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LIGHT DEFLECTION IN BINARY STARS

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ABSTRACT

The light deflection of one component of a binary system due to the gravitational field of the other component is investigated. While this relativistic effect has not been observed thus far, the question arises of whether this effect can become detectable in view of today's high-precision astrometry, which soon will reach the microarcsecond level of accuracy. The effect is studied and its observability is investigated. It turns out that in total there are about 10^3 binaries having orbital parameters such that the light deflection amounts to at least 1 μas. Two stringent criteria for the orbital parameters are presented, by means of which one can easily determine the maximal value of light deflection effect for a given binary system. It is found that for relevant binaries their orbital parameters must take rather extreme values in order to have a light deflection of the order of a few microarcseconds. Only in a very few and rather extreme binary systems might the light deflection effect be detectable by today's astrometry, but their existence is highly improbable. Thus, the detection of this subtle effect of relativity still remains a challenge for future astrometric missions.

Key words: astrometry – binaries: general

1. INTRODUCTION

Astrometric space missions, especially the ESA cornerstone mission *Gaia* (see, e.g., Perryman et al. 2001), are in preparation to attain a microarcsecond (μas) level of accuracy in absolute positional measurements of stars and other celestial objects. This unprecedented accuracy of astrometric observations makes it necessary to account for many subtle effects that were totally negligible before. A practical model for astrometric observations with an accuracy of 1 μas has been formulated by Klioner (2003), where the influence of the gravitational fields inside the solar system were taken into account. Furthermore, a number of additional effects potentially observable at this level of accuracy due to various gravitational fields generated outside of the solar system were also briefly discussed in that investigation. One of them is the gravitational light deflection of one companion of a binary system in the gravitational field of the other companion. This effect would change the apparent position of one component of a binary system. (Without the loss of generality, throughout the investigation the light deflection of component B at component A is considered, and hence, component A is considered to be the massive body, while component B is the light source). While it is clear that this effect is relatively small and even at the level of 1 μas observable only for edge-on binary systems, the huge amount of binaries generate some hope that there are relevant systems where this light deflection effect becomes detectable. For instance, *Gaia* will observe 10^9 stars brighter than the 20th apparent magnitude. Detailed numerical simulations predict the detection of about 10^8 (resolved, astrometric, eclipsing, and spectroscopic) binary systems by *Gaia* mission (see Zwitter & Munari 2004, which is a considerable increase compared to the 10^5 binary systems known so far see the Washington Double Star Catalog 2011).

Let us consider this argument a bit more quantitatively. For a simple estimate of the expected order of magnitude in light deflection, we apply the classical lens equation (in the form given by Equation (67) in Fritelli et al. 2000, Equation (24) in Bozza 2008, or Equation (23) in Zschocke 2011). In terms of orbital elements of a binary system, it can be written as follows:

$$\varphi = \frac{1}{2}\left(\sqrt{\frac{A^2}{r^2}\cos^2 i + 16\,\frac{m}{r}\,\frac{A}{r}\,\sin i} - \frac{A}{r}\,|\cos i|\right). \quad (1)$$

Here, φ is the light deflection angle, i is the inclination, A is the semimajor axis, and r is the distance of center of mass (CMS) of the binary system from the observer, and the Schwarzschild radius is $m = (G\,M/c^2)$, where G is the gravitational constant and c is the speed of light, and M is the stellar mass of component A of the binary system. From Equation (1) one obtains the maximal possible light deflection for edge-on binaries, i.e., attained for the case when the inclination is exactly 90°:

$$\varphi \leqslant 200\,\mu\text{as}\,\sqrt{\frac{M}{M_\odot}\,\frac{A}{\text{AU}}\,\frac{\text{pc}}{r}}\,. \quad (2)$$

Here, AU $= 1.496 \times 10^{11}$ m is the astronomical unit, pc $= 3.086 \times 10^{16}$ m stands for parallax of 1″, and $M_\odot$ is the solar mass. According to this formula, the choice of moderate values like $M = M_\odot$ and $A \sim 100$ AU would result in a significant light deflection effect at a microarcsecond level even at large distances of about $r \sim 100$ pc. A meaningful value for the density of stars in the solar neighborhood, 0.025 binaries pc^{-3}, already implies 10^5 binaries inside a sphere of $r = 100$ pc. Thus, one might conclude that there are a few relevant edge-on binary systems.

However, in order to estimate quantitatively the number of relevant systems, one needs to know the probability for the occurrence of such edge-on binaries that have a given light deflection depending on their orbital parameters such as inclination, mass, distance and semimajor axis. Such a relation between a given light deflection and orbital parameters is given by a so-called inclination formula. Since the inclinations of binary systems are of course randomly distributed it is meaningful to resolve such an inclination formula in terms of inclination. As has been shown by S. A. Klioner, F. Mignard, & M. Soffel (2003, private communication; see Appendix D for some basic steps, since their manuscript has not been published),[1] such an inclination formula can be obtained by means of the analytical solution of light deflection in standard post-Newtonian approximation, and is given as follows:

$$\left|\frac{\pi}{2} - i\right|_{\rm KMS} \leqslant 2\,\arctan\left(0.0197\,\frac{M}{M_\odot}\,\frac{\mu{\rm as}}{\varphi}\,\frac{\rm pc}{r}\right). \quad (3)$$

For a better illustration of the inclination formula, relation (3) is rewritten in terms of angular degrees instead of radians:

$$|\,90^\circ - i\,| \leqslant 2\overset{\circ}{.}25\,\frac{M}{M_\odot}\,\frac{\mu{\rm as}}{\varphi}\,\frac{\rm pc}{r}\,, \quad (4)$$

where $\arctan x = x + \mathcal{O}(x^3)$ has also been used. According to this relation, the inclination i of a binary system with stellar mass M and at distance r must not deviate from the edge-on value of 90° too much in order to have a given light deflection φ. For example, for a hypothetical binary star with $M = M_\odot$ situated at a distance of $r = 10\,{\rm pc}$ the light deflection effect attains $1\,\mu$as only if $|\,90^\circ - i\,| < 0\overset{\circ}{.}225$, which means that the probability of observing this binary at a favorable inclination is only about 0.2%.

But even this estimation is still much too optimistic. As a concrete example of today's high-precision astrometry, let us consider one important parameter about the astrometric accuracy of the *Gaia* mission: the accuracy of one individual positional measurement, which in the most ideal case (bright star, i.e., 10th magnitude) amounts to $25\,\mu$as, which implies $\varphi \geqslant 25\,\mu$as. Furthermore, inside a sphere of 10 pc around the Sun almost every star and binary system is known already by the data of the Research Consortium on Nearby Stars (RECONS 2012). Since inside that sphere there is no binary system having a light deflection on a microarcsecond level, one has to take at least $r > 10\,{\rm pc}$. By taking into account both of these points, one obtains $(\mu{\rm as}/\varphi)({\rm pc}/r) = 1/250$ in relation (4). Therefore, even in the best case one can conclude $|90^\circ - i| \leqslant 0\overset{\circ}{.}01\ M/M_\odot$, which means that for $M = M_\odot$ the probability of observing such a binary at a favorable inclination is practically only about 0.01%. And the binaries must be, in fact, almost edge-on in order to have a light deflection that can be detected by today's astrometry. Accordingly, while relation (2) triggers hope about the existence of many relevant binary systems, from relation (4) one concludes that the number of relevant binary systems is considerably reduced.

An estimation of the total amount of binaries depends on many different parameters such as mass, semimajor axis, inclination and distances of the binaries. Therefore, a simple estimation is not so straightforward as one might believe. Moreover, relation (2) has been obtained with the aid of a classical lens equation, while relation (4) has been obtain by means of the standard post-Newtonian approach. Both approaches have different regions of validity. However, a rigorous treatment of the problem of light deflection in binary systems implies the need of an analytical formula that is valid for such kinds of extreme astrometric configurations such as binary systems. Recently, a generalized lens equation has been derived by Zschocke (2011), which allows us to determine the light deflection of binary systems on a microarcsecond level. One aim of this study is, therefore, to reobtain criteria (2) and (4) as stringent conditions from one and the same approach, i.e., with the aid of a generalized lens equation. This is possible because in the corresponding limits the generalized lens equation agrees with the classical lens equation and the standard post-Newtonian solution. Another aim is to derive an inclination formula like Equation (3) for binary systems which allows us to determine the needed inclination for a given light deflection angle and as a function of the orbital parameters. For that, one has to take into account the distribution of stellar masses and the distribution of semimajor axes in binary systems. Finally, the aim of this study is to determine the total number of relevant binaries having a light deflection on a microarcsecond level, and to investigate the possibility of detecting this effect of light deflection with today's high-precision astrometry.

The article is organized as follows: in Section 2 some basics about orbital elements of binary systems are given. The generalized lens equation and the inclination formula are presented in Section 3. In Section 4 two stringent conditions on the orbital elements of binary systems (astrometric, spectroscopic, eclipsing, and resolved binaries) are presented, which allow us to determine whether or not the binary system will have the light deflection of a given magnitude. The total number of binaries which have a given light deflection for an infinite time of observation is estimated in Section 5, while the more practical case of a finite time of observation is considered in Section 6. The special case of resolved binaries is considered in Section 7. For that, the specific instrumentation of *Gaia* mission is considered in some detail as the most modern astrometric mission with the present-day highest possible accuracy. A summary is given in Section 8.

2. ORBITAL ELEMENTS OF A BINARY SYSTEM

Consider a binary system, component A with mass M_A at coordinate $\boldsymbol{r}_A$ and component B with mass M_B at coordinate $\boldsymbol{r}_B$. In order to express the light deflection effect in terms of orbital elements, spherical coordinates are introduced, illustrated by Figure 1.

The center of the coordinate system is located at the CMS, i.e.,

$$\boldsymbol{r}_{\rm CMS} = \frac{1}{M_A + M_B}\,(M_A\,\boldsymbol{r}_A + M_B\,\boldsymbol{r}_B)\,. \quad (5)$$

Thus, the vector $\boldsymbol{r}$, which points from CMS to the observer, is given by

$$\boldsymbol{r} = \begin{pmatrix} r\,\cos\omega\,\sin i \\ r\,\sin\omega\,\sin i \\ r\,\cos i \end{pmatrix}, \quad (6)$$

where $r = |\,\boldsymbol{r}\,|$, the argument of periapsis is denoted by ω, and i is the inclination (see Figure 1). The solution of the equation of motion yields for vectors $\boldsymbol{r}_A$ and $\boldsymbol{r}_B$ the expression given by Equations (A22)–(A26). Vector $\boldsymbol{x}_1$ points from the mass center of the massive body to the observer, and vector $\boldsymbol{x}_0$ points from the mass center of the massive body to the source (see also Figure 1). The coordinates of these vectors can be expressed by the orbital elements of the binary star as follows:

$$\boldsymbol{x}_1 = \boldsymbol{r} - \boldsymbol{r}_{\rm A} = \begin{pmatrix} r\,\cos\omega\,\sin i - \dfrac{A\,(\cos E - e)}{1 + \frac{M_A}{M_B}} \\ r\,\sin\omega\,\sin i - \dfrac{A\,\sqrt{1-e^2}\,\sin E}{1 + \frac{M_A}{M_B}} \\ r\,\cos i \end{pmatrix}, \quad (7)$$

[1] Klioner, S. A., Mignard, F., & Soffel, M. 2003, Astrometric Signature of Gravitational Microlensing on the Components of Edge-On Binary Systems, unpublished.

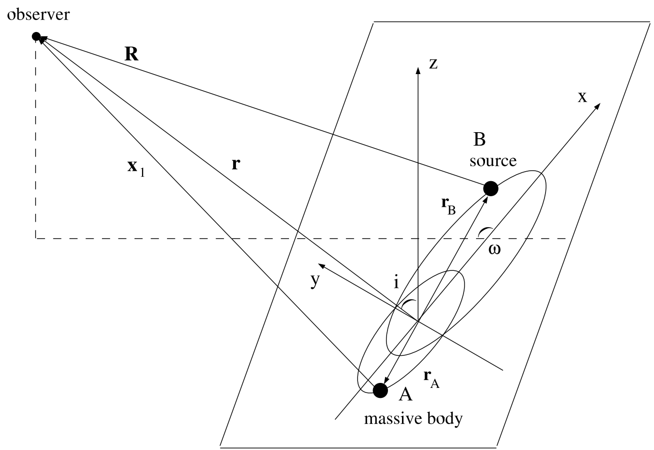

Figure 1. Seven orbital elements which define the orbit of a binary system: distance vector $\boldsymbol{r}$, semimajor axis A, inclination i, eccentricity e, eccentric anomaly E, periapsis ω, and mass ration M_A/M_B. The orbit of the binary system spans the (x, y)-plane and the z-axis is perpendicular to the orbital plane. The x-axis is oriented along the semimajor axis of the orbit of the binary system, while the y-axis is perpendicular to the x-axis. The vector $\boldsymbol{r}$ is directed from the center of mass (CMS) of the binary system, see Equation (5), to the observer. The center of spherical coordinate system is located at the CMS of binary system, i.e., $\boldsymbol{r}_{\rm CMS} = \boldsymbol{0}$. The inclination $0 \leqslant i \leqslant \pi$ is the angle between $\boldsymbol{r}$ and z-axis; $i = \pi/2$ is called edge-on and $i > \pi/2$ corresponds to retrograde orbit. The dotted line indicates the projection of $\boldsymbol{r}$ onto orbital (x, y)-plane, i.e., z-component of $\boldsymbol{r}$ equals zero. The angle between this projection and x-axis is called argument of periapsis $0 \leqslant \omega \leqslant \pi$. The orbital elements semimajor axis A, eccentricity $0 \leqslant e \leqslant 1$ and mass ratio M_A/M_B govern uniquely the geometric shape of both ellipses. The eccentric anomaly $0 \leqslant E \leqslant 2\pi$ (not plotted here), is defined in Equation (A16) of Appendix A and determines the actual position of the bodies A and B on their orbit.

$$\boldsymbol{x}_0 = \boldsymbol{r}_{\rm B} - \boldsymbol{r}_{\rm A} = - \begin{pmatrix} A(\cos E - e) \\ A\,\sqrt{1-e^2}\,\sin E \\ 0 \end{pmatrix}. \tag{8}$$

Here, A is the semimajor axis, e is the eccentricity, and E is the eccentric anomaly (see Appendix A). Vectors (7) and (8) will be used to express the light deflection in terms of orbital elements of the binary system.

3. INCLINATION FORMULA FROM GENERALIZED LENS EQUATION

A scheme of light propagation of a signal emitted at component B in the gravitational field of component A is shown in Figure 2. Vector $\boldsymbol{x}_1$ points from the mass center of the massive body to the observer, and vector $\boldsymbol{x}_0$ points from the mass center of the massive body to the source, and we define $\boldsymbol{R} = \boldsymbol{x}_0 - \boldsymbol{x}_1$, the absolute value $R = |\boldsymbol{R}|$, and the unit vector by $\boldsymbol{k} = \boldsymbol{R}/R$. Furthermore, the impact vector $\boldsymbol{d} = \boldsymbol{k} \times (\boldsymbol{x}_1 \times \boldsymbol{k})$ and its absolute value is denoted by $d = |\boldsymbol{d}|$. The Schwarzschild radius of the massive body, i.e., of component A of the binary system, is denoted by $m = (G\,M/c^2)$.

For determining the light deflection in weak gravitational fields there are two essential approaches: the standard post-Newtonian approach, e.g., Brumberg (1991), and the classical lens equation (e.g., Equation (67) in Fritelli et al. 2000, Equation (24) in Bozza 2008, or Equation (23) in Zschocke 2011). While the first approach is restricted by the condition $m \gg d$, the second approach is only valid in the case that the source and observer are far from the massive body, especially for $a = \boldsymbol{k} \cdot \boldsymbol{x}_1 \gg d$ and $b = -\boldsymbol{k} \cdot \boldsymbol{x}_0 \gg d$ (for a geometrical illustration of a, b and d see Figure 2). However, in binary systems extreme configurations are possible such as $d = 0$ or $b = 0$. Therefore, in order to investigate the light deflection in binary systems one needs a generalized lens equation that is valid in such extreme configurations where the standard post-Newtonian approach as well as the classical lens equation cannot be applied. Recently, Zschocke (2011) derived a generalized lens equation, which allows us to determine the light deflection in such extreme astrometric configurations as binary systems:

$$\varphi = \frac{1}{2}\left[\left(\frac{d^2}{x_1^2} + 8\,\frac{m}{x_1}\,\frac{x_0\,x_1 - \boldsymbol{x}_0\cdot\boldsymbol{x}_1}{R\,x_1}\right)^{1/2} - \frac{d}{x_1}\right]. \tag{9}$$

Actually, the lens equation has two solutions, but here only one solution is considered, while the second solution represents just the second image of one and the same source, which is not relevant in our investigation. The generalized lens equation is valid up to terms of the order of $\mathcal{O}(m^2/d\,'^2)$, and the absolute value of their total sum can be shown to be smaller or equal to $(15\,\pi/4)(m^2/d'^2)$. Here, $d\,' = L/E$ is Chandrasekhar's impact parameter (see Chandrasekhar 1983), with L being the orbital momentum and E the energy of the photon in the gravitational field of massive body. Basically, the light-ray of component B cannot be observed if $d\,'$ is smaller than the radius of massive body A. For stars, the radius is much larger than the Schwarzschild radius m, and hence $(m^2/d\,'^2) \ll 1$. Furthermore, the generalized lens equation (9) is finite for $d \rightarrow 0$ and $b = -\boldsymbol{k}\cdot\boldsymbol{x}_0 \rightarrow 0$, both of which are possible astrometric configurations in binary systems. Furthermore, in Zschocke (2011) it has been shown that the generalized lens equation (9) yields within the appropriate limits the correct standard post-Newtonian solution and the classical lens equation, and hence provides a bridge between these essential approaches.

In the following, Equation (9) is applied in order to determine the light deflection in binary systems. For that, the coordinates $\boldsymbol{x}_0$ and $\boldsymbol{x}_1$ are used in the form as given by Equations (7) and (8), respectively. A typical light curve of a binary system, calculated by means of generalized lens equation (9), is shown in Figure 3.

Then, an inclination formula is derived from the generalized lens equation (9). The impact of eccentricity is neglected, and circular orbits $e = 0$ are considered, implying $\omega = 0$. Thus, the coordinates $\boldsymbol{x}_0$ and $\boldsymbol{x}_1$ are simplified to the expressions (B1) and (B2) given in Appendix B. Furthermore, the maximal value of light deflection of a binary system is of interest, i.e., the astrometric configuration $E = 0$ is considered here. Then, by inserting these coordinates in the generalized lens equation (9) one obtains (see Equation (B14) in Appendix B) up to terms of the order of $\mathcal{O}(\sqrt{(m/r)(A/r)}A/r)$

$$\varphi = \frac{1}{2}\left[\left(\frac{A^2}{r^2}\,\cos^2 i + 8\,\frac{m}{r}\,\frac{A}{r}\,(1+\sin i)\right)^{1/2} - \frac{A}{r}\,|\cos i\,|\right]. \tag{10}$$

The minimal value $\varphi_{\rm min} = \varphi(i = 0)$ and maximal value $\varphi_{\rm max} = \varphi(i = \pi/2)$ of light deflection for the astrometric position $E = 0$ follow from Equation (10):

$$\varphi_{\rm min} = \frac{1}{2}\left[\left(\frac{A^2}{r^2} + 8\,\frac{m}{r}\,\frac{A}{r}\right)^{1/2} - \frac{A}{r}\right] \approx 2\,\frac{m}{r} = 0.0197\,\mu{\rm as}\,\frac{M}{M_\odot}\,\frac{\rm pc}{r}\,, \tag{11}$$

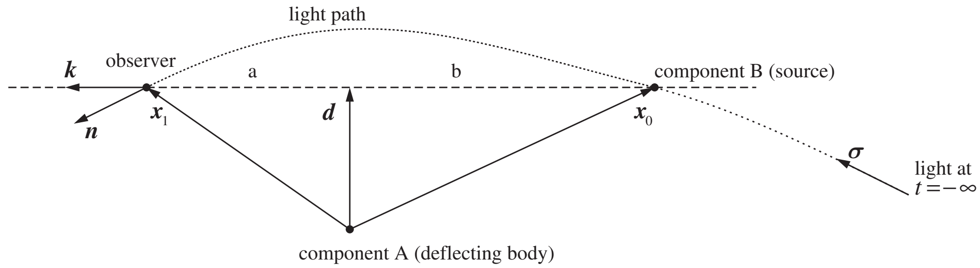

Figure 2. Binary star composed of component A as the massive body, and component B considered to be the light-source.

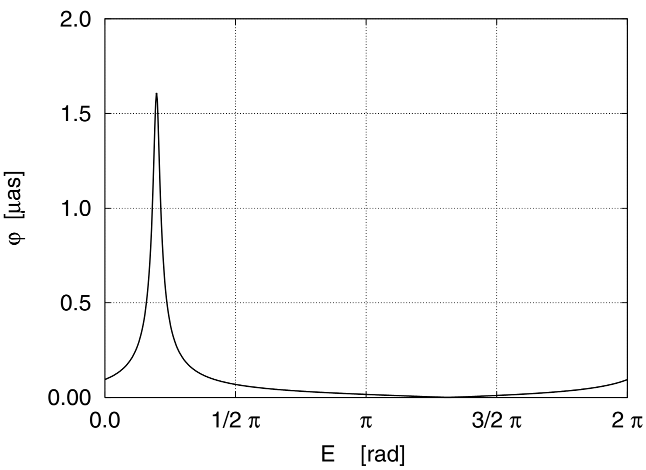

Figure 3. Typical light curve of a binary system, determined using generalized lens equation (9) or (B11), respectively. The parameters chosen are: distance $r = 1\,\mathrm{pc}$, semimajor axis $A = 100\,\mathrm{AU}$ inclination $i = 31/64\,\pi$, mass $M_A = 2\,M_\odot$, mass ratio $M_A/M_B = 2.0$, eccentricity $e = 0.25$, argument of periapsis $\omega = \pi/4$.

$$\varphi_{\max} = 2\,\frac{(m\,A)^{1/2}}{r} = 200\,\mu\mathrm{as}\,\left(\frac{M}{M_\odot}\,\frac{A}{\mathrm{AU}}\right)^{1/2}\,\frac{\mathrm{pc}}{r}\,, \quad (12)$$

where in Equation (11) terms of the order of $\mathcal{O}(m^2/r\,A)$ have been neglected. Expression (10) can be reconverted in terms of inclination (see Appendix C):

$$\left|\frac{\pi}{2} - i\right| = \arccos\left(-\frac{p}{2} + \sqrt{\frac{p^2}{4} - q}\right), \quad (13)$$

where

$$p = \frac{8\,m^2\,A - 4\,m\,r^2\,\varphi^2}{A(r^2\,\varphi^2 + 4\,m^2)}, \quad (14)$$

$$q = -\frac{A^2\,r^2\,\varphi^2 + 4\,m\,A\,r^2\,\varphi^2 - 4\,m^2\,A^2 - r^4\,\varphi^4}{A^2(r^2\,\varphi^2 + 4\,m^2)}. \quad (15)$$

The inclination formula (13) yields the upper limit for $|(\pi/2) - i|$ of a binary system in order to have a given value of light deflection φ. Note that the values of φ cannot be chosen arbitrarily, but they are restricted by $\varphi_{\min}$ and $\varphi_{\max}$ given by Equations (11) and (12), respectively.

The inclination formula (13) can considerably be simplified. From Equations (14) and (15) one obtains by series expansion

$$p = -4\,\frac{m}{A} + 8\,\frac{m^2}{r^2\,\varphi^2} + \mathcal{O}(m^3), \quad (16)$$

$$q = -1 - 4\,\frac{m}{A} + 8\,\frac{m^2}{r^2\,\varphi^2} - 4\,\frac{m^2}{A^2} + \frac{r^2\varphi^2}{A^2} + \mathcal{O}(m^3). \quad (17)$$

By means of Equation (12), the last term in Equation (17) can be estimated to be smaller than $4(m/A) \ll 1$. Here, it should be underlined that $(m/A) \ll (m/r\,\varphi)$ even at large distances of $r \simeq 10^3\,\mathrm{pc}$ and small values for the semimajor axis $A \simeq 1\,\mathrm{AU}$. Thus, one obtains

$$\left|\frac{\pi}{2} - i\right| \approx \arccos\left(1 - 8\,\frac{m^2}{r^2\,\varphi^2}\right) \approx 2\,\arctan\left(2\,\frac{m}{r\,\varphi}\right) \quad (18)$$

up to terms of the order of $\mathcal{O}(m^3/r^3\,\varphi^3)$ and $\mathcal{O}(m/A)$; here the relation $\arccos(1 - 8\,x^2) = 2\,\arctan 2\,x + \mathcal{O}(x^3)$ for $x \ll 1$ has been used. It should be underlined that the applicability of Equation (18) is restricted by the condition (D5) given in Appendix D and by $d \gg m$. Here, it should be noticed that $x = 2(m/r\,\varphi) \ll 1$, even in such an extreme case like $r = 1\,\mathrm{pc}$, $m = m_\odot$ and $\varphi \simeq 1\,\mu\mathrm{as}$ one obtains a small number of $x = 0.019$; for an analytical proof use the exact expression for $\varphi_{\min}$. Due to $(m/A) \ll (m/r\,\varphi)$, the impact of the semimajor axis is of a lower order and can be neglected in the inclination formula. The inclination formula (18) can be expressed in terms of dimensionless quantities as follows:

$$\left|\frac{\pi}{2} - i\right| \approx 2\,\arctan\left(0.0197\,\frac{M}{M_\odot}\,\frac{\mu\mathrm{as}}{\varphi}\,\frac{\mathrm{pc}}{r}\right). \quad (19)$$

Note that expression (19) agrees with the inclination formula (3) derived at the first time by S. A. Klioner, F. Mignard, & M. Soffel (2003, private communication) the arguments of their work are represented in Appendix D. The simplified inclination formula (19) is not only useful for straightforward estimations about the order of magnitude, but it's simple structure provides also an obvious comprehension about the interplay of the individual terms.

4. STRINGENT CONDITIONS ON ORBITAL PARAMETERS FOR BINARY SYSTEMS

In this section, two stringent conditions on the orbital elements are highlighted for binaries having a given light deflection. These strict conditions are valid for any binary system: astrometric, spectroscopic, eclipsing and resolved binaries.

The first stringent condition follows from the maximal light deflection angle (12), given by

$$\varphi \leqslant 200\,\mu\mathrm{as}\,\sqrt{\frac{M}{M_\odot}\,\frac{A}{\mathrm{AU}}\,\frac{\mathrm{pc}}{r}}. \quad (20)$$

It represents a strict criterion for the maximal light deflection of a binary system with a given Schwarzschild radius of component A, a given semimajor axis A, and a given distance r between the binary system and the observer.

The second stringent condition on the orbital elements follows from the inclination formula in the simplified form as given by Equation (19). For a better illustration, this condition is given in terms of angular degrees instead of radians. Using $\arctan x = x + \mathcal{O}(x^3)$ one obtains

$$|\,90^\circ - i\,| \leqslant 2\overset{\circ}{.}25\ \frac{M}{M_\odot}\ \frac{\mu\text{as}}{\varphi}\ \frac{\text{pc}}{r}\,. \tag{21}$$

According to this strict condition, the inclination i of a binary system with mass M (recall M is the stellar mass of component A) and at distance r must not exceed the given value in order to have a light deflection φ.

Both these stringent conditions, Equations (20) and (21), were already stated in the introductory section by Equations (2) and (4), respectively. However, is should be underlined here that both Equations (20) and (21) were obtained with the aid of one and the same approach, namely the generalized lens equation, while Equations (2) and (4) were obtained by means of the classical lens equation and the post-Newtonian solution, i.e., by two different approaches.

The observability of light deflection effect in binaries implies the realization of both these stringent conditions (20) and (21) simultaneously for a given binary system. But even if a given binary system satisfies both conditions, the observability of light deflection effect is not guaranteed, because the astrometric position $E = 0$ has to be reached during the time of observation. Nonetheless, as soon as the orbital elements r, A, m and i of the binaries are known, both stringent conditions (20) and (21) allow us to find a possible candidate for being a relevant binary system for a given light deflection φ depending on the instrumentation of the observer. However, as will be shown in Sections 5 and 6, the existence of such systems is highly improbable.

5. TOTAL NUMBER OF BINARIES WITH A GIVEN LIGHT DEFLECTION FOR INFINITE TIME OF OBSERVATION

In the previous section, the conditions on orbital parameters for a binary system have been determined in order to have a given magnitude of light deflection φ. In this section, the total number of such relevant binaries is determined. In order to estimate the total number of binaries having a given light deflection φ, the following formula is applied:

$$N\,(\varphi) = \int_{R_{\rm min}}^{R_{\rm max}} d^3 r\ \rho(r) \int_{A_{\rm min}}^{A_{\rm max}} dA\ f(A) \int_{\mu_{\rm min}}^{\mu_{\rm max}} d\mu\ f(\mu)\ P(i). \tag{22}$$

Here, $\rho(r)$ is the density of binaries, $f(A)$ is the semimajor axis distribution of binary systems, and $f(\mu)$ is the mass distribution of stars where $\mu = (M/M_\odot)$ is the mass-ratio of the massive body (component A) and solar mass. The probability distribution $P(i)$ of finding a binary system with a given inclination $0 \leqslant i \leqslant \pi$, is a function of distance r, semimajor axis A, mass ratio μ, and the given light deflection angle φ. According to Equation (13), the probability distribution $P(i)$ is given by (the inclination of binary systems is of course a random distribution)

$$P(i) = \frac{2}{\pi}\ \arccos\left(-\frac{p}{2} + \sqrt{\frac{p^2}{4} - q}\right), \tag{23}$$

where p and q are given by Equations (14) and (15).

For the minimal distance of a binary system from the Sun one can safely assume $R_{\rm min} = 1$ pc. From Equation (12) it follows that beyond a sphere with radius $R_{\rm max} = 2000$ pc only a very few exceptional binary systems might have a light deflection of more than 1 μas. The Sun is located in the Orion arm which has about 1000 pc across and approximately 3000 pc in length. For the estimate according to Equation (22), it is meaningful to assume that the stars are homogeneously distributed inside the Orion arm. For the uniform star density, $\rho_{\rm stars} = 0.1$ star pc^{-3}, a value which is in line with the data of RECONS (2012). Furthermore, a common presumption is that about 50% of all stars are components of a binary or multiple system (see Duquennoy & Mayor 1991 and Halbwachs et al. 2003). Then one obtains for the density of binaries

$$\rho(r) \simeq 0.025 \text{ binaries pc}^{-3}. \tag{24}$$

Let us now consider the distribution of semimajor axis A in binary systems. Statistical investigations show that the distribution of binary semimajor axis is flat in a logarithmic scale over the range of six orders of magnitude that is assumed to be valid in the large range $A_{\rm min} = 10\,R_\odot \leqslant A \leqslant 10^4$ AU $= A_{\rm max}$; see Kouwenhoven & de Grijs (2008). The lower limit $A_{\rm min}$ is determined by the semimajor axis at which Roche lobe overflow occurs, while the upper limit $A_{\rm max}$ depends on how large the average star density is. The logarithmic distribution is known as "Öpik's law" (1924) after its discoverer and given by $f(\text{A}) \sim (1/\text{A})$, a law which has also been confirmed by recent investigations, e.g., Poveda et al. (2006). This distribution is a consequence of the process of star formation as well as of the dynamical history of the binary system, and one can take Öpik this law as a given fact for numerical studies. Accordingly, see Appendix E:

$$f(\text{A}) = \frac{1}{\text{A}}\ \left(\ln \frac{A_{\rm max}}{A_{\rm min}}\right)^{-1}. \tag{25}$$

Furthermore, for the mass distribution $f(\mu)$ let us recall the initial mass function (IMF), which is the probability that a star is newly formed with a stellar mass M and is frequently assumed to be a power law $f(M) \sim M^{-\alpha}$. Originally, the IMF was introduced by Salpeter (1955) for solar neighborhood region who assumed the value $\alpha = 2.35$ and a validity region for stars with masses between $0.4\,M_\odot$ and $10\,M_\odot$. During the past decades the IMF has been refined by several investigations. In particular, the numerical values of slope parameter α and regions of validity have been proposed in subsequent investigations, e.g., Scalo (1986), Robin et al. (2003), Ninkovic & Trajkovska (2006), Ninkovic (1995), Kroupa (2001); for a review see Kroupa (2002). Moreover, the IMF does not necessarily coincide with the real mass distribution of stars because the IMF describes the mass distribution of a star formation, while the solar neighborhood mainly consists of evolved stars of main sequence. Here, for simplicity this distribution is used as a given fact with $\alpha = 2.35$ and the proposed region of validity is assumed to be $\mu_{\rm min} = 0.4$ and $\mu_{\rm max} = 10$. According to the IMF, one finds for $\alpha \neq 1$ (see Appendix E)

$$f(\mu) = \frac{(1-\alpha)\ \mu^{-\alpha}}{\mu_{\rm max}^{(1-\alpha)} - \mu_{\rm min}^{(1-\alpha)}}. \tag{26}$$

In order to motivate that distribution further, one can compare Equation (26) with the RECONS (2012) data where one finds a fair agreement. Using Equations (23)–(26), the results of the

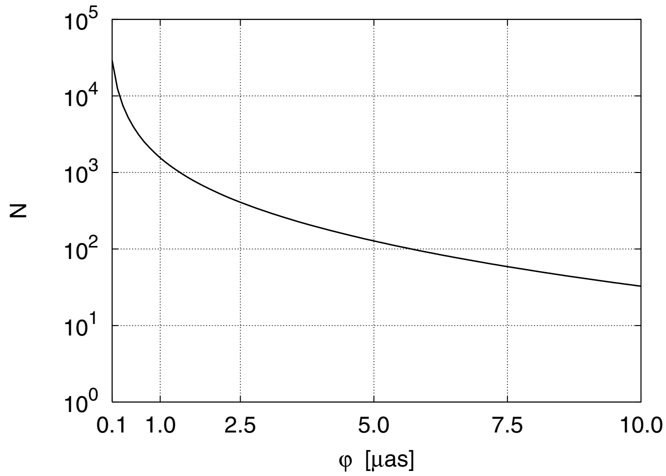

Figure 4. Total number of binaries according to Equation (22), having parameters such that the light deflection of component B at component A is larger than a given value for φ. Note that for one individual observation the astrometric accuracy of *Gaia* is about 25 μas and the end-of-mission accuracy is about 5 μas in the most ideal case (bright star).

estimate (22) are shown in Figure 4; recall that by evaluating this integral one has to take into account the boundary conditions given by Equations (11) and (12). According to Figure 4, in total there are about $N \sim 10^3$ binaries having a light deflection of at least $\varphi = 1\,\mu$as.

6. TOTAL NUMBER OF BINARIES WITH A GIVEN LIGHT DEFLECTION FOR FINITE TIME OF OBSERVATION

In Equation (22) the number of binaries with a given maximal possible light deflection φ has been determined, just by taking for eccentric anomaly the value $E = 0$, i.e., the ideal configuration where the light deflection takes its maximal value (note, the eccentricity $e = 0$). It is, however, obvious that during the most of the orbital motion one will have $E \neq 0$ and the light deflection will be much smaller than the maximum possible light deflection angle φ. On the other side, the orbital period T of relevant binaries, given by Equation (A21), will easily exceed the time of observation; for instance, the *Gaia* mission time is about $T_{\rm mission} \simeq 5$ years. Therefore, it is not very probable that the component B will be just at the relevant position near the value $E = 0$, where the light deflection becomes observable on the microarcsecond level. In order to determine that number of observable relevant binaries, one has to extend Equation (22) as follows:

$$N(\varphi) = \int_{R_{\rm min}}^{R_{\rm max}} d^3 r\ \rho(r) \int_{A_{\rm min}}^{A_{\rm max}} dA\ f(A) \times \int_{\mu_{\rm min}}^{\mu_{\rm max}} d\mu\ f(\mu)\ P(i)\ P(E). \tag{27}$$

Here, $P(E)$ is the probability for the binary system to be in the region E, where the light deflection is larger than a given value for φ.

In the very same way, as applied for the derivation of the inclination formula (13), one can reconvert Equations (B15) in terms of eccentric anomaly E and one finds the eccentric anomaly formula:

$$E = \pm \arccos\left(-\frac{p}{2} + \sqrt{\frac{p^2}{4} - q}\right), \tag{28}$$

where p and q are given by Equations (14) and (15). For a given value of light deflection φ, the formula (28) yields the value of eccentric anomaly E of a binary system characterized by semimajor axis A and Schwarzschild radius (or mass) m at a distance r. However, the values of φ cannot be chosen arbitrarily, instead they are restricted by $\varphi_{\rm min} = \varphi(E = \pm\pi/2)$ and $\varphi_{\rm max} = \varphi(E = 0)$ given by (of course, only astrometric positions with $0 \leqslant E \leqslant \pi/2$ are taken into account because for the area $\pi/2 \leqslant E \leqslant \pi$ the light deflection is negligible):

$$\varphi_{\rm min} = \frac{1}{2}\left(\sqrt{\frac{A^2}{r^2} + 8\,\frac{m}{r}\,\frac{A}{r}} - \frac{A}{r}\right) \approx 2\,\frac{m}{r} = 0.0197\,\mu{\rm as}\ \frac{M}{M_\odot}\ \frac{\rm pc}{r}, \tag{29}$$

$$\varphi_{\rm max} = 2\,\frac{\sqrt{m\,A}}{r} = 200\,\mu{\rm as}\ \sqrt{\frac{M}{M_\odot}\ \frac{A}{\rm AU}\ \frac{\rm pc}{r}}, \tag{30}$$

where in Equation (29) terms of the order of $\mathcal{O}(m^2/r\,A)$ have been neglected. These expressions resemble the corresponding expressions in Equations (11) and (12). According to Equation (28), the region where the binary system has a light deflection larger than or equal to φ is given by $2\,E$. One also has to take into account that during the *Gaia* mission time $T_{\rm mission}$ component B moves along the orbit and could move into the region $2\,E$. Therefore, the probability $P(E)$ that the binary system is during the *Gaia* mission time, at least once inside the relevant astrometric position with the value E in Equation (28), is given by

$$P(E) = \mathcal{P}_1\left(\frac{1}{\pi}\ \arccos\left(-\frac{p}{2} + \sqrt{\frac{p^2}{4} - q}\right) + \frac{T_{\rm mission}}{T}\right), \tag{31}$$

where the operator $\mathcal{P}_1$ is defined by

$$\mathcal{P}_1(x) = \begin{array}{ll} x & \text{if} \quad x < 1, \\ 1 & \text{if} \quad x \geqslant 1. \end{array} \tag{32}$$

The probability distribution (31) has to be implemented in Equation (27) in order to determine the number of binary systems having a given light deflection φ and to be observable during *Gaia* mission time $T_{\rm mission}$. The results of Equation (27) are shown in Figure 5; recall that by evaluating this integral one has to take into account the boundary conditions given by Equations (11) and (12). Accordingly, there are only very few binaries $\sim 10^2$ having a light deflection of at least $\varphi = 1\,\mu$as because only this number of binaries will reach the optimal astrometric position $E = 0$ during an assumed observation time of $T_{\rm mission} = 5$ years.

7. SPECIAL CASE: CONDITIONS ON ORBITAL PARAMETERS FOR RESOLVED BINARIES

In this section, the special case of a resolved binary system is considered. In order to investigate the detectability of the light deflection effect in binaries, today's most modern astrometric mission, the ESA cornerstone mission *Gaia*, and its instrumentation, which provides the highest possible astrometric accuracy at the moment, will be considered here as a concrete example.

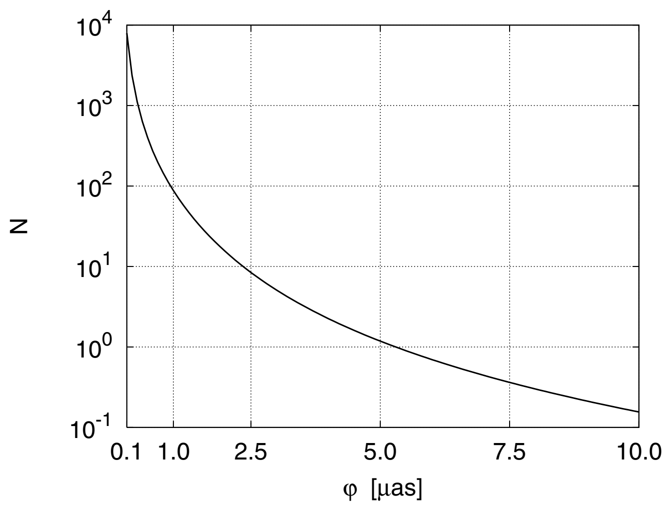

Figure 5. Total number of binaries according to Equation (27) where the binary system reaches the optimal configuration $E = 0$ during an assumed observation time of $T_{\rm mission} = 5$ yr, and having orbital parameters such that the light deflection of component B at component A is larger than a given value for φ. Note that for one individual observation the astrometric accuracy of *Gaia* is about 25 μas and the end-of-mission accuracy is about 5 μas in the most ideal case (bright star).

7.1. Resolving Power of Gaia

The core of *Gaia* optical instrumentation consists of two identical mirror telescopes, ASTRO-1 and ASTRO-2, with a rectangular pupil whose dimensions are $A = 0.50$ m, $B = 1.45$ m, and $f = 35$ m is the effective focal length. The intensity is given by (see Lattanzi et al. 1998 and Lindgren 1998):

$$I(z_{\rm A}, z_{\rm B}) = I_0 \left(\frac{\sin^2(z_{\rm A})}{z_{\rm A}^2} \; \frac{\sin^2(z_{\rm B})}{z_{\rm B}^2} \right), \tag{33}$$

where $z_{\rm A} = \pi\ {\rm A}/\lambda \sin\Theta_{\rm A}$, $z_{\rm B} = \pi\ {\rm B}/\lambda \sin\Theta_{\rm B}$, A and B are the width and length of rectangular mirror, λ is the wavelength of the incident light-ray, and $\Theta_{\rm A}$, $\Theta_{\rm B}$ are the angles of observation, i.e., the angle between the axis of the rectangular aperture and the line between aperture center and observation point, respectively. The intensity of incident light-ray at $\Theta_{\rm A} = 0, \Theta_{\rm B} = 0$ is denoted by I_0. The function $I(z_{\rm A}, z_{\rm B})/I_0$ in Equation (33) is the (by $\Theta_{\rm A} = \Theta_{\rm B} = 0$ normalized) point-spread function (PSF) for monochromatic incident light with wavelength λ for a rectangular aperture. The optical spectrum of stars is $\lambda = (350–750)$ nm. In Figure 6 the PSF for an incident monochromatic light-ray with $\lambda = 350$ nm is represented for *Gaia* telescopes.

Most of the light is concentrated in the central bright rectangular shaped pattern. The length $l_{\rm A}$ and width $l_{\rm B}$ of this rectangle is determined by the first zero-roots of Equation (33) at $z_{\rm A} \simeq \pi$ and $z_{\rm B} \simeq \pi$, respectively. From this it follows that $\sin\Theta_{\rm A} = \pi\ \lambda/(\pi\ {\rm A}) = \lambda/{\rm A}$ and $\sin\Theta_{\rm B} = \pi\ \lambda/(\pi\ {\rm B}) = \lambda/{\rm B}$. Furthermore, if the diffraction pattern is shown on a screen at a distance *f*, then the length and width are given by (see Hog et al. 1997):

$$L_{\rm A} = 2\,\frac{f\,\lambda}{\rm A}, \tag{34}$$

$$L_{\rm B} = 2\,\frac{f\,\lambda}{\rm B}, \tag{35}$$

where *f* is the focal length of the optic, i.e., of the rectangular *Gaia* mirror. Accordingly, the given numerical values

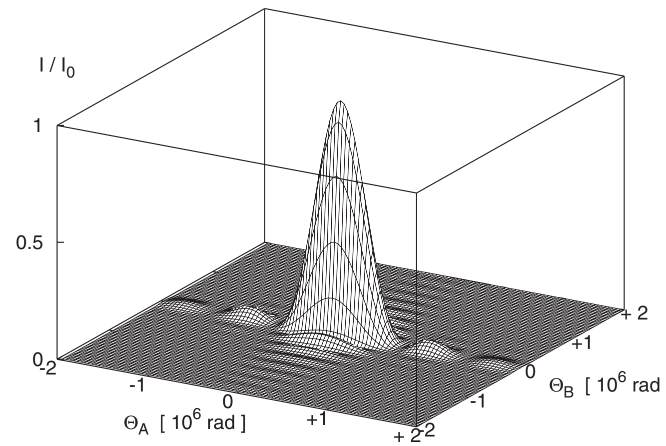

Figure 6. Point-spread function (PSF) for a rectangular telescope according to Equation (33). The incident monochromatic light-ray has a wavelength of $\lambda = 350$ nm. The parameters of the rectangular telescope are: $A = 0.5$ m, $B = 1.45$ m.

$A = 0.50$ m, $B = 1.45$ m, and $\lambda = 350$ nm result in $L_{\rm A} = 49.0\ \mu$m and $L_{\rm B} = 16.9\ \mu$m for the length and width of the "Airy rectangle" of the *Gaia* optics. Note that the "Airy rectangle" has the same order of magnitude as the pixel size ($10\ \mu$m $\times$ $30\ \mu$m) of the 110 CCD (Charge-Coupled Device) sensors of the astrometric field part of the focal plane. In order to separate two pointlike sources, the distance between their centers and the rectangle has to be larger than either $L_{\rm A}$ or $L_{\rm B}$. Since $L_{\rm B} < L_{\rm A}$, in this study the better resolution value $L_{\rm B}$ is used, which corresponds to a resolution angle of

$$\delta = \frac{L_{\rm B}}{2\,f} = \frac{\lambda}{\rm B}. \tag{36}$$

The resolving power is the minimal angular distance between two objects to be separable by *Gaia* instrumentation. With the parameters given above one obtains the resolving power δ of *Gaia* optics:

$$\delta = 0.24 \times 10^{-6}\ {\rm rad} = 49.7\ {\rm mas}. \tag{37}$$

In what follows this parameter is of fundamental importance in order to determine the ability of *Gaia* to determine the light deflection in binary systems.

7.2. Orbital Parameters of Resolved Binaries Observable by Gaia

In this section, the question is addressed of which and how many binary systems can be separated by *Gaia* instrumentation among all those relevant binaries found in the previous section; see Figure 5. On average, *Gaia* will observe each object 80 times, but will not constantly observe these objects during mission time. However, for simplicity the scanning law of *Gaia* is approximated by assuming a permanent observation of all objects during the whole mission time.

Furthermore, visual binaries are considered in this section, i.e., binaries which are separable by *Gaia* telescopes. Both of the largest telescopes of *Gaia* have a resolution angle δ discussed in the previous section (see Equations (36) and (37)). For binary systems, this resolution angle δ corresponds to a minimal distance between the components A and B to get separable within *Gaia* optics. Using Equations (B6) and (36) one obtains the condition

$$d = A\ |\cos i| \geqslant \delta r = \frac{\lambda}{B} r, \quad (38)$$

where r is the distance between the remote objects and *Gaia* observer. This condition is by far much more important than taking into account the effect of finite radius of the stars, which would imply $A\ |\cos i| \geqslant R_A$, where R_A is the radius of component A. By inserting the extreme case $A\ |\cos i| = \delta r$ into Equation (10), one obtains

$$\varphi = \frac{1}{2}\left(\sqrt{\delta^2 + 8\,(1 + \sin i)\ \frac{m}{r}\,\frac{A}{r}} - \delta\right) \simeq 4\,\frac{m}{r}\,\frac{A}{r}\,\frac{1}{\delta}, \quad (39)$$

where $\sin i \simeq 1$ has been used, and terms of higher order $\mathcal{O}(m^2)$ are neglected. Relation (39) is an expression for the maximal light deflection angle of a binary system when taking into account the resolving power of *Gaia*. Equation (39) is a much stricter restriction than the generalized lens equation (10) because Equation (39) determines the light deflection angle only of those binary systems having a resolution angle δ of *Gaia* optical instrumentation, while Equation (10) determines the light deflection angle of any possible binary system. Note that from Equation (39) follows the maximal possible distance of visual binaries for a given light deflection φ:

$$r \leqslant \left(4\,\frac{m\,A}{\varphi\,\delta}\right)^{1/2} = 0.18\,\mathrm{pc}\,\left(\frac{M}{M_\odot}\,\frac{A}{\mathrm{AU}}\right)^{1/2}, \quad (40)$$

where in the last expression the optimal values $\delta = 0.24 \times 10^{-6}$ rad and $\varphi = 25\,\mu$as have been used. The condition (40) can also be written as

$$A \geqslant 30\,\mathrm{AU}\,\frac{M_\odot}{M}\,\frac{r^2}{\mathrm{pc}^2}\,. \quad (41)$$

Both conditions (40) and (41) imply rather extreme orbital parameters on visual binaries. For instance, condition (40) implies a maximal distance of $r \leqslant 18$ pc for solar-mass-type binaries even with a huge semimajor axis of $A = 10^4$ AU, while condition (41) implies a large semimajor axis for solar-mass-type binaries outside a sphere of $r \geqslant 10$ pc. It is almost certain that such extreme parameters will not be realized in nature.

8. SUMMARY

In this study, the light deflection in binary systems has been considered. While there is absolutely no doubt about the existence of this relativistic effect, it has not been observed thus far. To investigate this effect of light deflection, an inclination formula (13) has been derived by means of the generalized lens equation (9) obtained recently by Zschocke (2011), and both equations are the theoretical basis for investigating the light deflection effect in binary systems. A simplified inclination formula has been presented by Equation (19) and its validity has been discussed in some detail. This simplified inclination formula has also been obtained by S. A. Klioner, F. Mignard, & M. Soffel (2003, private communication) by an independent approach. Furthermore, two stringent conditions on the orbital parameters have been given by Equations (20) and (21). Both stringent conditions are relations between the orbital elements of a (resolved, astrometric, eclipsing, spectroscopic) binary system for a given magnitude of light deflection, and allow us to find a relevant binary system in a straightforward way.

In Section 5, the total number of binaries with a given light deflection has been determined by means of the semimajor axis distribution according to "Öpik's law" and the mass distribution according to "Salpeter's mass distribution". Since the inclinations are randomly distributed, the inclination formula allows us to estimate the total number of relevant binaries with the aid of Equation (22). It turns out that in total about 10^3 binaries exist that have orbital parameters such that the light deflection amounts to at least 1 μas (see Figure 4).

In Section 6, a finite time of observation of five years (*Gaia* mission time) has been considered, which considerably reduces the total number of relevant binaries. Clearly, this case is of more practical importance, since a restricted time window of observation is in better agreement with reality than the first scenario. By taking into account the probability of finding the system in the ideal astrometric position $E = 0$ where the light deflection becomes maximal, it has been found by evaluating the corresponding integral (27) that there are no relevant binary systems in the ideal position $E = 0$ during *Gaia* mission time (see Figure 5). Thus, while in principle a few binaries will have a significant light deflection, the effect could not be detected due to the restricted finite time window of observation.

Furthermore, the special case of resolved binaries has been considered in Section 7. The astrometric instrumentation of the ESA cornerstone mission *Gaia* (see e.g., Perryman et al. 2001), has been considered in some detail in order to decide whether or not this subtle effect of light deflection can be observed. Two conditions for resolved binaries were presented in Equations (40) and (41) for such a special kind of binary systems. It has been shown, however, that even for the *Gaia* mission, which is an outstanding milestone of progress in astrometry, such binary systems must have rather extreme orbital parameters in order to reach today's level of detectability, i.e., on the microarcsecond level. The existence of such exotic binaries is, however, highly improbable.

In summary, the main results are presented by the inclination formulae (13) and its simplified version (19), the stringent conditions (20) and (21), and by the diagrams in Figures 4 and 5. Accordingly, one comes to the conclusion that the detectability of light deflection in binary systems reaches the technical limit of today's high-precision astrometry and might be detected only in case of a very few and highly exotic binary systems. It is, however, very unlikely that such extreme binaries might exist. It seems that the detection of the light deflection effect in binary systems needs an astrometric accuracy of better than about 0.1 μas. Thus, only astrometric missions of the next generation can accept the challenge of detecting this subtle effect of relativity.

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APPENDIX A

TWO-BODY PROBLEM

The calculations in this appendix mainly follow Landau & Lifshitz (1976). Consider two massive bodies, one component having a mass M_A and spatial coordinate $\boldsymbol{r}_A$, and a second component with a mass M_B and spatial coordinate $\boldsymbol{r}_B$, respectively. They orbit around their common center of mass $\boldsymbol{r}_{\rm CMS}$,

$$\boldsymbol{r}_{\rm CMS} = \frac{1}{M_A + M_B} \left(M_A\, \boldsymbol{r}_A + M_B\, \boldsymbol{r}_B\right). \tag{A1}$$

The Lagrangian $\mathcal{L}$ of the two-body problem is given by

$$\mathcal{L} = \frac{M_A}{2}\, \dot{\boldsymbol{r}}_A^2 + \frac{M_B}{2}\, \dot{\boldsymbol{r}}_B^2 - U(|\boldsymbol{r}_A - \boldsymbol{r}_B|), \tag{A2}$$

where U is the potential. With the aid of relative coordinate $\boldsymbol{r}_{AB} = \boldsymbol{r}_A - \boldsymbol{r}_B$ and reduced mass $\overline{M} = M_A\, M_B/(M_A + M_B)$, the two-body problem can be transformed into a one-body problem,

$$\mathcal{L} = \frac{\overline{M}}{2}\, \dot{\boldsymbol{r}}_{AB}^2 - U(r_{AB}). \tag{A3}$$

Polar coordinates (r_{AB}, ϕ) yield

$$\mathcal{L} = \frac{1}{2} \left(\overline{M}\, \dot{r}_{AB}^2 + r_{AB}^2\, \dot{\phi}^2\right) - U\,(r_{AB}). \tag{A4}$$

The orbital angular momentum L is conserved,

$$L = \overline{M}\, r_{AB}^2\, \dot{\phi} = \text{constant}, \tag{A5}$$

by means of which one obtains for the total energy of the two-body system the expression

$$E = \frac{\overline{M}}{2}\, \dot{r}_{AB}^2 + \frac{L^2}{2\, \overline{M}\, r_{AB}^2} + U\,(r_{AB}). \tag{A6}$$

From Equation (A6) one deduces

$$\dot{r}_{AB} = \left(\frac{2}{\overline{M}}\, [E - U\,(r_{AB})] - \frac{L^2}{\overline{M}^2\, r_{AB}^2}\right), \tag{A7}$$

and from Equation (A7) one obtains up to an integration constant

$$t = \int d r_{AB} \left(\frac{2}{\overline{M}}\, [E - U\,(r_{AB})] - \frac{L^2}{\overline{M}^2\, r_{AB}^2}\right)^{-1/2}, \tag{A8}$$

$$\phi = \int d r_{AB}\, \frac{\overline{M}}{r_{AB}^2} \left(2\, \overline{M}\, [E - U\,(r_{AB})] - \frac{L^2}{r_{AB}^2}\right)^{-1/2}, \tag{A9}$$

where in the second relation Equation (A5) has been used; note that Equation (A9) is the relation between r_{AB} and ϕ and is called the orbital equation. Equations (A8) and (A9) are the general integral solutions of a two-body problem. In order to integrate these equations ((A8) and (A9)) one has to specify the potential U. In the case of the Kepler problem one has

$$U(r) = -\frac{\alpha}{r_{AB}} \quad \text{with} \quad \alpha = G\, M_A\, M_B. \tag{A10}$$

Equation (A9) can be integrated and yields

$$\phi = \arccos \left(\frac{L}{r_{AB}} - \frac{\gamma\, \overline{M}\, M_A\, M_B}{L}\right) \times \left(2 \overline{M}\, E + \frac{\gamma^2\, \overline{M}^2\, M_A^2\, M_B^2}{L^2}\right)^{-1/2}, \tag{A11}$$

where the axes are chosen such that the mentioned integration constant vanishes. Furthermore, by introducing the eccentricity e (possible values of eccentricity are between $0 \leqslant e < 1$; $e = 0$ corresponds to a circular orbit),

$$e = \left(1 + \frac{2\, E\, L^2\, (M_A + M_B)}{\gamma^2\, M_A^3\, M_B^3}\right)^{1/2}, \tag{A12}$$

the solution (A11) can be written as

$$\frac{1}{r_{AB}}\, \frac{L^2}{\gamma\, \overline{M}\, M_A\, M_B} = 1 + e\, \cos\phi. \tag{A13}$$

Note the expressions of semimajor axis A and semiminor axis B,

$$A = \frac{L^2}{(1 - e^2)\, \gamma\, \overline{M}\, M_A\, M_B}, \tag{A14}$$

$$B = \frac{L^2}{\sqrt{1 - e^2}\, \gamma\, \overline{M}\, M_A\, M_B}. \tag{A15}$$

To solve the integral (A8), one substitutes

$$r_{AB} - A = -A\, e\, \cos E, \tag{A16}$$

where E is called eccentric anomaly. Then, one obtains for the integral in Equation (A8) the expression

$$t = \left(\frac{A^3}{\gamma\, (M_A + M_B)}\right)^{1/2} \int dE\, (1 - e\, \cos E), \tag{A17}$$

and the solution is given by

$$t = \left(\frac{A^3}{\gamma\, (M_A + M_B)}\right)^{1/2} (E - e\, \sin E), \tag{A18}$$

where the integration constant vanishes, i.e., the particle at $t = 0$ is in periastron. The Equations (A13) and (A18) are the general solutions of the two-body problem. They can be rewritten as

$$r_{AB} = A(1 - e\, \cos E), \tag{A19}$$

$$t = \left(\frac{A^3}{\gamma\, (M_A + M_B)}\right)^{1/2} (E - e\, \sin E). \tag{A20}$$

In the case of an ellipse, $E = 0$ in periastron, $E = \pi$ in apastron, and for a complete orbit E runs from $E = 0$ to $E = 2\pi$. Thus, one obtains for the orbital period the expression

$$T = 2\pi \left(\frac{A^3}{\gamma\, (M_A + M_B)}\right)^{1/2}. \tag{A21}$$

Note the solution $\boldsymbol{r}$ in Cartesian coordinates $x = r_{AB}\, \cos\phi$ and $y = r_{AB}\, \sin\phi$:

$$\boldsymbol{r}_{AB} = \begin{pmatrix} x \\ y \end{pmatrix}, \tag{A22}$$

$$x = A\, (\cos E - e), \tag{A23}$$

$$y = A(1 - e^2)^{1/2}\, \sin E. \tag{A24}$$

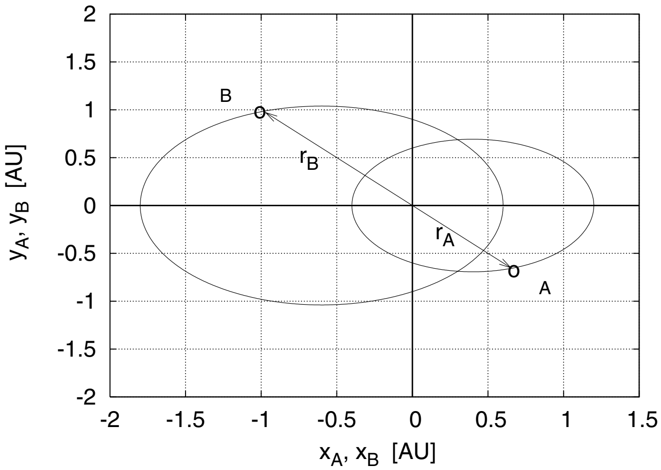

Figure 7. Geometrical representation of the coordinates of a binary star. In the example considered, the masses are $M_A = 1.5\ M_\odot$ and $M_B = 1.0\ M_\odot$, respectively. The semimajor axis of the binary system is chosen to be $A = 2$ AU and the eccentricity is taken to be $e = 0.5$. The coordinates of mass center are $\boldsymbol{r}_{\rm CMS} = \boldsymbol{0}$. The massive bodies A and B are always in opposition to each other.

The coordinates of bodies A and B, i.e., their orbits, are given by

$$\boldsymbol{r}_A = \boldsymbol{r}_{\rm CMS} + \frac{\boldsymbol{r}_{AB}}{1 + \frac{M_A}{M_B}}, \tag{A25}$$

$$\boldsymbol{r}_B = \boldsymbol{r}_{\rm CMS} - \frac{\boldsymbol{r}_{AB}}{1 + \frac{M_B}{M_A}}. \tag{A26}$$

Accordingly, the geometry of the orbit is determined by two orbital parameters: semimajor axis A and eccentricity e. In order to know the position of one celestial body, either component A or component B, two additional orbital parameters are needed, namely orbital period T and true anomaly ν. A geometrical representation of the coordinates of the components of a binary star is given in Figure 7 for the case of $M_A = 1.5\,M_\odot$, $M_B = 1.0\,M_\odot$, $e = 0.5$, $A = 2$ AU.

APPENDIX B

DERIVATION OF EQUATION (10)

For the inclination formula the impact of eccentricity on light deflection is neglected, thus $e = 0$, implying that $\omega = 0$ is assumed. Then, for the vectors from the massive body to observer $\boldsymbol{x}_1$ and from the massive body to source $\boldsymbol{x}_0$, one has

$$\boldsymbol{x}_1 = r \begin{pmatrix} \sin i - \epsilon_1 \cos E \\ -\epsilon_1 \sin E \\ \cos i \end{pmatrix}, \tag{B1}$$

$$\boldsymbol{x}_0 = -A \begin{pmatrix} \cos E \\ \sin E \\ 0 \end{pmatrix}, \tag{B2}$$

where the small parameter

$$\epsilon_1 = \frac{A}{r}\,\frac{m_B}{m_A + m_B} \ll 1 \tag{B3}$$

has been introduced. From Equations (B1) and (B2) one obtains for vector $\boldsymbol{k} = \boldsymbol{R}/R$, where $\boldsymbol{R} = \boldsymbol{x}_1 - \boldsymbol{x}_0$, the expression

$$\boldsymbol{k} = \left(1 + 2\,\epsilon_2\,\sin i\,\cos E + \epsilon_2^2\right)^{-1/2} \begin{pmatrix} \sin i + \epsilon_2 \cos E \\ \epsilon_2 \sin E \\ \cos i \end{pmatrix}, \tag{B4}$$

where the small parameter

$$\epsilon_2 = \frac{A}{r}\,\frac{m_A}{m_A + m_B} \ll 1 \tag{B5}$$

has been introduced. Note that Equations (B2) and (B4) yield

$$d = |\boldsymbol{k} \times \boldsymbol{x}_0| = A\,|\cos i|\,(1 + \mathcal{O}(\epsilon_2)). \tag{B6}$$

Using Equations (B1)–(B5), the generalized lens equation (9) reads

$$\varphi = \frac{1}{2}\,\frac{1}{T_1\,T_2}\left(\sqrt{W_1^2 + 8\,\frac{m}{r}\,\frac{A}{r}\,(T_0 + T_1 - \epsilon_1)\,T_2} - W_1\right), \tag{B7}$$

where $W_1 = A/r\sqrt{1 - T_0^2}$ and

$$T_0 = \sin i\,\cos E, \tag{B8}$$

$$T_1 = \left(1 - 2\,\epsilon_1\,\sin i\,\cos E + \epsilon_1^2\right)^{1/2}, \tag{B9}$$

$$T_2 = \left(1 + 2\,\epsilon_2\,\sin i\,\cos E + \epsilon_2^2\right)^{1/2}. \tag{B10}$$

By series expansion one obtains up to terms of the order of $\mathcal{O}((A/r)\sqrt{(m/r)(A/r)})$:

$$\varphi = \frac{1}{2}\left(\sqrt{W_2^2 + 8\,\frac{m}{r}\,\frac{A}{r}\,(1 + w)} - W_2\right), \tag{B11}$$

where $W_2 = \frac{A}{r}\sqrt{1 - w^2}$ with $w = \sin i\,\cos E$. The minimal and maximal light deflection angles are

$$\varphi_{\rm min} = \varphi\left(i = \frac{\pi}{2},\,E = \pi\right) = 0, \tag{B12}$$

$$\varphi_{\rm max} = \varphi\left(i = \frac{\pi}{2},\,E = 0\right) = 2\,\frac{\sqrt{m\,A}}{r}. \tag{B13}$$

In this study, the maximal possible light deflection effect is of interest. Accordingly, two configurations are relevant:

$$\varphi\,(E = 0) = \frac{1}{2}\left(\sqrt{\frac{A^2}{r^2}\,\cos^2 i + 8\,\frac{m}{r}\,\frac{A}{r}(1 + \sin i)} - \frac{A}{r}|\cos i|\right), \tag{B14}$$

up to the order of $\mathcal{O}((A/r)\sqrt{(m/r)(A/r)})$ which is just Equation (10), and

$$\varphi\left(i = \frac{\pi}{2}\right) = \frac{1}{2}\left(\sqrt{\frac{A^2}{r^2}\,\sin^2 E + 8\,\frac{m}{r}\,\frac{A}{r}\,(1 + \cos E)} - \frac{A}{r}\,|\sin E|\right), \tag{B15}$$

up to the order of $\mathcal{O}(A/r\sqrt{(m/r)(A/r)})$. Furthermore, it is useful to take into account only astrometric positions with $0 \leqslant E \leqslant (\pi/2)$, because otherwise the light deflection is certainly negligible.

APPENDIX C
DERIVATION OF EQUATION (13)

From Equation (B14) one obtains

$$\left(2\,\varphi + \frac{A}{r}\;|\cos i|\right)^2 = \frac{A^2}{r^2}\,\cos^2 i + 8\,\frac{m}{r}\,\frac{A}{r}\,(1+\sin i)\,. \quad \text{(C1)}$$

From Equation (C1) one obtains

$$\left(\varphi^2 + 4\,\frac{m^2}{r^2}\right)\frac{A^2}{r^2}\,\sin^2 i + 4\frac{m}{r}\,\frac{A}{r}\left(2\,\frac{m}{r}\frac{A}{r} - \varphi^2\right)\sin i = \left(\frac{A^2}{r^2} + 4\,\frac{m}{r}\frac{A}{r}\right)\varphi^2 - 4\frac{m^2}{r^2}\,\frac{A^2}{r^2} - \varphi^4. \quad \text{(C2)}$$

Equation (C2) represents an quadratic equation for the expression $|\sin i\,|$, which has the following both solutions for the inclination i:

$$\sin i = \left(-\frac{p}{2} \pm \sqrt{\frac{p^2}{4} - q}\right), \quad \text{(C3)}$$

where

$$p = \frac{8\,m^2\,A - 4\,m\,r^2\,\varphi^2}{A\left(r^2\,\varphi^2 + 4\,m^2\right)}, \quad \text{(C4)}$$

$$q = -\frac{A^2\,r^2\,\varphi^2 + 4\,m\,A\,r^2\,\varphi^2 - 4\,m^2\,A^2 - r^4\,\varphi^4}{A^2(r^2\,\varphi^2 + 4\,m^2)}. \quad \text{(C5)}$$

Equation (C3) represents two solutions; however, only the one with the plus sign is valid. This can be shown as follows. For the value $i = \pi/2$ the light deflection has to be $\varphi = \varphi_{\rm max} = 2\sqrt{m\,A}/r$, according to Equation (12). By inserting $\varphi_{\rm max}$ in Equations (C4) and (C5) one obtains $p = -2\,m/(A+m)$ and $q = -(A-m)/(A+m)$. If one inserts $i = \pi/2$ for p and q into Equation (C3) one obtains the relation

$$1 = \frac{m}{A+m} \pm \left(\frac{m^2}{(A+m)^2} + \frac{A-m}{A+m}\right)^{1/2} = \frac{m}{A+m} \pm \frac{A}{A+m}. \quad \text{(C6)}$$

Obviously, relation (C6) is only correct for the upper sign. A very similar proof can also be done using $\varphi_{\rm min}$, which also yields that the upper sign is the correct and only solution. Therefore, the inclination formula is given by (note that in the region under consideration $\sin i = \sin(\pi - i)$)

$$i = \begin{cases} \arcsin\left(-\frac{p}{2} + \sqrt{\frac{p^2}{4} - q}\right) & \text{for} \quad 0 \leqslant i \leqslant \frac{\pi}{2}, \\ \pi - \arcsin\left(-\frac{p}{2} + \sqrt{\frac{p^2}{4} - q}\right) & \text{for} \quad \frac{\pi}{2} < i \leqslant \pi. \end{cases} \quad \text{(C7)}$$

For the complete region $0 \leqslant i \leqslant \pi$ one obtains for the inclination formula the following expression:

$$\left|\frac{\pi}{2} - i\right| = \arccos\left(-\frac{p}{2} + \sqrt{\frac{p^2}{4} - q}\right), \quad \text{(C8)}$$

where p and q are given by Equations (C4) and (C5).

APPENDIX D
DERIVATION OF EQUATION (3)

In this appendix, a recalculation of some basic steps of S. A. Klioner, F. Mignard, & M. Soffel (2003, private communication) are given. A scheme of the light propagation in a binary system is presented by Figure 2. The light signal from component B, considered to be the light source, is deflected by component A, considered to be the massive body which curves the space-time. The vector $\boldsymbol{x}_1$ points from the mass center of the massive body to the observer, and vector $\boldsymbol{x}_0$ points from the mass center of massive body to the source; $\boldsymbol{R} = \boldsymbol{x}_0 - \boldsymbol{x}_1$ and its absolute value $R = |\boldsymbol{R}|$, and $m = GM/c^2$ is the Schwarzschild radius of massive body; the explicit label A is omitted. Furthermore, we define the impact vector $\boldsymbol{d} = \boldsymbol{k} \times (\boldsymbol{x}_1 \times \boldsymbol{k})$, and its absolute value $d = |\boldsymbol{d}|$; see also Figure 2.

According to Klioner & Zschocke (2010), the transformation of $\boldsymbol{k}$ to the unit tangent vector $\boldsymbol{n}$ of the light-trajectory at the observer is in standard post-Newtonian approach given by

$$\boldsymbol{n} = \boldsymbol{k} - 2\,m\,\frac{\boldsymbol{k} \times (\boldsymbol{x}_0 \times \boldsymbol{x}_1)}{x_1\,(x_0\,x_1 + \boldsymbol{x}_0 \cdot \boldsymbol{x}_1)} + \mathcal{O}(m^2). \quad \text{(D1)}$$

Note, the PPN parameter of the parameterized post-Newtonian formalism, which characterizes a possible deviation of the physical reality from the general theory of gravity, is set equal to 1 here for simplicity. This expression is valid as long as $d \gg m$, but diverges for $d \to 0$. Thus, it is not valid for all possible binary configurations, but instead one has take care to consider only such astrometric configurations with $d \gg m$. By means of Equation (D1) one obtains for the light deflection angle φ, i.e., for the angle between $\boldsymbol{n}$ and $\boldsymbol{k}$, the expression

$$\varphi = 2\,\frac{m}{r}\,\tan\frac{\psi}{2}, \quad \text{(D2)}$$

where $\sin\psi/(1+\cos\psi) = \tan\psi/2$, $x_1 = r + \mathcal{O}(A)$ has been used, and ψ is the angle between $\boldsymbol{x}_0$ and $\boldsymbol{x}_1$. The expression (D2) diverges for $\psi \to \pi$, which corresponds with the mentioned divergence of Equation (D1) for $d \to 0$. Obviously, $\psi \leqslant i + \pi/2$ (from Equation (28) it is obvious that eccentric anomaly E of binary system should be very close to zero for the light deflection effect to be observable at the microarcsecond level, i.e., we actually could even assume $\psi \simeq i + \pi/2$) and one obtains

$$\varphi \leqslant 2\,\frac{m}{r}\,\tan\left(\frac{i}{2} + \frac{\pi}{4}\right) = 2\,\frac{m}{r}\,\cot\left(\frac{\pi}{4} - \frac{i}{2}\right), \quad \text{(D3)}$$

where $\tan(\alpha + \pi/2) = -\cot\alpha$, $\cot\alpha = \tan^{-1}\alpha$, and the asymmetry of function $\cot\alpha$ has been used. From Equation (D3) one obtains

$$\left|\frac{\pi}{2} - i\right|_{\rm KMS} \leqslant 2\,\arctan\left(0.0197\,\frac{M}{M_\odot}\,\frac{\mu\text{as}}{\varphi}\,\frac{\text{pc}}{r}\right), \quad \text{(D4)}$$

where $(m/m_\odot) = (M/M_\odot)$ has been used (recall M is the mass of component A and $M_\odot$ is the solar mass), and the numerical values $m_\odot \simeq 1.476 \times 10^3$ m, $1\,\mu$as $\simeq 4.848 \times 10^{-12}$ rad and 1 pc $\simeq 3.086 \times 10^{16}$ m have been inserted, so that $2\,m_\odot/(\mu\text{as pc}) \simeq 0.0197$. The simplified inclination formula (D4) has first been obtained by S. A. Klioner, F. Mignard, & M. Soffel (2003, private communication); here we notice the fact that due to the divergence of the post-Newtonian solution (D1) for $d \to 0$, which corresponds to $\psi \to \pi$, the applicability of Equation (D4) is restricted by the condition $d \gg m$. Using $d = A\ |\cos i|$ one obtains the validity condition for the applicability of Equation (D4):

$$\left|\frac{\pi}{2} - i\right|_{\rm KMS} \gg \arcsin\frac{m}{A}\ . \tag{D5}$$

Equation (D4) provides a relation between inclination *i* and light deflection φ for a binary system characterized by its distance *r* from the observer and its stellar mass *M*; note that (D4) agrees with the simplified inclination formula given by Equation (19).

APPENDIX E

PROBABILITY DISTRIBUTION

Assume a probability distribution of quantity x is given by $f(x)$. The probability P of finding a system in the interval $x_i \leqslant x \leqslant x_i + \Delta x$ is given by

$$P(x_i \leqslant x \leqslant x_i + \Delta x) = \frac{\int_{x_i}^{x_i+\Delta x} dz\ f(z)}{\int_{x_{\rm min}}^{x_{\rm max}} dz\ f(z)}, \tag{E1}$$

where the region of validity of probability distribution $f(x)$ is given by $x_{\rm min}$ and $x_{\rm max}$. In the infinitesimal limit $\Delta x \to dx$, one obtains by series expansion the following explicit form for the probability distributions used here: for a power law $f(x) \sim x^{-\alpha}$ with $\alpha \neq 1$ one finds

$$f(x) = \frac{(1-\alpha)\ x^{-\alpha}}{x_{\rm max}^{(1-\alpha)} - x_{\rm min}^{(1-\alpha)}}, \tag{E2}$$

and for a logarithmic law $f(x) \sim x^{-1}$ one has

$$f(x) = \frac{1}{x}\ \left(\ln\frac{x_{\rm max}}{x_{\rm min}}\right)^{-1}. \tag{E3}$$

The normalization is $\int_{x_{\rm min}}^{x_{\rm max}} f(x)\,dx = 1$ and the averaged value $\overline{x} = \int_{x_{\rm min}}^{x_{\rm max}} f(x)\,x\,dx$.

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