcal{M}$ assumed to be embedded in R^{N} $(\mathrm{N} > \mathrm{n})$
- basis in R^{N}: $\bar{\mathbf{e}}_{(\mu)} \cdot \bar{\mathbf{e}}_{(\nu)} = \eta_{\mu\nu} = \mathrm{diag}\underbrace{(-1, +1, \ldots, +1)}_{\mathrm{N}}$
- $\bar{\mathbf{b}}_{(\mu)}$ expanded in terms of $\bar{\mathbf{e}}_{(\mu)}$, so $\bar{\mathbf{b}}_{(\mu)} \cdot \bar{\mathbf{b}}_{(\nu)}$ and $\bar{\mathbf{b}}_{(\mu)} \otimes \bar{\mathbf{b}}_{(\nu)}$ defined in terms of $\bar{\mathbf{e}}_{(\mu)} \cdot \bar{\mathbf{e}}_{(\nu)}$ and $\bar{\mathbf{e}}_{(\mu)} \otimes \bar{\mathbf{e}}_{(\nu)}$, respectively

Some comments are in order:

- embedding of manifold $\mathcal{M}$ in R^{N} is always possible:
  (a) Riemann manifolds: Whitney(1936), Nash(1956)
  (b) semi-Riemann manifolds: Clarke(1970), Greene(1970)
- embedding of $\mathcal{M}$ in R^{N} is a theoretical construction and then tensor components w.r.t. basis vectors $\overline{\mathbf{b}}_{(\mu)}$ and $\overline{\mathbf{b}}^{(\mu)}$
- however: manifold $\mathcal{M}$ exists without embedding in R^{N} and then tensor components w.r.t. basis vectors $\overline{\boldsymbol{\partial}}_{(\mu)}$ and $\overline{\mathbf{d}}x^{(\mu)}$
- both approaches, either $\overline{\mathbf{b}}_{(\mu)}$, $\overline{\mathbf{b}}^{(\mu)}$ or $\overline{\boldsymbol{\partial}}_{(\mu)}$, $\overline{\mathbf{d}}x^{(\mu)}$, lead to the same transformation law of tensor components as given by the second equation in Section 1.5.2
- Ricci calculus (starting in Section 3) does not refer to basis vectors explicitly and does not use embedding, but just applies this transformation law of tensor components

1.4 The natural and dual basis

1.4.1 Example: 2-dimensional space



natural basis

$\overline{\mathbf{b}}_{(\mu)}$ $\mu = 1,2$

basis index (which basis vector)

dual basis

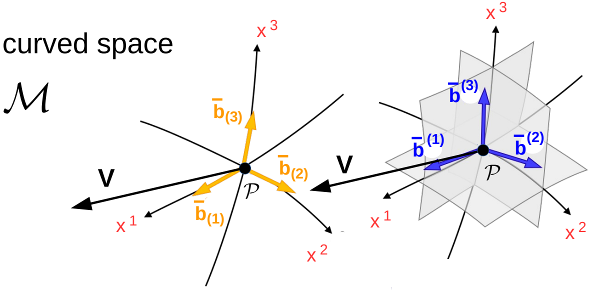
$\overline{\mathbf{b}}^{(\mu)}$ $\mu = 1,2$

basis index (which basis vector)

$$\boxed{\overline{\mathbf{b}}_{(\mu)} \cdot \overline{\mathbf{b}}^{(\nu)} = \delta_\mu^\nu \quad \text{where} \quad \mu, \nu = 1, 2} \tag{3}$$

- $\overline{\mathbf{b}}_{(\mu)} \in \mathrm{T}_{\mathcal{P}}\mathcal{M}$ ... tangent space at $\mathcal{P} \in \mathcal{M}$
- $\overline{\mathbf{b}}^{(\mu)} \in \mathrm{T}^*_{\mathcal{P}}\mathcal{M}$ ... dual tangent space at $\mathcal{P} \in \mathcal{M}$

1.4.2 Example: 3-dimensional space



natural basis

$\bar{\mathbf{b}}_{(\mu)}$ $\mu = 1,2,3$

basis index (which basis vector)

dual basis

$\bar{\mathbf{b}}^{(\mu)}$ $\mu = 1,2,3$

basis index (which basis vector)

$$\boxed{\bar{\mathbf{b}}_{(\mu)} \cdot \bar{\mathbf{b}}^{(\nu)} = \delta^{\nu}_{\mu} \quad \text{where} \quad \mu, \nu = 1, 2, 3} \tag{4}$$

- $\bar{\mathbf{b}}_{(\mu)} \in \mathrm{T}_{\mathcal{P}}\mathcal{M}$ ... tangent space at $\mathcal{P} \in \mathcal{M}$
- $\bar{\mathbf{b}}^{(\mu)} \in \mathrm{T}^{*}_{\mathcal{P}}\mathcal{M}$ ... dual tangent space at $\mathcal{P} \in \mathcal{M}$

- in general case: $\overline{\mathbf{b}}_{(\mu)}$ and $\overline{\mathbf{b}}^{(\mu)}$ are not unit vectors
- $\mathbf{V}$ developed in natural basis and dual basis

$$\boxed{\mathbf{V} = V^{\mu}\,\overline{\mathbf{b}}_{(\mu)} = V_{\mu}\,\overline{\mathbf{b}}^{(\mu)}} \tag{5}$$

- in oblique and curvilinear coordinate systems:

  natural and dual basis different $\overline{\mathbf{b}}_{(\mu)} \neq \overline{\mathbf{b}}^{(\mu)}$

  contravariant and covariant components different $V^{\mu} \neq V_{\mu}$

- only in Cartesian coordinate systems:

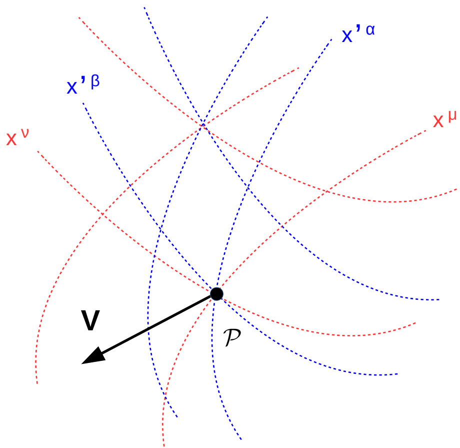
  natural and dual basis coincide $\overline{\mathbf{b}}_{(\mu)} = \overline{\mathbf{b}}^{(\mu)}$

  contravariant and covariant components coincide $V^{\mu} = V_{\mu}$

1.5 Coordinate transformations from $\{x\}$ to $\{x'\}$

- How transform basis and vector components?

curved
space-time
$\mathcal{M}$



1.5.1 Transformation of basis and vector components

- natural basis and contravariant components

$$\boxed{\overline{\mathbf{b}}_{(\beta')} = B^{\nu}{}_{\beta'}\, \overline{\mathbf{b}}_{(\nu)} \qquad V^{\alpha'} = A^{\alpha'}{}_{\mu}\, V^{\mu}} \tag{6}$$

- dual basis and covariant components

$$\boxed{\overline{\mathbf{b}}^{(\alpha')} = A^{\alpha'}{}_{\mu}\, \overline{\mathbf{b}}^{(\mu)} \qquad V_{\beta'} = B^{\nu}{}_{\beta'}\, V_{\nu}} \tag{7}$$

- Jacobian and inverse Jacobian

$$\boxed{A^{\alpha'}{}_{\mu} = \left(\frac{\partial x^{\alpha'}}{\partial x^{\mu}}\right) \quad \text{and} \quad B^{\nu}{}_{\beta'} = \left(\frac{\partial x^{\nu}}{\partial x^{\beta'}}\right)} \tag{8}$$

1.5.2 Transformation of tensor components

- tensor $\mathbf{T}$ is a generalization of vector $\mathbf{V}$ in Eq. (5)
- $\mathbf{T}$ developed in natural basis and dual basis

$$\boxed{\mathbf{T} = \underbrace{T^{\mu_1 \ldots \mu_k}{}_{\nu_1 \ldots \nu_l}}_{\text{tensor components}} \underbrace{\overline{\mathbf{b}}_{(\mu_1)} \otimes \cdots \otimes \overline{\mathbf{b}}_{(\mu_k)}}_{\text{natural basis}} \underbrace{\overline{\mathbf{b}}^{(\nu_1)} \otimes \cdots \otimes \overline{\mathbf{b}}^{(\nu_l)}}_{\text{dual basis}}}$$

- transformation of components of $\mathbf{T}$

$$\boxed{T^{\alpha'_1 \ldots \alpha'_k}{}_{\beta'_1 \ldots \beta'_l} = A^{\alpha'_1}{}_{\mu_1} \ldots A^{\alpha'_k}{}_{\mu_k} \; B^{\nu_1}{}_{\beta'_1} \ldots B^{\nu_l}{}_{\beta'_l} \; T^{\mu_1 \ldots \mu_k}{}_{\nu_1 \ldots \nu_l}}$$

- $\mathbf{T}$ are called tensors of rank (k, l) (geometrical objects)

1.5.3 Usefulness of covariant and contravariant components

1. complete tensor contraction $(k = l)$ yields scalars

$$\boxed{\begin{aligned} S'_1 &= T^{\alpha'_1}{}_{\alpha'_1} &&= T^{\mu_1}{}_{\mu_1} &&= S_1 \\ S'_2 &= T^{\alpha'_1\alpha'_2}{}_{\alpha'_1\alpha'_2} &&= T^{\mu_1\mu_2}{}_{\mu_1\mu_2} &&= S_2 \\ &\vdots \\ S'_k &= T^{\alpha'_1\ldots\alpha'_k}{}_{\alpha'_1\ldots\alpha'_k} &&= T^{\mu_1\ldots\mu_k}{}_{\mu_1\ldots\mu_k} &&= S_k \end{aligned}} \tag{9}$$

2. incomplete tensor contraction yields new tensors

$$\boxed{T^{\mu_1\ldots\mu_n\ldots\mu_k}{}_{\nu_1\ldots\mu_n\ldots\nu_l}} \tag{10}$$

3. tensor relations valid in any coordinate system, e.g.

$$\boxed{U^{\alpha'_1\alpha'_2}{}_{\beta'_1} = W^{\alpha'_1\alpha'_2}{}_{\beta'_1} \quad \Longleftrightarrow \quad U^{\mu_1\mu_2}{}_{\nu_1} = W^{\mu_1\mu_2}{}_{\nu_1}} \tag{11}$$

2. The metric tensor

2.1 Definition of metric tensor by line element

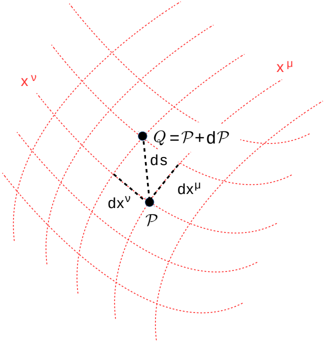
- definition of line element

$$\boxed{ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu} \tag{12}$$

curved

space-time

$\mathcal{M}$



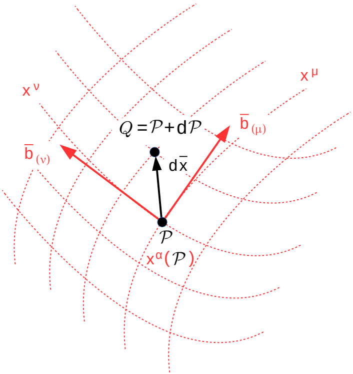
- How to get $g_{\mu\nu}$?

- consider line element as norm $ds^2 = d\bar{\mathbf{x}} \cdot d\bar{\mathbf{x}}$ of vector $d\bar{\mathbf{x}}$

curved

space-time

$\mathcal{M}$



- $d\bar{\mathbf{x}}$ ... four-vector with components dx^μ with $\mu = 0, 1, 2, 3$
- $\bar{\mathbf{b}}_{(\mu)}$ ... four basis vectors with $\mu = 0, 1, 2, 3$ at $\mathcal{P} \in \mathcal{M}$

$$\boxed{d\bar{\mathbf{x}} = dx^\mu \, \bar{\mathbf{b}}_{(\mu)}} \tag{13}$$

- then the line element is given by

$$\boxed{ds^2 = \overline{\mathbf{b}}_{(\mu)} \cdot \overline{\mathbf{b}}_{(\nu)} \; dx^\mu \; dx^\nu = \overline{\mathbf{b}}^{(\mu)} \cdot \overline{\mathbf{b}}^{(\nu)} \; dx_\mu \; dx_\nu} \qquad (14)$$

- metric tensor components (Eq. (44) in [1])

$$\boxed{g_{\mu\nu} = \overline{\mathbf{b}}_{(\mu)} \; \cdot \; \overline{\mathbf{b}}_{(\nu)} \quad \text{and} \quad g^{\mu\nu} = \overline{\mathbf{b}}^{(\mu)} \; \cdot \; \overline{\mathbf{b}}^{(\nu)}} \qquad (15)$$

note that $g^{\mu\nu} = g^{-1}_{\mu\nu}$ i.e. $g^{\mu\alpha} \, g_{\alpha\nu} = \delta^\mu_\nu$

- metric tensor (Eq. (47) in [1])

$$\boxed{\mathbf{g} = g_{\mu\nu} \; \overline{\mathbf{b}}^{(\mu)} \; \otimes \; \overline{\mathbf{b}}^{(\nu)} = g^{\mu\nu} \; \overline{\mathbf{b}}_{(\mu)} \; \otimes \; \overline{\mathbf{b}}_{(\nu)}} \qquad (16)$$

often it is not distinguished between ds^2 and $g_{\mu\nu}$ and $\mathbf{g}$

- metric has $n\left(n+1\right)/2$ independent components in n-dimensional space because of symmetry $g_{\mu\nu} = g_{\nu\mu}$

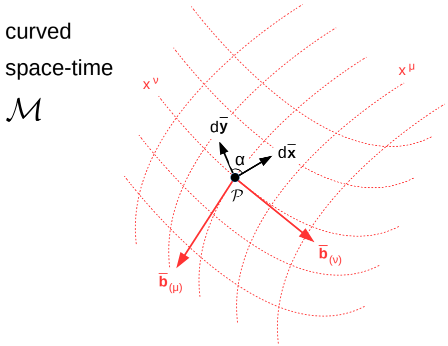
- e.g.: 10 independent components in 4-dimensional space

$$\boxed{g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{10} & g_{11} & g_{12} & g_{13} \\ g_{20} & g_{21} & g_{22} & g_{23} \\ g_{30} & g_{31} & g_{32} & g_{33} \end{pmatrix}} \tag{17}$$

2.2 The metric tensor and angles

- consider two infinitesimal vectors $d\overline{\mathbf{x}}, d\overline{\mathbf{y}} \in \mathrm{T}_{\mathcal{P}}\,\mathcal{M}$

$$\boxed{d\overline{\mathbf{x}} = dx^{\mu}\,\overline{\mathbf{b}}_{(\mu)} \quad \text{and} \quad d\overline{\mathbf{y}} = dy^{\nu}\,\overline{\mathbf{b}}_{(\nu)}} \tag{18}$$



$$\boxed{\cos\alpha = \frac{d\overline{\mathbf{x}} \cdot d\overline{\mathbf{y}}}{\|d\overline{\mathbf{x}}\|\;\|d\overline{\mathbf{y}}\|} = \frac{g_{\mu\nu}\,dx^{\mu}dy^{\nu}}{\sqrt{g_{\alpha\beta}\,dx^{\alpha}dx^{\beta}}\;\sqrt{g_{\alpha\beta}\,dy^{\alpha}dy^{\beta}}}} \tag{19}$$

2.3 The metric tensor and converting components

- contravariant in covariant components by metric tensor

$$\boxed{\begin{aligned} V_\mu &= g_{\mu\alpha}\; V^\alpha \\ T_{\mu\nu} &= g_{\mu\alpha}\, g_{\nu\beta}\; T^{\alpha\beta} \end{aligned}} \qquad (20)$$

- covariant in contravariant components by metric tensor

$$\boxed{\begin{aligned} V^\mu &= g^{\mu\alpha}\; V_\alpha \\ T^{\mu\nu} &= g^{\mu\alpha}\, g^{\nu\beta}\; T_{\alpha\beta} \end{aligned}} \qquad (21)$$

- mathematical foundation behind Eqs. (20) and (21)
  Musical Isomorphism between $\mathrm{T}_{\mathcal{P}}\mathcal{M}$ and $\mathrm{T}^*_{\mathcal{P}}\mathcal{M}$ [2]

2.4 Examples for the metric tensor

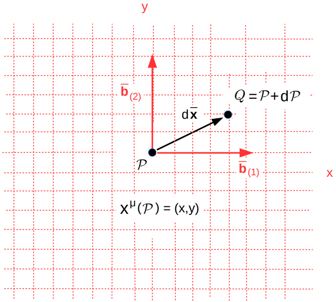
2.4.1 Metric tensor of flat space R^2 in Cartesian coordinates

- Cartesian coordinates: $(x^1, x^2) = (x, y)$

2-dimensional space
$\mathcal{M} = \mathrm{R}^2$

$\bar{\mathbf{b}}_{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \bar{\mathbf{e}}_x$

$\bar{\mathbf{b}}_{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \bar{\mathbf{e}}_y$



- metric tensor

$$\boxed{g_{\mu\nu} = \bar{\mathbf{b}}_{(\mu)} \cdot \bar{\mathbf{b}}_{(\nu)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} \tag{22}$$

- line element

$$\boxed{ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu = dx^2 + dy^2} \tag{23}$$

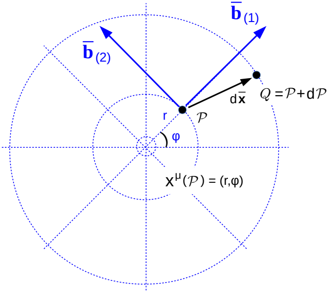
2.4.2 Metric tensor of flat space R^2 in Polar coordinates

- Polar coordinates: $(x^1, x^2) = (r, \varphi)$

2-dimensional space
$\mathcal{M} = \mathrm{R}^2$

$$\bar{\mathbf{b}}_{(1)} = \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix} = \bar{\mathbf{e}}_r$$

$$\bar{\mathbf{b}}_{(2)} = \begin{pmatrix} -r\sin\varphi \\ r\cos\varphi \end{pmatrix} = r\,\bar{\mathbf{e}}_\varphi$$



- metric tensor

$$\boxed{g_{\mu\nu} = \bar{\mathbf{b}}_{(\mu)} \cdot \bar{\mathbf{b}}_{(\nu)} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}} \tag{24}$$

- line element

$$\boxed{ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu = dr^2 + r^2\, d\varphi^2} \tag{25}$$

2.4.3 Some important conclusions

(1) metric components $g_{\mu\nu}$ different in different coordinates

$$\boxed{g_{\mu\nu} \neq g_{\mu\nu}} \tag{26}$$

but distance in Eq. (23) is the same as in Eq. (25)

$$\boxed{ds_g^2\,(P, Q) = ds_g^2\,(P, Q)} \tag{27}$$

i.e. distance is independent of chosen coordinates

(2) therefore: metric $\mathbf{g}$ as geometrical object remains the same under (passive) change of coordinates

$$\boxed{\mathbf{g} = \mathbf{g}} \tag{28}$$

- conclusions **(1)** and **(2)** are valid in general

2.4.4 Metric tensor of sphere S^2 in spherical coordinates

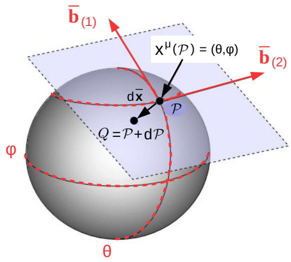
- Spherical coordinates: $(x^1, x^2) = (\theta, \varphi)$

2-dimensional space

$\mathcal{M} = S^2$

$$\bar{\mathbf{b}}_{(1)} = \begin{pmatrix} R\cos\theta\ \cos\varphi \\ R\cos\theta\ \sin\varphi \\ -R\ \sin\theta \end{pmatrix} = R\ \bar{\mathbf{e}}_\theta$$

$$\bar{\mathbf{b}}_{(2)} = \begin{pmatrix} -R\sin\theta\ \sin\varphi \\ R\sin\theta\cos\varphi \\ 0 \end{pmatrix} = R\sin\theta\ \bar{\mathbf{e}}_\varphi$$



- metric tensor

$$\boxed{g_{\mu\nu} = \bar{\mathbf{b}}_{(\mu)} \cdot \bar{\mathbf{b}}_{(\nu)} = \begin{pmatrix} R^2 & 0 \\ 0 & R^2 \sin^2\theta \end{pmatrix}} \tag{29}$$

- line element

$$\boxed{ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu = R^2\, d\theta^2 + R^2 \sin^2\theta\, d\varphi^2} \tag{30}$$

3. The field equations of gravity

3.1 Einstein's field equations of gravity

- metric tensor $g_{\alpha\beta}$ is determined by the field equations

$$\boxed{\underbrace{R_{\alpha\beta} - \frac{1}{2}\, g_{\alpha\beta}\, R}_{\text{curvature of space}} = \underbrace{\frac{8\,\pi\,G}{c^4}\; T_{\alpha\beta}}_{\text{matter}}} \tag{31}$$

- Ricci tensor ("Ricci curvature" of space-time)

$$\boxed{R_{\alpha\beta} = \Gamma^{\mu}_{\alpha\beta\,,\,\mu} - \Gamma^{\mu}_{\alpha\mu\,,\,\beta} + \Gamma^{\mu}_{\mu\nu}\,\Gamma^{\nu}_{\alpha\beta} - \Gamma^{\nu}_{\alpha\mu}\,\Gamma^{\mu}_{\nu\beta}} \tag{32}$$

- Ricci scalar

$$\boxed{R = R_{\alpha\beta}\, g^{\alpha\beta}} \tag{33}$$

- Riemann-Christoffel tensor (curvature of space-time)

$$\boxed{R^{\mu}{}_{\alpha\nu\beta} = \Gamma^{\mu}_{\alpha\beta\,,\,\nu} - \Gamma^{\mu}_{\alpha\nu\,,\,\beta} + \Gamma^{\mu}_{\nu\rho}\,\Gamma^{\rho}_{\alpha\beta} - \Gamma^{\rho}_{\alpha\nu}\,\Gamma^{\mu}_{\rho\beta}} \tag{34}$$

- stress-energy tensor of matter $T_{\alpha\beta}$
- $T_{\alpha\beta} = T_{\beta\alpha}$ hence only 10 independent components

$$\boxed{T_{\alpha\beta} = \begin{pmatrix} T_{00} & T_{01} & T_{02} & T_{03} \\ T_{10} & T_{11} & T_{12} & T_{13} \\ T_{20} & T_{21} & T_{22} & T_{23} \\ T_{30} & T_{31} & T_{32} & T_{33} \end{pmatrix}} \tag{35}$$

- T_{00} ... energy-density
- T_{0j} ... energy-flux in x^j-direction
- T_{jk} ... flux of x^j-component of momentum in x^k-direction

- Eqs. (31) represent 10 equations for 10 components of $g_{\alpha\beta}$ but they are not independent of each other
- 4 Bianchi identities

$$\boxed{\left(R^{\alpha\beta} - \frac{1}{2}\, g^{\alpha\beta}\, R\right)_{;\,\beta} = 0 \qquad \Longrightarrow \qquad T^{\alpha\beta}{}_{;\,\beta} = 0} \tag{36}$$

covariant derivative for scalar S, vector V^{α}, tensor $T^{\alpha\beta}$

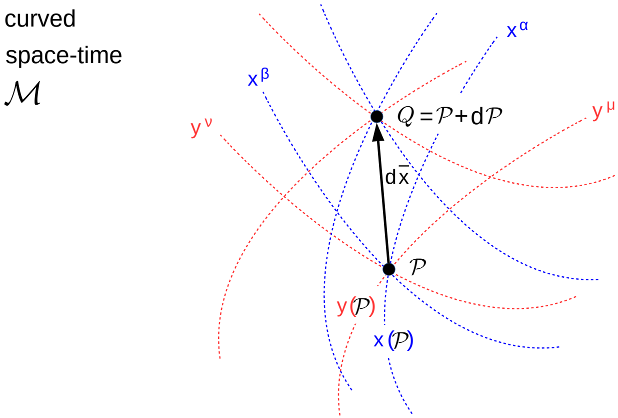
$$\boxed{\begin{aligned} S_{;\,\mu} &= S_{,\,\mu} \\ V^{\alpha}_{\;;\,\mu} &= V^{\alpha}_{\;,\,\mu} + \Gamma^{\alpha}_{\mu\nu}\, V^{\nu} \\ T^{\alpha\beta}{}_{;\,\mu} &= T^{\alpha\beta}{}_{,\,\mu} + \Gamma^{\alpha}_{\mu\nu}\, T^{\nu\beta} + \Gamma^{\beta}_{\mu\nu}\, T^{\alpha\nu} \end{aligned}} \tag{37}$$

Therefore:

- Eqs. (31) represent only 6 independent equations
- Eqs. (31) determine $g_{\alpha\beta}$ up to (4 passive or 4 active) coordinate transformations

3.2 Invariance of GR by passive coordinate transformations

- keep points of $\mathcal{M}$ fixed and change coordinates



- passive coordinate transformation implies four equations

$$\boxed{x(\mathcal{P}) \quad \Longrightarrow \quad y(\mathcal{P})} \tag{38}$$

- passive coordinate transformations do not change ds^2 of points $P, Q \in \mathcal{M}$

$$\boxed{ds^2 = g_{\alpha\beta}\left(x\left(\mathcal{P}\right)\right)\; dx^\alpha dx^\beta = g_{\mu\nu}\left(y\left(\mathcal{P}\right)\right)\; dy^\mu dy^\nu} \tag{39}$$

- Eq. (39) means that these sets are physically equivalent (i.e. these sets describe the very same physical system)

$$\boxed{\left(\mathcal{M}\;,\; g_{\alpha\beta}\left(x\right)\right) \Longleftrightarrow \left(\mathcal{M}\;,\; g_{\mu\nu}\left(y\right)\right)} \tag{40}$$

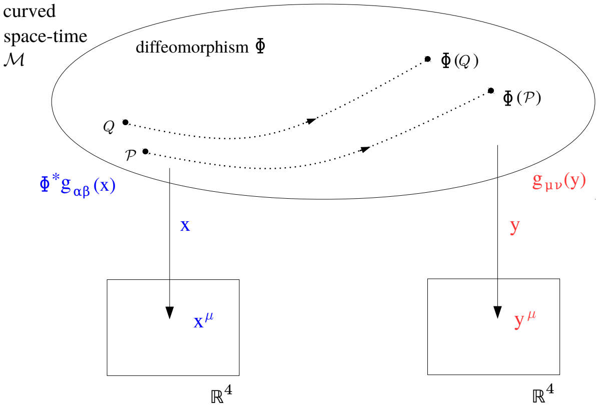
- Eq. (39) implies transformation

$$\boxed{g_{\alpha\beta}\left(x\left(\mathcal{P}\right)\right) = \frac{\partial y^\mu}{\partial x^\alpha}\frac{\partial y^\nu}{\partial x^\beta}\; g_{\mu\nu}\left(y\left(\mathcal{P}\right)\right)} \tag{41}$$

- $g_{\alpha\beta}$ and $g_{\mu\nu}$ components of same metric: $\mathbf{g} = \mathbf{g}$

3.3 Invariance of GR by active coordinate transformations

- keep coordinates fixed and change points of $\mathcal{M}$



- active coordinate transformation implies four equations

$$\boxed{x(\mathcal{P}) \quad \Longrightarrow \quad x(\Phi(\mathcal{P})) = y(\mathcal{P}) \qquad \forall \mathcal{P} \in \mathcal{M}} \qquad (42)$$

- active coordinate transformations do not change ds^2 of points $\mathcal{P}, \mathrm{Q} \in \mathcal{M}$ and their images $\Phi(\mathcal{P}), \Phi(\mathrm{Q}) \in \mathcal{M}$

$$\boxed{ds^2 = \Phi^* g_{\alpha\beta}(\mathcal{P}, \mathrm{Q})\, dx^\alpha dx^\beta = g_{\mu\nu}(\Phi(\mathcal{P}), \Phi(\mathrm{Q}))\, dy^\mu dy^\nu} \quad (43)$$

- where $\Phi^* g_{\alpha\beta}$ ... pulled-back metric ($\Phi^* \mathbf{g} = \mathbf{g}$)
- $g_{\alpha\beta}$ and $g_{\mu\nu}$ components of distinct metrics: $\mathbf{g} \neq \mathbf{g}$
- Eq. (43) means that these sets are isometric (p.227 in [3]) i.e.: they are physically equivalent (*Leibniz equivalence*)

$$\boxed{(\mathcal{M}\ ,\ \Phi^* g_{\alpha\beta}(x)) \Longleftrightarrow (\mathcal{M}\ ,\ g_{\mu\nu}(y))} \quad (44)$$

- Eq. (43) implies transformation

$$\boxed{\Phi^* g_{\alpha\beta}(x(\mathcal{P})) = \frac{\partial y^\mu}{\partial x^\alpha} \frac{\partial y^\nu}{\partial x^\beta}\, g_{\mu\nu}(y(\mathcal{P}))} \quad (45)$$

- if Φ proceeds along congruence of Killing vector field then $\mathbf{g} = \mathbf{g}$ and in this case Φ is an isometry (p.43 in [3])
- one has carefully to distinguish isometric and isometry

3.4 Landau-Lifschitz formulation of gravity

- exact reformulation of Eqs. (31) by Landau-Lifschitz [4, 5]

$$\boxed{H^{\alpha\mu\beta\nu}{}_{,\,\mu\nu} = \frac{16\,\pi\,G}{c^4}\,(-g)\left(T^{\alpha\beta} + t^{\alpha\beta}_{\mathrm{LL}}\right)} \tag{46}$$

- super potential

$$\boxed{H^{\alpha\mu\beta\nu} = \mathfrak{g}^{\alpha\beta}\,\mathfrak{g}^{\mu\nu} - \mathfrak{g}^{\alpha\nu}\,\mathfrak{g}^{\beta\mu}} \tag{47}$$

- metric density

$$\boxed{\mathfrak{g}^{\alpha\beta} = \sqrt{-g}\;g^{\alpha\beta}} \tag{48}$$

- $g = \det\left(g_{\mu\nu}\right)$ ... determinant of metric tensor
- $t^{\alpha\beta}_{\mathrm{LL}}$ ... Landau-Lifschitz pseudo-tensor
  given by Eq. (6.5) in Poisson and Will, *Gravity* (2014)

- LL formulation (i.e. Eq. (46)) is a reformulation of GR as a non-linear field theory in flat background space-time $\mathcal{M}_0$ (diagrammatical representation is given in Section 4.1)
- cf. text in

  D. Keppel, D.A. Nichols, Y. Chen, K.S. Thorne,

  Physical Review D **80** (2009) 124015:

  "... one reformulates the Einstein equations as a nonlinear field theory in the space of that flat auxiliary metric..."

  "... Landau-Lifshitz formulation of general relativity as a nonlinear field theory in flat space-time..."

- general-covariant LL formulation as non-linear field theory in flat background space-time has been developed in [5]

- Eq. (46) is valid in any curvilinear coordinates which cover the flat background manifold $\mathcal{M}_0$
- Eq. (46) represents 10 equations for 10 components of $\overline{g}^{\alpha\beta}$ but they are not independent of each other
- 4 identity relations

$$\boxed{H^{\alpha\mu\beta\nu}{}_{,\,\mu\nu\beta} = 0 \quad \Longrightarrow \quad \underbrace{\left[(-g)\left(T^{\alpha\beta} + t^{\alpha\beta}_{\rm LL}\right)\right]_{,\,\beta} = 0}_{\text{local law of conservation}}} \qquad (49)$$

- Eq. (46) represents only 6 independent equations
- Eq. (46) determines $\overline{g}^{\alpha\beta}$ up to (4 passive or 4 active) coordinate transformations

- Eq. (49) related to global energy-momentum conservation as it will be discussed in Section 3.5

- metric density

$$\boxed{\overline{g}^{\,\alpha\beta} = \sqrt{-g}\, g^{\alpha\beta} \quad \text{with} \quad g = \det\left(g_{\mu\nu}\right)} \tag{50}$$

- orthogonality relation

$$\boxed{\overline{g}^{\alpha\mu}\, \overline{g}_{\mu\beta} = \delta^{\alpha}_{\beta}} \tag{51}$$

allows to switch between upper and lower components

- metric tensor

$$\boxed{g^{\,\alpha\beta} = \sqrt{-\overline{g}}\, \overline{g}^{\alpha\beta} \quad \text{with} \quad \overline{g} = \det\left(\overline{g}_{\mu\nu}\right)} \tag{52}$$

- orthogonality relation

$$\boxed{g^{\alpha\mu}\, g_{\mu\beta} = \delta^{\alpha}_{\beta}} \tag{53}$$

allows to switch between upper and lower components

3.5 The energy-momentum conservation

- local conservation law (49) admits global conservation law of energy-momentum for isolated systems

$$\frac{dP^\alpha}{dt} = \underbrace{\frac{d}{dt}\int d^3x\,(-g)\left(T^{\alpha\beta} + t_{\mathrm{LL}}^{\alpha\beta}\right) = 0}_{\text{global law of conservation}} \qquad (54)$$

- statements valid if (54) Minkowskian at spatial infinity:

1. integral (54) is convergent
2. integral (54) is coordinate-independent
3. integral (54) is global energy-momentum conservation

(i) E. Poisson, C. Will "*Gravity*" (Box 6.1)

(ii) C. Misner, K. Thorne, J. Wheeler "*Gravitation*" (§20.5)

4. Field equations of gravity in flat space

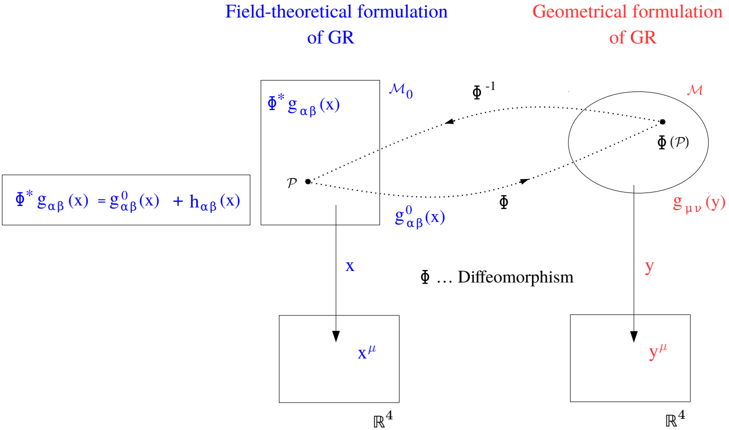
4.1 Einstein's field equations of gravity in flat space

- as mentioned LL is reformulation of GR in flat space-time
- separation of $g_{\alpha\beta}$ in flat metric $g^0_{\alpha\beta}$ and perturbation $h_{\alpha\beta}$

$$\boxed{g_{\alpha\beta}(x) = g^0_{\alpha\beta}(x) + h_{\alpha\beta}(x)} \qquad (55)$$

- $h_{\alpha\beta}$ propagates in flat background space-time
- many physicists developed field-theoretical formulation:
  M. Fierz, N. Rosen, A. Papapetrou, S.N. Gupta, S. Deser, R. Kraichnan, W. Thirring, F.J. Belifante, L.D. Landau, J.M. Lifschitz, R. Feynman, S. Weinberg, S.W. Hawking, S.V. Babak, L.P. Grishchuk, A.N. Petrov, A.D. Popova,
- an excellent historical overview is given by:
  J. Brian Pitts, W.C. Schieve (2018) in gr-qc/0111004
  Null Cones in Lorentz-Covariant General Relativity

- Eq. (55) in language of differential geometry



$$\boxed{\Phi^* g_{\alpha\beta}(x) = \frac{\partial y^\mu}{\partial x^\alpha} \frac{\partial y^\nu}{\partial x^\beta} g_{\mu\nu}(y)} \qquad (56)$$

- active coordinate transformations do not change ds^2

$$\boxed{\begin{aligned} ds^2 &= \underbrace{g_{\mu\nu}(y)\;\; dy^\mu dy^\nu}_{\text{in } \mathcal{M}} = \underbrace{\Phi^* g_{\alpha\beta}(x)\;\; dx^\alpha\, dx^\beta}_{\text{in } \mathcal{M}_0} \\ &= \underbrace{g^0_{\alpha\beta}(x)\;\; dx^\alpha dx^\beta}_{ds_0^2 \text{ in } \mathcal{M}_0} + \underbrace{h_{\alpha\beta}(x)}_{\text{fields in } \mathcal{M}_0}\;\; \underbrace{dx^\alpha\, dx^\beta}_{\text{in } \mathcal{M}_0} \end{aligned}} \tag{57}$$

- Eq. (57) means that these sets

$$\boxed{(\mathcal{M}_0\;,\;\; \Phi^* g_{\alpha\beta}(x)) \Longleftrightarrow (\mathcal{M}\;,\;\; g_{\mu\nu}(y))} \tag{58}$$

describe the same physical system equivalently
in spite that manifolds $\mathcal{M}_0$ and $\mathcal{M}$ are not isometric

$$\boxed{g^0_{\alpha\beta}(x) \neq \frac{\partial y^\mu}{\partial x^\alpha}\frac{\partial y^\nu}{\partial x^\beta}\, g_{\mu\nu}(y)} \tag{59}$$

4.2 Landau-Lifschitz formulation in flat space

- instead to insert (55) into Einstein's field equations (31) Landau-Lifschitz formulation (46) is more appropriate to get field-theoretical formulation of GR in closed form

- separation of metric $g_{\alpha\beta}$ into flat metric $g^0_{\alpha\beta} = \eta_{\alpha\beta}$ and perturbation $h_{\alpha\beta}$ implies in terms of metric density:

$$\boxed{\overline{g}^{\,\alpha\beta} = \eta^{\alpha\beta} - \overline{h}^{\,\alpha\beta}} \tag{60}$$

- harmonic gauge

$$\boxed{\overline{h}^{\,\alpha\beta}{}_{,\,\beta} = 0 \quad \Longleftrightarrow \quad \Box_g x^\alpha = 0} \tag{61}$$

- curved d'Alembert: $\Box_g = (-g)^{-1/2}\,\partial_\mu\,\left((-g)^{1/2}\,g^{\mu\nu}\right)\partial_\nu$
- curved d'Alembert in harmonic coordinates: $\Box_g = g^{\mu\nu}\,\partial_\mu\,\partial_\nu$

- inserting (60) and (61) into (46) yields non-linear wave-equation in flat background manifold $\mathcal{M}_0$

$$\boxed{\Box\, \overline{h}^{\alpha\beta} = -\frac{16\,\pi\, G}{c^4}\,\left(\tau^{\alpha\beta} + t^{\alpha\beta}\right)\ \text{with}\ \Box = \eta^{\mu\nu}\,\partial_\mu\,\partial_\nu} \qquad (62)$$

- Eq. (62) so-called relaxed Einstein's field equations
- $\tau^{\alpha\beta} = (-g)\, T^{\alpha\beta}$
- $t^{\alpha\beta} = (-g)\left(t^{\alpha\beta}_{\mathrm{LL}} + t^{\alpha\beta}_{\mathrm{H}}\right)$
- $t^{\alpha\beta}_{\mathrm{H}}$ ... harmonic gauge term given by Eq. (6.53) in Poisson and Will, *Gravity* (2014)

- $\overline{h}^{\alpha\beta}{}_{,\,\beta} = 0$ equivalent to local conservation law Eq. (49)

$$\boxed{\left(\tau^{\alpha\beta} + t^{\alpha\beta}\right)_{,\,\beta} = 0 \quad \Longleftrightarrow \quad \overline{h}^{\alpha\beta}{}_{,\,\beta} = 0} \qquad (63)$$

5. The residual gauge transformation
5.1 The class of harmonic coordinates

- harmonic coordinates not uniquely determined by Eq. (61)

$$\boxed{\Box_g \, x^\alpha = 0}$$

- consider a coordinate transformation of the form

$$\boxed{x'^{\,\alpha} = x^\alpha + \varphi^\alpha(x)} \qquad (64)$$

- these new coordinates $\{x'\}$ are also harmonic if

$$\boxed{\Box_g \, \varphi^\alpha(x) = 0} \qquad (65)$$

- Eq. (61) selects a class of infinitely many harmonic systems

- it is advantageous to adopt the following convention:

$$\boxed{x'^{\alpha} = x^{\alpha} + \varphi^{\alpha}(x)}$$

- $\{x'\}$ are curvilinear harmonic coordinates which map $\mathcal{M}_0$

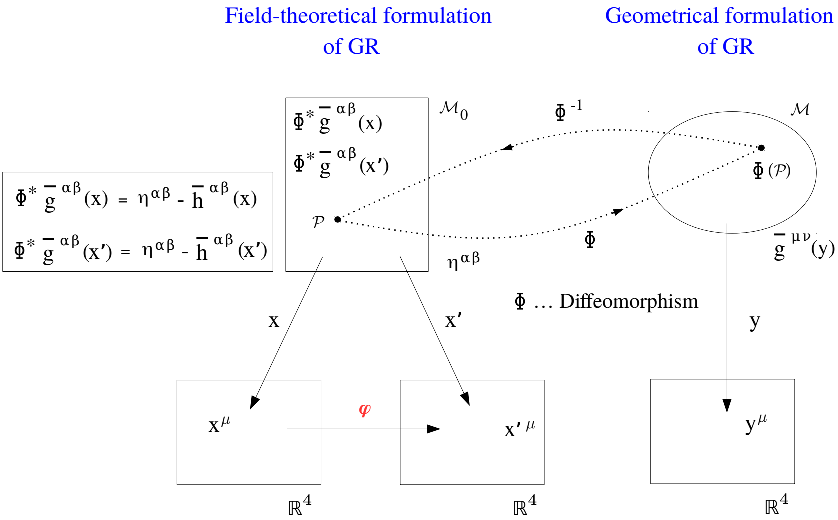
- $\{x\}$ are Minkowskian coordinates which map $\mathcal{M}_0$

- $\varphi^{\alpha}(x)$ are gauge functions in Minkowskian coordinates

- note that Eq. (65) implies $\overline{g}^{\mu\nu}\,\partial_{\mu}\,\partial_{\nu}\,\varphi^{\alpha} = 0$
- hence Eq. (65) using Eq. (60) can be written in the form

$$\boxed{\Box\,\varphi^{\alpha}(x) - \overline{h}^{\mu\nu}\,\partial_{\mu}\,\partial_{\nu}\,\varphi^{\alpha}(x) = 0} \qquad (66)$$

5.2 Diagramatical representation of Eq. (60) and Eq. (64)



5.3 The residual gauge transformation of metric density

- change of metric density under coordinate transformations

$$\boxed{\overline{g}'^{\,\alpha\beta}\left(x'\right) = \frac{1}{\left|J\left(x\right)\right|}\,\frac{\partial x'^{\,\alpha}}{\partial x^{\mu}}\,\frac{\partial x'^{\,\beta}}{\partial x^{\nu}}\,\overline{g}^{\mu\nu}\left(x\right)} \tag{67}$$

- where J is Jacobian determinant of Eq. (64)
- series expansion yields in Minkowskian system $\{x\}$

$$\boxed{\begin{aligned} \overline{g}'^{\,\alpha\beta} &= \overline{g}^{\alpha\beta} + \left(\frac{1}{|J|} - 1\right)\overline{g}^{\alpha\beta} \\ &+ \frac{1}{|J|}\left(\varphi^{\alpha}{}_{,\mu}\,\overline{g}^{\mu\beta} + \varphi^{\beta}{}_{,\nu}\,\overline{g}^{\nu\alpha} + \varphi^{\alpha}{}_{,\mu}\,\varphi^{\beta}{}_{,\nu}\,\overline{g}^{\mu\nu}\right) \\ &- \sum_{n=1}^{\infty}\frac{1}{n!}\,\overline{g}'^{\,\alpha\beta}{}_{,\mu_1\ldots\mu_n}\,\varphi^{\mu_1}\ldots\varphi^{\mu_n} \end{aligned}} \tag{68}$$

- the gauge terms have no impact on observables

5.4 The residual gauge transformation of metric tensor

- change of metric tensor under coordinate transformations

$$\boxed{g_{\alpha\beta}\,(x) = \frac{\partial x'^{\,\mu}}{\partial x^{\alpha}}\,\frac{\partial x'^{\,\nu}}{\partial x^{\beta}}\,g'_{\mu\nu}\,(x')} \tag{69}$$

- series expansion yields in Minkowskian system $\{x\}$

$$\boxed{\begin{aligned} g_{\alpha\beta} = {} & g'_{\alpha\beta} + \varphi^{\mu}{}_{,\,\alpha}\,g'_{\mu\beta} + \varphi^{\nu}{}_{,\,\beta}\,g'_{\nu\alpha} + \varphi^{\mu}{}_{,\,\alpha}\,\varphi^{\nu}{}_{,\,\beta}\,g'_{\mu\nu} \\ & + \left(\delta^{\mu}_{\alpha} + \varphi^{\mu}{}_{,\,\alpha}\right)\left(\delta^{\nu}_{\beta} + \varphi^{\nu}{}_{,\,\beta}\right)\sum_{n=1}^{\infty}\frac{1}{n!}\,g'_{\mu\nu\,,\,\mu_1\ldots\mu_n}\,\varphi^{\mu_1}\ldots\varphi^{\mu_n} \end{aligned}} \tag{70}$$

- the gauge terms have no impact on observables

6. Post-Minkowskian formalism

6.1 Post-Minkowskian expansion of field equations

- exact field equations in Eq. (62) were given by:

$$\boxed{\Box\, \overline{h}^{\alpha\beta} = -\frac{16\,\pi\,G}{c^4}\,\left(\tau^{\alpha\beta} + t^{\alpha\beta}\right)} \tag{71}$$

- perturbation of metric and metric density in powers of G

$$\boxed{h_{\alpha\beta} = \sum_{n=1}^{\infty} G^n\, h_{\alpha\beta}^{(\mathrm{nPM})} \quad \text{and} \quad \overline{h}^{\alpha\beta} = \sum_{n=1}^{\infty} G^n\, \overline{h}^{\alpha\beta}_{(\mathrm{nPM})}} \tag{72}$$

- "energy-momentum tensors" in powers of G

$$\boxed{\begin{aligned} \tau^{\alpha\beta} &= T^{\alpha\beta} + \sum_{n=1}^{\infty} G^n\, \tau^{\alpha\beta}_{(\mathrm{nPM})} \\ t^{\alpha\beta} &= \sum_{n=1}^{\infty} G^n\, t^{\alpha\beta}_{(\mathrm{nPM})} \end{aligned}} \tag{73}$$

- yields hierarchy of field equations in flat background manifold $\mathcal{M}_0$ covered by Cartesian coordinates $\{x\}$

$$\boxed{\begin{aligned} &\Box\, \overline{h}^{\alpha\beta}_{(1\mathrm{PM})} = -\frac{16\,\pi}{c^4}\; T^{\alpha\beta} \\ &\Box\, \overline{h}^{\alpha\beta}_{(2\mathrm{PM})} = -\frac{16\,\pi}{c^4}\left(\tau^{\alpha\beta}_{(1\mathrm{PM})} + t^{\alpha\beta}_{(1\mathrm{PM})}\right) \\ &\qquad\quad \vdots \\ &\Box\, \overline{h}^{\alpha\beta}_{(\mathrm{nPM})} = -\frac{16\,\pi}{c^4}\left(\tau^{\alpha\beta}_{((\mathrm{n}-1)\mathrm{PM})} + t^{\alpha\beta}_{((\mathrm{n}-1)\mathrm{PM})}\right) \end{aligned}} \tag{74}$$

- Eqs. (74) solved by iteration
- $T^{\alpha\beta}$ is stress-energy of matter in special relativity
- harmonic gauge must be satisfied order by order

$$\boxed{\overline{h}^{\alpha\beta}_{(\mathrm{nPM})\,,\,\beta} = 0} \tag{75}$$

ensures local law of conservation due to Eq. (63)

6.2 Post-Minkowskian expansion of residual gauge fields

- post-Minkowskian series of residual gauge fields

$$\boxed{\varphi^{\alpha}(x) = \sum_{n=1}^{\infty} G^n \, \varphi^{\alpha}_{(\mathrm{nPM})}(x)} \tag{76}$$

- inserting Eq. (76) and Eq. (72) into Eq. (66) yields

$$\boxed{\begin{aligned} \Box\, \varphi^{\alpha\,(\mathrm{1PM})}(x) &= 0 \\ \Box\, \varphi^{\alpha\,(\mathrm{2PM})}(x) &= \overline{h}^{\mu\nu}_{(\mathrm{1PM})}\, \varphi^{\alpha\,(\mathrm{1PM})}_{,\,\mu\nu} \\ &\vdots \\ \Box\, \varphi^{\alpha\,(\mathrm{nPM})}(x) &= \sum_{m=1}^{n-1} \overline{h}^{\mu\nu}_{((\mathrm{n-m})\mathrm{PM})}\, \varphi^{\alpha\,(\mathrm{mPM})}_{,\,\mu\nu} \end{aligned}} \tag{77}$$

6.2.1 The residual gauge transformation of metric tensor

- inserting (72) and (76) into (70) yields in $\{x\}$

$$\boxed{\sum_{n=1}^{\infty} G^n \, h_{\alpha\beta}^{(\rm nPM)} = \sum_{n=1}^{\infty} G^n \left(h_{\alpha\beta}^{\prime\,(\rm nPM)} + \partial\varphi_{\alpha\beta}^{(\rm nPM)} + \Omega_{\alpha\beta}^{(\rm nPM)} \right)} \quad (78)$$

- linear gauge terms for metric tensor

$$\boxed{\partial\varphi_{\alpha\beta}^{(\rm nPM)} = \varphi^{\mu\,(\rm nPM)}_{\;,\,\alpha} \, \eta_{\mu\beta} + \varphi^{\mu\,(\rm nPM)}_{\;,\,\beta} \, \eta_{\mu\alpha}} \quad (79)$$

- non-linear gauge terms for metric tensor

$$\boxed{\Omega_{\alpha\beta}^{(\rm nPM)} = \Omega_{\alpha\beta}^{(\rm nPM)} \left[\varphi^{\mu\,(\rm mPM)} \right] \quad \text{with} \quad m < n} \quad (80)$$

6.2.2 The residual gauge transformation of metric density

- inserting (72) and (76) into (68) yields in $\{x\}$

$$\boxed{\sum_{n=1}^{\infty} G^n \overline{h}^{\alpha\beta}_{(\mathrm{nPM})} = \sum_{n=1}^{\infty} G^n \left(\overline{h}^{\prime\,\alpha\beta}_{(\mathrm{nPM})} + \partial\overline{\varphi}^{\alpha\beta}_{(\mathrm{nPM})} + \overline{\Omega}^{\alpha\beta}_{(\mathrm{nPM})} \right)} \qquad (81)$$

- linear gauge terms for metric density

$$\boxed{\partial\overline{\varphi}^{\alpha\beta}_{(\mathrm{nPM})} = \varphi^{\alpha\,(\mathrm{nPM})}_{\;\;,\mu}\,\eta^{\mu\beta} + \varphi^{\beta\,(\mathrm{nPM})}_{\;\;,\mu}\,\eta^{\mu\alpha} - \varphi^{\mu\,(\mathrm{nPM})}_{\;\;,\mu}\,\eta^{\alpha\beta}} \qquad (82)$$

- non-linear gauge terms for metric density

$$\boxed{\overline{\Omega}^{\alpha\beta}_{(\mathrm{nPM})} = \overline{\Omega}^{\alpha\beta}_{(\mathrm{nPM})}\left[\varphi^{\mu\,(\mathrm{mPM})}\right] \quad \text{with} \quad m < n} \qquad (83)$$

7. MPM formalism

1. iterative approach to solve Eq. (74) outside isolated source in terms of symmetric tracefree multipoles
2. simplification by gauge transformation Eq. (81)

- pioneering work: K. Thorne (1980) [6]
- further developed: L. Blanchet and T. Damour (1986) [7]

- subsequent developments (1986 - 2008)
  T. Damour, L. Blanchet, B. Iyer, G. Faye, P. Jaranowski, G. Esposito-Farese, S. Sinha, S. Kopeikin, G. Schäfer

7.1 Definition of an isolated source of matter

1. compact source of matter inside sphere with radius r_0

$$\boxed{T^{\alpha\beta}(t, \boldsymbol{x}) = 0 \quad \text{for} \quad r > r_0} \tag{84}$$

where $r = |\boldsymbol{x}|$

2. Fock-Sommerfeld boundary conditions:
   (a) asymptotically Minkowski space

$$\boxed{\lim_{\substack{r\to\infty \\ t+\frac{r}{c}=\mathrm{const}}} \overline{h}^{\alpha\beta}(t, \boldsymbol{x}) = 0} \tag{85}$$

   (b) no-incoming radiation

$$\boxed{\lim_{\substack{r\to\infty \\ t+\frac{r}{c}=\mathrm{const}}} \left(\frac{\partial}{\partial r} r\, \overline{h}^{\alpha\beta}(t, \boldsymbol{x}) + \frac{\partial}{\partial ct}\, r\, \overline{h}^{\alpha\beta}(t, \boldsymbol{x}) \right) = 0} \tag{86}$$

7.2 General MPM solution of metric density

- general solution of metric density

$$\boxed{\overline{g}^{\mathrm{gen}\,\alpha\beta} = \eta^{\alpha\beta} - \sum_{n=1}^{\infty} G^n\, \overline{h}^{\mathrm{gen}\,\alpha\beta}_{(\mathrm{nPM})}\left[I_L, J_L, W_L, X_L, Y_L, Z_L\right]} \qquad (87)$$

- simplification by gauge transformation

$$\boxed{x^{\alpha}_{\mathrm{can}} = x^{\alpha}_{\mathrm{gen}} + \sum_{n=1}^{\infty} G^n\, \varphi^{\alpha}_{(\mathrm{nPM})}\left(x_{\mathrm{gen}}\right)} \qquad (88)$$

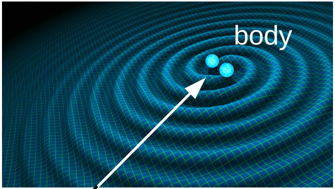
$$\boxed{\overline{g}^{\mathrm{can}\,\alpha\beta} = \overline{g}^{\mathrm{gen}\,\alpha\beta} + \sum_{n=1}^{\infty} G^n\, \partial\overline{\varphi}^{\alpha\beta}_{(\mathrm{nPM})} + \sum_{n=1}^{\infty} G^n\, \overline{\Omega}^{\alpha\beta}_{(\mathrm{nPM})}} \qquad (89)$$

- canonical solution of metric density

$$\boxed{\overline{g}^{\mathrm{can}\,\alpha\beta} = \eta^{\alpha\beta} - \sum_{n=1}^{\infty} G^n\, \overline{h}^{\mathrm{can}\,\alpha\beta}_{(\mathrm{nPM})}\left[M_L, S_L\right]} \qquad (90)$$

7.3 Why MPM is focussed on metric density?

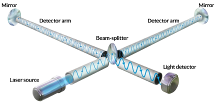
- determination of gravitational waves in far-zone



far wave zone
of gravitational system
just plane waves

huge distance (~ 10^{25} m)
between body and observer

observer

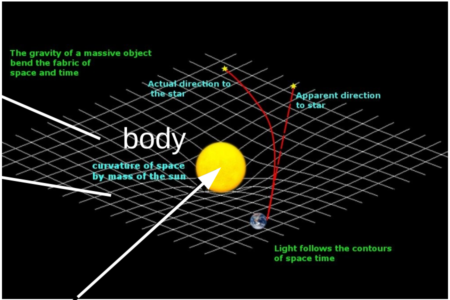


TT projection of metric relevant

$$\boxed{h^{TT}_{\alpha\beta} = \overline{h}^{TT}_{\alpha\beta}}$$

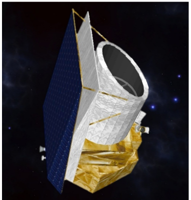
7.4 Why do we need metric tensor?

- determination of light trajectories in near-zone



near-zone
of gravitational system

not simply plane waves



observer

small distance (~ 10^{12} m)
between body and observer

entire metric relevant

$$\boxed{h_{\alpha\beta} \neq \bar{h}_{\alpha\beta}}$$

7.5 General MPM solution of metric tensor

- general solution of metric tensor

$$\boxed{g_{\mathrm{gen}\,\alpha\beta} = \eta_{\alpha\beta} + \sum_{n=1}^{\infty} G^n \, h^{(\mathrm{nPM})}_{\mathrm{gen}\,\alpha\beta} \left[I_L, J_L, W_L, X_L, Y_L, Z_L\right]} \quad (91)$$

- simplification by gauge transformation

$$\boxed{x^{\alpha}_{\mathrm{can}} = x^{\alpha}_{\mathrm{gen}} + \sum_{n=1}^{\infty} G^n \, \varphi^{\alpha}_{(\mathrm{nPM})} \left(x_{\mathrm{gen}}\right)} \quad (92)$$

$$\boxed{g_{\mathrm{can}\,\alpha\beta} = g_{\mathrm{gen}\,\alpha\beta} - \sum_{n=1}^{\infty} G^n \, \partial\varphi^{(\mathrm{nPM})}_{\alpha\beta} - \sum_{n=1}^{\infty} G^n \, \Omega^{(\mathrm{nPM})}_{\alpha\beta}} \quad (93)$$

- canonical solution of metric tensor

$$\boxed{g_{\mathrm{can}\,\alpha\beta} = \eta_{\alpha\beta} + \sum_{n=1}^{\infty} G^n \, h^{(\mathrm{nPM})}_{\mathrm{can}\,\alpha\beta} \left[M_L, S_L\right]} \quad (94)$$

8. MPM formalism in 1PM approximation

8.1 The field equations

- field equation and gauge condition

$$\boxed{\Box\, \overline{h}^{\alpha\beta}_{(1\mathrm{PM})} = -\frac{16\,\pi}{c^4}\, T^{\alpha\beta} \text{ and } \overline{h}^{\alpha\beta}_{(1\mathrm{PM})\,,\,\beta} = 0} \tag{95}$$

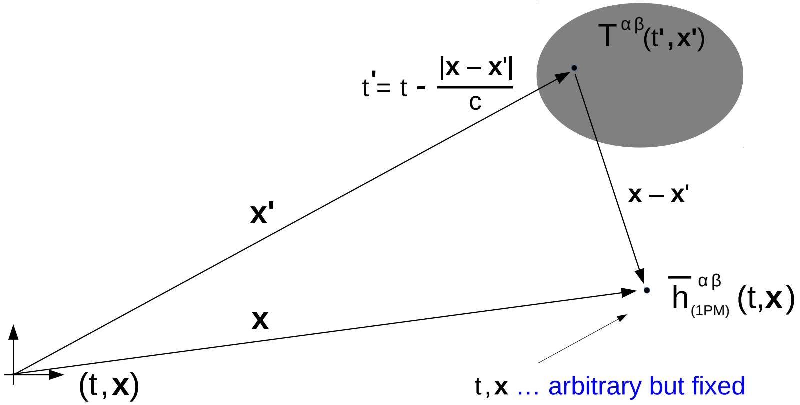
- solution

$$\boxed{\overline{h}^{\alpha\beta}_{(1\mathrm{PM})}\,(t, \boldsymbol{x}) = -\frac{16\,\pi}{c^4} \Box^{-1}_{\mathrm{R}}\, T^{\alpha\beta}\,(t\,,\ \boldsymbol{x})} \tag{96}$$

- inverse d'Alembert operator

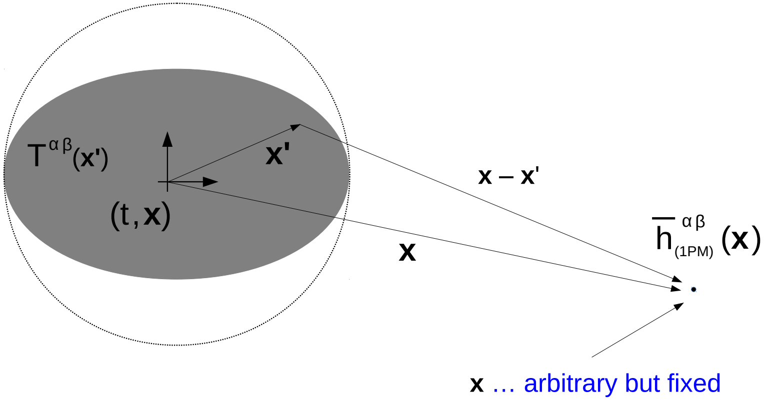
$$\boxed{\Box^{-1}_{\mathrm{R}} f\,(t, \boldsymbol{x}) = -\frac{1}{4\,\pi} \int d^3x' \frac{f\,(t', \boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|}} \tag{97}$$

- graphical representation of Eq. (96)
- variable $\boldsymbol{x}'$ runs over three-dimensional space of source



8.2 Solution in terms of time-independent multipoles

- for motivation consider simple case: $T^{\alpha\beta}(\boldsymbol{x}') = \mathrm{const}$
- origin of spatial coordinates has to be near body's CoM
- sphere with radius r_0 and $r = |\boldsymbol{x}|$, $r' = |\boldsymbol{x}'|$
- body enclosed in that sphere: $T^{\alpha\beta}(\boldsymbol{x}') = 0$ for $r' > r_0$



- series expansion valid for $r' < r$

$$\boxed{\frac{1}{|\boldsymbol{x} - \boldsymbol{x}'|} = \sum_{l=0}^{\infty} \frac{(-1)^l}{l!}\, x'_L\, \partial_L\, \frac{1}{r}} \tag{98}$$

- $x'_L = x'_{a_1} \ldots x'_{a_l}$ and $\partial_L = \partial_{a_1 \ldots a_l}$
- $x'_L\, \partial_L \frac{1}{r} = \sum_{a_1=1}^{3} \sum_{a_2=1}^{3} \cdots \sum_{a_l=1}^{3} x'_{a_1} \ldots x'_{a_l}\, \partial_{a_1 \ldots a_l}\, \frac{1}{r}$
- some examples reveal STF structure:

$$\boxed{\begin{aligned} \partial_{a_1}\, \frac{1}{r} &= (-1)^1\, \frac{x_{a_1}}{r^3} \\ \partial_{a_1 a_2}\, \frac{1}{r} &= (-1)^2 \left(3\, \frac{x_{a_1} x_{a_2}}{r^5} - \frac{\delta_{a_1 a_2}}{r^3} \right) \\ &\vdots \\ \partial_L\, \frac{1}{r} &= (-1)^l\, \frac{(2\,l - 1)!!}{r^{l+1}}\, \frac{x_{\langle a_1 \ldots a_l \rangle}}{r^l} \end{aligned}} \tag{99}$$

- from $n'_L \, \hat{n}_L = \hat{n}'_L \, \hat{n}_L$ follows

$$\boxed{x'_L \, \partial_L \, \frac{1}{r} = \hat{x}'_L \, \partial_L \, \frac{1}{r}} \tag{100}$$

- where $\hat{x}'_L = x'_{<a_1} \dots x'_{a_l>}$ are STF with respect to $a_1 \dots a_l$
- one obtains metric density in terms of STF multipoles

$$\boxed{\overline{h}^{\alpha\beta}_{(1\mathrm{PM})}(\boldsymbol{x}) = \frac{4}{c^4} \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \, \partial_L \, \frac{\hat{F}^{\alpha\beta}_L}{r}} \tag{101}$$

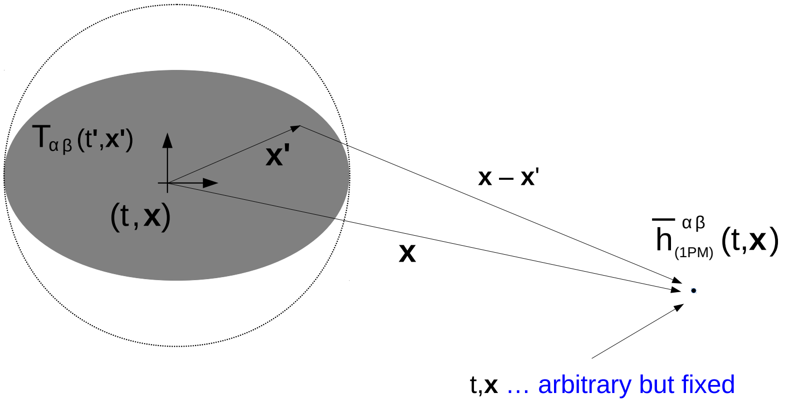
- these 10 time-independent STF multipoles are given by:

$$\boxed{\hat{F}^{\alpha\beta}_L = \int d^3x' \, \hat{x}'_L \, T^{\alpha\beta}(\boldsymbol{x}')} \tag{102}$$

- multipoles $\hat{F}^{\alpha\beta}_L$ are STF with respect to $a_1 \dots a_l$
- from now on: simpler notation for multipoles $\hat{F}^{\alpha\beta}_L \equiv F^{\alpha\beta}_L$

8.3 Solution in terms of time-dependent multipoles

- sphere with radius r_0 and $r = |\boldsymbol{x}|$, $r' = |\boldsymbol{x}'|$
- body enclosed in that sphere: $T^{\alpha\beta}(\boldsymbol{x}', t') = 0$ for $r' > r_0$



- case of time-dependent multipoles is complicated [6, 7]

$$\boxed{\overline{h}^{\alpha\beta}_{(1\mathrm{PM})}\left(t,\boldsymbol{x}\right) = \frac{4}{c^4}\sum_{l=0}^{\infty}\frac{\left(-1\right)^l}{l!}\,\partial_L\,\frac{F_L^{\alpha\beta}\left(s\right)}{r}} \tag{103}$$

retarded time $s = t - |\boldsymbol{x}|/c$

- these 10 time-dependent STF multipoles are given by:

$$\boxed{F_L^{\alpha\beta}\left(s\right) = \int d^3x'\,\hat{x}'_L\int\limits_{-1}^{+1} dz\,\delta_l\left(z\right)\;T^{\alpha\beta}\left(\frac{s+z\,r'}{c}\,,\,\boldsymbol{x}'\right)} \tag{104}$$

- Eqs. (103) - (104) given in [10]
- detailed proof of Eqs. (103) - (104) in my manuscript [9]

8.4 Decomposition in irreducible STF multipoles

- metric density in Eq. (103) in terms of 10 multipoles

$$
\boxed{
\begin{aligned}
\overline{h}^{00}_{(1\mathrm{PM})}(t,\boldsymbol{x}) &= \frac{4}{c^4}\sum_{l=0}^{\infty}\frac{(-1)^l}{l!}\,\partial_L\,\frac{F_L(s)}{r}\\
\overline{h}^{0i}_{(1\mathrm{PM})}(t,\boldsymbol{x}) &= \frac{4}{c^4}\sum_{l=0}^{\infty}\frac{(-1)^l}{l!}\,\partial_L\,\frac{G_{iL}(s)}{r}\\
\overline{h}^{ij}_{(1\mathrm{PM})}(t,\boldsymbol{x}) &= \frac{4}{c^4}\sum_{l=0}^{\infty}\frac{(-1)^l}{l!}\,\partial_L\,\frac{H_{ijL}(s)}{r}
\end{aligned}
}
\tag{105}
$$

- F_L is irreducible (STF in L)
- G_{iL} is reducible (STF in L but not STF in iL)
- H_{ijL} is reducible (STF in L but not STF in ijL)

- multipoles are integrals over stress-energy tensor

$$\boxed{\begin{aligned} F_L(s) &= \int d^3x'\, \hat{x}'_L \int_{-1}^{+1} dz\, \delta_l(z)\;\; T^{00}\left(\frac{s+z\,r'}{c}\,,\,\boldsymbol{x}'\right) \\ G_{iL}(s) &= \int d^3x'\, \hat{x}'_L \int_{-1}^{+1} dz\, \delta_l(z)\;\; T^{0i}\left(\frac{s+z\,r'}{c}\,,\,\boldsymbol{x}'\right) \\ H_{ijL}(s) &= \int d^3x'\, \hat{x}'_L \int_{-1}^{+1} dz\, \delta_l(z)\;\; T^{ij}\left(\frac{s+z\,r'}{c}\,,\,\boldsymbol{x}'\right) \end{aligned}} \qquad (106)$$

- $T^{00} = T_{00}$ and $T^{0i} = T^{0}{}_{i} = -T_{0i}$ and $T^{ij} = T_{ij}$

- metric density in terms of 10 irreducible multipoles [10]

$$\boxed{\overline{h}^{00} = \frac{4\,G}{c^4}\sum_{l=0}^{\infty}\frac{(-1)^l}{l!}\partial_L\frac{\mathcal{A}_L}{r}} \tag{107}$$

$$\boxed{\overline{h}^{0i} = \frac{4\,G}{c^4}\sum_{l=0}^{\infty}\frac{(-1)^l}{l!}\left(\partial_{iL}\frac{\mathcal{B}_L}{r} + \partial_{iL-1}\frac{\mathcal{C}_{iL-1}}{r} + \epsilon_{iab}\partial_{aL-1}\frac{\mathcal{D}_{bL-1}}{r}\right)} \tag{108}$$

$$\boxed{\begin{aligned}\overline{h}^{ij} = \frac{4\,G}{c^4}\sum_{l=0}^{\infty}\frac{(-1)^l}{l!}\Bigg(&\partial_{ijL}\frac{\mathcal{E}_L}{r} + \delta_{ij}\,\partial_L\frac{\mathcal{I}_L}{r} + \partial_{L-1(i}\frac{\mathcal{G}_{j)L-1}}{r}\\ &+ \epsilon_{ab(i}\partial_{j)aL-1}\frac{\mathcal{H}_{bL-1}}{r} + \partial_{L-2}\frac{\mathcal{J}_{ijL-2}}{r} + \partial_{aL-2}\frac{\epsilon_{ab(i}\mathcal{T}_{j)bL-2}}{r}\Bigg)\end{aligned}} \tag{109}$$

10 irreducible multipoles: $\mathcal{A}_L, \mathcal{B}_L, \mathcal{C}_L, \mathcal{D}_L, \mathcal{E}_L, \mathcal{I}_L, \mathcal{G}_L, \mathcal{H}_L, \mathcal{J}_L, \mathcal{T}_L$

- some examples:

$$\boxed{\begin{aligned}
\mathcal{A}_L &= \int d^3x' \, \hat{x}'_L \int_{-1}^{+1} dz \, \delta_l \; T^{00} \\
\mathcal{B}_L &= -\frac{1}{l+1} \, \frac{2l+1}{2l+3} \int d^3x' \, \hat{x}'_{aL} \int_{-1}^{+1} dz \, \delta_{l+1} \; T^{0a} \\
&\vdots \\
\mathcal{E}_L &= \frac{1}{l+1} \, \frac{1}{l+2} \, \frac{2l+1}{2l+5} \int d^3x' \, \hat{x}'_{abL} \int_{-1}^{+1} dz \, \delta_{l+2} \; T^{ab} \\
&\vdots
\end{aligned}} \tag{110}$$

8.5 The local law of conservation (gauge condition)

- these 10 multipoles in Eqs. (107) - (109) not independent
- 4 relations of 1PM local conservation law (cf. Eq. (63))

$$\boxed{T^{\alpha\beta}{}_{,\beta} = 0 \quad \Longleftrightarrow \quad \overline{h}^{\alpha\beta}_{(1\mathrm{PM})\,,\beta} = 0} \tag{111}$$

$$\boxed{\begin{array}{lll} \mathcal{C}_L = -\dot{\mathcal{A}}_L - \ddot{\mathcal{B}}_L & \text{for} & l \geq 1 \\ \mathcal{G}_L = -2\,\dot{\mathcal{B}}_L - 2\,\ddot{\mathcal{E}}_L - 2\,\mathcal{I}_L & \text{for} & l \geq 1 \\ \mathcal{J}_L = 2\,\dot{\mathcal{A}}_L + 4\,\ddot{\mathcal{B}}_L + 2\,\ddot{\mathcal{E}}_L + 2\,\dot{\mathcal{I}}_L & \text{for} & l \geq 2 \\ \mathcal{T}_L = -2\,\dot{\mathcal{D}}_L - \ddot{\mathcal{H}}_L & \text{for} & l \geq 2 \end{array}} \tag{112}$$

- only 6 independent STF multipoles: $\mathcal{A}_L, \mathcal{B}_L, \mathcal{D}_L, \mathcal{E}_L, \mathcal{I}_L, \mathcal{H}_L$
- detailed proof of Eqs. (112) is given in my manuscript [11]

8.6 Definition of new multipoles

- definition of new irreducible multipoles

$$\boxed{\begin{aligned}
I_L &= -\left(\mathcal{A}_L + 2\,\dot{\mathcal{B}}_L + \mathcal{I}_L\right) && \text{for} \quad l \geq 0 \\
J_L &= +\left(\mathcal{D}_L + \frac{1}{2}\,\dot{\mathcal{H}}_L\right) && \text{for} \quad l \geq 1 \\
W_L &= -\left(\mathcal{B}_L + \frac{1}{2}\,\dot{\mathcal{E}}_L\right) && \text{for} \quad l \geq 0 \\
X_L &= -\frac{1}{2}\,\dot{\mathcal{E}}_L && \text{for} \quad l \geq 0 \\
Y_L &= +\left(\dot{\mathcal{B}}_L + \ddot{\mathcal{E}}_L + \mathcal{I}_L\right) && \text{for} \quad l \geq 0 \\
Z_L &= -\frac{1}{2}\,\dot{\mathcal{H}}_L && \text{for} \quad l \geq 1
\end{aligned}} \tag{113}$$

6 (new) independent STF multipoles: $I_L, J_L, W_L, X_L, Y_L, Z_L$

8.7 The general 1PM solution of metric density

- general solution of 1PM metric density (cf. Eq. (87))

$$\boxed{\overline{g}^{\,\mathrm{gen}\,\alpha\beta}_{(1\mathrm{PM})} = \eta^{\alpha\beta} - G^1\,\overline{h}^{\,\mathrm{gen}\,\alpha\beta}_{(1\mathrm{PM})}\left[I_L, J_L, W_L, X_L, Y_L, Z_L\right]} \qquad (114)$$

- gauge transformation in 1PM approximation $\left(\overline{\Omega}^{\alpha\beta}_{(1\mathrm{PM})} = 0\right)$

$$\boxed{x^{\alpha}_{\mathrm{can}} = x^{\alpha}_{\mathrm{gen}} + G^1\,\varphi^{\alpha}_{(1\mathrm{PM})}\left(x_{\mathrm{gen}}\right)}$$

$$\boxed{\overline{g}^{\,\mathrm{can}\,\alpha\beta}_{(1\mathrm{PM})} = \overline{g}^{\,\mathrm{gen}\,\alpha\beta}_{(1\mathrm{PM})} + G^1\,\partial\overline{\varphi}^{\alpha\beta}_{(1\mathrm{PM})}}$$

- canonical 1PM metric density (cf. Eq. (90))

$$\boxed{\overline{g}^{\,\mathrm{can}\,\alpha\beta}_{(1\mathrm{PM})} = \eta^{\alpha\beta} - G^1\,\overline{h}^{\,\mathrm{can}\,\alpha\beta}_{(1\mathrm{PM})}\left[M_L, S_L\right]} \qquad (115)$$

- explicit form of 1PM canonical metric density perturbation:

$$
\boxed{
\begin{aligned}
\overline{h}^{\,\mathrm{can}\,00}_{(\mathrm{1PM})}\left(t,\boldsymbol{x}\right) &= +\frac{4}{c^2}\sum_{l=0}^{\infty}\frac{\left(-1\right)^l}{l!}\,\partial_L\,\frac{M_L\left(s\right)}{r} \\
\overline{h}^{\,\mathrm{can}\,0i}_{(\mathrm{1PM})}\left(t,\boldsymbol{x}\right) &= -\frac{4}{c^3}\sum_{l=1}^{\infty}\frac{\left(-1\right)^l}{l!}\,\partial_{L-1}\,\frac{\dot{M}_{iL-1}\left(s\right)}{r} \\
&\quad - \frac{4}{c^3}\sum_{l=1}^{\infty}\frac{\left(-1\right)^l\,l}{\left(l+1\right)!}\,\epsilon_{iab}\,\partial_{aL-1}\,\frac{S_{bL-1}\left(s\right)}{r} \\
\overline{h}^{\,\mathrm{can}\,ij}_{(\mathrm{1PM})}\left(t,\boldsymbol{x}\right) &= +\frac{4}{c^4}\sum_{l=2}^{\infty}\frac{\left(-1\right)^l}{l!}\,\partial_{L-2}\,\frac{\ddot{M}_{ijL-2}\left(s\right)}{r} \\
&\quad + \frac{8}{c^4}\sum_{l=2}^{\infty}\frac{\left(-1\right)^l\,l}{\left(l+1\right)!}\,\partial_{aL-2}\,\frac{\epsilon_{ab<i}\,\dot{S}_{j>bL-2}\left(s\right)}{r}
\end{aligned}
}
\qquad (116)
$$

- 1PM gauge terms for metric density:

$$\boxed{\partial\overline{\varphi}^{\alpha\beta}_{(1\mathrm{PM})} = \varphi^{\alpha\,(1\mathrm{PM})}_{,\mu}\,\eta^{\mu\beta} + \varphi^{\beta\,(1\mathrm{PM})}_{,\mu}\,\eta^{\mu\alpha} - \varphi^{\mu\,(1\mathrm{PM})}_{,\mu}\,\eta^{\alpha\beta}} \tag{117}$$

- 1PM gauge functions for metric density:

$$\boxed{\begin{aligned} \varphi^{0}_{(1\mathrm{PM})} &= +\sum_{l=0}^{\infty}\partial_L\frac{W_L}{r} \\ \varphi^{i}_{(1\mathrm{PM})} &= +\sum_{l=0}^{\infty}\partial_{iL}\frac{X_L}{r} + \sum_{l=1}^{\infty}\partial_{L-1}\frac{Y_{iL-1}}{r} \\ &\quad + \sum_{l=1}^{\infty}\epsilon_{iab}\partial_{aL-1}\frac{Z_{bL-1}}{r} \end{aligned}} \tag{118}$$

8.8 The general 1PM solution of metric tensor

- general solution of 1PM metric tensor (cf. Eq. (91)):

$$\boxed{g^{(1\mathrm{PM})}_{\mathrm{gen}\,\alpha\beta} = \eta_{\alpha\beta} + G^1\; h^{(1\mathrm{PM})}_{\mathrm{gen}\,\alpha\beta}\left[I_L, J_L, W_L, X_L, Y_L, Z_L\right]} \qquad (119)$$

- gauge transformation in 1PM approximation $\left(\Omega^{(1\mathrm{PM})}_{\alpha\beta} = 0\right)$

$$\boxed{x^{\alpha}_{\mathrm{can}} = x^{\alpha}_{\mathrm{gen}} + G^1\; \varphi^{\alpha}_{(1\mathrm{PM})}\left(x_{\mathrm{gen}}\right)} \qquad (120)$$

$$\boxed{g^{(1\mathrm{PM})}_{\mathrm{can}\,\alpha\beta} = g^{(1\mathrm{PM})}_{\mathrm{gen}\,\alpha\beta} + G^1\; \partial\varphi^{(1\mathrm{PM})}_{\alpha\beta}} \qquad (121)$$

- canonical 1PM metric tensor (cf. Eq. (94)):

$$\boxed{g^{(1\mathrm{PM})}_{\mathrm{can}\,\alpha\beta} = \eta_{\alpha\beta} + G^1\; h^{(1\mathrm{PM})}_{\mathrm{can}\,\alpha\beta}\left[M_L, S_L\right]} \qquad (122)$$

- explicit form of 1PM canonical metric tensor perturbation:

$$\boxed{\begin{aligned}
h^{(\mathrm{1PM})}_{\mathrm{can}\,00}\left(t,\boldsymbol{x}\right) &= +\frac{2}{c^2}\sum_{l=0}^{\infty}\frac{(-1)^l}{l!}\,\partial_L\,\frac{M_L\left(s\right)}{r}\\
h^{(\mathrm{1PM})}_{\mathrm{can}\,0i}\left(t,\boldsymbol{x}\right) &= +\frac{4}{c^3}\sum_{l=1}^{\infty}\frac{(-1)^l}{l!}\,\partial_{L-1}\,\frac{\dot{M}_{iL-1}\left(s\right)}{r}\\
&\quad+\frac{4}{c^3}\sum_{l=1}^{\infty}\frac{(-1)^l\;l}{(l+1)!}\,\epsilon_{iab}\,\partial_{aL-1}\,\frac{S_{bL-1}\left(s\right)}{r}\\
h^{(\mathrm{1PM})}_{\mathrm{can}\,ij}\left(t,\boldsymbol{x}\right) &= +\frac{2}{c^2}\,\delta_{ij}\sum_{l=0}^{\infty}\frac{(-1)^l}{l!}\,\partial_L\,\frac{M_L\left(s\right)}{r}\\
&\quad+\frac{4}{c^4}\sum_{l=2}^{\infty}\frac{(-1)^l}{l!}\,\partial_{L-2}\,\frac{\ddot{M}_{ijL-2}\left(s\right)}{r}\\
&\quad+\frac{8}{c^4}\sum_{l=2}^{\infty}\frac{(-1)^l\;l}{(l+1)!}\,\partial_{aL-2}\,\frac{\epsilon_{ab(i}\dot{S}_{j)bL-2}\left(s\right)}{r}
\end{aligned}} \qquad (123)$$

- 1PM gauge terms for metric tensor:

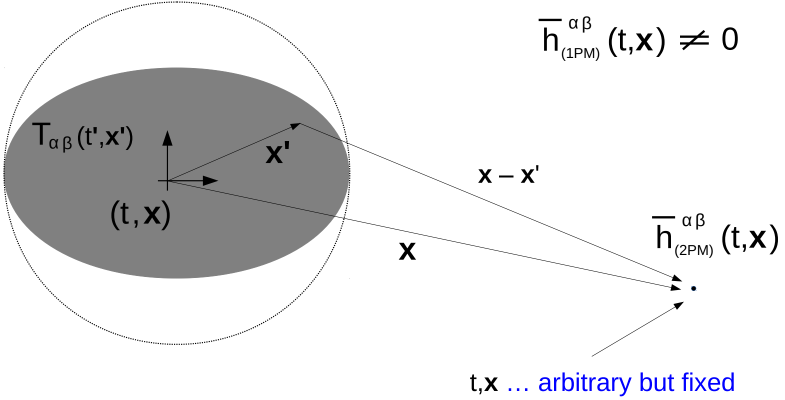
$$\boxed{\partial\varphi^{(1\mathrm{PM})}_{\alpha\beta} = \varphi^{\mu\,(1\mathrm{PM})}_{,\,\alpha}\,\eta_{\mu\beta} + \varphi^{\mu\,(1\mathrm{PM})}_{,\,\beta}\,\eta_{\mu\alpha}} \tag{124}$$

- 1PM gauge functions for metric density (cf. Eq. (118)):

$$\boxed{\begin{aligned} \varphi^{0}_{(1\mathrm{PM})} &= +\sum_{l=0}^{\infty}\partial_L\frac{W_L}{r} \\ \varphi^{i}_{(1\mathrm{PM})} &= +\sum_{l=0}^{\infty}\partial_{iL}\frac{X_L}{r} + \sum_{l=1}^{\infty}\partial_{L-1}\frac{Y_{iL-1}}{r} \\ &\quad + \sum_{l=1}^{\infty}\epsilon_{iab}\partial_{aL-1}\frac{Z_{bL-1}}{r} \end{aligned}} \tag{125}$$

9. MPM formalism in 2PM approximation

- sphere with radius r_0 and $r = |\boldsymbol{x}|$, $r' = |\boldsymbol{x}'|$
- body enclosed in that sphere: $T^{\alpha\beta}(\boldsymbol{x}', t') = 0$ for $r' > r_0$



9.1 The field equations

- field equations and gauge condition

$$\boxed{\Box \overline{h}^{\alpha\beta}_{(2\mathrm{PM})} = -\frac{16\pi}{c^4} \left(\tau^{\alpha\beta}_{(1\mathrm{PM})} + t^{\alpha\beta}_{(1\mathrm{PM})} \right) \text{ and } \overline{h}^{\alpha\beta}_{(2\mathrm{PM})\,,\,\beta} = 0} \tag{126}$$

- formal solution

$$\boxed{\overline{h}^{\alpha\beta}_{(2\mathrm{PM})}\left(t, \boldsymbol{x}\right) = -\frac{16\,\pi}{c^4} \Box^{-1}_{\mathrm{R}} \left(\tau^{\alpha\beta}_{(1\mathrm{PM})} + t^{\alpha\beta}_{(1\mathrm{PM})} \right) \left(t\,,\, \boldsymbol{x}\right)} \tag{127}$$

- inverse d'Alembert operator ($\boldsymbol{x}'$ runs over entire space)

$$\boxed{\Box^{-1}_{\mathrm{R}} f\left(t, \boldsymbol{x}\right) = -\frac{1}{4\,\pi} \int d^3x' \frac{f\left(t', \boldsymbol{x}'\right)}{\left|\boldsymbol{x} - \boldsymbol{x}'\right|}} \tag{128}$$

- $\tau^{\alpha\beta}_{(1\mathrm{PM})}$ is non-zero for $r \le r_0$

$$\boxed{\tau^{\alpha\beta}_{(1\mathrm{PM})} = \eta_{\mu\nu}\,\overline{h}^{\mu\nu}_{(1\mathrm{PM})}\;T^{\alpha\beta}} \tag{129}$$

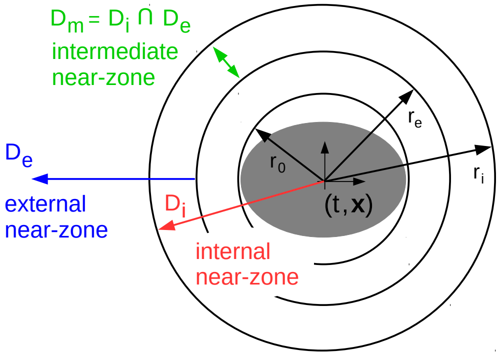
- $t^{\alpha\beta}_{(1\mathrm{PM})}$ is non-zero for $r \le r_0$

$$\boxed{\begin{aligned} t^{\alpha\beta}_{(1\mathrm{PM})} = {} & +\overline{h}^{\alpha\mu}_{(1\mathrm{PM}),\nu}\;\overline{h}^{\beta\nu}_{(1\mathrm{PM}),\mu} - \overline{h}^{\alpha\beta}_{(1\mathrm{PM}),\mu\nu}\;\overline{h}^{\mu\nu}_{(1\mathrm{PM})} \\ & + \frac{1}{2}\,\overline{h}^{(1\mathrm{PM}),\alpha}_{\mu\nu}\;\overline{h}^{\mu\nu\,,\,\beta}_{(1\mathrm{PM})} - \frac{1}{4}\eta^{\rho\sigma}\eta^{\mu\nu}\overline{h}^{(1\mathrm{PM}),\alpha}_{\mu\nu}\;\overline{h}^{(1\mathrm{PM}),\beta}_{\rho\sigma} \\ & + \overline{h}^{\alpha\mu}_{(1\mathrm{PM}),\nu}\;\overline{h}^{\beta\,,\,\nu}_{(1\mathrm{PM})\mu} + \frac{1}{8}\eta^{\alpha\beta}\eta^{\rho\sigma}\eta^{\mu\nu}\overline{h}^{(1\mathrm{PM}),\omega}_{\mu\nu}\;\overline{h}^{(1\mathrm{PM})}_{\rho\sigma\,,\,\omega} \\ & - \frac{1}{4}\eta^{\alpha\beta}\overline{h}^{(1\mathrm{PM})}_{\mu\nu\,,\,\omega}\;\overline{h}^{\mu\nu\,,\,\omega}_{(1\mathrm{PM})} + \frac{1}{2}\eta^{\alpha\beta}\overline{h}^{(1\mathrm{PM})}_{\nu\rho\,,\,\mu}\;\overline{h}^{\mu\rho\,,\,\nu}_{(1\mathrm{PM})} \\ & - \overline{h}^{(1\mathrm{PM})\,,\,\alpha}_{\mu\nu}\;\overline{h}^{\beta\nu\,,\,\mu}_{(1\mathrm{PM})} - \overline{h}^{(1\mathrm{PM})\,,\,\beta}_{\mu\nu}\;\overline{h}^{\alpha\nu\,,\,\mu}_{(1\mathrm{PM})} \end{aligned}} \tag{130}$$

- problem: thus far $\overline{h}^{\alpha\beta}_{(1\mathrm{PM})}$ only determined for $r > r_0$
  therefore: treatment of 2PM different from 1PM

9.2 Separation of spatial space

- separation of spatial space into three areas



- $\mathrm{D_e}$: post-Minkowskian expansion $\overline{h}_{\mathrm{e}}^{\alpha\beta}$ (in vacuum)
- $\mathrm{D_i}$: post-Newtonian expansion $\overline{h}_{\mathrm{i}}^{\alpha\beta}$ (with matter)
- $\mathrm{D_m}$: matching both solutions $\overline{h}_{\mathrm{e}}^{\alpha\beta}$ and $\overline{h}_{\mathrm{i}}^{\alpha\beta}$ valid

9.3 The 2PM solution in $\mathrm{D_e}$

- 2PM field equations in vacuum

$$\boxed{\Box\, \overline{h}^{\alpha\beta}_{(2\mathrm{PM})} = -\frac{16\,\pi}{c^4}\; t^{\alpha\beta}_{(1\mathrm{PM})}} \tag{131}$$

- formal 2PM solution in vacuum

$$\boxed{\overline{h}^{\alpha\beta}_{(2\mathrm{PM})}\,(t, \boldsymbol{x}) = -\frac{16\,\pi}{c^4}\Box^{-1}_{\mathrm{R}}\; t^{\alpha\beta}_{(1\mathrm{PM})}\,(t\,,\,\boldsymbol{x})} \tag{132}$$

- inverse d'Alembertian $\Box^{-1}_{\mathrm{R}}$ in (132) runs over entire space
  i.e. one needs $t^{\alpha\beta}_{(1\mathrm{PM})}$ in entire space
  i.e. one needs $\overline{h}^{\alpha\beta}_{(1\mathrm{PM})}$ in entire space

- 1PM field equations in vacuum

$$\boxed{\Box\,\overline{h}^{\alpha\beta}_{(1\mathrm{PM})} = 0} \tag{133}$$

- 1PM solution in vacuum in entire space $(r \neq 0)$ as function of 10 field multipoles ($F_L^{\alpha\beta}$ are not integrals over $T^{\alpha\beta}$) [6]

$$\boxed{\overline{h}^{\alpha\beta}_{(1\mathrm{PM})}\left(t,\boldsymbol{x}\right) = \frac{4}{c^4}\sum_{l=0}^{\infty}\frac{\left(-1\right)^l}{l!}\,\partial_L\,\frac{F_L^{\alpha\beta}\left(s\right)}{r}} \tag{134}$$

- 1PM solution in vacuum in entire space $(r \neq 0)$ as function function of 6 STF field multipoles by Eqs. (105) - (118)

$$\boxed{\overline{h}^{\alpha\beta}_{(1\mathrm{PM})}\left(t,\boldsymbol{x}\right) = \overline{h}^{\alpha\beta}_{(1\mathrm{PM})}\left[I_L, J_L, W_L, X_L, Y_L, Z_L\right]} \tag{135}$$

- i.e. $I_L, J_L, W_L, X_L, Y_L, Z_L$ are not integrals over $T^{\alpha\beta}$

- (135) into (132) via (130) yields divergent integrals since (135) is valid in $\mathrm{R}^3_* \times \mathrm{R}$ (entire space-time with $r \neq 0$)

- finite integrals by Hadamard technique to cut $r = 0$ cf. inverse d'Alembertian in Eq. (128)

$$\boxed{\mathrm{FP}_{B=0}\left(\Box_{\mathrm{R}}^{-1} f\right)(t, \boldsymbol{x}) = -\frac{1}{4\,\pi} \lim_{B \rightarrow 0} \int d^3x' \left(\frac{r'}{r_0}\right)^B \frac{f\left(t', \boldsymbol{x}'\right)}{\left|\boldsymbol{x} - \boldsymbol{x}'\right|}} \tag{136}$$

- 2PM solution in vacuum as function of 6 field multipoles

$$\boxed{\overline{h}^{\alpha\beta}_{(\mathrm{2PM})}\left(t, \boldsymbol{x}\right) = -\frac{16\,\pi}{c^4} \mathrm{FP}_{B=0}\, \Box_{\mathrm{R}}^{-1}\, t^{\alpha\beta}_{(\mathrm{1PM})}\left(t\,,\ \boldsymbol{x}\right)} \tag{137}$$

where $t^{\alpha\beta}_{(\mathrm{1PM})}$ is given by Eq. (130) with $h^{\alpha\beta}_{(\mathrm{1PM})}$ in Eq. (135)

- from Eqs. (135) and (137) one obtains for special case $\mathrm{D_e}$

$$\boxed{\overline{h}^{\alpha\beta}_{\mathrm{e}} = G^1 \overline{h}^{\alpha\beta}_{\mathrm{e\,(1PM)}} + G^2 \overline{h}^{\alpha\beta}_{\mathrm{e\,(2PM)}} + \mathcal{O}\left(G^3\right)} \tag{138}$$

- with 1PM perturbation in $\mathrm{D_e}$

$$\boxed{\overline{h}^{\alpha\beta}_{\mathrm{e\,(1PM)}}\left(t, \boldsymbol{x}\right) = \overline{h}^{\alpha\beta}_{\mathrm{e\,(1PM)}}\left[I_L, J_L, W_L, X_L, Y_L, Z_L\right]} \tag{139}$$

- with 2PM perturbation in $\mathrm{D_e}$

$$\boxed{\overline{h}^{\alpha\beta}_{\mathrm{e\,(2PM)}}\left(t, \boldsymbol{x}\right) = \overline{h}^{\alpha\beta}_{\mathrm{e\,(2PM)}}\left[I_L, J_L, W_L, X_L, Y_L, Z_L\right]} \tag{140}$$

where $I_L, J_L, W_L, X_L, Y_L, Z_L$ are STF field multipoles, i.e. they are not integrals over $T^{\alpha\beta}$ but general functions of s

9.4 The 2PN solution in $\mathrm{D_i}$

- exact field equations in Eq. (62) were given by:

$$\boxed{\Box\,\overline{h}^{\alpha\beta} = -\frac{16\,\pi\,G}{c^4}\,\left(\tau^{\alpha\beta} + t^{\alpha\beta}\right)} \tag{141}$$

- post-Newtonian expansion $v \ll c$ of Eq. (141) yields [12]

$$\boxed{\begin{aligned} \Box\overline{h}^{00}_{\mathrm{i}} &= -\frac{16\pi G}{c^4}\left(1 - \frac{4}{c^2}V\right)T^{00} + \frac{14}{c^4}V_{,k}V_{,k} + \mathcal{O}\,(6) \\ \Box\overline{h}^{0i}_{\mathrm{i}} &= -\frac{16\pi G}{c^4}\,T^{0i} + \mathcal{O}\,(5) \\ \Box\overline{h}^{ij}_{\mathrm{i}} &= -\frac{16\pi G}{c^4}\,T^{ij} - \frac{4}{c^4}\left(V_{,i}V_{,j} - V_{,k}V_{,k}\frac{\delta_{ij}}{2}\right) + \mathcal{O}\,(6) \end{aligned}} \tag{142}$$

- Eqs. (142) are 2PN approximation in MPM formalism [12]

- 2PN solution of Eqs. (142) [12]

$$\boxed{\begin{aligned} \overline{h}_{\mathrm{i}}^{00} &= \frac{4}{c^2}\, V - \frac{4}{c^4}\left(W_{kk} - 2\, V^2\right) + \mathcal{O}\left(6\right) \\ \overline{h}_{\mathrm{i}}^{0i} &= \frac{4}{c^3}\, V_i + \mathcal{O}\left(5\right) \\ \overline{h}_{\mathrm{i}}^{ij} &= \frac{4}{c^4}\, W_{ij} + \mathcal{O}\left(6\right) \end{aligned}} \tag{143}$$

- where the potentials are given by [12]

$$\boxed{\begin{aligned} V &= -4\,\pi\, G\, \Box_{\mathrm{R}}^{-1}\, \frac{T^{00} + T^{ii}}{c^2} \\ V_i &= -4\,\pi\, G\, \Box_{\mathrm{R}}^{-1}\, \frac{T^{0i}}{c} \\ W_{ij} &= -4\,\pi\, G\, \Box_{\mathrm{R}}^{-1}\left[\frac{T^{ij}}{c^2} + \frac{1}{G}\left(V_{,\,i}\, V_{,\,j} - V_{,\,k}\, V_{,\,k}\, \frac{\delta_{ij}}{2}\right)\right] \end{aligned}} \tag{144}$$

9.5 Matching

- matching: field multipoles (general functions of s) into source multipoles (integrals over $T^{\alpha\beta}$)

- matching condition in $\mathrm{D_m}$ (cf. Eq. (2.28) in [8])

$$\boxed{\overline{\mathcal{M}(\overline{h}_e^{\alpha\beta})} = \mathcal{M}(\overline{\overline{h}_i^{\alpha\beta}})}$$

post-Newtonian expansion of MPM expansion of $\overline{h}_e^{\alpha\beta}$

multipole-expansion of post-Newtonian expansion of $\overline{h}_i^{\alpha\beta}$

- $\overline{h}_e^{\alpha\beta}$ is given by Eqs. (138) - (140)
- $\overline{h}_i^{\alpha\beta}$ is given by Eqs. (143) - (144)

9.6 General 2PM solution of metric density

- general 2PM solution of metric density [12]

$$\boxed{\begin{aligned}\overline{g}^{\mathrm{gen}\,\alpha\beta}_{(2\mathrm{PM})} = \eta^{\alpha\beta} &- G^1\,\overline{h}^{\mathrm{gen}\,\alpha\beta}_{(1\mathrm{PM})}\left[I_L, J_L, W_L, X_L, Y_L, Z_L\right] \\ &- G^2\,\overline{h}^{\mathrm{gen}\,\alpha\beta}_{(2\mathrm{PM})}\left[I_L, J_L, W_L, X_L, Y_L, Z_L\right]\end{aligned}} \tag{145}$$

- gauge transformation in 2PM approximation

$$\boxed{x^{\alpha}_{\mathrm{can}} = x^{\alpha}_{\mathrm{gen}} + G^1\,\varphi^{\alpha}_{(1\mathrm{PM})}\left(x_{\mathrm{gen}}\right) + G^2\,\varphi^{\alpha}_{(2\mathrm{PM})}\left(x_{\mathrm{gen}}\right)}$$

$$\boxed{\overline{g}^{\mathrm{can}\,\alpha\beta}_{(2\mathrm{PM})} = \overline{g}^{\mathrm{gen}\,\alpha\beta}_{(1\mathrm{PM})} + G^1 \partial\overline{\varphi}^{\alpha\beta}_{(1\mathrm{PM})} + G^2\left(\partial\overline{\varphi}^{\alpha\beta}_{(2\mathrm{PM})} + \overline{\Omega}^{\alpha\beta}_{(2\mathrm{PM})}\right)}$$

- canonical metric density in 2PM approximation

$$\boxed{\begin{aligned}\overline{g}^{\mathrm{can}\,\alpha\beta}_{(2\mathrm{PM})} = \eta^{\alpha\beta} &- G^1\,\overline{h}^{\mathrm{can}\,\alpha\beta}_{(1\mathrm{PM})}\left[M_L, S_L\right] \\ &- G^2\,\overline{h}^{\mathrm{can}\,\alpha\beta}_{(2\mathrm{PM})}\left[M_L, S_L\right]\end{aligned}} \tag{146}$$

- canonical metric density perturbation [12]
  in the order $\mathcal{O}(6,5,6)$

$$\boxed{\begin{aligned}
&\overline{h}^{\,\mathrm{can}\,00}_{(\mathrm{2PM})}(t,\boldsymbol{x}) = +\frac{7}{c^4}\left(\sum_{l=0}^{\infty}\frac{(-1)^l}{l!}\,\partial_L\,\frac{M_L(s)}{r}\right)^2 \\
&\overline{h}^{\,\mathrm{can}\,0i}_{(\mathrm{2PM})}(t,\boldsymbol{x}) = 0 \\
&\overline{h}^{\,\mathrm{can}\,ij}_{(\mathrm{2PM})}(t,\boldsymbol{x}) = +\frac{1}{c^4}\,\delta_{ij}\left(\sum_{l=0}^{\infty}\frac{(-1)^l}{l!}\,\partial_L\,\frac{M_L(s)}{r}\right)^2 \\
&-\frac{4}{c^4}\mathrm{FP}_{B=0}\Box_R^{-1}\left(\partial_i\sum_{l=0}^{\infty}\frac{(-1)^l}{l!}\partial_L\frac{M_L(s)}{r}\right)\left(\partial_j\sum_{l=0}^{\infty}\frac{(-1)^l}{l!}\partial_L\frac{M_L(s)}{r}\right)
\end{aligned}}$$

- $\overline{h}^{\,\mathrm{can}\,ij}_{(\mathrm{2PM})}$ associated with Hadamard technique in Eq. (136)
- solution for time-dependent quadrupole-quadrupole [12]

9.7 General 2PM solution of metric tensor

- general solution of 2PM metric in my manuscript [13]

$$\boxed{\begin{aligned} g^{(2\mathrm{PM})}_{\mathrm{gen}\,\alpha\beta} = \eta_{\alpha\beta} &+ G^1\; h^{(1\mathrm{PM})}_{\mathrm{gen}\,\alpha\beta}\left[I_L, J_L, W_L, X_L, Y_L, Z_L\right] \\ &+ G^2\; h^{(2\mathrm{PM})}_{\mathrm{gen}\,\alpha\beta}\left[I_L, J_L, W_L, X_L, Y_L, Z_L\right] \end{aligned}} \qquad (147)$$

- gauge transformation in 2PM approximation

$$\boxed{x^{\alpha}_{\mathrm{can}} = x^{\alpha}_{\mathrm{gen}} + G^1\,\varphi^{\alpha}_{(1\mathrm{PM})}\left(x_{\mathrm{gen}}\right) + G^2\,\varphi^{\alpha}_{(2\mathrm{PM})}\left(x_{\mathrm{gen}}\right)}$$

$$\boxed{g^{(2\mathrm{PM})}_{\mathrm{can}\,\alpha\beta} = g^{(2\mathrm{PM})}_{\mathrm{gen}\,\alpha\beta} + G^1\partial\varphi^{(1\mathrm{PM})}_{\alpha\beta} + G^2\left(\partial\varphi^{(2\mathrm{PM})}_{\alpha\beta} + \Omega^{(2\mathrm{PM})}_{\alpha\beta}\right)}$$

- canonical metric tensor in 2PM approximation

$$\boxed{\begin{aligned} g^{(1\mathrm{PM})}_{\mathrm{can}\,\alpha\beta} = \eta_{\alpha\beta} &+ G^1\; h^{(1\mathrm{PM})}_{\mathrm{can}\,\alpha\beta}\left[M_L, S_L\right] \\ &+ G^2\; h^{(2\mathrm{PM})}_{\mathrm{can}\,\alpha\beta}\left[M_L, S_L\right] \end{aligned}} \qquad (148)$$

- canonical metric perturbation in my manuscript [13] in the order $\mathcal{O}\left(6,5,6\right)$

$$
\boxed{
\begin{aligned}
& h^{(\mathrm{2PM})}_{\mathrm{can}\,00}\left(t,\boldsymbol{x}\right) = -\frac{2}{c^4}\left(\sum_{l=0}^{\infty}\frac{\left(-1\right)^l}{l!}\;\partial_L\;\frac{M_L\left(s\right)}{r}\right)^2 \\
& h^{(\mathrm{2PM})}_{\mathrm{can}\,0i}\left(t,\boldsymbol{x}\right) = 0 \\
& h^{(\mathrm{2PM})}_{\mathrm{can}\,ij}\left(t,\boldsymbol{x}\right) = +\frac{2}{c^4}\,\delta_{ij}\left(\sum_{l=0}^{\infty}\frac{\left(-1\right)^l}{l!}\;\partial_L\;\frac{M_L\left(s\right)}{r}\right)^2 \\
& -\frac{4}{c^4}\mathrm{FP}_{B=0}\Box^{-1}_R\left(\partial_i\sum_{l=0}^{\infty}\frac{\left(-1\right)^l}{l!}\partial_L\frac{M_L\left(s\right)}{r}\right)\left(\partial_j\sum_{l=0}^{\infty}\frac{\left(-1\right)^l}{l!}\partial_L\frac{M_L\left(s\right)}{r}\right)
\end{aligned}
}
$$

- $h^{(\mathrm{2PM})}_{\mathrm{can}\,ij}$ associated with Hadamard technique in Eq. (136)
- solution for time-independent quadrupole-quadrupole is given in detail in my manuscript [13]

9.8 Impact of 2PN effects on light deflection and time delay

- 2PN light deflection (e.g. grazing ray at Jupiter)

$$\boxed{\begin{aligned} \varphi_{\rm 2PN}^{M_A \times M_A} &\le 16\, \frac{G^2 M_A^2}{c^4 P_A^2}\, \frac{x_1}{P_A}\, \frac{x_0\, x_1}{\left(x_0 + x_1\right)^2} \le 16\,\mu{\rm as} \\ \varphi_{\rm 2PN}^{M_A \times M_A^{ab}} &= 4\, \varphi_{\rm 2PN}^{M_A \times M_A} \left|J_2^A\right| = 0.95\,\mu{\rm as} \qquad [16] \end{aligned}} \tag{149}$$

- micro-arcsecond . . . $1\,\mu{\rm as} \simeq 4.85 \times 10^{-12}\,{\rm radians}$
- M_A and P_A . . . mass and radius of body A
- J_2^A . . . second zonal harmonics of body A
- x_0 and x_1 . . . distance body-source and body-observer
- ESA astrometry mission Gaia: launched December 2013
  aimed precision in angular measurements: $\varphi \sim 5\,\mu{\rm as}$
  Near-future astrometry able to detect 2PN effects beyond monopole structure
- [16] S. Zschocke, Physical Review D **105** (2022) 024040

- 2PN time delay (e.g. grazing ray at Jupiter)

$$\boxed{\begin{aligned} \Delta t_{\rm 2PN}^{M_A \times M_A} &\le 8\,\frac{G^2 M_A^2}{c^4 P_A^2}\,\frac{x_1}{c}\,\frac{x_0}{x_0 + x_1} \le 9.3\,{\rm ps} \\ \Delta t_{\rm 2PN}^{M_A \times M_A^{ab}} &\sim \Delta t_{\rm 2PN}^{M_A \times M_A}\left|J_2^A\right| \sim 0.6\,{\rm ps} \quad \text{(guess)} \end{aligned}} \tag{150}$$

- pico-second ... $1\,{\rm ps} = 10^{-12}\,{\rm seconds}$
- atomic clocks on Earth (optical clocks): $\Delta t / t = 10^{-18}$
- atomic clocks in Space (DSAC): $\Delta t / t = 10^{-15}$

  e.g. precision for a signal $t = 10^4\,{\rm s}$: $\Delta t = (0.01 - 10)\,{\rm ps}$

1. ps-level in time-delay measurements achieved by VLBI [14] (about subsequent discussions see also my manuscript [15])
2. note that today's precision in distance Earth-Moon $10^{-3}\,{\rm m}$ by LLR corresponds to precision in time of $\Delta t = 3\,{\rm ps}$

- Today's VLBI facilities and atomic clocks are almost able to detect 2PN effects beyond monopole structure

10. Summary

- MPM is an approach to determine metric density $\overline{g}^{\alpha\beta}$
- from metric density $\overline{g}^{\alpha\beta}$ one may obtain metric $g_{\alpha\beta}$

- MPM makes use of field-theoretical formulation of GR

- general solution depends on 10 irreducible multipoles $\mathcal{A}_L, \mathcal{B}_L, \mathcal{C}_L, \mathcal{D}_L, \mathcal{E}_L, \mathcal{I}_L, \mathcal{G}_L, \mathcal{H}_L, \mathcal{J}_L, \mathcal{T}_L$
- 6 irreducible multipoles independent (local law of conservation) $I_L, J_L, W_L, X_L, Y_L, Z_L$
- 2 multipoles physically relevant (residual gauge) M_L, S_L

- In foreseeable future 2PN effects beyond monopole structure are detectable

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