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Numerical versus analytical accuracy of the formulae for light propagation

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Abstract
Numerical integration of the differential equations of light propagation in the Schwarzschild metric shows that in some situations relevant for practical observations the well-known post-Newtonian solution for light propagation has an error up to 16 μas. The aim of this work is to demonstrate this fact, identify the reason for this error and to derive an analytical formula accurate at the level of 1 μas as needed for high-accuracy astrometric projects (e.g., Gaia). An analytical post-post-Newtonian solution for the light propagation for both Cauchy and boundary problems is given for the Schwarzschild metric augmented by the parametrized post-Newtonian and post-linear parameters β, γ and ϵ. Using analytical upper estimates of each term we investigate which post-post-Newtonian terms may play a role for an observer in the solar system at the level of 1 μas and conclude that only one post-post-Newtonian term remains important for this numerical accuracy. In this way, an analytical solution for the boundary problem for light propagation is derived. That solution contains terms of both post-Newtonian and post-post-Newtonian order, but is valid for the given numerical level of 1 μas. The derived analytical solution has been verified using the results of a high-accuracy numerical integration of differential equations of light propagation and found to be correct at the level well below 1 μas for an arbitrary observer situated within the solar system. Furthermore, the origin of the post-post-Newtonian terms relevant for the microarcsecond accuracy is elucidated. We demonstrate that these terms result from an inadequate choice of the impact parameter in the standard post-Newtonian formulae. Introducing another impact parameter, that can be called ‘coordinate independent’, we demonstrate that all these terms disappear from the formulae.

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1. Introduction

It is well known that adequate relativistic modeling is indispensable for the success of microarcsecond space astrometry. One of the most important relativistic effects for astrometric observations in the solar system is the gravitational light deflection. The largest contribution in the light deflection comes from the spherically symmetric (Schwarzschild) parts of the gravitational fields of each solar system body. Although the planned astrometric satellites Gaia, SIM, etc will not observe very close to the Sun, they can observe very close to the giant planets also producing significant light deflection. This poses the problem of modeling this light deflection with a numerical accuracy of better than 1 μas.

The exact differential equations of motion for a light ray in the Schwarzschild field can be solved numerically as well as analytically. However, the exact analytical solution is given in terms of elliptic integrals, implying numerical efforts comparable with direct numerical integration, so that approximate analytical solutions are usually used. In fact, the standard parametrized post-Newtonian (PPN) solution is sufficient in many cases and has been widely applied. So far, there was no doubt that the post-Newtonian order of approximation is sufficient for astrometric missions even up to microarcsecond level of accuracy, besides astrometric observations close to the edge of the Sun. However, a direct comparison reveals a deviation between the standard post-Newtonian approach and the high-accuracy numerical solution of the geodetic equations. In particular, we have found a difference of up to 16 μas in light deflection for solar system objects observed close to giant planets. This error has triggered detailed numerical and analytical investigation of the problem.

Usually, in the framework of general relativity or the PPN formalism analytical orders of smallness of various terms are considered. Here the role of the small parameter is played by c^{-1}, where c is the light velocity. Standard post-Newtonian and post-post-Newtonian solutions are derived by retaining terms of relevant analytical orders of magnitude. On the other hand, for practical calculations only numerical magnitudes of various terms are relevant. In this work we attempt to close this gap and combine the analytical parametrized post-post-Newtonian solution with exact analytical estimates of the numerical magnitudes of various terms. In this way we derive a compact analytical solution for light propagation where all terms are indeed relevant at the level of 1 μas. The derived analytical solution is then verified using the high-accuracy numerical integration of the differential equations of light propagation and found to be correct at the level well below 1 μas.

We use fairly standard notations as follows.

- G is the Newtonian constant of gravitation.
- c is the velocity of light.
- β and γ are the parameters of the PPN formalism which characterize possible deviation of the physical reality from general relativity theory ($\beta = \gamma = 1$ in general relativity).
- Lower case Latin indices $i, j, \ldots$ take values 1, 2, 3.
- Lower case Greek indices $\mu, \nu, \ldots$ take values 0, 1, 2, 3.
- Repeated indices imply Einstein's summation irrespective of their positions (e.g. $a^i b^i = a^1 b^1 + a^2 b^2 + a^3 b^3$ and $a^\alpha b^\alpha = a^0 b^0 + a^1 b^1 + a^2 b^2 + a^3 b^3$).
- A dot over any quantity designates the total derivative with respect to the coordinate time of the corresponding reference system e.g. $\dot{a} = \frac{\mathrm{d}a}{\mathrm{d}t}$.
- The three-dimensional coordinate quantities ('3-vectors') referred to the spatial axes of the corresponding reference system are set in boldface: $\boldsymbol{a} = a^i$.
- The absolute value (Euclidean norm) of a '3-vector' $\boldsymbol{a}$ is denoted as $|\boldsymbol{a}|$ or, simply, a and can be computed as $a = |\boldsymbol{a}| = (a^1 a^1 + a^2 a^2 + a^3 a^3)^{1/2}$.

- The scalar product of any two '3-vectors' $\boldsymbol{a}$ and $\boldsymbol{b}$ with respect to the Euclidean metric δ_{ij} is denoted by $\boldsymbol{a} \cdot \boldsymbol{b}$ and can be computed as $\boldsymbol{a} \cdot \boldsymbol{b} = \delta_{ij} a^i b^j = a^i b^i$.
- The vector product of any two '3-vectors' $\boldsymbol{a}$ and $\boldsymbol{b}$ is designated by $\boldsymbol{a} \times \boldsymbol{b}$ and can be computed as $(\boldsymbol{a} \times \boldsymbol{b})^i = \varepsilon_{ijk} a^j b^k$, where $\varepsilon_{ijk} = (i-j)(j-k)(k-i)/2$ is the fully antisymmetric Levi-Civita symbol.
- For any two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, the angle between them is designated as $\delta(\boldsymbol{a}, \boldsymbol{b})$. Clearly, for an angle between two vectors one has $0 \leqslant \delta(\boldsymbol{a}, \boldsymbol{b}) \leqslant \pi$. The angle $\delta(\boldsymbol{a}, \boldsymbol{b})$ can be computed in many ways, for example, as $\delta(\boldsymbol{a}, \boldsymbol{b}) = \arccos \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{ab}$.

This paper is a concise exposition of the work performed in the framework of the ESA project Gaia and published in a series of preprints [11, 12, 21, 22]. The paper is organized as follows. In section 2 we present the exact differential equations for light propagation in the Schwarzschild field in harmonic gauge. High-accuracy numerical integrations of these equations are discussed in section 3. In section 4 we discuss the standard post-Newtonian approximation and demonstrate its errors by a direct comparison with numerical results. In section 5 the analytical post-post-Newtonian solution for light propagation is given. Section 6 is devoted to the boundary problem for light propagation in the post-post-Newtonian approximation. Investigations of the post-post-Newtonian terms in the formulae for the light deflection reveal that these terms can be divided into two groups: 'regular' (those which can be estimated as const $\cdot \frac{m^2}{d^2}$, where m is the Schwarzschild radius of the deflecting body and $\boldsymbol{d}$ is the impact parameter) and 'enhanced' (those which cannot be estimated like this and may become substantially larger than the 'regular' terms). In section 7 we clarify the physical origin of the 'enhanced' post-post-Newtonian terms. The results are summarized in section 8.

2. Schwarzschild metric and null geodesics in harmonic coordinates

We need a tool to calculate the real numerical accuracy of some analytical formulae for light propagation in various situations. To this end, we consider the exact Schwarzschild metric and its null geodesics in harmonic gauge. Those exact differential equations for the null geodesics will be solved numerically with high accuracy (see below) and that numerical solution provides the required reference.

2.1. Metric tensor

In the harmonic gauge

$$\frac{\partial(\sqrt{-g}\, g^{\alpha\beta})}{\partial x^\beta} = 0, \tag{1}$$

the components of the covariant metric tensor of the Schwarzschild solution are given by

$$\begin{aligned} g_{00} &= -\frac{1-a}{1+a}, \\ g_{0i} &= 0, \\ g_{ij} &= (1+a)^2 \delta_{ij} + \frac{a^2}{x^2} \frac{1+a}{1-a} x^i x^j, \end{aligned} \tag{2}$$

where

$$a = \frac{m}{x}, \tag{3}$$

and $m = \frac{GM}{c^2}$ is the Schwarzschild radius of a body with mass M. The contravariant components of the metric read

$$g^{00} = \frac{1+a}{1-a},\quad g^{0i} = 0,\quad g^{ij} = \frac{1}{(1+a)^2}\delta_{ij} - \frac{a^2}{x^2}\frac{1}{(1+a)^2}x^i x^j. \tag{4}$$

Considering that the determinant of the metric can be computed as

$$g = -(1+a)^4, \tag{5}$$

one can easily check that this metric satisfies the harmonic conditions (1).

2.2. *Christoffel symbols*

The Christoffel symbols of second kind are defined as

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2}g^{\mu\nu}\left(\frac{\partial g_{\nu\alpha}}{\partial x^\beta} + \frac{\partial g_{\nu\beta}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^\nu}\right). \tag{6}$$

Using (2) and (4) one gets

$$\Gamma^0_{0i} = \frac{a}{x^2}\frac{1}{1-a^2}x^i,\quad \Gamma^i_{00} = \frac{a}{x^2}\frac{1-a}{(1+a)^3}x^i,\quad \Gamma^i_{jk} = \frac{a}{x^2}x^i\delta_{jk} - \frac{a}{x^2}\frac{1}{1+a}(x^j\delta_{ik} + x^k\delta_{ij}) - \frac{a^2}{x^4}\frac{2-a}{1-a^2}x^i x^j x^k. \tag{7}$$

All other Christoffel symbols vanish.

2.3. *Isotropic condition*

The conditions that a photon follows an isotropic geodesic can be formulated as an equation for the four components of the coordinate velocity $\dot{x}^\alpha$ of that photon:

$$g_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\beta}{\mathrm{d}\lambda} = 0, \tag{8}$$

λ being the canonical parameter, or

$$g_{00} + \frac{2}{c}g_{0i}\dot{x}^i + \frac{1}{c^2}g_{ij}\dot{x}^i\dot{x}^j = 0, \tag{9}$$

where $\dot{x}^i = \mathrm{d}x^i/\mathrm{d}t$ is the coordinate velocity of the photon. Equation (9) is a first integral of motion for the differential equation for light propagation and must be valid for any point of an isotropic geodesic. Substituting the ansatz $\dot{\boldsymbol{x}} = cs\boldsymbol{\mu}$, where $\boldsymbol{\mu}$ is a unit coordinate direction of light propagation ($\boldsymbol{\mu}\cdot\boldsymbol{\mu} = 1$) and $s = |\dot{\boldsymbol{x}}|/c$, into (9) one gets for metric (2)

$$s = \frac{1-a}{1+a}\left(1 - a^2 + \frac{a^2}{x^2}(\boldsymbol{x}\cdot\boldsymbol{\mu})^2\right)^{-1/2}. \tag{10}$$

This formula allows one to compute the absolute value of the coordinate velocity of light in the chosen reference system if the position of the photon x^i and the coordinate direction of its propagation $\boldsymbol{\mu}$ are given.

2.4. Equation of isotropic geodesics

The geodetic equations

$$\frac{\mathrm{d}^2x^\mu}{\mathrm{d}\lambda^2}+\Gamma^\mu_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\beta}{\mathrm{d}\lambda}=0 \tag{11}$$

can be re-parametrized by coordinate time t to give

$$\ddot{x}^i=-c^2\Gamma^i{}_{00}-2c\Gamma^i{}_{0j}\dot{x}^j-\Gamma^i_{jk}\dot{x}^j\dot{x}^k+\dot{x}^i\left(c\Gamma^0{}_{00}+2\Gamma^0{}_{0j}\dot{x}^j+\frac{1}{c}\Gamma^0_{jk}\dot{x}^j\dot{x}^k\right). \tag{12}$$

Substituting the Christoffel symbols one gets the differential equations for light propagation in metric (2):

$$\ddot{\boldsymbol{x}}=\frac{a}{x^2}\left[-c^2\frac{1-a}{(1+a)^3}-\dot{\boldsymbol{x}}\cdot\dot{\boldsymbol{x}}+a\frac{2-a}{1-a^2}\left(\frac{\boldsymbol{x}\cdot\dot{\boldsymbol{x}}}{x}\right)^2\right]\boldsymbol{x}+2\frac{a}{x^2}\frac{2-a}{1-a^2}(\boldsymbol{x}\cdot\dot{\boldsymbol{x}})\dot{\boldsymbol{x}}. \tag{13}$$

Equation (10) for the isotropic condition together with $\dot{\boldsymbol{x}}\cdot\dot{\boldsymbol{x}}=c^2s^2$ could be used to avoid the term containing $\dot{\boldsymbol{x}}\cdot\dot{\boldsymbol{x}}$, but this does not simplify the equations and we prefer not to do this here.

3. Numerical integration of the equations of light propagation

Our goal is to integrate (13) numerically to get a solution for the trajectory of a light ray with an accuracy much higher than the goal accuracy of 1 μas $\approx 4.8\times10^{-12}$ radians. For these numerical integrations a simple FORTRAN 95 code using quadruple (128 bit) arithmetic has been written. Numerical integrator ODEX [7] has been adapted to the quadruple precision. ODEX is an extrapolation algorithm based on the explicit midpoint rule. It has automatic order selection, local accuracy control and dense output. Using forth and back integration to estimate the accuracy, each numerical integration is automatically checked to achieve a numerical accuracy of at least 10^{-24} in the components of both position and velocity of the photon at each moment of time.

The numerical integration is first used to solve the initial-value (Cauchy) problem for differential equations (13). Equation (10) should be used to choose the initial conditions. The problem of light propagation has thus only five degrees of freedom: three degrees of freedom correspond to the position of the photon and two other degrees of freedom correspond to the unit direction of light propagation (of course, in the Schwarzschild field with its symmetry one also has further integrals of motion, but here we ignore this; see section 5.4 below). The absolute value of the coordinate light velocity can be computed from (10). Fixing initial position of the photon $\boldsymbol{x}(t_0)$ and initial (unit) direction of propagation $\boldsymbol{\mu}$ one gets the initial velocity of the photon as a function of $\boldsymbol{\mu}$ and s computed for given $\boldsymbol{\mu}$ and $\boldsymbol{x}$ as given by (10):

$$\begin{aligned}\boldsymbol{x}(t_0)&=\boldsymbol{x}_0,\\ \dot{\boldsymbol{x}}(t_0)&=cs\boldsymbol{\mu}.\end{aligned} \tag{14}$$

The numerical integration yields the position $\boldsymbol{x}$ and velocity $\dot{\boldsymbol{x}}$ of the photon as a function of the time t. The dense output of ODEX allows one to obtain the position and velocity of the photon on a selected grid of moments of time. Equation (10) must hold for any moment of time as soon as it is satisfied by the initial conditions. Therefore, equation (10) can also be used to check the accuracy of the numerical integration.

For the purposes of this work we need to have an accurate solution of two-value boundary problem. That is, a solution of equation (13) with boundary conditions

$$\begin{aligned}\boldsymbol{x}(t_0)&=\boldsymbol{x}_0,\\ \boldsymbol{x}(t_1)&=\boldsymbol{x}_1,\end{aligned} \tag{15}$$

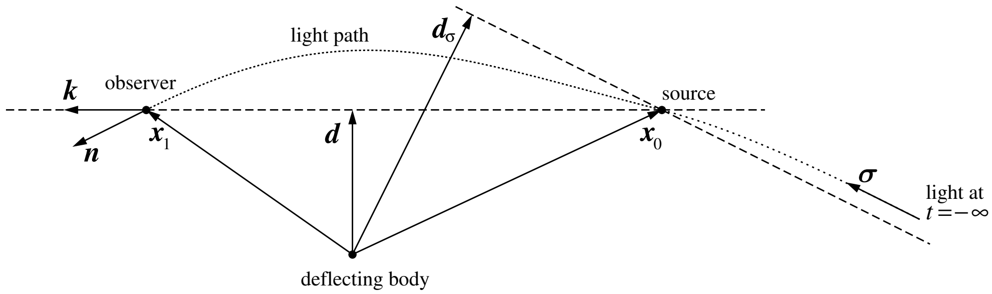

Figure 1. Definitions of the vectors $\boldsymbol{x}_1$, $\boldsymbol{x}_0$, $\boldsymbol{k}$, $\boldsymbol{n}$, $\boldsymbol{\sigma}$. The vectors $\boldsymbol{d}$ (defined in section 4.1) and $\boldsymbol{d}_\sigma$ (defined in section 5.4) are also shown.

where $\boldsymbol{x}_0$ and $\boldsymbol{x}_1$ are two given constants, t_0 is assumed to be fixed and t_1 is unknown and should be determined by solving (13). Instead of using some numerical methods to solve this boundary problem directly, we generate solutions of a family of boundary problems from our solution of the initial value problem (14). Each intermediate result computed during the numerical integration with initial conditions (14) gives us a high-accuracy solution of the corresponding two-value boundary problem (15): t_1 and $\boldsymbol{x}_1$ are simply taken from the numerical integration.

As discussed in [10], light propagation is characterized by three unit vectors (see figure 1): the coordinate direction $\boldsymbol{n}$ of light propagation at the point of reception

$$\boldsymbol{n} = \frac{\dot{\boldsymbol{x}}(t_1)}{|\dot{\boldsymbol{x}}(t_1)|}, \tag{16}$$

the coordinate direction $\boldsymbol{\sigma}$ of light propagation for time going to minus infinity:

$$\boldsymbol{\sigma} = \lim_{t\to-\infty} \frac{1}{c}\dot{\boldsymbol{x}}(t), \tag{17}$$

and the coordinate direction $\boldsymbol{k}$ from the point of light emission to the point of reception:

$$\boldsymbol{k} = \frac{\boldsymbol{R}}{R}, \qquad \boldsymbol{R} = \boldsymbol{x}_1 - \boldsymbol{x}_0. \tag{18}$$

In the following discussion we will compare predictions of various analytical models for $\boldsymbol{n}$ in the framework of the boundary problem (15). The reference value for these comparisons can be computed using (16) and $\dot{\boldsymbol{x}}(t_1)$ from the numerical integration. The accuracy of this $\boldsymbol{n}$ computed from our numerical integrations is guaranteed to be of the order of 10^{-24} radians and can be considered as exact for our purposes.

4. The deficiency of the standard post-Newtonian approach

Let us now demonstrate that the standard post-Newtonian formulae for light propagation have too large numerical errors when compared to the accurate numerical solution of the geodetic equations described in the previous section.

4.1. Equations of the post-Newtonian approach

The well-known equations of light propagation in the first post-Newtonian approximation with PPN parameters have been discussed by many authors (see, for example, [3, 17, 20]). Let us here summarize the standard post-Newtonian formulae. The differential equations for the light rays read (see also section 5.1.4 below)

$$\ddot{\boldsymbol{x}} = -(c^2 + \gamma \dot{x}^k \dot{x}^k) \frac{a\boldsymbol{x}}{x^2} + 2(1+\gamma) \frac{a\dot{\boldsymbol{x}}(\dot{x}^k x^k)}{x^2} + \mathcal{O}(c^{-2}). \tag{19}$$

The analytical solution of (19) can be written in the form

$$\boldsymbol{x}(t) = \boldsymbol{x}_{\text{pN}} + \mathcal{O}(c^{-4}), \tag{20}$$

$$\boldsymbol{x}_{\text{pN}} = \boldsymbol{x}_0 + c(t - t_0)\boldsymbol{\sigma} + \Delta\boldsymbol{x}(t), \tag{21}$$

where

$$\Delta\boldsymbol{x}(t) = -(1+\gamma)m \left(\boldsymbol{\sigma} \times (\boldsymbol{x}_0 \times \boldsymbol{\sigma}) \left(\frac{1}{x - \boldsymbol{\sigma}\cdot\boldsymbol{x}} - \frac{1}{x_0 - \boldsymbol{\sigma}\cdot\boldsymbol{x}_0} \right) + \boldsymbol{\sigma} \log \frac{x + \boldsymbol{\sigma}\cdot\boldsymbol{x}}{x_0 + \boldsymbol{\sigma}\cdot\boldsymbol{x}_0} \right). \tag{22}$$

Solution (20)–(22) satisfies the following initial conditions:

$$\begin{aligned} &\boldsymbol{x}(t_0) = \boldsymbol{x}_0, \\ &\lim_{t\to-\infty} \dot{\boldsymbol{x}}(t) = c\boldsymbol{\sigma}. \end{aligned} \tag{23}$$

From (20)–(22) it is easy to derive the following expression for the unit tangent vector at the observer's position $\boldsymbol{x}_1$ for the boundary problem (15) (the standard technique to do this is given, e.g., in [3] and used below in section 6 in the post-post-Newtonian approximation):

$$\boldsymbol{n}_{\text{pN}} = \boldsymbol{k} - (1+\gamma)m \frac{\boldsymbol{d}}{d^2} \frac{x_0 x_1 - \boldsymbol{x}_0\cdot\boldsymbol{x}_1}{x_1 R}, \tag{24}$$

where $\boldsymbol{R}$ and $\boldsymbol{k}$ are defined by (18), and $\boldsymbol{d} = \boldsymbol{k} \times (\boldsymbol{x}_0 \times \boldsymbol{k}) = \boldsymbol{k} \times (\boldsymbol{x}_1 \times \boldsymbol{k})$ is the impact parameter of the straight line connecting $\boldsymbol{x}_0$ and $\boldsymbol{x}_1$.

4.2. Comparison of the post-Newtonian formula and the numerical solution

In order to investigate the accuracy of the standard post-Newtonian formulae we have compared the post-Newtonian predictions of the light deflection with the results of the numerical solution of geodetic equations. Here, we calculate the angle between the unit tangent vector $\boldsymbol{n}_{\text{pN}}$ defined by (24) and the vector $\boldsymbol{n}$ computed using (16) from the numerical integration of (13).

Having performed extensive tests, we have found that, in the real solar system, the error of $\boldsymbol{n}_{\text{pN}}$ for observations made by an observer situated in the vicinity of the Earth attains 16 μas. These results are illustrated by table 1 and figure 2. Table 1 contains the parameters we have used in our numerical simulations as well as the maximal angular deviation between $\boldsymbol{n}_{\text{pN}}$ and $\boldsymbol{n}$ in each set of simulations. We have performed simulations with different bodies of the solar systems assuming that the minimal impact distance d is equal to the radius of the corresponding body, and the maximal distance $x_1 = |\boldsymbol{x}_1|$ between the gravitating body and the observer is given by the maximal distance between the gravitating body and the Earth. The simulation shows that the error of $\boldsymbol{n}_{\text{pN}}$ is generally increasing for larger x_1 and decreasing for larger d. The dependence of the error of $\boldsymbol{n}_{\text{pN}}$ for fixed d and x_1 and increasing distance x_0 between the gravitating body and the source is given in figure 2 for the case of Jupiter, where the minimal d and maximal x_1 (according to table 1) were used. Moreover, the error of $\boldsymbol{n}_{\text{pN}}$ is found to be proportional to m^2 which leads us to the necessity to deal with the post-post-Newtonian approximation for light propagation.

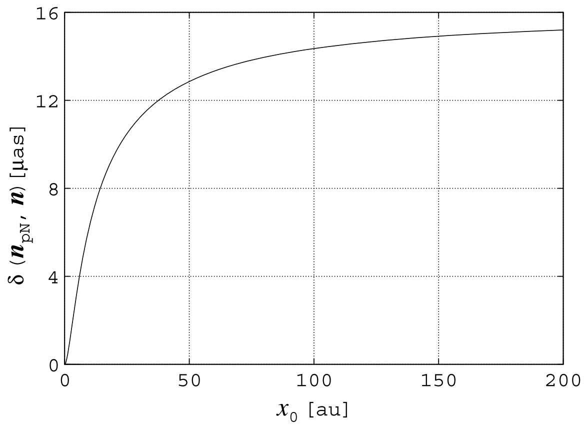

Figure 2. The angle between $\boldsymbol{n}_{\rm pN}$ and $\boldsymbol{n}$ for Jupiter. The vector $\boldsymbol{n}_{\rm pN}$ is evaluated by means of the standard post-Newtonian formula (24), while $\boldsymbol{n}$ is taken from the numerical integration as described in section 3. The impact parameter d is taken to be the radius of Jupiter and the distance x_1 between Jupiter and the observer is 6 au.

Table 1. Numerical parameters of the Sun and giant planets are taken from [8, 19]. $d_{\rm min}$ is the minimal value of the impact parameter d that was used in the simulations. For each body, $d_{\rm min}$ is equal to the radius of the body. For the Sun at 45° the impact parameter is computed as $d = \sin 45^\circ \times 1$au. $x_1^{\rm max}$ is the maximal absolute value of the distance x_1 between the gravitating body and the observer that was used in the simulations. $\delta_{\rm max}$ is the maximal angle between $\boldsymbol{n}_{\rm pN}$ and $\boldsymbol{n}$ found in our numerical simulations.

	Sun	Sun at 45°	Jupiter	Saturn	Uranus	Neptune
$m = GM/c^2$ (m)	1476.6	1476.6	1.409 87	0.422 15	0.064 473	0.076 067
$d_{\rm min}$ (10^6 m)	696.0	105 781.7	71.492	60.268	25.559	24.764
$x_1^{\rm max}$ (au)	1	1	6	11	21	31
$\delta_{\rm max}$ (μas)	3187.8	6.32×10^{-4}	16.13	4.42	2.58	5.84

5. Analytical post-post-Newtonian solution

The goal of this section is to derive a rigorous analytical post-post-Newtonian solution for light propagation in the gravitational field of one spherically symmetric body in the framework of the PPN formalism extended by a non-linear parameter for the terms of order c^{-4} in g_{ij}. The geodetic equation for the light ray in the Schwarzschild metric can in principle be integrated exactly [4]. However, such an analytical solution is given in terms of elliptic integrals and is not very suitable for massive calculations. Besides that, only the trajectory of the photon is readily available from the literature, but not the position and velocity of a photon as functions of time. Fortunately, in many cases of interest approximate solutions are sufficient. The standard way to solve the geodetic equation is the well-known post-Newtonian approximation scheme. Normally, in practical applications of relativistic light propagation, the first post-Newtonian solution is used. Post-post-Newtonian effects have also been sometimes considered [9, 13], but in a way which cannot be called self-consistent since no rigorous solution in the post-post-Newtonian approximation has been used. Such a rigorous post-post-Newtonian analytical solution for light propagation in the Schwarzschild metric has been derived in [2, 3] in general relativity in a class of gauges. However, the parametrization in [2, 3] does not allow one to consider alternative theories of gravity and therefore, a post-post-Newtonian solution for light propagation within the PPN formalism and its extension to the second post-Newtonian approximation is not known. However, it is clearly necessary to have such a solution. Therefore, our goal is to generalize the post-post-Newtonian solution of [2] and to extend it for the boundary problem for light propagation.

5.1. Differential equations of light propagation and their integral

The first part of the problem is to derive the differential equations of light propagation with PPN and post-linear parameters.

5.1.1. Metric tensor in the parametrized post-post-Newtonian approximation. Expanding metric (2) in powers of c^{-1}, retaining only the terms relevant for the post-post-Newtonian solution for the light propagation, and introducing the PPN parameters β and γ [20] and the post-linear parameter ϵ one gets

$$\begin{aligned} g_{00} &= -1 + 2a - 2\beta a^2 + \mathcal{O}(c^{-6}),\\ g_{0i} &= 0,\\ g_{ij} &= \delta_{ij} + 2\gamma a\delta_{ij} + \epsilon\left(\delta_{ij} + \frac{x^i x^j}{x^2}\right)a^2 + \mathcal{O}(c^{-6}), \end{aligned} \tag{25}$$

a being again defined by (3). In general relativity one has $\beta = \gamma = \epsilon = 1$. The parameter ϵ should be considered as a formal way to trace, in the following calculations, the terms coming from the terms c^{-4} in g_{ij}. No physical meaning of ϵ is claimed here. However, this parameter is equivalent to the parameter Λ of [14–16] and the parameter ϵ of [6].

The corresponding contravariant components of the metric tensor can be deduced from (25) and are given by

$$\begin{aligned} g^{00} &= -1 - 2a + 2(\beta - 2)a^2 + \mathcal{O}(c^{-6}),\\ g^{0i} &= 0,\\ g^{ij} &= \delta_{ij} - 2\gamma a\delta_{ij} + \left((4\gamma^2 - \epsilon)\delta_{ij} - \epsilon\frac{x^i x^j}{x^2}\right)a^2 + \mathcal{O}(c^{-6}). \end{aligned} \tag{26}$$

The determinant of the metric tensor reads

$$g = -1 - 2(3\gamma - 1)a - 2(\beta + 2\epsilon + 6\gamma(\gamma - 1))a^2 + \mathcal{O}(c^{-6}), \tag{27}$$

$$\sqrt{-g} = 1 + (3\gamma - 1)a + (2\beta + 4\epsilon - 1 + 3\gamma(\gamma - 2))a^2 + \mathcal{O}(c^{-6}). \tag{28}$$

Metric (25) is obviously harmonic for $\gamma = \beta = \epsilon = 1$ since the harmonic conditions (1) take the form

$$\begin{aligned} \frac{\partial(\sqrt{-g}g^{0\alpha})}{\partial x^\alpha} &= 0,\\ \frac{\partial(\sqrt{-g}g^{i\alpha})}{\partial x^\alpha} &= (1-\gamma)\frac{a x^i}{x^2} + ((1+\gamma)^2 - 2\beta - 2\epsilon)\frac{a^2 x^i}{x^2} + \mathcal{O}(c^{-6}). \end{aligned} \tag{29}$$

5.1.2. Christoffel symbols. The Christoffel symbols of second kind defined by (6) can be derived from metric (25)–(26):

$$\Gamma^0{}_{00} = 0, \tag{30}$$

$$\Gamma^0{}_{0i} = \frac{a x^i}{x^2} + (1-\beta)\frac{2a^2 x^i}{x^2} + \mathcal{O}(c^{-6}), \tag{31}$$

$$\Gamma^0{}_{ik} = 0, \tag{32}$$

$$\Gamma^i{}_{00} = \frac{a x^i}{x^2} - (\beta+\gamma)\frac{2a^2 x^i}{x^2} + \mathcal{O}(c^{-6}), \tag{33}$$

$$\Gamma^i{}_{0k} = 0, \tag{34}$$

$$\Gamma^i{}_{kl} = \gamma\,(x^i\delta_{kl} - x^k\delta_{il} - x^l\delta_{ik})\frac{a}{x^2} + \left(2(\epsilon-\gamma^2)x^i\delta_{kl} - (\epsilon - 2\gamma^2)(x^k\delta_{il} + x^l\delta_{ik}) - 2\epsilon\frac{x^i x^k x^l}{x^2}\right)\frac{a^2}{x^2} + \mathcal{O}(c^{-6}). \tag{35}$$

5.1.3. Isotropic condition for the null geodetic. From now on, x^α denote the coordinates of a photon, x^i denote the spatial coordinates of the photon and $x = |\boldsymbol{x}|$ is the distance of the photon from the gravitating body that is situated at the origin of the used reference system. As it was discussed in section 2.3, equation (9) allows one to compute the absolute value of the coordinate velocity of light if the position of the photon x^i and the unit coordinate direction of its propagation μ^i ($\boldsymbol{\mu}\cdot\boldsymbol{\mu} = 1$) are given. Using (25) for $s = |\dot{\boldsymbol{x}}|/c$ one gets

$$s = 1 - (1+\gamma)a + \frac{1}{2}\left(-1 + 2\beta - \epsilon + \gamma(2+3\gamma) - \epsilon\left(\frac{\boldsymbol{\mu}\cdot\boldsymbol{x}}{x}\right)^2\right)a^2 + \mathcal{O}(c^{-6}). \tag{36}$$

5.1.4. Differential equations of light propagation. Inserting the Christoffel symbols (30)–(35) into (12), one gets the following equations of light propagation in the post-post-Newtonian approximation:

$$\ddot{\boldsymbol{x}} = -(c^2 + \gamma\,\dot{\boldsymbol{x}}\cdot\dot{\boldsymbol{x}})\frac{a\boldsymbol{x}}{x^2} + 2(1+\gamma)\frac{a\dot{\boldsymbol{x}}(\dot{\boldsymbol{x}}\cdot\boldsymbol{x})}{x^2} + 2((\beta+\gamma)c^2 + (\gamma^2-\epsilon)(\dot{\boldsymbol{x}}\cdot\dot{\boldsymbol{x}}))\frac{a^2\boldsymbol{x}}{x^2} + 2\epsilon\frac{a^2\boldsymbol{x}(\dot{\boldsymbol{x}}\cdot\boldsymbol{x})^2}{x^4} + 2(2(1-\beta)+\epsilon-2\gamma^2)\frac{a^2\dot{\boldsymbol{x}}(\dot{\boldsymbol{x}}\cdot\boldsymbol{x})}{x^2} + \mathcal{O}(c^{-4}). \tag{37}$$

Here, for estimating the analytical order of smallness of the terms we take into account that $|\dot{\boldsymbol{x}}| = \mathcal{O}(c)$. Using (36) and $\dot{\boldsymbol{x}}\cdot\dot{\boldsymbol{x}} = c^2 s^2$ one can simplify (37) to get

$$\ddot{\boldsymbol{x}} = -(1+\gamma)c^2\frac{a\boldsymbol{x}}{x^2} + 2(1+\gamma)\frac{a\dot{\boldsymbol{x}}(\dot{\boldsymbol{x}}\cdot\boldsymbol{x})}{x^2} + 2c^2(\beta - \epsilon + 2\gamma(1+\gamma))\frac{a^2\boldsymbol{x}}{x^2} + 2\epsilon\frac{a^2\boldsymbol{x}(\dot{\boldsymbol{x}}\cdot\boldsymbol{x})^2}{x^4} + 2(2(1-\beta)+\epsilon-2\gamma^2)\frac{a^2\dot{\boldsymbol{x}}(\dot{\boldsymbol{x}}\cdot\boldsymbol{x})}{x^2} + \mathcal{O}(c^{-4}). \tag{38}$$

5.1.5. Equations of light propagation with the additional trace parameter α. For our purposes it is advantageous to have one more parameter that can be used to trace terms in the following calculations which come from the post-post-Newtonian terms in the equations of motion of a photon. We denote this parameter α and introduce it in the above equation simply as a factor for all the post-post-Newtonian terms on the right-hand side:

$$
\begin{aligned}
\ddot{\boldsymbol{x}} = &-(1+\gamma)c^2\frac{a\boldsymbol{x}}{x^2} + 2(1+\gamma)\frac{a\dot{\boldsymbol{x}}(\dot{\boldsymbol{x}}\cdot\boldsymbol{x})}{x^2} \\
&+ 2c^2\alpha(\beta-\epsilon+2\gamma(1+\gamma))\frac{a^2\boldsymbol{x}}{x^2} + 2\alpha\epsilon\frac{a^2\boldsymbol{x}(\dot{\boldsymbol{x}}\cdot\boldsymbol{x})^2}{x^4} \\
&+ 2\alpha(2(1-\beta)+\epsilon-2\gamma^2)\frac{a^2\dot{\boldsymbol{x}}(\dot{\boldsymbol{x}}\cdot\boldsymbol{x})}{x^2} + \mathcal{O}(c^{-4}).
\end{aligned}
\tag{39}
$$

Setting $\alpha = 0$ in the solution of (39) one can formally get a second-order solution for the post-Newtonian equations of light propagation. The merit of this parameter will be clear below.

5.2. Initial value problem

Let us now solve analytically an initial value problem for the derived equations. For initial conditions (23) using the same approach as in [2, 3], one gets

$$\frac{1}{c}\dot{\boldsymbol{x}}_N = \boldsymbol{\sigma}, \tag{40}$$

$$\boldsymbol{x}_N = \boldsymbol{x}_0 + c(t-t_0)\boldsymbol{\sigma}, \tag{41}$$

$$\frac{1}{c}\dot{\boldsymbol{x}}_{\rm pN} = \boldsymbol{\sigma} + m\boldsymbol{A}_1(\boldsymbol{x}_N), \tag{42}$$

$$\boldsymbol{x}_{\rm pN} = \boldsymbol{x}_N + m(\boldsymbol{B}_1(\boldsymbol{x}_N) - \boldsymbol{B}_1(\boldsymbol{x}_0)), \tag{43}$$

$$\frac{1}{c}\dot{\boldsymbol{x}}_{\rm ppN} = \boldsymbol{\sigma} + m\boldsymbol{A}_1(\boldsymbol{x}_{\rm pN}) + m^2\boldsymbol{A}_2(\boldsymbol{x}_N), \tag{44}$$

$$\boldsymbol{x}_{\rm ppN} = \boldsymbol{x}_N + m(\boldsymbol{B}_1(\boldsymbol{x}_{\rm pN}) - \boldsymbol{B}_1(\boldsymbol{x}_0)) + m^2(\boldsymbol{B}_2(\boldsymbol{x}_N) - \boldsymbol{B}_2(\boldsymbol{x}_0)), \tag{45}$$

where

$$\boldsymbol{A}_1(\boldsymbol{x}) = -(1+\gamma)\left(\frac{\boldsymbol{\sigma}\times(\boldsymbol{x}\times\boldsymbol{\sigma})}{x(x-\boldsymbol{\sigma}\cdot\boldsymbol{x})} + \frac{\boldsymbol{\sigma}}{x}\right), \tag{46}$$

$$\boldsymbol{B}_1(\boldsymbol{x}) = -(1+\gamma)\left(\frac{\boldsymbol{\sigma}\times(\boldsymbol{x}\times\boldsymbol{\sigma})}{x-\boldsymbol{\sigma}\cdot\boldsymbol{x}} + \boldsymbol{\sigma}\log(x+\boldsymbol{\sigma}\cdot\boldsymbol{x})\right), \tag{47}$$

$$
\begin{aligned}
\boldsymbol{A}_2(\boldsymbol{x}) = &-\frac{1}{2}\alpha\epsilon\frac{\boldsymbol{\sigma}\cdot\boldsymbol{x}}{x^4}\boldsymbol{x} + 2(1+\gamma)^2\frac{\boldsymbol{\sigma}\times(\boldsymbol{x}\times\boldsymbol{\sigma})}{x^2(x-\boldsymbol{\sigma}\cdot\boldsymbol{x})} + (1+\gamma)^2\frac{\boldsymbol{\sigma}\times(\boldsymbol{x}\times\boldsymbol{\sigma})}{x(x-\boldsymbol{\sigma}\cdot\boldsymbol{x})^2} \\
&- (1+\gamma)^2\frac{\boldsymbol{\sigma}}{x(x-\boldsymbol{\sigma}\cdot\boldsymbol{x})} + \left(2(1-\alpha+\gamma)(1+\gamma)+\alpha\beta-\frac{1}{2}\alpha\epsilon\right)\frac{\boldsymbol{\sigma}}{x^2} \\
&- \frac{1}{4}(8(1+\gamma-\alpha\gamma)(1+\gamma)-4\alpha\beta+3\alpha\epsilon)(\boldsymbol{\sigma}\cdot\boldsymbol{x})\frac{\boldsymbol{\sigma}\times(\boldsymbol{x}\times\boldsymbol{\sigma})}{x^2|\boldsymbol{\sigma}\times\boldsymbol{x}|^2} \\
&- \frac{1}{4}(8(1+\gamma-\alpha\gamma)(1+\gamma)-4\alpha\beta+3\alpha\epsilon)\frac{\boldsymbol{\sigma}\times(\boldsymbol{x}\times\boldsymbol{\sigma})}{|\boldsymbol{\sigma}\times\boldsymbol{x}|^3}(\pi-\delta(\boldsymbol{\sigma},\boldsymbol{x})),
\end{aligned}
\tag{48}
$$

$$\begin{aligned}\boldsymbol{B}_2(\boldsymbol{x}) = &-(1+\gamma)^2\frac{\boldsymbol{\sigma}}{x-\boldsymbol{\sigma}\cdot\boldsymbol{x}} + (1+\gamma)^2\frac{\boldsymbol{\sigma}\times(\boldsymbol{x}\times\boldsymbol{\sigma})}{(x-\boldsymbol{\sigma}\cdot\boldsymbol{x})^2} + \frac{1}{4}\alpha\epsilon\frac{\boldsymbol{x}}{x^2}\\ &-\frac{1}{4}\alpha(8(1+\gamma)-4\beta+3\epsilon)\frac{\boldsymbol{\sigma}}{|\boldsymbol{\sigma}\times\boldsymbol{x}|}\left(\frac{\pi}{2}-\delta(\boldsymbol{\sigma},\boldsymbol{x})\right)\\ &-\frac{1}{4}(8(1+\gamma-\alpha\gamma)(1+\gamma)-4\alpha\beta+3\alpha\epsilon)(\boldsymbol{\sigma}\cdot\boldsymbol{x})\frac{\boldsymbol{\sigma}\times(\boldsymbol{x}\times\boldsymbol{\sigma})}{|\boldsymbol{\sigma}\times\boldsymbol{x}|^3}(\pi-\delta(\boldsymbol{\sigma},\boldsymbol{x})),\end{aligned} \tag{49}$$

or, alternatively, for $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$

$$\boldsymbol{B}_1(\boldsymbol{x}) = -(1+\gamma)\left(\frac{\boldsymbol{\sigma}\times(\boldsymbol{x}\times\boldsymbol{\sigma})}{x-\boldsymbol{\sigma}\cdot\boldsymbol{x}} - \boldsymbol{\sigma}\log(x-\boldsymbol{\sigma}\cdot\boldsymbol{x})\right), \tag{50}$$

$$\begin{aligned}\boldsymbol{B}_2(\boldsymbol{x}) = &+(1+\gamma)^2\frac{\boldsymbol{\sigma}}{x-\boldsymbol{\sigma}\cdot\boldsymbol{x}} + (1+\gamma)^2\frac{\boldsymbol{\sigma}\times(\boldsymbol{x}\times\boldsymbol{\sigma})}{(x-\boldsymbol{\sigma}\cdot\boldsymbol{x})^2} + \frac{1}{4}\alpha\epsilon\frac{\boldsymbol{x}}{x^2}\\ &-\frac{1}{4}\alpha(8(1+\gamma)-4\beta+3\epsilon)\frac{\boldsymbol{\sigma}}{|\boldsymbol{\sigma}\times\boldsymbol{x}|}\left(\frac{\pi}{2}-\delta(\boldsymbol{\sigma},\boldsymbol{x})\right)\\ &-\frac{1}{4}(8(1+\gamma-\alpha\gamma)(1+\gamma)-4\alpha\beta+3\alpha\epsilon)(\boldsymbol{\sigma}\cdot\boldsymbol{x})\frac{\boldsymbol{\sigma}\times(\boldsymbol{x}\times\boldsymbol{\sigma})}{|\boldsymbol{\sigma}\times\boldsymbol{x}|^3}(\pi-\delta(\boldsymbol{\sigma},\boldsymbol{x})).\end{aligned} \tag{51}$$

With these definitions the solution of (39) reads

$$\begin{aligned}&\boldsymbol{x}(t) = \boldsymbol{x}_{\rm ppN}(t) + \mathcal{O}(c^{-6}),\\ &\frac{1}{c}\dot{\boldsymbol{x}}(t) = \frac{1}{c}\dot{\boldsymbol{x}}_{\rm ppN}(t) + \mathcal{O}(c^{-6}).\end{aligned} \tag{52}$$

It is easy to check that the solution for the coordinate velocity of light $\dot{\boldsymbol{x}}_{\rm ppN}$ satisfies integral (36). In order to demonstrate this fact, it is important to understand that the position $\boldsymbol{x}$ in (36) lies on the trajectory of the photon and must be therefore considered as $\boldsymbol{x}_{\rm pN}$ in the post-Newtonian terms and as $\boldsymbol{x}_{\rm N}$ in the post-post-Newtonian terms of (44).

5.3. *Vector $\boldsymbol{n}$ in the initial value problem*

Using (44) one gets

$$\boldsymbol{n} = \boldsymbol{\sigma} + m\boldsymbol{C}_1(\boldsymbol{x}_{\rm pN}) + m^2\boldsymbol{C}_2(\boldsymbol{x}_{\rm N}) + \mathcal{O}(c^{-6}), \tag{53}$$

where

$$\boldsymbol{C}_1(\boldsymbol{x}) = \boldsymbol{A}_1(\boldsymbol{x}) - \boldsymbol{\sigma}(\boldsymbol{\sigma}\cdot\boldsymbol{A}_1(\boldsymbol{x})) = -(1+\gamma)\frac{\boldsymbol{\sigma}\times(\boldsymbol{x}\times\boldsymbol{\sigma})}{x(x-\boldsymbol{\sigma}\cdot\boldsymbol{x})},$$

$$\begin{aligned}\boldsymbol{C}_2(\boldsymbol{x}) &= \boldsymbol{A}_2(\boldsymbol{x}) - \boldsymbol{A}_1(\boldsymbol{x})(\boldsymbol{\sigma}\cdot\boldsymbol{A}_1(\boldsymbol{x})) - \frac{1}{2}\boldsymbol{\sigma}(\boldsymbol{A}_1(\boldsymbol{x})\cdot\boldsymbol{A}_1(\boldsymbol{x}))\\ &\quad - \boldsymbol{\sigma}(\boldsymbol{\sigma}\cdot\boldsymbol{A}_2(\boldsymbol{x})) + \frac{3}{2}\boldsymbol{\sigma}(\boldsymbol{\sigma}\cdot\boldsymbol{A}_1(\boldsymbol{x}))^2\\ &= -\frac{1}{2}\alpha\epsilon\frac{\boldsymbol{\sigma}\cdot\boldsymbol{x}}{x^4}\boldsymbol{\sigma}\times(\boldsymbol{x}\times\boldsymbol{\sigma}) + (1+\gamma)^2\frac{\boldsymbol{\sigma}\times(\boldsymbol{x}\times\boldsymbol{\sigma})}{x^2(x-\boldsymbol{\sigma}\cdot\boldsymbol{x})}\\ &\quad + (1+\gamma)^2\frac{\boldsymbol{\sigma}\times(\boldsymbol{x}\times\boldsymbol{\sigma})}{x(x-\boldsymbol{\sigma}\cdot\boldsymbol{x})^2} - \frac{1}{2}(1+\gamma)^2\frac{\boldsymbol{\sigma}}{x^2}\frac{x+\boldsymbol{\sigma}\cdot\boldsymbol{x}}{x-\boldsymbol{\sigma}\cdot\boldsymbol{x}}\\ &\quad - \frac{1}{4}(8(1+\gamma-\alpha\gamma)(1+\gamma)-4\alpha\beta+3\alpha\epsilon)(\boldsymbol{\sigma}\cdot\boldsymbol{x})\frac{\boldsymbol{\sigma}\times(\boldsymbol{x}\times\boldsymbol{\sigma})}{x^2|\boldsymbol{\sigma}\times\boldsymbol{x}|^2}\\ &\quad - \frac{1}{4}(8(1+\gamma-\alpha\gamma)(1+\gamma)-4\alpha\beta+3\alpha\epsilon)\frac{\boldsymbol{\sigma}\times(\boldsymbol{x}\times\boldsymbol{\sigma})}{|\boldsymbol{\sigma}\times\boldsymbol{x}|^3}(\pi-\delta(\boldsymbol{\sigma},\boldsymbol{x})).\end{aligned} \tag{54}$$

5.4. Impact parameters

As we have seen in sections 4.1 and 5.2 the usual analytical solutions are expressed through one of the following two impact parameters:

$$\boldsymbol{d}_\sigma = \boldsymbol{\sigma} \times (\boldsymbol{x}_0 \times \boldsymbol{\sigma}), \tag{55}$$

$$\boldsymbol{d} = \boldsymbol{k} \times (\boldsymbol{x}_0 \times \boldsymbol{k}) = \boldsymbol{k} \times (\boldsymbol{x}_1 \times \boldsymbol{k}), \tag{56}$$

where $\boldsymbol{x}_0$ is the initial point in both Cauchy and boundary problems given by (23) and (15), respectively, while $\boldsymbol{x}_1$ is the final position in the boundary problem. Both these impact parameters naturally arise in practical calculations of light propagation when positions of the source and the observer are given in some reference system (e.g., in the BCRS [10]). However, these parameters are clearly coordinate dependent and have no profound physical meaning. One can expect that formulae involving these impact parameters contain some spurious, non-physical terms obscuring the physical meaning of the formulae. As we will see below it is indeed the case. Now, we introduce another impact parameter

$$\boldsymbol{d}' = \lim_{t\to-\infty} \frac{1}{c}\dot{\boldsymbol{x}}(t) \times \left(\boldsymbol{x}(t) \times \frac{1}{c}\dot{\boldsymbol{x}}(t)\right) = \lim_{t\to-\infty} \boldsymbol{\sigma} \times (\boldsymbol{x}(t) \times \boldsymbol{\sigma}). \tag{57}$$

For a similar impact parameter defined at $t \to +\infty$

$$\boldsymbol{d}'' = \lim_{t\to+\infty} \frac{1}{c}\dot{\boldsymbol{x}}(t) \times \left(\boldsymbol{x}(t) \times \frac{1}{c}\dot{\boldsymbol{x}}(t)\right) = \lim_{t\to+\infty} \boldsymbol{\nu} \times (\boldsymbol{x}(t) \times \boldsymbol{\nu}), \tag{58}$$

where $\boldsymbol{\nu} = \lim_{t\to+\infty} \frac{1}{c}\dot{\boldsymbol{x}}(t)$, one has $|\boldsymbol{d}'| = |\boldsymbol{d}''|$. It is also clear that the angle between $\boldsymbol{d}'$ and $\boldsymbol{d}''$ is equal to the full light deflection (see below). Since both $\boldsymbol{d}'$ and $\boldsymbol{d}''$ reside at time-like infinity and since the metric under study is asymptotically flat, these parameters can be called coordinate independent.

One can show that $d' = d''$ coincides with the impact parameter D introduced, e.g., by equation (215) of section 20 of [4] in terms of full energy and the angular momentum of the photon (see also [1] for a useful discussion). Indeed, in the polar coordinates (x, φ) Chandrasekhar's impact parameter $D = f(x)x^2\dot{\varphi}$, where $\lim_{x\to\infty} f(x) = 1$. Clearly, $x^2\dot{\varphi} = |\dot{\boldsymbol{x}}(t) \times \boldsymbol{x}(t)|$ and it is obvious that $d' = d'' = D$. Interestingly, this discussion allows one to find an exact integral of the equations of motion for a photon in the Schwarzschild field. The equations of light propagation (13) in the Schwarzschild metric (2) in harmonic coordinates have an integral

$$\boldsymbol{D} = \frac{(1+a)^3}{1-a}\frac{1}{c}\dot{\boldsymbol{x}}(t) \times \boldsymbol{x}(t) = \text{const}, \tag{59}$$

while for the parametrized post-post-Newtonian equations of motion given by (39) one has

$$\begin{aligned} \boldsymbol{D} &= \exp(2(1+\gamma)a + \alpha(2(1-\beta) + \epsilon - 2\gamma^2)a^2)\frac{1}{c}\dot{\boldsymbol{x}}(t) \times \boldsymbol{x}(t) \\ &= (1 + 2(1+\gamma)a + (2(1+\gamma)^2 + \alpha(2(1-\beta) + \epsilon - 2\gamma^2))a^2)\frac{1}{c}\dot{\boldsymbol{x}}(t) \times \boldsymbol{x}(t) + \mathcal{O}(c^{-6}) \\ &= \text{const}. \end{aligned} \tag{60}$$

The first line of (60) represents an *exact* integral of the (approximate) equations of motion (39). In both cases Chandrasekhar's D is the absolute value of $\boldsymbol{D}$ as given above.

Let us stress that the impact parameter $\boldsymbol{d}'$ is not convenient for practical calculations, but we will use it below to understand the physical origin of various terms in the formulae describing light propagation. Therefore, we need to have a relation between impact parameters

(55)–(57). The relation between $\boldsymbol{d}'$ and $\boldsymbol{d}_\sigma$ can be derived using the post-Newtonian solution for light propagation given above:

$$\boldsymbol{d}' = \boldsymbol{d}_\sigma \left(1 + (1+\gamma)\frac{m}{d_\sigma^2}(x_0 + \boldsymbol{\sigma}\cdot\boldsymbol{x}_0)\right) + \mathcal{O}(c^{-4}). \tag{61}$$

The relation of $\boldsymbol{d}'$ and $\boldsymbol{d}$ can be derived using the formulae of section 4.1:

$$\boldsymbol{d}' = \boldsymbol{d}\left(1 + (1+\gamma)\frac{m}{d^2}\frac{x_1+x_0}{R}\frac{R^2-(x_1-x_0)^2}{2R}\right) - (1+\gamma)m\boldsymbol{k}\frac{x_1-x_0+R}{R} + \mathcal{O}(c^{-4}). \tag{62}$$

Now we are ready to proceed to the analysis of the post-post-Newtonian equations of light propagation.

5.5. Total light deflection

In order to derive the total light deflection, we have to consider the limits of the coordinate light velocity $\dot{\boldsymbol{x}}$ for $t \to \pm\infty$. Using formulae of section 5.2 one gets

$$\lim_{t\to-\infty}\frac{1}{c}\dot{\boldsymbol{x}}(t) = \boldsymbol{\sigma}, \tag{63}$$

$$\begin{aligned}\lim_{t\to+\infty}\frac{1}{c}\dot{\boldsymbol{x}}(t) &\equiv \boldsymbol{\nu}\\ &= \boldsymbol{\sigma} - 2(1+\gamma)m\frac{\boldsymbol{\sigma}\times(\boldsymbol{x}_0\times\boldsymbol{\sigma})}{|\boldsymbol{x}_0\times\boldsymbol{\sigma}|^2} - 2(1+\gamma)^2m^2\frac{\boldsymbol{\sigma}}{|\boldsymbol{x}_0\times\boldsymbol{\sigma}|^2}\\ &\quad - \frac{1}{4}\pi(8(1+\gamma-\alpha\gamma)(1+\gamma) - 4\alpha\beta + 3\alpha\epsilon)m^2\frac{\boldsymbol{\sigma}\times(\boldsymbol{x}_0\times\boldsymbol{\sigma})}{|\boldsymbol{x}_0\times\boldsymbol{\sigma}|^3}\\ &\quad + 2(1+\gamma)^2m^2(x_0 + \boldsymbol{\sigma}\cdot\boldsymbol{x}_0)\frac{\boldsymbol{\sigma}\times(\boldsymbol{x}_0\times\boldsymbol{\sigma})}{|\boldsymbol{x}_0\times\boldsymbol{\sigma}|^4} + \mathcal{O}(c^{-6}).\end{aligned} \tag{64}$$

Therefore, the total light deflection reads

$$\begin{aligned}|\boldsymbol{\sigma}\times\boldsymbol{\nu}| &= 2(1+\gamma)m\frac{1}{|\boldsymbol{x}_0\times\boldsymbol{\sigma}|} - 2(1+\gamma)^2m^2(x_0+\boldsymbol{\sigma}\cdot\boldsymbol{x}_0)\frac{1}{|\boldsymbol{x}_0\times\boldsymbol{\sigma}|^3}\\ &\quad + \frac{1}{4}(8(1+\gamma-\alpha\gamma)(1+\gamma) - 4\alpha\beta + 3\alpha\epsilon)\pi m^2\frac{1}{|\boldsymbol{x}_0\times\boldsymbol{\sigma}|^2} + \mathcal{O}(c^{-6}).\end{aligned} \tag{65}$$

Equation (65) defines the sine of the angle of the total light deflection in the post-post-Newtonian approximation. The first term in (65) is the post-Newtonian expression of the total light deflection. The other two terms are the post-post-Newtonian corrections. Using d' defined by (57) and related to d_σ by (61) one can rewrite (65) as

$$|\boldsymbol{\sigma}\times\boldsymbol{\nu}| = 2(1+\gamma)\frac{m}{d'} + \frac{1}{4}(8(1+\gamma-\alpha\gamma)(1+\gamma) - 4\alpha\beta + 3\alpha\epsilon)\pi\frac{m^2}{d'^2} + \mathcal{O}(c^{-6}). \tag{66}$$

This result with $\alpha = 1$ coincides with equation (4) of [6] and also agrees with the results of [2, 5, 14, 18] in the corresponding limits. It is now clear that the second term on the right-hand side of (65) 'corrects' the main post-Newtonian term converting it to $2(1+\gamma)m/d'$. Note that the total light deflection $|\boldsymbol{\sigma}\times\boldsymbol{\nu}|$ is a coordinate-independent quantity and (66) expresses it through coordinate-independent quantities while (65) does not.

6. Post-post-Newtonian solution of the boundary problem

For practical modeling of observations it is not sufficient to consider the initial value problem for light propagation. The two-point boundary value problem given by (15) is important here.

This section is devoted to a derivation of the post-post-Newtonian solution of this boundary problem for (39).

6.1. Formal expressions

An iterative solution of (40)–(45) for the propagation time $\tau = t_1 - t_0$ and unit direction $\boldsymbol{\sigma}$ reads

$$c\tau = R - m\boldsymbol{k}\cdot[\boldsymbol{B}_1(\boldsymbol{x}_1) - \boldsymbol{B}_1(\boldsymbol{x}_0)] - m^2\boldsymbol{k}\cdot[\boldsymbol{B}_2(\boldsymbol{x}_1) - \boldsymbol{B}_2(\boldsymbol{x}_0)] + \frac{m^2}{2R}|\boldsymbol{k}\times(\boldsymbol{B}_1(\boldsymbol{x}_1) - \boldsymbol{B}_1(\boldsymbol{x}_0))|^2 + \mathcal{O}(c^{-6}), \tag{67}$$

$$\begin{aligned}\boldsymbol{\sigma} = \boldsymbol{k} &+ m\frac{1}{R}(\boldsymbol{k}\times[\boldsymbol{k}\times(\boldsymbol{B}_1(\boldsymbol{x}_1) - \boldsymbol{B}_1(\boldsymbol{x}_0))]) + m^2\frac{1}{R}(\boldsymbol{k}\times[\boldsymbol{k}\times(\boldsymbol{B}_2(\boldsymbol{x}_1) - \boldsymbol{B}_2(\boldsymbol{x}_0))])\\ &+ m^2\frac{1}{R^2}(\boldsymbol{B}_1(\boldsymbol{x}_1) - \boldsymbol{B}_1(\boldsymbol{x}_0))\times[\boldsymbol{k}\times(\boldsymbol{B}_1(\boldsymbol{x}_1) - \boldsymbol{B}_1(\boldsymbol{x}_0))]\\ &- \frac{3}{2}m^2\frac{1}{R^2}\boldsymbol{k}|\boldsymbol{k}\times(\boldsymbol{B}_1(\boldsymbol{x}_1) - \boldsymbol{B}_1(\boldsymbol{x}_0))|^2 + \mathcal{O}(c^{-6}).\end{aligned} \tag{68}$$

These expressions are still implicit since in order to achieve the post-post-Newtonian accuracy one should use the post-Newtonian relation between $\boldsymbol{\sigma}$ and $\boldsymbol{k}$ to represent $\boldsymbol{\sigma}$ in $\boldsymbol{B}_1$ appearing in the post-Newtonian terms. That relation can be again obtained from (68) by neglecting all terms of order $\mathcal{O}(c^{-4})$. In contrast, in the terms of the order of $\mathcal{O}(c^{-4})$ in (67) and (68) one can use the Newtonian relation $\boldsymbol{\sigma} = \boldsymbol{k}$.

6.2. The propagation time $c\tau$

Substituting (47) and (49) into (67) one can derive an explicit formula for the time of light propagation as a function of the given boundary conditions $\boldsymbol{x}_0$ and $\boldsymbol{x}_1$:

$$\begin{aligned}
\text{N}\Big|\quad & c\tau = R\\
\text{pN}\Big|\quad & + (1+\gamma)m\log\frac{x_1 + x_0 + R}{x_1 + x_0 - R}\\
\Delta\text{pN}\Big|\quad & + \frac{1}{2}(1+\gamma)^2 m^2\frac{R}{|\boldsymbol{x}_1\times\boldsymbol{x}_0|^2}((x_1 - x_0)^2 - R^2)\\
\text{ppN}\Big|\quad & + \frac{1}{8}\alpha\epsilon\frac{m^2}{R}\left(\frac{x_0^2 - x_1^2 - R^2}{x_1^2} + \frac{x_1^2 - x_0^2 - R^2}{x_0^2}\right)\\
\text{ppN}\Big|\quad & + \frac{1}{4}\alpha(8(1+\gamma) - 4\beta + 3\epsilon)m^2\frac{R}{|\boldsymbol{x}_1\times\boldsymbol{x}_0|}\delta(\boldsymbol{x}_1, \boldsymbol{x}_0)\\
& + \mathcal{O}(c^{-6}).
\end{aligned} \tag{69}$$

Here, we have used that $\delta(\boldsymbol{k}, \boldsymbol{x}_0) - \delta(\boldsymbol{k}, \boldsymbol{x}_1) = \delta(\boldsymbol{x}_1, \boldsymbol{x}_0)$. Here and below we classify the character of the individual terms by the labels N (Newtonian), pN (post-Newtonian), ppN (post-post-Newtonian) and ΔpN (terms that are formally of the post-post-Newtonian order $\mathcal{O}(c^{-4})$, but may numerically become significantly larger than other post-post-Newtonian terms; see below). Using $|\boldsymbol{x}_1\times\boldsymbol{x}_0| = Rd$, where d is the impact parameter defined by (56), and assuming general-relativistic values of all parameters $\alpha = \beta = \gamma = \epsilon = 1$ one gets the following estimates of the sums of the terms labeled by 'ΔpN' and 'ppN', respectively (the proofs are given in [21, 22]):

$$|c\delta\tau_{\Delta\mathrm{pN}}| \leqslant 2\frac{m^2}{d^2}R\frac{4x_1x_0}{(x_1+x_0)^2} \leqslant 2\frac{m^2}{d^2}R, \tag{70}$$

$$|c\delta\tau_{\mathrm{ppN}}| \leqslant \frac{15}{4}\pi\frac{m^2}{d}. \tag{71}$$

These estimates and all estimates we give below are reachable for some values of parameters and, in this sense, cannot be improved. From these estimates we can conclude that among the post-post-Newtonian terms, $c\delta\tau_{\Delta\mathrm{pN}}$ can become significantly larger compared to the other post-post-Newtonian terms. For this reason we will call such terms 'enhanced' post-post-Newtonian terms. The physical origin and properties of the 'enhanced' post-post-Newtonian terms will be discussed in section 7.

The effect of $|c\delta\tau_{\mathrm{ppN}}|$ for the Sun is less than 3.7 cm for arbitrary boundary conditions and can be neglected for any current and planned observations. Therefore, the formula for the time of light propagation between two given points can be simplified by taking only the relevant terms:

$$\begin{aligned} c\tau &= R + (1+\gamma)m\log\frac{x_1+x_0+R}{x_1+x_0-R} \\ &\quad - \frac{1}{2}(1+\gamma)^2 m^2 \frac{R}{|\boldsymbol{x}_1\times\boldsymbol{x}_0|^2}(R^2-(x_1-x_0)^2) + \mathcal{O}\left(\frac{m^2}{d}\right) + \mathcal{O}(m^3). \end{aligned} \tag{72}$$

This expression can be written in an elegant form

$$c\tau = R + (1+\gamma)m\log\frac{x_1+x_0+R+(1+\gamma)m}{x_1+x_0-R+(1+\gamma)m} + \mathcal{O}\left(\frac{m^2}{d}\right) + \mathcal{O}(m^3) \tag{73}$$

that has been already derived in [13] in an inconsistent way (see section 8.3.1.1 and equation (8-54) of [13]). As a criterion if the additional post-post-Newtonian term is required for a given situation, one can use (70) giving the upper boundary of the additional term.

6.3. *Transformation from $\boldsymbol{k}$ to $\boldsymbol{\sigma}$*

Substituting (47) and (49) into (68) one gets

$$\begin{aligned}
&\text{N}\quad \boldsymbol{\sigma} = \boldsymbol{k} \\
&\text{pN}\quad + (1+\gamma)m\frac{x_1-x_0+R}{|\boldsymbol{x}_1\times\boldsymbol{x}_0|^2}\boldsymbol{k}\times(\boldsymbol{x}_0\times\boldsymbol{x}_1) \\
&\Delta\text{pN}\quad + \frac{(1+\gamma)^2}{2}m^2\boldsymbol{k}\times(\boldsymbol{x}_0\times\boldsymbol{x}_1)\frac{(x_1+x_0)(x_1-x_0-R)(x_1-x_0+R)^2}{|\boldsymbol{x}_1\times\boldsymbol{x}_0|^4} \\
&\text{scaling}\quad - \frac{(1+\gamma)^2}{2}m^2\frac{(x_1-x_0+R)^2}{|\boldsymbol{x}_1\times\boldsymbol{x}_0|^2}\boldsymbol{k} \\
&\text{ppN}\quad + m^2\boldsymbol{k}\times(\boldsymbol{x}_0\times\boldsymbol{x}_1)\left[-\frac{1}{4}\alpha\epsilon\frac{1}{R^2}\left(\frac{1}{x_1^2}-\frac{1}{x_0^2}\right)\right. \\
&\text{ppN}\quad + \frac{1}{8}(8(1+\gamma-\alpha\gamma)(1+\gamma)-4\alpha\beta+3\alpha\epsilon)\frac{1}{|\boldsymbol{x}_1\times\boldsymbol{x}_0|^3}
\end{aligned}$$

$$\Big|_{\mathrm{ppN}} \quad \times \left(2R^2(\pi-\delta(\boldsymbol{k},\boldsymbol{x}))+\left(x_1^2-x_0^2-R^2\right)\delta(\boldsymbol{x}_1,\boldsymbol{x}_0)\right)\Big]$$

$$+\mathcal{O}(c^{-6}). \tag{74}$$

This formula allows one to compute $\boldsymbol{\sigma}$ for the given boundary conditions $\boldsymbol{x}_0$ and $\boldsymbol{x}_1$. Let us estimate the magnitude of the individual terms in (74) in the angle $\delta(\boldsymbol{\sigma},\boldsymbol{k})$ between $\boldsymbol{\sigma}$ and $\boldsymbol{k}$. This angle can be computed from the vector product $\boldsymbol{\rho}=\boldsymbol{k}\times\boldsymbol{\sigma}$, and, therefore, the term in (74) proportional to $\boldsymbol{k}$ and labeled as 'scaling' plays no role. Here and below, terms proportional to $\boldsymbol{k}$ do not influence the directions in the considered approximation. These terms are only necessary to keep the involved vectors to have unit length. Now, we represent the vector product $\boldsymbol{\rho}$ as the sum of three kinds of terms: $\boldsymbol{\rho}=\boldsymbol{\rho}_{\mathrm{pN}}+\boldsymbol{\rho}_{\Delta\mathrm{pN}}+\boldsymbol{\rho}_{\mathrm{ppN}}$ where each term is the vector product of $\boldsymbol{k}$ and the sum of the correspondingly labeled terms in (74). Using

$$|\boldsymbol{k}\times[\boldsymbol{k}\times(\boldsymbol{x}_0\times\boldsymbol{x}_1)]|=|\boldsymbol{k}\times(\boldsymbol{x}_0\times\boldsymbol{x}_1)|=Rd, \tag{75}$$

and general-relativistic values of the parameters $\alpha=\beta=\gamma=\epsilon=1$ one gets (the proofs can be found in [21, 22])

$$|\boldsymbol{\rho}_{\mathrm{pN}}|\leqslant\frac{4m}{d}\begin{bmatrix}1, & x_0\leqslant x_1,\\ \dfrac{x_1}{x_1+x_0}, & x_0>x_1\end{bmatrix}\leqslant\frac{4m}{d}, \tag{76}$$

$$|\boldsymbol{\rho}_{\Delta\mathrm{pN}}|\leqslant 16\frac{m^2}{d^3}\begin{bmatrix}\dfrac{4}{27}(x_1+x_0), & \dfrac{1}{2}x_1\leqslant x_0\leqslant x_1,\\ \dfrac{x_1^2x_0}{(x_1+x_0)^2}, & x_0<\dfrac{1}{2}x_1 \text{ or } x_0>x_1,\end{bmatrix} \tag{77}$$

$$|\boldsymbol{\rho}_{\mathrm{ppN}}|\leqslant\frac{15}{4}\pi\frac{m^2}{d^2}. \tag{78}$$

Note that $\boldsymbol{\rho}_{\mathrm{pN}}$ and $\boldsymbol{\rho}_{\Delta\mathrm{pN}}$ themselves as well as their estimates are not continuous for $\boldsymbol{x}_1\to\boldsymbol{x}_0$ since in this limit an infinitely small change of $\boldsymbol{x}_1$ leads to big changes in $\boldsymbol{k}$. Discontinuity of the same origin appears for many other terms. The limit $\boldsymbol{x}_1\to\boldsymbol{x}_0$ and the corresponding discontinuity have, clearly, no physical importance.

We see that among the terms of order $\mathcal{O}(m^2)$ only $|\boldsymbol{\rho}_{\Delta\mathrm{pN}}|$ cannot be estimated as const $\times\, m^2/d^2$. The sum of the three other terms labeled as 'ppN' can be estimated as given by (78). In most cases these terms can be neglected at the level of 1 μas. Indeed, it is easy to see that $|\boldsymbol{\rho}_{\mathrm{ppN}}|$ can exceed 1 μas only for observations within about 3.3 angular radii from the Sun. Accordingly, we obtain a simplified formula for the transformation from $\boldsymbol{k}$ to $\boldsymbol{\sigma}$ keeping only the post-Newtonian and 'enhanced' post-post-Newtonian terms labeled as 'pN' and 'ΔpN' in (74):

$$\boldsymbol{\sigma}=\boldsymbol{k}+\boldsymbol{d}S\left(1-S\frac{1}{2}(x_1+x_0)\left(1+\frac{x_0-x_1}{R}\right)\right)+\mathcal{O}\left(\frac{m^2}{d^2}\right)+\mathcal{O}(m^3), \tag{79}$$

$$S=(1+\gamma)\frac{m}{d^2}\left(1-\frac{x_0-x_1}{R}\right). \tag{80}$$

Equation (77) can be used as a criterion if the post-post-Newtonian term in (79) is necessary for a given accuracy and configuration.

6.4. Transformation from $\boldsymbol{\sigma}$ to $\boldsymbol{n}$

The transformation between $\boldsymbol{n}$ and $\boldsymbol{\sigma}$ is given by (53) and (54). We need, however, to express the relativistic terms in (53) as the functions of $\boldsymbol{k}$. To this end we note that $\boldsymbol{x}_{\mathrm{pN}}=\boldsymbol{x}_1+\mathcal{O}(c^{-4})$

and $\boldsymbol{x}_1 = \boldsymbol{x}_0 + R\boldsymbol{k}$, use (74) for $\boldsymbol{\sigma}$ in $\boldsymbol{C}_1(\boldsymbol{x}_{\rm pN})$, and get

$$
\begin{aligned}
&{\scriptstyle \rm N}\Big| \quad \boldsymbol{n} = \boldsymbol{\sigma} \\
&{\scriptstyle \rm pN}\Big| \quad - (1+\gamma) m \boldsymbol{k} \times (\boldsymbol{x}_0 \times \boldsymbol{x}_1) \frac{R}{|\boldsymbol{x}_1 \times \boldsymbol{x}_0|^2} \left(1 + \frac{\boldsymbol{k}\cdot\boldsymbol{x}_1}{x_1}\right) \\
&{\scriptstyle \rm scaling}\Big| \quad + \frac{1}{4}(1+\gamma)^2 m^2 \frac{\boldsymbol{k}}{|\boldsymbol{x}_1 \times \boldsymbol{x}_0|^2} \frac{R}{x_1} \left(1 + \frac{\boldsymbol{k}\cdot\boldsymbol{x}_1}{x_1}\right) (3x_1 - x_0 - R)(x_1 - x_0 + R) \\
&{\scriptstyle \Delta\rm pN}\Big| \quad + m^2 \boldsymbol{k} \times (\boldsymbol{x}_0 \times \boldsymbol{x}_1) \Bigg[(1+\gamma)^2 \frac{x_1 + x_0}{|\boldsymbol{x}_1 \times \boldsymbol{x}_0|^2} \left(1 + \frac{\boldsymbol{k}\cdot\boldsymbol{x}_1}{x_1}\right) \frac{R(R^2 - (x_1 - x_0)^2)}{2|\boldsymbol{x}_1 \times \boldsymbol{x}_0|^2} \\
&{\scriptstyle \rm ppN}\Big| \qquad + (1+\gamma)^2 \frac{R}{|\boldsymbol{x}_1 \times \boldsymbol{x}_0|^2} \left(1 + \frac{\boldsymbol{k}\cdot\boldsymbol{x}_1}{x_1}\right) \frac{1}{x_1} \\
&{\scriptstyle \rm ppN}\Big| \qquad + \frac{1}{2}(1+\gamma)^2 \frac{R^2}{|\boldsymbol{x}_1 \times \boldsymbol{x}_0|^4} \left(1 + \frac{\boldsymbol{k}\cdot\boldsymbol{x}_1}{x_1}\right) \left(1 - \frac{x_1 + x_0}{R}\right) (R^2 - (x_1 - x_0)^2) \\
&{\scriptstyle \rm ppN}\Big| \qquad - \frac{1}{2}\alpha\epsilon \frac{\boldsymbol{k}\cdot\boldsymbol{x}_1}{R x_1^4} \\
&{\scriptstyle \rm ppN}\Big| \qquad - \frac{1}{4}(8(1+\gamma-\alpha\gamma)(1+\gamma) - 4\alpha\beta + 3\alpha\epsilon) \frac{\boldsymbol{k}\cdot\boldsymbol{x}_1}{x_1^2} \frac{R}{|\boldsymbol{x}_1 \times \boldsymbol{x}_0|^2} \\
&{\scriptstyle \rm ppN}\Big| \qquad - \frac{1}{4}(8(1+\gamma-\alpha\gamma)(1+\gamma) - 4\alpha\beta + 3\alpha\epsilon) \frac{R^2}{|\boldsymbol{x}_1 \times \boldsymbol{x}_0|^3} (\pi - \delta(\boldsymbol{k}, \boldsymbol{x}_1)) \Bigg] \\
&\quad + \mathcal{O}(c^{-6}).
\end{aligned} \tag{81}
$$

This expression allows one to compute the difference between the vectors $\boldsymbol{n}$ and $\boldsymbol{\sigma}$ starting from the boundary conditions $\boldsymbol{x}_0$ and $\boldsymbol{x}_1$. Let us estimate the magnitude of the individual terms in (81) in the angle $\delta(\boldsymbol{\sigma}, \boldsymbol{n})$ between $\boldsymbol{n}$ and $\boldsymbol{\sigma}$. This angle can be computed from the vector product $\boldsymbol{\varphi} = \boldsymbol{\sigma} \times \boldsymbol{n}$. Again the term in (81) proportional to $\boldsymbol{k}$ and labeled as 'scaling' plays no role since $\boldsymbol{\sigma} \times \boldsymbol{k} = \mathcal{O}(c^{-2})$. In order to estimate the effects of the other terms in (81), we split $\boldsymbol{\varphi} = \boldsymbol{\varphi}_{\rm pN} + \boldsymbol{\varphi}_{\Delta\rm pN} + \boldsymbol{\varphi}_{\rm ppN}$ similarly as we did with $\boldsymbol{\rho}$ above, take into account that $|\boldsymbol{\sigma} \times (\boldsymbol{k} \times (\boldsymbol{x}_0 \times \boldsymbol{x}_1))| = Rd + \mathcal{O}(c^{-2})$, assume again $\alpha = \beta = \gamma = \epsilon = 1$ and get [21, 22]

$$
|\boldsymbol{\varphi}_{\rm pN}| = 2m |\boldsymbol{\sigma} \times [\boldsymbol{k} \times (\boldsymbol{x}_0 \times \boldsymbol{x}_1)]| \frac{R}{|\boldsymbol{x}_1 \times \boldsymbol{x}_0|^2} \left(1 + \frac{\boldsymbol{k}\cdot\boldsymbol{x}_1}{x_1}\right) \leqslant 4 \frac{m}{d}, \tag{82}
$$

$$
\begin{aligned}
|\boldsymbol{\varphi}_{\Delta\rm pN}| &= 4m^2 |\boldsymbol{\sigma} \times [\boldsymbol{k} \times (\boldsymbol{x}_0 \times \boldsymbol{x}_1)]| \left(1 + \frac{\boldsymbol{k}\cdot\boldsymbol{x}_1}{x_1}\right) \frac{R(x_1 + x_0)}{|\boldsymbol{x}_1 \times \boldsymbol{x}_0|^4} \frac{R^2 - (x_1 - x_0)^2}{2} \\
&\leqslant 4 \frac{m^2}{d^2} \frac{4 x_1 x_0}{d(x_1 + x_0)} \leqslant 16 \frac{m^2}{d^2} \frac{x_1}{d},
\end{aligned} \tag{83}
$$

$$
|\boldsymbol{\varphi}_{\rm ppN}| \leqslant \frac{15}{4} \pi \frac{m^2}{d^2}. \tag{84}
$$

Equation (84) shows that the 'ppN' terms can attain 1 μas only if one observes within approximately 3.3 angular radii from the Sun. In many cases these terms can be neglected. Accordingly, we obtain a simplified formula for the transformation from $\boldsymbol{\sigma}$ to $\boldsymbol{n}$ keeping only the post-Newtonian and 'enhanced' post-post-Newtonian terms labeled as 'pN' and 'ΔpN' in (81):

$$\boldsymbol{n} = \boldsymbol{\sigma} + \boldsymbol{d}T\left(1 + T x_1 \frac{R + x_0 - x_1}{R + x_0 + x_1}\right) + \mathcal{O}\left(\frac{m^2}{d^2}\right) + \mathcal{O}(m^3), \tag{85}$$

$$T = -(1+\gamma)\frac{m}{d^2}\left(1 + \frac{\boldsymbol{k}\cdot\boldsymbol{x}_1}{x_1}\right). \tag{86}$$

Equation (83) can be used as a criterion if the additional post-post-Newtonian term in (85) is necessary for a given accuracy and configuration.

6.5. Transformation from $\boldsymbol{k}$ *to* $\boldsymbol{n}$

Finally, a direct relation between the vectors $\boldsymbol{k}$ and $\boldsymbol{n}$ should be derived. To this end, we combine (74) and (81) to get

$$\begin{aligned}
{}_{\rm N}\Big|\quad & \boldsymbol{n} = \boldsymbol{k} \\
{}_{\rm pN}\Big|\quad & - (1+\gamma) m \frac{\boldsymbol{k}\times(\boldsymbol{x}_0\times\boldsymbol{x}_1)}{x_1(x_1x_0 + \boldsymbol{x}_1\cdot\boldsymbol{x}_0)} \\
{}_{\Delta\rm pN}\Big|\quad & + (1+\gamma)^2 m^2 \frac{\boldsymbol{k}\times(\boldsymbol{x}_0\times\boldsymbol{x}_1)}{(x_1x_0 + \boldsymbol{x}_1\cdot\boldsymbol{x}_0)^2}\frac{x_1+x_0}{x_1} \\
{}_{\rm scaling}\Big|\quad & - \frac{1}{8}(1+\gamma)^2\frac{m^2}{x_1^2}\boldsymbol{k}\frac{((x_1-x_0)^2 - R^2)^2}{|\boldsymbol{x}_1\times\boldsymbol{x}_0|^2} \\
{}_{\rm ppN}\Big|\quad & + m^2 \boldsymbol{k}\times(\boldsymbol{x}_0\times\boldsymbol{x}_1)\Bigg[\frac{1}{2}(1+\gamma)^2\frac{R^2 - (x_1-x_0)^2}{x_1^2|\boldsymbol{x}_1\times\boldsymbol{x}_0|^2} \\
{}_{\rm ppN}\Big|\quad & \qquad + \frac{1}{4}\alpha\epsilon\frac{1}{R}\left(\frac{1}{Rx_0^2} - \frac{1}{Rx_1^2} - 2\frac{\boldsymbol{k}\cdot\boldsymbol{x}_1}{x_1^4}\right) \\
{}_{\rm ppN}\Big|\quad & \qquad - \frac{1}{4}(8(1+\gamma-\alpha\gamma)(1+\gamma) - 4\alpha\beta + 3\alpha\epsilon)R\frac{\boldsymbol{k}\cdot\boldsymbol{x}_1}{x_1^2|\boldsymbol{x}_1\times\boldsymbol{x}_0|^2} \\
{}_{\rm ppN}\Big|\quad & \qquad + \frac{1}{8}(8(1+\gamma-\alpha\gamma)(1+\gamma) - 4\alpha\beta + 3\alpha\epsilon)\frac{x_1^2 - x_0^2 - R^2}{|\boldsymbol{x}_1\times\boldsymbol{x}_0|^3}\delta(\boldsymbol{x}_1,\boldsymbol{x}_0)\Bigg] \\
& + \mathcal{O}(c^{-6}).
\end{aligned} \tag{87}$$

This formula allows one to compute the unit coordinate direction of light propagation $\boldsymbol{n}$ at the point of reception starting from the positions of the source $\boldsymbol{x}_0$ and the observer $\boldsymbol{x}_1$.

As in other cases our goal now is to estimate the effect of the individual terms in (87) on the angle $\delta(\boldsymbol{k},\boldsymbol{n})$ between $\boldsymbol{k}$ and $\boldsymbol{n}$. This angle can be computed from the vector product $\boldsymbol{\omega} = \boldsymbol{k}\times\boldsymbol{n}$. The term in (87) proportional to $\boldsymbol{k}$ and labeled by 'scaling' obviously plays no role here and can be ignored. For the other terms in $\boldsymbol{\omega} = \boldsymbol{\omega}_{\rm pN} + \boldsymbol{\omega}_{\Delta\rm pN} + \boldsymbol{\omega}_{\rm ppN}$ taking into account (75) and considering the general-relativistic values $\alpha = \beta = \gamma = \epsilon = 1$ one gets [21, 22]

$$|\boldsymbol{\omega}_{\rm pN}| = 2m\frac{1}{x_1}\frac{|\boldsymbol{k}\times(\boldsymbol{x}_0\times\boldsymbol{x}_1)|}{x_1x_0 + \boldsymbol{x}_1\cdot\boldsymbol{x}_0} \leqslant 4\frac{m}{d}\frac{x_0}{x_1+x_0} \leqslant 4\frac{m}{d}, \tag{88}$$

$$\begin{aligned}
|\boldsymbol{\omega}_{\Delta\rm pN}| &= 4m^2\frac{x_1+x_0}{x_1}\frac{|\boldsymbol{k}\times(\boldsymbol{x}_0\times\boldsymbol{x}_1)|}{(x_1x_0 + \boldsymbol{x}_1\cdot\boldsymbol{x}_0)^2} \\
&\leqslant 16\frac{m^2}{d^3}\frac{Rx_1x_0^2}{(x_1+x_0)^3} \leqslant 16\frac{m^2}{d^3}\frac{x_1x_0^2}{(x_1+x_0)^2} \leqslant 16\frac{m^2}{d^2}\frac{x_1}{d},
\end{aligned} \tag{89}$$

or, alternatively,

$$|\boldsymbol{\omega}_{\Delta\text{pN}}| \leqslant \frac{64}{27}\frac{m^2}{d^2}\frac{R}{d}. \tag{90}$$

We give four possible estimates of $|\boldsymbol{\omega}_{\Delta\text{pN}}|$. These estimates can be useful in different situations. Note that the last estimate in (89) and the estimate in (90) cannot be related to each other and reflect different properties of $|\boldsymbol{\omega}_{\Delta\text{pN}}|$ as a function of multiple variables.

The effect of all the 'ppN' terms in (87) can be estimated as

$$|\boldsymbol{\omega}_{\text{ppN}}| \leqslant \frac{15}{4}\pi\frac{m^2}{d^2}. \tag{91}$$

Again these terms can attain 1 μas only for observations within about 3.3 angular radii from the Sun and can be neglected. Accordingly, we obtain a simplified formula for the transformation from $\boldsymbol{k}$ to $\boldsymbol{n}$ keeping only the post-Newtonian and 'enhanced' post-post-Newtonian terms labeled as 'pN' and 'ΔpN' in (87):

$$\boldsymbol{n} = \boldsymbol{k} + \boldsymbol{d}P\left(1 + P x_1 \frac{x_0 + x_1}{R}\right) + \mathcal{O}\left(\frac{m^2}{d^2}\right) + \mathcal{O}(m^3), \tag{92}$$

$$P = -(1+\gamma)\frac{m}{d^2}\left(\frac{x_0 - x_1}{R} + \frac{\boldsymbol{k}\cdot\boldsymbol{x}_1}{x_1}\right). \tag{93}$$

Let us also note that the post-post-Newtonian term in (92) is maximal for sources at infinity:

$$\begin{aligned} |\boldsymbol{\omega}_{\Delta\text{pN}}| \leqslant \lim_{x_0\to\infty} |\boldsymbol{\omega}_{\Delta\text{pN}}| &= \lim_{x_0\to\infty} (1+\gamma)^2 m^2 \frac{x_0 + x_1}{x_1} \frac{\boldsymbol{k}\times(\boldsymbol{x}_0\times\boldsymbol{x}_1)|}{(x_1 x_0 + \boldsymbol{x}_1\cdot\boldsymbol{x}_0)^2} \\ &= (1+\gamma)^2 (1-\cos\Phi)^2 \frac{m^2}{d^2}\frac{x_1}{d}, \end{aligned} \tag{94}$$

where $\Phi = \delta(\boldsymbol{x}_0, \boldsymbol{x}_1)$ is the angle between the vectors $\boldsymbol{x}_0$ and $\boldsymbol{x}_1$. Several useful estimates of this term are given by (89) and (90). These estimates can be used as a criterion which allows one to decide if the post-post-Newtonian correction is important for a particular situation.

6.6. Transformation from $\boldsymbol{k}$ *to* $\boldsymbol{n}$ *for stars and quasars*

In principle, the formulae for the boundary problem given above are also valid for stars and quasars. However, for sufficiently large x_0 the formulae could be simplified. It is the purpose of this section to derive the formulae for this case.

6.6.1. Transformation from $\boldsymbol{k}$ *to* $\boldsymbol{\sigma}$*.* First, let us show that for stars and quasars the approximation

$$\boldsymbol{\sigma} = \boldsymbol{k} \tag{95}$$

is valid for an accuracy of 1 μas. Using estimates (76) and (77) for the two terms in (79) one can see that for $x_0 \gg x_1$ the angle $\delta(\boldsymbol{\sigma}, \boldsymbol{k})$ can be estimated as

$$\delta(\boldsymbol{\sigma}, \boldsymbol{k}) \leqslant 4\frac{m}{d}\frac{x_1}{x_1 + x_0}\left(1 + 4\frac{m}{d}\frac{x_1}{d}\frac{x_0}{x_1 + x_0}\right). \tag{96}$$

Clearly, $\delta(\boldsymbol{\sigma}, \boldsymbol{k})$ goes to zero for $x_0 \to \infty$. The numerical values of this upper estimate are given in table 2 for x_0 equal to 1, 10 and 100 pc. The angle $\delta(\boldsymbol{\sigma}, \boldsymbol{k})$ is smaller for stars at larger distances. However, for objects with $x_0 < 1$ pc the difference between $\boldsymbol{\sigma}$ and $\boldsymbol{k}$ must

Table 2. Numerical values of estimate (96) in μas for the angle between $\boldsymbol{\sigma}$ and $\boldsymbol{k}$ due to the solar system bodies for various values of x_0.

x_0 (pc)	Sun	Sun at 45°	Jupiter	Saturn	Uranus	Neptune
1	8.506	0.056	0.473	0.309	0.212	0.382
10	0.851	0.006	0.047	0.031	0.021	0.038
100	0.085	0.001	0.004	0.003	0.002	0.004

be explicitly taken into account. From the point of view of the relativistic model these objects should be treated in the same way as solar system objects.

6.6.2. Transformation from σ to $\boldsymbol{n}$. As soon as we accept the equality of $\boldsymbol{\sigma}$ and $\boldsymbol{k}$ for stars the only relevant step is the transformation between $\boldsymbol{\sigma}$ and $\boldsymbol{n}$. This transformation in the post-post-Newtonian approximation is given by (53) and (54). In the framework of the relativistic light deflection model, the distances to stars and quasars are assumed to be unknown and so large that they can be considered infinitely large. For such sources it is natural to use the observer's position $\boldsymbol{x}_1$ as initial position denoted in (23) as $\boldsymbol{x}_0$. Therefore, in (55) and (61) one should formally replace $\boldsymbol{x}_0$ by $\boldsymbol{x}_1$. For example, the impact parameter $\boldsymbol{d}_\sigma$ is defined as

$$\boldsymbol{d}_\sigma = \boldsymbol{\sigma} \times (\boldsymbol{x}_1 \times \boldsymbol{\sigma}). \tag{97}$$

We can rewrite (53) and (54) as

$$\begin{aligned}
\text{N}\Big|\quad & \boldsymbol{n} = \boldsymbol{\sigma} \\
\text{pN}\Big|\quad & - (1+\gamma) m \frac{\boldsymbol{d}_\sigma}{d_\sigma^2}\left(1 + \frac{\boldsymbol{\sigma}\cdot\boldsymbol{x}_1}{x_1}\right) \\
\Delta\text{pN}\Big|\quad & + (1+\gamma)^2 m^2 \frac{\boldsymbol{d}_\sigma}{d_\sigma^3}\frac{x_1}{d_\sigma}\left(1 + \frac{\boldsymbol{\sigma}\cdot\boldsymbol{x}_1}{x_1}\right)^2 \\
\text{scaling}\Big|\quad & - \frac{1}{2} m^2 (1+\gamma)^2 \frac{\boldsymbol{\sigma}}{d_\sigma^2}\left(1 + \frac{\boldsymbol{\sigma}\cdot\boldsymbol{x}_1}{x_1}\right)^2 \\
\text{ppN}\Big|\quad & - \frac{1}{2} m^2 \alpha\epsilon \frac{\boldsymbol{\sigma}\cdot\boldsymbol{x}_1}{x_1^4}\boldsymbol{d}_\sigma \\
\text{ppN}\Big|\quad & + (1+\gamma)^2 m^2 \frac{\boldsymbol{d}_\sigma}{d_\sigma^2}\frac{1}{x_1}\left(1 + \frac{\boldsymbol{\sigma}\cdot\boldsymbol{x}_1}{x_1}\right) \\
\text{ppN}\Big|\quad & - \frac{1}{4}(8(1+\gamma-\alpha\gamma)(1+\gamma) - 4\alpha\beta + 3\alpha\epsilon) m^2 \frac{\boldsymbol{d}_\sigma}{d_\sigma^2}\frac{\boldsymbol{\sigma}\cdot\boldsymbol{x}_1}{x_1^2} \\
\text{ppN}\Big|\quad & - \frac{1}{4}(8(1+\gamma-\alpha\gamma)(1+\gamma) - 4\alpha\beta + 3\alpha\epsilon) m^2 \frac{\boldsymbol{d}_\sigma}{d_\sigma^3}(\pi - \delta(\boldsymbol{\sigma}, \boldsymbol{x}_1)) \\
& + \mathcal{O}(m^3),
\end{aligned} \tag{98}$$

where $d_\sigma = |\boldsymbol{d}_\sigma| = |\boldsymbol{\sigma}\times\boldsymbol{x}_1|$. Now we need to estimate the effect of the individual terms in (98) on the angle $\delta(\boldsymbol{\sigma}, \boldsymbol{n})$ between $\boldsymbol{\sigma}$ and $\boldsymbol{n}$. This angle can be computed from the vector product $\boldsymbol{\psi} = \boldsymbol{\sigma} \times \boldsymbol{n}$. The term in (98) proportional to $\boldsymbol{\sigma}$ and labeled as 'scaling' obviously plays no role and can be ignored. For the other terms in $\boldsymbol{\psi} = \boldsymbol{\psi}_{\rm pN} + \boldsymbol{\psi}_{\Delta\rm pN} + \boldsymbol{\psi}_{\rm ppN}$ taking into account

Table 3. Numerical values of the analytical upper estimates of the post-post-Newtonian terms of the order of $\mathcal{O}(m^2/d)$ in (69) and $\mathcal{O}(m^2/d^2)$ in (74), (81), (87) and (98).

	Sun	Sun at 45°	Jupiter	Saturn	Uranus	Neptune
$\lvert c\delta\tau_{\rm ppN}\rvert$ (10^{-6} m)	36 906.0	242.9	0.328	0.036	0.002	0.003
$\lvert\boldsymbol{\rho}_{\rm ppN}\rvert$, $\lvert\varphi_{\rm ppN}\rvert$, $\lvert\boldsymbol{\omega}_{\rm ppN}\rvert$, $\lvert\psi_{\rm ppN}\rvert$ (10^{-3} μas)	10 937.4	0.474	0.945	0.120	0.016	0.023

that $|\boldsymbol{\sigma} \times \boldsymbol{d}_\sigma| = d_\sigma$ and considering the general-relativistic values $\alpha = \beta = \gamma = \epsilon = 1$ we get [21, 22]

$$|\psi_{\rm pN}| = 2m\frac{|\boldsymbol{\sigma} \times \boldsymbol{d}_\sigma|}{d_\sigma^2}\left(1 + \frac{\boldsymbol{\sigma}\cdot\boldsymbol{x}_1}{x_1}\right) \leqslant 4\frac{m}{d_\sigma}, \tag{99}$$

$$|\psi_{\Delta\rm pN}| = 4m^2\frac{|\boldsymbol{\sigma} \times \boldsymbol{d}_\sigma|}{d_\sigma^3}\frac{x_1}{d_\sigma}\left(1 + \frac{\boldsymbol{\sigma}\cdot\boldsymbol{x}_1}{x_1}\right)^2 \leqslant 16\frac{m^2}{d_\sigma^2}\frac{x_1}{d_\sigma}, \tag{100}$$

$$|\psi_{\rm ppN}| \leqslant \frac{15}{4}\pi\frac{m^2}{d_\sigma^2}. \tag{101}$$

The estimate shows that the 'ppN' terms can be neglected at the level of 1 μas except for the observations within about 3.3 angular radii from the Sun. Omitting these terms one gets an expression valid at the level of 1 μas in all other cases:

$$\boldsymbol{n} = \boldsymbol{\sigma} + \boldsymbol{d}_\sigma\, Q(1 + Qx_1) + \mathcal{O}\left(\frac{m^2}{d_\sigma^2}\right) + \mathcal{O}(m^3), \tag{102}$$

$$Q = -(1+\gamma)\frac{m}{d_\sigma^2}\left(1 + \frac{\boldsymbol{\sigma}\cdot\boldsymbol{x}_1}{x_1}\right). \tag{103}$$

This coincides with (92) and (93) and with (85) and (86) for $x_0 \to \infty$. This formula together with $\boldsymbol{\sigma} = \boldsymbol{k}$ can be applied for sources at distances larger than 1 pc to attain the accuracy of 1 μas. Alternatively, equations (92) and (93) can be used for the same purpose giving slightly better accuracy for very close stars. However, distance information (parallax) is necessary to use (92) and (93).

6.7. Numerical estimates and Monte-Carlo simulations

Table 3 contains numerical values of the 'regular' post-post-Newtonian terms of order $\mathcal{O}(m^2/d)$ in (69) and of order $\mathcal{O}(m^2/d^2)$ in (74), (81), (87) and (98). The analytical estimates are given by (71), (78), (84), (91) and (101), respectively. One can see that at the level of 10 cm in distances and 1 μas in angles these terms are irrelevant except for the observations within 3.3 angular radii from the Sun.

A series of additional Monte-Carlo simulations using randomly chosen boundary conditions has been performed to verify the given estimates of the post-post-Newtonian terms numerically. The results of these simulations fully confirm all our estimates.

Using estimate (94) and the parameters of the solar system bodies given in table 1, one can compute the maximal values of the 'enhanced' post-post-Newtonian term in the transformation from $\boldsymbol{k}$ to $\boldsymbol{n}$. For grazing rays one can apply $\cos\Phi \simeq -1$, while for the Sun at 45° one can apply $\cos\Phi \simeq -1/\sqrt{2}$. The results are shown in table 4. Comparing these values with those in the last line of table 1 one sees that the 'enhanced' post-post-Newtonian terms match the

Table 4. Maximal numerical value (94) of the 'enhanced' post-post-Newtonian term in (92) for the solar system bodies with the parameters given in table 1.

	Sun	Sun at 45°	Jupiter	Saturn	Uranus	Neptune
max $\lvert\omega_{\Delta\mathrm{pN}}\rvert$ (μas)	3192.8	6.63×10^{-4}	16.11	4.42	2.58	5.83

error of the standard post-Newtonian formula. The deviation for a grazing ray to the Sun is a few μas and originates from the post-post-Newtonian terms neglected in (92).

The vector $\boldsymbol{n}$ computed using (92) can be denoted as $\boldsymbol{n}'_{\mathrm{pN}}$. The numerical validity of $\boldsymbol{n}'_{\mathrm{pN}}$ can be confirmed by the direct comparisons of $\boldsymbol{n}'_{\mathrm{pN}}$ and vector $\boldsymbol{n}$ computed using numerical integrations of the geodetic equations as discussed in section 4.2. For example, the results for Jupiter show that the error of $\boldsymbol{n}'_{\mathrm{pN}}$ does not exceed 0.04 μas. The origin of this small deviation is well understood and will be discussed elsewhere.

7. Physical origin of the 'enhanced' post-post-Newtonian terms

We have found above the estimates of various terms in the transformations between the units vectors $\boldsymbol{\sigma}$, $\boldsymbol{n}$, and $\boldsymbol{k}$ characterizing light propagation. These estimates reveal that in each transformation 'enhanced' post-post-Newtonian terms exist that can become much larger than other 'regular' post-post-Newtonian terms. In each case the sum of the 'regular' post-post-Newtonian terms can be estimated as $\frac{15}{4}\pi\frac{m^2}{d^2}$. The 'enhanced' terms can be much larger, being, however, of analytical order m^2. In this section we clarify the physical origin of the 'enhanced' terms.

First, let us note that the 'enhanced' post-post-Newtonian terms in (74), (81), (87) and (98) contain only the parameter γ. It is clear that these terms come from the post-Newtonian terms in the metric and in the equations of motion (parameter α does not appear in these terms; see section 5.1.5). Therefore, their origin is the formal second-order (post-post-Newtonian) solution of the first-order (post-Newtonian) equations given by the first line of (39).

Now let us demonstrate that the 'enhanced' terms result from an inadequate choice of the impact parameter $\boldsymbol{d}$ or $\boldsymbol{d}_\sigma$ in the standard post-Newtonian formulae. Indeed, we can demonstrate that the 'enhanced' terms disappear if the light deflection formulae are expressed through the coordinate-independent impact parameter $\boldsymbol{d}'$ defined by (57). Equations (79), (80), (85), (86), (92), (93), (102) and (103) can be written as

$$\boldsymbol{\sigma} = \boldsymbol{k} + \boldsymbol{d}' S' + \mathcal{O}\left(\frac{m^2}{d^2}\right) + \mathcal{O}(m^3), \tag{104}$$

$$S' = (1+\gamma)\frac{m}{d'^2}\left(1 - \frac{x_0 - x_1}{R}\right), \tag{105}$$

$$\boldsymbol{n} = \boldsymbol{\sigma} + \boldsymbol{d}' T' + \mathcal{O}\left(\frac{m^2}{d^2}\right) + \mathcal{O}(m^3), \tag{106}$$

$$T' = -(1+\gamma)\frac{m}{d'^2}\left(1 + \frac{\boldsymbol{k}\cdot\boldsymbol{x}}{x}\right), \tag{107}$$

$$\boldsymbol{n} = \boldsymbol{k} + \boldsymbol{d}' P' + \mathcal{O}\left(\frac{m^2}{d^2}\right) + \mathcal{O}(m^3), \tag{108}$$

$$P' = -(1+\gamma)\frac{m}{d'^2}\left(\frac{x_0 - x}{R} + \frac{\boldsymbol{k}\cdot\boldsymbol{x}}{x}\right), \tag{109}$$

$$\boldsymbol{n} = \boldsymbol{\sigma} + \boldsymbol{d}' Q' + \mathcal{O}\left(\frac{m^2}{d_\sigma^2}\right) + \mathcal{O}(m^3), \tag{110}$$

$$Q' = -(1+\gamma)\frac{m}{d'^2}\left(1 + \frac{\boldsymbol{\sigma}\cdot\boldsymbol{x}}{x}\right), \tag{111}$$

respectively. Therefore, in each case the 'enhanced' post-post-Newtonian terms only correct the post-Newtonian terms that use inadequate impact parameter. Let us stress, however, that for practical calculations (79), (80), (85), (86), (92), (93), (102) and (103) are more convenient.

8. Summary and concluding remarks

In this paper the numerical accuracy of the post-Newtonian and post-post-Newtonian formulae for light propagation in the parametrized Schwarzschild field has been investigated. Analytical formulae have been compared with high-accuracy numerical integrations of the geodetic equations. In this way we demonstrate that the standard post-Newtonian formulae for the boundary problem (light propagation between two given points) cannot be used at the accuracy level of 1 μas for observations performed by an observer situated within the solar system. The error of the standard formula may attain $\sim$16 μas. Detailed analysis has shown that the error is of post-post-Newtonian order $\mathcal{O}(m^2)$. On the other hand, the post-post-Newtonian terms are often thought to be of order m^2/d^2 and can be estimated to be much smaller than 1 μas in this case. To clarify this contradiction we have derived and investigated the explicit analytical post-post-Newtonian solution for light propagation. For each individual term in the relevant formulae exact analytical upper estimates have been found. It turns out that in each case there exist post-post-Newtonian terms that can become much larger than the other ones and cannot be estimated as const $\times\, m^2/d^2$. We call these terms 'enhanced' post-post-Newtonian terms. These terms depend only on γ and come from the second-order solution of the post-Newtonian equations of light propagation (equation (39) with $\alpha = 0$). For this reason one could argue that the 'enhanced' post-post-Newtonian terms should not be called 'post-post-Newtonian', but better 'm^2-terms' or similarly. The physical origin of the 'enhanced' terms is discussed in the previous section. The derived analytical solution shows that no 'regular' post-post-Newtonian terms are relevant for the accuracy of 1 μas in the conditions of planned astrometric missions (Gaia, SIM, etc). Most of the 'regular' terms come from the post-post-Newtonian terms in the metric tensor. It is not the post-Newtonian equation of light propagation (equation (39) with $\alpha = 0$) itself, but the standard analytical way to solve this equation that is responsible for the numerical error of 16 μas mentioned above.

The compact formulae for light propagation time and for the transformations between the directions $\boldsymbol{\sigma}$, $\boldsymbol{n}$ and $\boldsymbol{k}$ have been derived. The formulae are given by (73), (79), (80), (85), (86), (92), (93), (102) and (103). These formulae contain only terms (both post-Newtonian and post-post-Newtonian) that are numerically relevant at the level of 10 cm for the Shapiro delay and 1 μas for the directions for any observer situated in the solar system and not observing closer than 3.3 angular radii of the Sun.

Let us finally note that the post-post-Newtonian term in (92) and (93) is closely related to the gravitational lens formula. Here we only note that all the formulae for the boundary problem given in this paper are not valid for $d = 0$ (d always appears in the denominators of these formulae). On the other hand, the standard post-Newtonian lens equation successfully treats this case known as the Einstein ring solution. The relation between the lens approximation and the standard post-Newtonian expansion is a different topic which will be considered in a subsequent paper.

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References

[1] Bodenner J and Will C M 2003 *Am. J. Phys.* **71** 770
[2] Brumberg V A 1987 Kinematica i physika nebesnykh tel **3** 8 (in Russian)
[3] Brumberg V A 1991 *Essential Relativistic Celestial Mechanics* (Bristol: Adam Hilger)
[4] Chandrasekhar S 1983 *The Mathematical Theory of Black Holes* (Oxford: Clarendon)
[5] Cowling S A 1984 *Mon. Not. R. Astron. Soc.* **209** 415
[6] Epstein R and Shapiro I I 1980 *Phys. Rev.* D **22** 2947
[7] Hairer E, Norsett S P and Wanner G 1993 *Solving Ordinary Differential Equations 1. Nonstiff Problems* (Berlin: Springer) the software is available from http://www.unige.ch/hairer/software.html
[8] IERS Conventions 2003 *Frankfurt am Main: Verlag des Bundesamts für Kartographie und Geodäsie, 2004* (*IERS Technical Note* 32) ed D D McCarthy and G Petit 127 pp
[9] Hellings R W 1986 *Astron. J.* **91** 650
[10] Klioner S A 2003 *Astron. J.* **125** 1580
[11] Klioner S A and Zschocke S 2007 arXiv:0902.4206
[12] Klioner S A and Zschocke S 2009 arXiv:0911.2170
[13] Moyer T D 2003 *Formulation for Observed and Computed Values of Deep Space Network Data Types for Navigation* (Hoboken, NJ: Wiley) (also available as Deep Space Communications and Navigation Series, JPL Publication 00-7, 2000)
[14] Richter G W and Matzner R A 1982 *Phys. Rev.* D **26** 1219
[15] Richter G W and Matzner R A 1982 *Phys. Rev.* D **26** 2549
[16] Richter G W and Matzner R A 1983 *Phys. Rev.* D **28** 3007
[17] Soffel M H 1989 *Relativity in Astrometry, Celestial Mechanics and Geodesy* (Berlin: Springer)
[18] Teyssandier P and Le Poncin-Lafitte Chr 2008 *Class. Quantum Grav.* **25** 145020
[19] Weissman P R, McFadden L-A and Johnson T V 1999 *Encyclopedia of the Solar System* (San Diego, CA: Academic)
[20] Will C M 1993 *Theory and Experiment in Gravitational Physics* (Cambridge: Cambridge University Press)
[21] Zschocke S and Klioner S A 2009 arXiv:0904.3704
[22] Zschocke S and Klioner S A 2009 arXiv:0907.4281