On the post-linear metric of a solar system body

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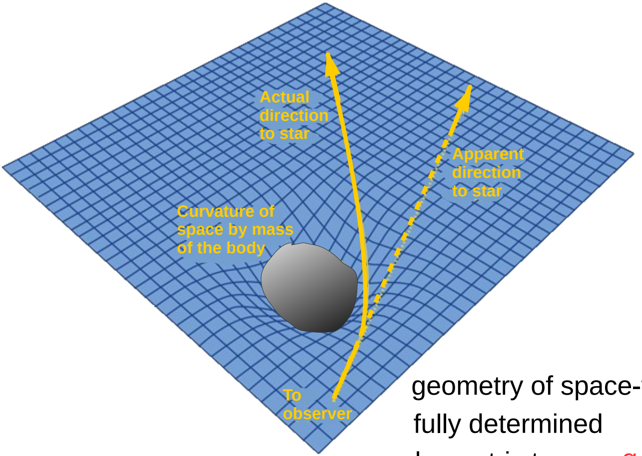
1. Introduction

- astrometry needs to determine light trajectory $\boldsymbol{x}(t)$ from light source through curved space-time of solar system



geometry of space-time
fully determined
by metric tensor $g_{\alpha\beta}$

- in mathematical terms:
  light trajectory $\boldsymbol{x}\,(t)$ is governed by geodesic equation

$$\boxed{\frac{\ddot{x}^i}{c^2} + \Gamma^i_{\mu\nu}\,\frac{\dot{x}^\mu}{c}\,\frac{\dot{x}^\nu}{c} - \Gamma^0_{\mu\nu}\,\frac{\dot{x}^\mu}{c}\,\frac{\dot{x}^\nu}{c}\,\frac{\dot{x}^i}{c} = 0} \qquad (1)$$

- $\Gamma^\alpha_{\mu\nu}$ ... Christoffel symbols

$$\boxed{\Gamma^\alpha_{\mu\nu} = \frac{1}{2}\,g^{\alpha\beta}\left(g_{\beta\mu\,,\,\nu} + g_{\beta\nu\,,\,\mu} - g_{\mu\nu\,,\,\beta}\right)} \qquad (2)$$

- $g_{\alpha\beta}$ ... metric tensor of fundamental importance for astrometry

2. The field equations of gravity

- metric tensor $g_{\alpha\beta}$ is determined by the field equations

$$\boxed{\underbrace{R_{\alpha\beta} - \frac{1}{2}\, g_{\alpha\beta}\, R}_{\text{curvature of space}-\text{time}} \quad = \underbrace{\frac{8\,\pi\, G}{c^4}\, T_{\alpha\beta}}_{\text{matter}}} \tag{3}$$

- Ricci tensor

$$\boxed{R_{\alpha\beta} = \Gamma^{\mu}_{\alpha\beta\,,\,\mu} - \Gamma^{\mu}_{\alpha\mu\,,\,\beta} + \Gamma^{\mu}_{\mu\nu}\,\Gamma^{\nu}_{\alpha\beta} - \Gamma^{\nu}_{\alpha\mu}\,\Gamma^{\mu}_{\nu\beta}} \tag{4}$$

- Ricci scalar

$$\boxed{R = R_{\alpha\beta}\, g^{\alpha\beta}} \tag{5}$$

- stress-energy tensor of matter $T_{\alpha\beta}$

- Landau-Lifschitz: gravitational theory in harmonic coordinates

flat d'Alembert

$$\boxed{\Box\, \overline{g}^{\alpha\beta} = \frac{16\pi G}{c^4}\left(\tau^{\alpha\beta} + t^{\alpha\beta}\right)} \qquad (1^*)$$

metric density

related to energy-momentum of matter

related to energy-momentum of gravitational field

- LL operates with **metric density** $\overline{g}^{\alpha\beta}$ instead of **metric** $g_{\alpha\beta}$

$$\boxed{\overline{g}^{\alpha\beta} = \sqrt{-\det\left(g_{\mu\nu}\right)}\; g^{\alpha\beta}} \qquad (6)$$

3. Multipolar Post-Minkowskian formalism

- iterative approach to solve Eq. (1^*) outside the body

- introduced by K. Thorne (1980)
- important advancements by L. Blanchet and T. Damour (1986)

- subsequent developments (1989 - 2008)
  T. Damour, L. Blanchet, B. Iyer, G. Faye, P. Jaranowski,
  G. Esposito-Farese, S. Sinha, S. Kopeikin, G. Schäfer

- exact solution of **metric density** for massive body:

$$\boxed{\overline{g}^{\alpha\beta} = \eta^{\alpha\beta} - \sum_{n=1}^{\infty} G^n \, \overline{h}^{\alpha\beta}_{(\mathrm{nPM})} \left[M_L, S_L\right] + \underbrace{\text{gauge terms}}_{\text{unphysical}}} \tag{7}$$

- multipoles are integrals over stress-energy tensor of body

(a) mass-multipoles M_L (shape, inner structure, oscillations)
(b) spin-multipoles S_L (rotational motions, inner currents)

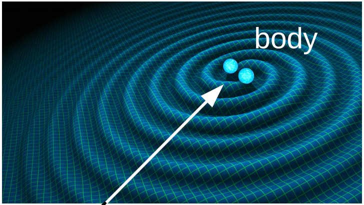
Physics is governed by **metric tensor** $g_{\alpha\beta}$

- **metric** determines line element
- **metric** determines proper time
- **metric** determines proper length
- **metric** determines angular distance
- **metric** determines scalar curvature
- **metric** determines light trajectory

$\vdots$ etc.

Why for MPM sufficient to consider **metric density** $\overline{g}_{\alpha\beta}$?

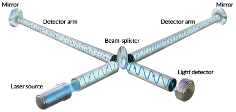
- MPM developed for understanding gravitational waves



far wave zone
of gravitational system

just plane waves

observer

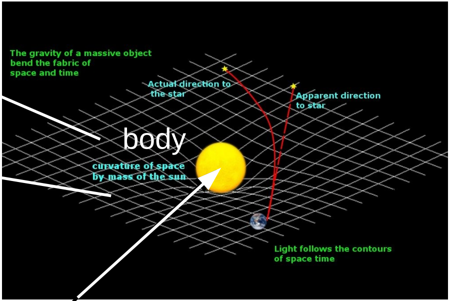


huge distance (~ 10^{25} m)
between body and observer

TT projection of metric relevant

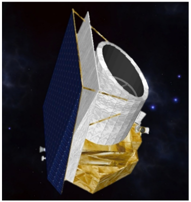
$$\boxed{g^{TT}_{\alpha\beta} = \bar{g}^{TT}_{\alpha\beta}}$$

- we are interested in light deflection in near-zone of solar system



near-zone
of gravitational system

not simply plane waves



observer

small distance (~ 10^{12} m)
between body and observer

entire metric relevant

$$\boxed{g_{\alpha\beta} \neq \bar{g}_{\alpha\beta}}$$

4. The metric tensor

- knowledge of **metric density** allows to obtain **metric tensor**

- exact solution of **metric tensor** for massive body

$$\boxed{\begin{aligned} g_{\alpha\beta} = \eta_{\alpha\beta} \; + \; & \sum_{n=1}^{\infty} G^n \, h_{\alpha\beta}^{(\mathrm{nPM})} \left[M_L, S_L \right] \\ & + \; \underbrace{\text{gauge terms}}_{\text{unphysical}} \end{aligned}} \tag{8}$$

4.1 The linear term of metric perturbation

$$
\boxed{
\begin{aligned}
h_{00}^{(1\mathrm{PM})}(t,\boldsymbol{x}) &= +\frac{2}{c^2}\sum_{l=0}^{\infty}\frac{(-1)^l}{l!}\,\partial_L\,\frac{M_L(s)}{r} \\
h_{0i}^{(1\mathrm{PM})}(t,\boldsymbol{x}) &= +\frac{4}{c^3}\sum_{l=1}^{\infty}\frac{(-1)^l}{l!}\,\partial_{L-1}\,\frac{\dot{M}_{iL-1}(s)}{r} \\
&\quad+\frac{4}{c^3}\sum_{l=1}^{\infty}\frac{(-1)^l\;l}{(l+1)!}\,\epsilon_{iab}\,\partial_{aL-1}\,\frac{S_{bL-1}(s)}{r} \\
h_{ij}^{(1\mathrm{PM})}(t,\boldsymbol{x}) &= +\frac{2}{c^2}\,\delta_{ij}\sum_{l=0}^{\infty}\frac{(-1)^l}{l!}\,\partial_L\,\frac{M_L(s)}{r} \\
&\quad+\frac{4}{c^4}\sum_{l=2}^{\infty}\frac{(-1)^l}{l!}\,\partial_{L-2}\,\frac{\ddot{M}_{ijL-2}(s)}{r} \\
&\quad+\frac{8}{c^4}\sum_{l=2}^{\infty}\frac{(-1)^l\;l}{(l+1)!}\,\partial_{aL-2}\,\frac{\epsilon_{ab(i}\dot{S}_{j)bL-2}(s)}{r}
\end{aligned}
}
\tag{9}
$$

4.2 The post-linear term of metric perturbation

$$\boxed{\begin{aligned} h_{00}^{(2\mathrm{PM})}(t,\boldsymbol{x}) &= -\frac{2}{c^4}\left(\sum_{l=0}^{\infty}\frac{(-1)^l}{l!}\,\partial_L\,\frac{M_L(s)}{r}\right)^2 \\ h_{0i}^{(2\mathrm{PM})}(t,\boldsymbol{x}) &= 0 \\ h_{ij}^{(2\mathrm{PM})}(t,\boldsymbol{x}) &= +\frac{2}{c^4}\,\delta_{ij}\left(\sum_{l=0}^{\infty}\frac{(-1)^l}{l!}\,\partial_L\,\frac{M_L(s)}{r}\right)^2 \\ &-\frac{4}{c^4}\,\square_R^{-1}\left(\partial_i\sum_{l=0}^{\infty}\frac{(-1)^l}{l!}\partial_L\frac{M_L(s)}{r}\right)\left(\partial_j\sum_{l=0}^{\infty}\frac{(-1)^l}{l!}\partial_L\frac{M_L(s)}{r}\right)\end{aligned}} \qquad (10)$$

- post-linear terms associated with integration procedure: $\square_R^{-1}$
- post-linear spin-multipole terms $S_L(s)$ irrelevant for sub-μas

5. Summary

- MPM is an approach to determine **metric density** $\overline{g}^{\alpha\beta}$
- from MPM one may obtain **metric** $g_{\alpha\beta}$
- **metric** $g_{\alpha\beta}$ required for geodesic equation
- **post-linear metric** $g_{\alpha\beta}$ relevant for $\mu\mathrm{as}$ and sub-$\mu\mathrm{as}$

References

[1] L.D. Landau, E.M. Lifschitz, *The Classical Theory of Fields*, Third English Edition, Pergamon Press, Oxford, 1971.

[2] K.S. Thorne, *Multipole expansions of gravitational radiation*, Rev. Mod. Phys. **52** (1980) 299.

[3] L. Blanchet, T. Damour, *Radiative gravitational fields in general relativity*, Phil. Trans. R. Soc. London A **320** (1986) 379.

[4] T. Damour, B.R. Iyer, *Multipole analysis for electromagnetism and linearized gravity with irreducible Cartesian tensors*, Phys. Rev. D **43** (1991) 3259.

[5] S. Zschocke, *A detailed proof of the fundamental theorem of STF multipole expansion in linearized gravity*, Int. J. Mod. Phys. D **23** (2014) 1450003.

[6] S. Zschocke, *Post-linear metric of a compact source of matter*, Phys. Rev. D **100** (2019) 084005.

Backup Slides

Fock-Sommerfeld boundary conditions

1. flatness of the metric tensor at spatial infinity

$$\boxed{\lim_{\substack{r\to\infty \\ t+\frac{r}{c}=\text{const}}} \overline{h}^{\alpha\beta}(t,\boldsymbol{x}) = 0} \tag{11}$$

2. no-incoming radiation condition

$$\boxed{\lim_{\substack{r\to\infty \\ t+\frac{r}{c}=\text{const}}} \left(\frac{\partial}{\partial r} r\,\overline{h}^{\mu\nu}(t,\boldsymbol{x}) + \frac{\partial}{\partial ct}\, r\,\overline{h}^{\mu\nu}(t,\boldsymbol{x})\right) = 0} \tag{12}$$

Formal solution of Landau-Lifschitz field equation

$$\boxed{\overline{h}^{\alpha\beta}(t,\boldsymbol{x}) = -\frac{16\,\pi\,G}{c^4}\,\left(\Box_{\mathrm{R}}^{-1}\left(\tau^{\alpha\beta} + t^{\alpha\beta}\right)\right)(t,\boldsymbol{x})} \tag{13}$$

- with the inverse d'Alembert operator

$$\boxed{\left(\Box_{\mathrm{R}}^{-1} f\right)(t,\boldsymbol{x}) = -\frac{1}{4\,\pi}\int d^3x'\,\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}\,f\left(u\,,\,\boldsymbol{x}'\right)} \tag{14}$$

- and the retarded time u

$$\boxed{u = t - \frac{|\boldsymbol{x}-\boldsymbol{x}'|}{c}} \tag{15}$$

- Eq. (6) is implicit because $\tau^{\alpha\beta}$ and $t^{\alpha\beta}$ depend on $\overline{h}^{\alpha\beta}$

Multipolar Post-Minkowskian formalism

- MPM: iterative approach to solve Eq. (6) outside the body

$$
\boxed{
\begin{aligned}
\overline{h}^{\alpha\beta}_{(1\mathrm{PM})} &= -\frac{16\,\pi}{c^4}\,\Box^{-1}_{\mathrm{R}}\,T^{\alpha\beta} \\
\overline{h}^{\alpha\beta}_{(2\mathrm{PM})} &= -\frac{16\,\pi}{c^4}\,\Box^{-1}_{\mathrm{R}}\left(\tau^{\alpha\beta}_{(1\mathrm{PM})} + t^{\alpha\beta}_{(1\mathrm{PM})}\right) \\
&\;\;\vdots \\
\overline{h}^{\alpha\beta}_{(\mathrm{nPM})} &= -\frac{16\,\pi}{c^4}\,\Box^{-1}_{\mathrm{R}}\left(\tau^{\alpha\beta}_{((\mathrm{n}-1)\mathrm{PM})} + t^{\alpha\beta}_{((\mathrm{n}-1)\mathrm{PM})}\right)
\end{aligned}
}
\tag{16}
$$

- complicated integrals since $\tau^{\alpha\beta}_{(\mathrm{nPM})}$, $t^{\alpha\beta}_{(\mathrm{nPM})}$ depend on $\overline{h}^{\alpha\beta}_{(\mathrm{nPM})}$

- MPM determines **metric density** $\overline{g}_{\alpha\beta}$ but not **metric** $g_{\alpha\beta}$

How to get the metric tensor from MPM

- post-Minkowskian expansion of metric density

$$\boxed{\overline{g}^{\alpha\beta} = \eta^{\alpha\beta} - \underbrace{G^1\, \overline{h}^{\alpha\beta}_{(1\mathrm{PM})}}_{\mathrm{linear\ \ term}} - \underbrace{G^2\, \overline{h}^{\alpha\beta}_{(2\mathrm{PM})}}_{\mathrm{post-linear\ \ term}} - \ldots} \tag{17}$$

- post-Minkowskian expansion of metric tensor

$$\boxed{g_{\alpha\beta} = \eta_{\alpha\beta} + \underbrace{G^1\, h^{(1\mathrm{PM})}_{\alpha\beta}}_{\mathrm{linear\ \ term}} + \underbrace{G^2\, h^{(2\mathrm{PM})}_{\alpha\beta}}_{\mathrm{post-linear\ \ term}} + \ldots} \tag{18}$$

- knowledge of **metric density** allows to determine **metric**

$$\boxed{\begin{array}{ccc} h^{(1\mathrm{PM})}_{\alpha\beta} & \longleftrightarrow & \overline{h}^{\alpha\beta}_{(1\mathrm{PM})} \\ h^{(2\mathrm{PM})}_{\alpha\beta} & \longleftrightarrow & \overline{h}^{\alpha\beta}_{(2\mathrm{PM})} \end{array}} \tag{19}$$

Relations between metric density and metric

- relation in 1PM approximation

$$\boxed{h_{\alpha\beta}^{(1\mathrm{PM})} = \overline{h}^{\mu\nu}_{(1\mathrm{PM})}\,\eta_{\alpha\mu}\,\eta_{\beta\nu} - \frac{1}{2}\,\overline{h}_{(1\mathrm{PM})}\,\eta_{\alpha\beta}} \tag{20}$$

- relation in 2PM approximation

$$\boxed{\begin{aligned} h_{\alpha\beta}^{(2\mathrm{PM})} &= \overline{h}^{\mu\nu}_{(2\mathrm{PM})}\,\eta_{\alpha\mu}\,\eta_{\beta\nu} - \frac{1}{2}\,\overline{h}_{(2\mathrm{PM})}\,\eta_{\alpha\beta} + \frac{1}{8}\,\overline{h}^{2}_{(1\mathrm{PM})}\,\eta_{\alpha\beta} \\ &- \frac{1}{2}\,\overline{h}_{(1\mathrm{PM})}\,\overline{h}^{\mu\nu}_{(1\mathrm{PM})}\,\eta_{\alpha\mu}\,\eta_{\beta\nu} + \overline{h}^{\rho\nu}_{(1\mathrm{PM})}\,\overline{h}^{\mu\sigma}_{(1\mathrm{PM})}\,\eta_{\mu\nu}\,\eta_{\alpha\rho}\,\eta_{\beta\sigma} \\ &- \frac{1}{4}\,\overline{h}^{\mu\nu}_{(1\mathrm{PM})}\,\overline{h}^{\rho\sigma}_{(1\mathrm{PM})}\,\eta_{\mu\rho}\,\eta_{\nu\sigma}\,\eta_{\alpha\beta} \end{aligned}} \tag{21}$$

Multipoles

- the multipoles are integrals over the stress-energy tensor

$$\boxed{\hat{F}_L^{\alpha\beta}(s) = \int d^3x' \, \hat{x}'_L \int_{-1}^{+1} dz \, \delta_l(z) \;\; T^{\alpha\beta}\left(\frac{s + z\, r'}{c}, \boldsymbol{x}'\right)} \tag{22}$$

- where $\hat{x}_L$ are STF tensors, e.g. $\hat{x}_{a_1 a_2} = x_{a_1} x_{a_2} - \frac{1}{3} r \, \delta_{a_1 a_2}$
- and $r = |\boldsymbol{x}|$ and $r' = |\boldsymbol{x}'|$
- and $\partial_L = \partial_{a_1 \ldots a_l}$ are l spatial derivatives
- and s is the retarded time

$$\boxed{s = t - \frac{|\boldsymbol{x}|}{c}} \tag{23}$$

- and δ_l are the following functions

$$\boxed{\delta_l = \frac{(2l+1)!!}{2^{l+1} \; l!} \left(1 - z^2\right)^l} \tag{24}$$

Landau-Lifschitz formulation

- Then the field equations are of considerably simpler structure

$$\boxed{\Box\,\overline{g}^{\alpha\beta} = 2\,\kappa\,\left(\tau^{\alpha\beta} + t^{\alpha\beta}\right)} \tag{25}$$

- $\Box$ ... flat d'Alembert operator
- with the metric density

$$\boxed{\overline{g}^{\alpha\beta} = \sqrt{-g}\,g^{\alpha\beta}} \tag{26}$$

- related to "stress-energy" of matter

$$\boxed{\tau^{\alpha\beta} = -g\;T^{\alpha\beta}} \tag{27}$$

- related to "stress-energy" of gravitational field

$$\boxed{t^{\alpha\beta} = -g\;t^{\alpha\beta}_{\mathrm{LL}} + \frac{1}{2\,\kappa}\left(\overline{g}^{\alpha\mu}{}_{,\,\nu}\,\overline{g}^{\beta\nu}{}_{,\,\mu} - \overline{g}^{\alpha\beta}{}_{,\,\mu\nu}\,\overline{g}^{\mu\nu}\right)} \tag{28}$$

Fundamental Theorem of MPM

- formal solution in 1PM approximation

$$\boxed{\overline{h}^{\alpha\beta}_{(1\mathrm{PM})}\,(t,\boldsymbol{x}) = -\frac{16\,\pi}{c^4}\left(\Box_R^{-1}\; T^{\alpha\beta}\right)(t,\boldsymbol{x})} \tag{29}$$

- given in terms of 10 symmetric tracefree (STF) multipoles

$$\boxed{\overline{h}^{\alpha\beta}_{(1\mathrm{PM})}\,(t,\boldsymbol{x}) = \frac{4}{c^4}\,\sum_{l=0}^{\infty}\frac{(-1)^l}{l!}\;\partial_L\;\frac{F_L^{\alpha\beta}\,(s)}{r}} \tag{30}$$

- $s = t - \dfrac{|\boldsymbol{x}|}{c}$ ... retarded time
- multipoles $F_L^{\alpha\beta}$ ... integrals over stress-energy tensor of matter

Post-Minkowskian expansion

- post-Minkowskian expansion yields hierarchy of field equations

$$\boxed{\begin{aligned} \Box\, \overline{h}^{\alpha\beta}_{(1\mathrm{PM})} &= -\frac{16\,\pi}{c^4}\; T^{\alpha\beta} \\ \Box\, \overline{h}^{\alpha\beta}_{(2\mathrm{PM})} &= -\frac{16\,\pi}{c^4}\left(\tau^{\alpha\beta}_{(1\mathrm{PM})} + t^{\alpha\beta}_{(1\mathrm{PM})}\right) \\ &\vdots \\ \Box\, \overline{h}^{\alpha\beta}_{(\mathrm{nPM})} &= -\frac{16\,\pi}{c^4}\left(\tau^{\alpha\beta}_{((\mathrm{n}-1)\mathrm{PM})} + t^{\alpha\beta}_{((\mathrm{n}-1)\mathrm{PM})}\right) \end{aligned}} \tag{31}$$

Diffeomorphism between physical and background manifold