A generalized lens equation for light deflection in weak gravitational fields

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Abstract
A generalized lens equation for weak gravitational fields of Schwarzschild metric and valid for finite distances of source and observer from the light deflecting body is suggested. The magnitude of neglected terms in the generalized lens equation is estimated to be smaller than or equal to $\frac{15\pi}{4}\frac{m^2}{d'^2}$, where m is the Schwarzschild radius of the massive body and d' is Chandrasekhar's impact parameter. The main applications of this generalized lens equation are extreme astrometrical configurations, where the *standard post-Newtonian approach* as well as *the classical lens equation* cannot be applied. It is shown that in the appropriate limits, the proposed lens equation yields the known post-Newtonian terms, 'enhanced' post-post-Newtonian terms and the classical lens equation, thus providing a link between both these essential approaches for determining the light deflection.

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1. Introduction

Today's astrometry necessitates theoretical predictions of light deflection by massive bodies on microarcsecond (μas) level, e.g. astrometric missions SIM (NASA) or GAIA (ESA). In principle, an astrometric precision on microarcsecond level can be achieved by the numerical integration of the geodesic equation of light propagation. On the other side, modern astrometric missions like GAIA determine the positions and proper motions of approximately 1 billion objects, each of which is observed about 100 times. The data reduction of such a huge amount of observations implies the need of analytical solutions, because the numerical investigation of the geodesic equation is by far too time consuming.

The metric of a massive body can be expanded in terms of multipoles, i.e. monopole term, quadrupole term and higher multipoles [2, 17]. Usually, the largest contributions of light deflection originate from the spherically symmetric part (Schwarzschild) of the massive body under consideration. The exact analytical solution of light propagation in the Schwarzschild

metric [7] inherits elliptic integrals, but their evaluation becomes comparable with the time effort needed for the numerical integration of the geodesic equation. Thus, approximative analytical solutions valid on the microarcsecond level of accuracy are indispensible for highly time-efficient data reduction.

In the same way, exact lens equations of light deflection have been obtained in [8, 9, 14]. However, these exact lens equations are also given in terms of elliptic integrals. Therefore, approximations of these exact solutions are also needed for a time-efficient data reduction. An excellent overview of such approximative lens equations has recently been presented in [4].

Basically, two essential approximative approaches are known in order to determine the light deflection in weak gravitational fields:

The first one is the standard parameterized post-Newtonian approach (PPN) [6, 13] which is of the order $\mathcal{O}(m)$. Over the last few decades, it has been the common understanding that the higher order terms $\mathcal{O}(m^2)$ are negligible even on microarcsecond level, except for observations in the vicinity of the Sun. However, recent investigations [1, 11, 15, 16] have revealed that the post-post-Newtonian approximation [5, 6], which is of the order $\mathcal{O}(m^2)$, is needed for such high accuracy. Both approximations are applicable for $d \gg m$, where d is the impact parameter of the unperturbed light ray.

The second one is the standard weak-field approximative lens equation, which usually is called the classical lens equation, see equation (67) in [9] or equation (24) in [4]. One decisive advantage of the classical lens equation is its validity for arbitrarily small values of the impact vector d. The classical lens equation is valid for astrometrical configurations where the source and the observer are far enough from the lens, especially in the case of $a \gg d$ and $b \gg d$, where $a = \boldsymbol{k} \cdot \boldsymbol{x}_1$ and $b = -\boldsymbol{k} \cdot \boldsymbol{x}_0$, where $\boldsymbol{x}_0$ and $\boldsymbol{x}_1$ are the three-vectors from the center of the massive body to the source and observer, respectively, and $\boldsymbol{k}$ is the unit vector from the source to the observer. However, the classical lens equation is not applicable for determining the light deflection of moons of giant planets in the solar system, because astrometrical configurations with $b = 0$ are possible.

Moreover, there are astrometric configurations where neither the standard post-Newtonian approach nor the classical lens equation is applicable, for instance, binary systems. In order to investigate the light deflection in such systems, a link between both these approaches is needed. Such a link can be provided by a generalized lens equation which, in the appropriate limits, coincides with the standard post-Newtonian approach and the classical lens equation. Accordingly, the aim of our investigation is an analytical expression for the generalized lens equation having a form very similar to the classical lens equation. We formulate the following conditions under which our generalized lens equation should be applicable:

(1) valid for $d = 0$, $a = x_1 \gg m$, $b = x_0 \gg m$;
(2) valid for $a = 0$, $d \gg m$, $b \neq 0$;
(3) valid for $b = 0$, $d \gg m$, $a \neq 0$.

These conditions imply that the light path is always far enough from the lens, and thus inherit weak gravitational fields, i.e. small light-deflection angles. In order to control the numerical accuracy, the generalized lens equation is compared with the numerical solution of the exact geodesic equation in the Schwarzschild metric (throughout the paper, we work in harmonic gauge):

$$g_{00} = -\frac{1-a}{1+a}, \qquad g_{i0} = 0,$$
$$g_{ij} = (1+a)^2 \delta_{ij} + \frac{a^2}{x^2} \frac{1+a}{1-a} x^i x^j . \qquad (1)$$
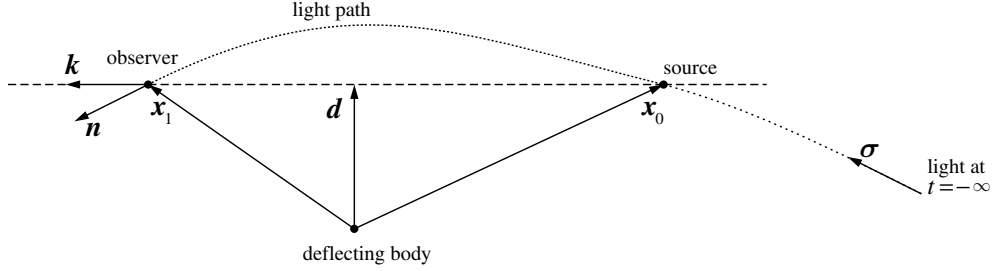
Figure 1. A geometrical representation of the boundary problem under consideration for light propagation from the source to the observer.

Here, $a = \frac{m}{x}$ where $m = \frac{GM}{c^2}$ is the Schwarzschild radius and M is the mass of the light deflecting body, G is the Newtonian constant of gravitation and c is the speed of light. Latin indices take values 1, 2, 3, and the Euclidean metric $\delta_{ij} = 1(0)$ for $i = j$ ($i \neq j$). The absolute value of a three-vector is denoted by $x = |\boldsymbol{x}| = \sqrt{x_1^2 + x_2^2 + x_3^2}$. The exact geodesic equation in the Schwarzschild metric reads, cf [11],

$$\ddot{\boldsymbol{x}} = \frac{a}{x^2}\left[-c^2\frac{1-a}{(1+a)^3} - \dot{\boldsymbol{x}}\cdot\dot{\boldsymbol{x}} + a\frac{2-a}{1-a^2}\left(\frac{\boldsymbol{x}\cdot\dot{\boldsymbol{x}}}{x}\right)^2\right]\boldsymbol{x} + 2\frac{a}{x^2}\frac{2-a}{1-a^2}(\boldsymbol{x}\cdot\dot{\boldsymbol{x}})\,\dot{\boldsymbol{x}}, \tag{2}$$

where the dot denotes the time derivative with respect to the coordinate time t, and $\boldsymbol{x}$ is the three-vector pointing from the center of mass of the massive body to the photon trajectory at the time moment t. The scalar product of two three-vectors with respect to the Euclidean metric δ_{ij} is $\boldsymbol{a}\cdot\boldsymbol{b} = \delta_{ij}\,a^i b^j$. The numerical solution of this equation will be used in order to determine the accuracy of approximative solutions. We abbreviate the angle between two three-vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ by $\delta(\boldsymbol{a},\boldsymbol{b})$, which can be computed by means of $\delta(\boldsymbol{a},\boldsymbol{b}) = \arccos\frac{\boldsymbol{a}\cdot\boldsymbol{b}}{a\,b}$.

Our paper is organized as follows. In section 2, the standard post-Newtonian approach is discussed. In section 3, the steps of the post-post-Newtonian approach relevant for this investigation are shown, and I will briefly summarize the main results of our article [11]. The generalized lens equation is obtained in section 4 and discussed in section 5. A summary is given in section 6.

2. Post-Newtonian approximation

Let us consider the trajectory of a light signal in the post-Newtonian Schwarzschild metric:

$$g_{00} = -1 + 2a + \mathcal{O}(c^{-4}), \quad g_{i0} = 0,$$
$$g_{ij} = \delta_{ij} + 2\,\gamma\,a\,\delta_{ij} + \mathcal{O}(c^{-4}). \tag{3}$$

Here, γ is the parameter of the PPN formalism, which characterizes the possible deviation of the physical reality from general relativity theory where $\gamma = 1$. The light ray is being emitted at a position $\boldsymbol{x}_0$ at the time moment t_0 and received at the position $\boldsymbol{x}_1$ at the time moment t_1, see figure 1.

Light propagation is governed by the geodesic equation, in post-Newtonian order given by

$$\ddot{\boldsymbol{x}} = -(1+\gamma)\,c^2\,\frac{a\,\boldsymbol{x}}{x^2} + 2\,(1+\gamma)\,\frac{a\,\dot{\boldsymbol{x}}\,(\dot{\boldsymbol{x}}\cdot\boldsymbol{x})}{x^2} + \mathcal{O}(c^{-2})\,. \tag{4}$$

The unit tangent vector at the point of observation is $n = \frac{\dot{x}(t_1)}{|\dot{x}(t_1)|}$, and the unit tangent vector $k = \frac{R}{R}$, where $R = x_1 - x_0$ and the absolute value is $R = |R|$. Furthermore, we define the unit tangent vector at remote past: $\sigma = \lim_{t \to -\infty} \frac{\dot{x}(t)}{c}$.

Up to post-Newtonian order, the differential equation (4) can be solved analytically. The solution for the transformation between n and k reads

$$n = k - (1 + \gamma) \frac{m}{d'} \frac{d'}{d'} \frac{x_0 x_1 - x_0 \cdot x_1}{R x_1} + \mathcal{O}(m^2), \tag{5}$$

in terms of the coordinate-independent impact vector d', cf equation (57) of [11]:

$$d' = \lim_{t \to -\infty} \sigma \times (x(t) \times \sigma). \tag{6}$$

This impact parameter is identical to Chandrasekhar's impact parameter [11, 12], which means that in vectorial form, $d' = \frac{L}{E}$, where L is the orbital three-momentum and E is the energy of the photon on the light trajectory; cf equation (215) in chapter 20 of [7].

By means of $\sin \varphi = |n \times k|$, we find the light-deflection angle $\varphi = \delta(n, k)$ in the post-Newtonian approximation:

$$\varphi = (1 + \gamma) \frac{m}{d'} \frac{x_0 x_1 - x_0 \cdot x_1}{R x_1} + \mathcal{O}(m^2). \tag{7}$$

Note that $\frac{x_0 x_1 - x_0 \cdot x_1}{R x_1} \leqslant 2$, and therefore $\varphi \leqslant \frac{4m}{d'}$. One problem of the post-Newtonian solution (5) or (7) is that one can only state that the neglected terms are of order $\mathcal{O}(m^2)$, but their magnitude remains unclear. In order to make a statement about the upper estimate of the higher order terms, one needs to consider the geodesic equation in the post-post-Newtonian approximation.

3. Post-post Newtonian approximation

Now we will consider the trajectory of a light signal in the post-post-Newtonian Schwarzschild metric:

$$g_{00} = -1 + 2a - 2\beta a^2 + \mathcal{O}(c^{-6}), \quad g_{i0} = 0,$$
$$g_{ij} = \delta_{ij} + 2\gamma a \delta_{ij} + \epsilon \left(\delta_{ij} + \frac{x^i x^j}{x^2} \right) a^2 + \mathcal{O}(c^{-6}). \tag{8}$$

The geodesic equation of light propagation in the post-post-Newtonian approximation is given by [11]

$$\ddot{x} = -(1 + \gamma) c^2 \frac{a x}{x^2} + 2(1 + \gamma) \frac{a \dot{x}(\dot{x} \cdot x)}{x^2} + 2c^2 (\beta - \epsilon + 2\gamma(1 + \gamma)) \frac{a^2 x}{x^2}$$
$$+ 2\epsilon \frac{a^2 x (\dot{x} \cdot x)^2}{x^4} + 2(2(1 - \beta) + \epsilon - 2\gamma^2) \frac{a^2 \dot{x}(\dot{x} \cdot x)}{x^2} + \mathcal{O}(c^{-4}). \tag{9}$$

The parameters β, γ and ϵ characterize the possible deviation of physical reality from general relativity theory (in general relativity $\beta = \gamma = \epsilon = 1$). The solution of (9) and the transformation between the unit vectors n and k in post-post-Newtonian order have been given in [11], cf equations (108) and (109) ibid., and read

$$n = k - (1 + \gamma) \frac{m}{d'} \frac{d'}{d'} \frac{x_0 x_1 - x_0 \cdot x_1}{R x_1} + \mathcal{O}\left(\frac{m^2}{d'^2} \right). \tag{10}$$

The terms of the order $\mathcal{O}\left(\frac{m^2}{d'^2} \right)$ can be estimated to be smaller than or equal to $\frac{15\pi}{4} \frac{m^2}{d'^2}$. From equation (10) we obtain the expression

$$\varphi = (1 + \gamma) \frac{m}{d'} \frac{x_0 x_1 - x_0 \cdot x_1}{R x_1} + \mathcal{O}\left(\frac{m^2}{d'^2} \right). \tag{11}$$

Solutions (10) and (11) are identical to the post-Newtonian solutions (5) and (7), respectively. This fact means that the post-post-Newtonian terms in metric (8) and also the post-post-Newtonian terms in the geodesic equation (9) contribute only terms which can be estimated to be smaller than or equal to $\frac{15\pi}{4}\frac{m^2}{d'^2}$. Therefore, the only difference between (11) and (7) here is that the post-post-Newtonian approximation allows us to make a statement about the upper magnitude of the regular post-post-Newtonian terms.

4. Generalized lens equation

Usually, in practical astrometry the position of the observer x_1 and the position of the light-deflecting body are known (here, the center of the massive body coincides with the coordinate center), but the impact parameter d' is not accessible. Therefore, solutions (10) or (11) are not applicable in the form presented. Instead, one has to rewrite these solutions in terms of the impact vector of the unperturbed light ray

$$d = k \times (x_1 \times k).\tag{12}$$

For that one needs a relation between the impact vector d' defined in equation (6) and the impact vector d defined in equation (12). Such a relation has been given in [11], cf (62) ibid., and reads (note $d' = d + \mathcal{O}(m)$)

$$d' = d + (1+\gamma)\frac{m}{d'}\frac{x_0 + x_1}{R}\frac{x_0 x_1 - x_0 \cdot x_1}{R} + \mathcal{O}(m^2).\tag{13}$$

Equation (13) represents a quadratic equation for d', and both these solutions correspond to the two possible light trajectories. A comparison of (13) with (11) yields the relation

$$d' = d + x_1 \varphi + \frac{x_0 + x_1 - R}{R}x_1 \varphi + \mathcal{O}(m^2),\tag{14}$$

where φ is given by equation (11) and we have separated a term $\frac{x_0+x_1-R}{R}x_1\varphi$ which can be shown to contribute to the light deflection φ only to order $\mathcal{O}\!\left(\frac{m^2}{d'^2}\right)$. By inserting (14) into (11), we obtain a quadratic equation which has the solution

$$\varphi_{1,2} = \frac{1}{2}\left(\sqrt{\frac{d^2}{x_1^2} + 4(1+\gamma)\frac{m}{x_1}\frac{x_0 x_1 - x_0 \cdot x_1}{R\, x_1}} \mp \frac{d}{x_1}\right) + \mathcal{O}\left(\frac{m^2}{d'^2}\right).\tag{15}$$

The solution with the upper (lower) sign is denoted by φ_1 (φ_2). For astrometry the solution φ_1 can be considered to be the more relevant solution, because φ_2 represents the second image of one and the same source. One can show that the terms $\mathcal{O}\!\left(\frac{m^2}{d'^2}\right)$ are smaller than or equal to $\frac{15\pi}{4}\frac{m^2}{d'^2}$. Equation (15) represents the generalized lens equation. This equation is applicable not only for any configurations where the post-Newtonian approach, the post-post-Newtonian approach, or the classical lens equation is valid, but also for all those extreme configurations given by points (1)–(3) in the introductory section. In particular, it allows an analytical investigation of light deflection in binary systems [20]. In the following section, we show that formula (15) represents a link between the standard post-Newtonian approach and the classical lens equation.

5. Discussion of the generalized lens equation

5.1. Comparison with standard post-Newtonian and post-post-Newtonian approach

In this section, we compare the generalized lens equation (15) with the standard post-Newtonian and post-post-Newtonian approach of light deflection. A series expansion of

the solution φ_1 in equation (15) for $d \gg m$ yields

$$\varphi_1 = \varphi_{\mathrm{pN}} + \varphi_{\mathrm{ppN}} + \mathcal{O}(m^3) + \mathcal{O}\left(\frac{m^2}{d'^2}\right), \tag{16}$$

with

$$\varphi_{\mathrm{pN}} = (1 + \gamma)\, \frac{m}{d}\, \frac{x_0\, x_1 - \boldsymbol{x}_0 \cdot \boldsymbol{x}_1}{R\, x_1} \leqslant 4\, \frac{m}{d}, \tag{17}$$

$$\varphi_{\mathrm{ppN}} = -(1 + \gamma)^2\, \frac{m^2}{d^2}\, \frac{(x_0\, x_1 - \boldsymbol{x}_0 \cdot \boldsymbol{x}_1)^2}{R^2\, d\, x_1} \leqslant 16\, \frac{m^2}{d^2}\, \frac{x_1}{d}. \tag{18}$$

Expression (17) is called the *standard post-Newtonian solution*, cf equation (24) in [11]. Expression (18) is just the 'enhanced' post-post-Newtonian term, cf equations (3) and (4) in [19]. The 'enhanced term' can be arbitrarily large for small d and large x_1. That is the reason why the standard post-Newtonian and post-post-Newtonian solution is not applicable for extreme configurations like binary stars. The term $\mathcal{O}(m^3)$ will be discussed below, see equation (27); here it is only essential to realize that this term can be larger than the neglected terms $\mathcal{O}\left(\frac{m^2}{d'^2}\right)$.

5.2. Comparison of the generalized lens equation and the classical lens equation

The standard weak-field lens equation is usually called a *classical lens equation* and given, for instance, in equation (67) in [9] or equation (24) in [4]. Let us briefly reconsider the classical lens equation. According to the scheme in figure 2, we obtain the following geometrical relations:

$$\varphi + \psi = \delta, \tag{19}$$

$$a\, \tan\varphi = b\, \tan\psi. \tag{20}$$

Here, the angles are $\psi = \delta(\boldsymbol{\mu}, \boldsymbol{k})$ and $\delta = \delta(\boldsymbol{n}, \boldsymbol{\mu})$, where $\boldsymbol{\mu} = \frac{\dot{\boldsymbol{x}}(t_0)}{|\dot{\boldsymbol{x}}(t_0)|}$ is the unit tangent vector at the position of the source in the direction of the propagation of the light signal.

If the source and the observer are infinitely far from the massive body, then the total light-deflection angle $\delta = \delta\,(\boldsymbol{n}, \boldsymbol{\mu})$ in the Schwarzschild metric reads [3]

$$\delta = 2\,(1 + \gamma)\, \frac{m}{d'} + \mathcal{O}\left(\frac{m^2}{d'^2}\right), \tag{21}$$

which is a coordinate-independent result. The terms of the order $\mathcal{O}\left(\frac{m^2}{d'^2}\right)$ can be estimated to be smaller than or equal to $\frac{15\pi}{4}\, \frac{m^2}{d'^2}$, see [3]. In a classical lens approach, the approximation $d' \simeq d + a\, \tan\varphi$ is used, see figure 2. Inserting this relation into (21), by means of geometrical relations (19) and (20), and using $\tan\varphi = \varphi + \mathcal{O}(\varphi^3)$ and $\tan\psi = \psi + \mathcal{O}(\varphi^3)$, we obtain the quadratic equation

$$\varphi^2 + \frac{d}{a}\, \varphi - 2\,(1 + \gamma)\, \frac{m}{a}\, \frac{b}{a + b} = 0. \tag{22}$$

The solution of equation (22) is the classical lens equation

$$\varphi_{1,2}^{\mathrm{class}} = \frac{1}{2}\left(\sqrt{\frac{d^2}{a^2} + 8(1 + \gamma)\, \frac{m}{a}\, \frac{b}{a + b}} \mp \frac{d}{a}\right), \tag{23}$$

which is valid in the case of $a, b \gg d$; the solution with the upper (lower) sign is denoted by $\varphi_1^{\mathrm{class}}\,\left(\varphi_2^{\mathrm{class}}\right)$.
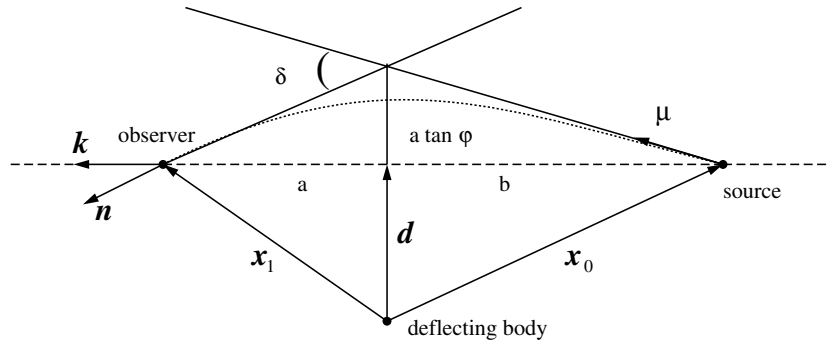
Figure 2. A geometrical representation of the classical lens.

It should be noticed that in (23) not only the light deflection angle φ is assumed to be small, but also the source and observer are assumed to be far from the massive body, i.e. $\delta(\boldsymbol{x}_0, \boldsymbol{x}_1) \simeq \pi$; note that due to that fact equation (23) agrees with the classical lens equation (67) in [8]. Therefore, the classical lens equation is not applicable for extreme configurations like binary systems or light deflection of moons at their giant planets of solar system.

It can easily be shown that the classical lens equation (23) follows straightforward from the generalized lens equation (15). That means, if we rewrite (15) in terms of $a = \boldsymbol{k} \cdot \boldsymbol{x}_1$ and $b = -\boldsymbol{k} \cdot \boldsymbol{x}_0$ and perform a corresponding series expansion of the generalized lens equation (15), then we just obtain the classical lens equation (23) as the leading term in this series.

Furthermore, in the limit $d \to 0$, known as the Einstein ring solution, the generalized lens equation (15) and the classical lens equation (23) yield the same result:

$$\lim_{d \to 0} \varphi_{1,2} = \lim_{d \to 0} \varphi_{1,2}^{\text{class}} = \sqrt{2 (1 + \gamma) \frac{m}{x_1} \frac{x_0}{x_0 + x_1}} \,. \tag{24}$$

Finally, we note that in the extreme configuration $b = 0$ (in this limit φ_2 does not exist), we obtain from (15) the result

$$\lim_{b \to 0} \varphi_1 = \frac{1}{2} \left(\sqrt{\frac{d^2}{x_1^2} + 4 (1 + \gamma) \frac{m}{x_1} \frac{d \, a}{(x_1 + d) \, x_1}} - \frac{d}{x_1} \right) \leqslant \sqrt{(1 + \gamma) \frac{m}{x_1}}, \tag{25}$$

while the classical lens equation yields simply $\varphi_1^{\text{class}} = 0$. Obviously, in the limit $a \to 0$, expression (25) yields zero as it has to be because in this limit the distance between source and observer vanishes, that means no light deflection.

5.3. Comparison with the exact solution

The accuracy of (15) and the stated estimate that the neglected terms are smaller than or equal to $\frac{15 \pi}{4} \frac{m^2}{d'^2}$ have also been confirmed by a comparison with the exact numerical solution of (2). For that, we have solved the geodesic equation (2) in the Schwarzschild metric by the numerical integrator ODEX [10] for several extreme astrometrical configurations. Using forth and back integration, a numerical accuracy of at least 10^{-24} in the components of position and velocity of the photon is guaranteed. Thus, the numerical integration can be considered as an exact solution of the geodesic equation which is denoted by φ_{num}. This numerical approach has been described in some detail in [11]. In all considered extreme configurations, the validity

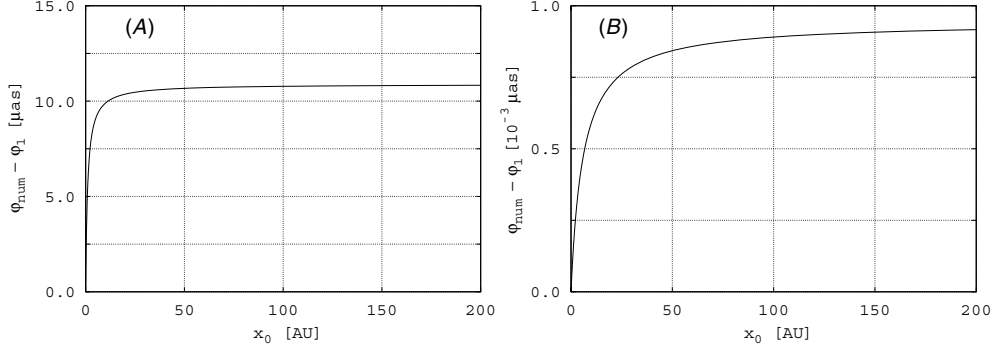**Figure 3.** Comparison of the solution φ_1 of the generalized lens equation (15) with the exact numerical solution φ_{num} for the case of a grazing ray at Sun (A) ($\boldsymbol{x}_1 = (-1.0 \text{ a.u.}, 0, 0)$, $m_\odot = 1476.6$ m, $d' = 696.0 \times 10^6$ m) and Jupiter (B) ($\boldsymbol{x}_1 = (-6.0 \text{ a.u.}, 0, 0)$, $m_{\text{\huge ↑}} = 1.40987$ m, $d' = 71.492 \times 10^6$ m), where a.u. $= 1.496 \times 10^{11}$, m denotes the astronomical unit.

of (15) and the given estimate of neglected terms have been confirmed. As example, in figure 3 we present the results for the light deflection of a grazing ray at Sun and Jupiter. These examples elucidate the fact that the accuracy of the generalized lens equation (15) is much beyond the microarcsecond level of accuracy in the case of light deflection at giant planets. The reason for this fact is that $\frac{15\pi}{4} \frac{m^2}{d'^2} \ll 1 \, \mu\text{as}$ for giant planets of the solar system; only in the vicinity of the Sun we have $\frac{15\pi}{4} \frac{m^2}{d'^2} \sim 11 \, \mu\text{as}$.

The accuracy shown in figure 3(B) is considerably better than the post-post-Newtonian solution investigated in detail in [11, 18], cf figure 3 with figure 2 in [18]. In order to understand the numerical difference between figure 3 and figure 2 in [18], we perform a further series expansion of equation (15) up to the terms of order m^4, which means

$$\varphi_1 = \varphi_{\text{pN}} + \varphi_{\text{ppN}} + \varphi_{\text{pppN}} + \mathcal{O}(m^4) + \mathcal{O}\left(\frac{m^2}{d'^2}\right), \tag{26}$$

where the 'enhanced' terms beyond post-post-Newtonian terms are

$$\varphi_{\text{pppN}} = 2\,(1+\gamma)^3 \, \frac{m^3}{d^3} \, \frac{(x_0\,x_1 - \boldsymbol{x}_0 \cdot \boldsymbol{x}_1)^3}{R^3\,d^2\,x_1} \leqslant 128\,\frac{m^3}{d^3}\,\frac{x_1^2}{d^2}. \tag{27}$$

The given estimation in (27) shows that for large x_1, this term can be considerably larger than the neglected terms of order $\mathcal{O}(\frac{m^2}{d'^2})$. Moreover, the numerical difference between figure 3 and figure 2 in [18] is just given by the term in (27).

6. Summary

Modern astrometry has achieved a microarcsecond level of accuracy, e.g. astrometric missions SIM (NASA) or GAIA (ESA). A time-efficient data reduction implies the need of approximative and highly precise solutions for light deflection at this level of accuracy. In our investigation, we have suggested a generalized lens equation (15) for weak gravitational fields of Schwarzschild metric and valid for finite distances of source and observer from the light deflecting body. The derivation is based on the solution of the geodesic equation (11) in the post-Newtonian metric and Chandrasekhar's coordinate-independent impact parameter d' (6) and its relation to the light-deflection angle φ given in (14). The neglected terms in

(15) can be estimated to be smaller than or equal to $\frac{15\pi}{4}\frac{m^2}{d'^2}$. The accuracy of the generalized lens equation (15) is considerably better than the standard post-Newtonian and post-post-Newtonian approach, which has been investigated in some detail in [11, 18] and the reason for this fact has been pointed out.

The generalized lens equation (15) satisfies three conditions formulated in the introductory section. Moreover, we have shown that in the appropriate limits we obtain the post-Newtonian terms, 'enhanced' post-post-Newtonian terms and the classical lens equation. Thus, the generalized lens equation (15) also provides a link between these essential approaches to determine the light deflection. Numerical investigations have confirmed the analytical results obtained.

The generalized lens equation (15) allows an analytical understanding and investigation of light deflection in extreme astrometric configurations. In particular, the determination of light deflection in binary systems using the generalized lens equation (15) has been investigated in [20].

Acknowledgments

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