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On the efficient computation of the quadrupole light deflection

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Abstract

Although the formulas for the light deflection due to the quadrupole gravitational field of deflecting bodies are well known, the formulas are rather complicated, so that massive computations of quadrupole light deflection (e.g. in the framework of astrometric survey missions like Gaia) are time-consuming. Considering an observer situated within a few million kilometers from the Earth (clearly the most practical case), we derive the simplest possible form of the relevant formulas still having a numerical accuracy of 1 μas. This form leads to simple upper estimates for the quadrupole light deflection in various cases allowing one to relate the magnitude of the actual quadrupole deflection with the corresponding monopole deflection due to the same body. These upper estimates can be used to decide if, for a given configuration, the actual quadrupole deflection should be computed for a given accuracy goal.

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1. Introduction

In the near future astrometric observations will reach an accuracy of 1 microarcsecond (μas) level. This level of accuracy requires a precise modelling of light propagation. In particular, the light deflection due to the quadrupole gravitational field of deflecting bodies should be taken into account [5]. On the other hand, the accuracy of radio and laser radar links of future missions like BepiColombo or Juno requires modelling of the light travel time at the level of millimeters. Therefore, also the Shapiro delay due to quadrupole fields is of practical interest.

Analytical formulas for quadrupole light deflection are well known. Analytical solutions of light deflection in a quadrupole gravitational field have been investigated by many authors [1, 2, 4–8, 14]. For the first time the full analytical solution for the light trajectory in a quadrupole field has been obtained in [6]. These results were confirmed by a different approach in [13]. Various generalization (higher-order multipole moments, time dependence, etc) were derived in [10–12]. The formulas suitable for high-accuracy data reduction are given e.g. in [5]. These formulas are rather complicated, so that massive computations of quadrupole light deflection are time-consuming. This represents a problem for data processing of astrometric surveys. For instance, the ESA mission Gaia (to be launched in 2012) will have to process about 10^{12} individual observations of about 10^9 distinct celestial objects. It is, therefore, obvious that efficient analytical algorithms to compute the quadrupole light deflection are mandatory. Furthermore, for observations in the solar system, the quadrupole light deflection reaches the microarcsecond level only for objects at a relatively small angular distance from giant planets. Accordingly, it is highly useful to find simple analytical formulas by means of which one can decide whether or not the quadrupole field needs to be taken into account for a given accuracy and a given geometrical configuration.

In this paper we assume that the observer is located within a few million kilometers from the Earth's orbit which is clearly the most practical case (e.g. Gaia will have a Lissajous-like orbit around the Lagrange point L_2 of the system Earth–Sun). This allows one to simplify the formulas for the quadrupole light deflection considerably. Besides these simplified formulas we derive simple analytical estimates of the quadrupole deflection allowing one to decide if the effect should be computed and taken into account for a given accuracy. We also give a strict upper estimate for the quadrupole Shapiro delay.

The paper is organized as follows. In section 2 we summarize some basics about light deflection and introduce the notation. In section 3 the full quadrupole formula in the post-Newtonian order, the simplest possible expression for stars and quasars still having an accuracy of 1 μas and simple upper estimates of the latter are presented. In section 4 the full quadrupole formula in the post-Newtonian order, the simplest possible expression for solar system objects still having an accuracy of 1 μas and criteria are presented. An improved estimation of the quadrupole Shapiro effect is given in section 5. The efficiency and correctness of the upper estimates and simplified quadrupole formulas have been investigated numerically and analytically, and the results are discussed in section 6. A summary of the findings is given in section 7.

2. Some basic formulas of light propagation

Let us summarize some basic formulas of light propagation in the post-Newtonian approximation. The geodetic equations in the post-Newtonian order are linear with respect to the metric components and, therefore, the coordinates of a photon and the derivative with respect to coordinate time t are given by [5]

$$\boldsymbol{x}(t) = \boldsymbol{x}(t_0) + c\,\boldsymbol{\sigma}\,(t - t_0) + \sum_i \Delta\boldsymbol{x}_i(t) + \mathcal{O}(c^{-4}), \qquad (1)$$

$$\dot{\boldsymbol{x}}(t) = c\,\boldsymbol{\sigma} + \sum_i \Delta\dot{\boldsymbol{x}}_i(t) + \mathcal{O}(c^{-4}). \qquad (2)$$

The sum runs over individual terms in the metric of various physical origins (e.g. the monopole gravitational field of various bodies, quadrupole fields, higher-order multipole fields, etc). Here, t_0 is the moment of emission, $\boldsymbol{x}_0 = \boldsymbol{x}(t_0)$ is the position of source and $\boldsymbol{\sigma} = \lim_{t\to-\infty} \frac{\dot{\boldsymbol{x}}(t)}{c}$ is the unit tangent vector of the light path at infinitely past. The position of the observer is $\boldsymbol{x}_1 = \boldsymbol{x}(t_1)$ and t_1 is the moment of observation. The unit coordinate direction of the light propagation at the moment of observation reads $\boldsymbol{n} = \frac{\dot{\boldsymbol{x}}(t_1)}{|\dot{\boldsymbol{x}}(t_1)|}$. In the post-Newtonian order the transformation $\boldsymbol{\sigma}$ to $\boldsymbol{n}$ reads [5, 6]

$$\boldsymbol{n} = \boldsymbol{\sigma} + \sum_i \delta\boldsymbol{\sigma}_i + \mathcal{O}(c^{-4}), \qquad \delta\boldsymbol{\sigma}_i = \boldsymbol{\sigma} \times (c^{-1}\Delta\dot{\boldsymbol{x}}_i(t_1) \times \boldsymbol{\sigma}). \tag{3}$$

The spherical symmetric part (monopole contribution) due to one massive body A and its absolute value are given by (cf equation (102) in [9])

$$\delta\boldsymbol{\sigma}^A_{\rm pN} = -(1+\gamma)\frac{G}{c^2}M_A\frac{\boldsymbol{d}_A}{d_A^2}\left(1 + \frac{\boldsymbol{\sigma}\cdot\boldsymbol{r}^A_1}{r^A_1}\right), \qquad \left|\delta\boldsymbol{\sigma}^A_{\rm pN}\right| = (1+\gamma)\frac{G}{c^2}\frac{M_A}{d_A}\left(1 + \frac{\boldsymbol{\sigma}\cdot\boldsymbol{r}^A_1}{r^A_1}\right). \tag{4}$$

Here γ is the PPN parameter, M_A is the mass of body A, c is the speed of light, G is the gravitational constant and $\boldsymbol{d}_A = \boldsymbol{\sigma} \times \left(\boldsymbol{r}^A_1 \times \boldsymbol{\sigma}\right)$ is the impact parameter, $d_A = |\boldsymbol{d}_A|$, $\boldsymbol{r}^A_1 = \boldsymbol{x}(t_1) - \boldsymbol{x}_A$, $r^A_1 = \left|\boldsymbol{r}^A_1\right|$ and $\boldsymbol{x}_A$ is the position of massive body A.

In order to consider light propagation between two given points $\boldsymbol{x}_0$ and $\boldsymbol{x}_1$ (as it is needed for the data processing for solar system objects) let us define vectors $\boldsymbol{r}^A_0 = \boldsymbol{x}(t_0) - \boldsymbol{x}_A$ and $\boldsymbol{R} = \boldsymbol{x}_1 - \boldsymbol{x}_0 = \boldsymbol{r}^A_1 - \boldsymbol{r}^A_0$ with absolute values $R = |\boldsymbol{R}|$ and $r^A_0 = \left|\boldsymbol{r}^A_0\right|$, and unit vector $\boldsymbol{k} = \boldsymbol{R}/R$. In the post-Newtonian approximation, the transformation $\boldsymbol{k}$ to $\boldsymbol{n}$ reads [5]

$$\boldsymbol{n} = \boldsymbol{k} + \sum_i \delta\boldsymbol{k}_i + \mathcal{O}(c^{-4}), \qquad \delta\boldsymbol{k}_i = \boldsymbol{k} \times ([c^{-1}\Delta\dot{\boldsymbol{x}}_i(t_1) - R^{-1}\Delta\boldsymbol{x}_i(t_1)] \times \boldsymbol{k}). \tag{5}$$

The impact parameter $\boldsymbol{d}_A$ can be computed as $\boldsymbol{d}_A = \boldsymbol{k} \times \left(\boldsymbol{r}^A_1 \times \boldsymbol{k}\right) + \mathcal{O}(c^{-2}) = \boldsymbol{k} \times \left(\boldsymbol{r}^A_0 \times \boldsymbol{k}\right) + \mathcal{O}(c^{-2})$. The spherical symmetric part (monopole contribution) due to one massive body A and its absolute value are given by (cf equation (70) in [5] or equation (24) in [9])

$$\delta\boldsymbol{k}^A_{\rm pN} = -(1+\gamma)\frac{G}{c^2}\frac{M_A}{r^A_1}\frac{\boldsymbol{k} \times \left(\boldsymbol{r}^A_0 \times \boldsymbol{r}^A_1\right)}{r^A_0 r^A_1 + \boldsymbol{r}^A_0\cdot\boldsymbol{r}^A_1}, \qquad \left|\delta\boldsymbol{k}^A_{\rm pN}\right| = (1+\gamma)\frac{G}{c^2}\frac{M_A}{r^A_1}\frac{\left|\boldsymbol{r}^A_0 \times \boldsymbol{r}^A_1\right|}{r^A_0 r^A_1 + \boldsymbol{r}^A_0\cdot\boldsymbol{r}^A_1}. \tag{6}$$

3. The quadrupole light deflection for stars and quasars

Using the expression $\Delta\dot{\boldsymbol{x}}_{\rm Q}(t_1)$ given by equation (44) of [5] one gets [8, 17]

$$\delta\boldsymbol{\sigma}_{\rm Q} = \sum_A \delta\boldsymbol{\sigma}^A_{\rm Q}, \qquad \delta\boldsymbol{\sigma}^A_{\rm Q} = \frac{1+\gamma}{2}\frac{G}{c^2}\left[\boldsymbol{\alpha}'_A\frac{\dot{\mathcal{U}}_A}{c} + \boldsymbol{\beta}'_A\frac{\dot{\mathcal{E}}_A}{c} + \boldsymbol{\gamma}'_A\frac{\dot{\mathcal{F}}_A}{c} + \boldsymbol{\delta}'_A\frac{\dot{\mathcal{V}}_A}{c}\right]. \tag{7}$$

The scalar functions and vectorial coefficients are given by equations (A.1)–(A.8) of appendix A.1. The last three terms in equation (7) can be estimated as

$$T^A_1 = \frac{1+\gamma}{2}\frac{G}{c^2}\left|\boldsymbol{\beta}'_A\frac{\dot{\mathcal{E}}_A}{c} + \boldsymbol{\gamma}'_A\frac{\dot{\mathcal{F}}_A}{c} + \boldsymbol{\delta}'_A\frac{\dot{\mathcal{V}}_A}{c}\right| \leqslant 13\frac{G}{c^2}\frac{M_A\left|J^A_2\right|P^2_A}{\left(r^{A\,\rm min}_1\right)^3}. \tag{8}$$

Here, P_A is the equatorial radius, J^A_2 is the second zonal harmonics of a massive body A and $r^{A\,\rm min}_1$ is the minimal distance between the massive body and observer. The proof of this estimation is given in [17].

It is well known (see table 1 of [5]) that the quadrupole light deflection in the solar system can achieve the level of 1μas only for the giant planets (and, possibly, the Sun). Using the parameters in table 1 we obtain from (8) that $T^A_1 \leqslant 10^{-6}\,\mu$as for all these bodies. Furthermore, numerical simulations have confirmed the correctness of the values given in table 2. Accordingly, these terms in (8) can safely be neglected at the level of a microarcsecond. Accordingly, the simplest possible expression of quadrupole light deflection for stars and

Table 1. Numerical parameters of the giant planets taken from [16]. In this table the values of $r_1^{A\,\min}$ are given under assumption that the observer is in the vicinity of the Earth's orbit. The value J_2 for the Sun is taken from [3].

Parameter	Sun	Jupiter	Saturn	Uranus	Neptune
GM_A/c^2 (m)	1476	1.40987	0.42215	0.064 473	0.076 067
J_2^A (10^{-3})	0.0002	14.697	16.331	3.516	3.538
P_A (10^6 m)	696	71.492	60.268	25.559	24.764
$r_1^{A\,\min}$ (10^{12} m)	0.147	0.59	1.20	2.59	4.31
$GM_A\,J_2^A\,P_A^2/c^2$ (10^{15} m^3)	0.143	0.106	0.025	0.000 148	0.000 165

Table 2. Maximal numerical values of the neglected terms as given by (8) and (15).

Parameter	Sun	Jupiter	Saturn	Uranus	Neptune
T_1^A (μas)	$<10^{-6}$ for Sun and giant planets				
T_2^A (μas)	1.86×10^{-3}	3.26×10^{-2}	5.32×10^{-3}	8.11×10^{-4}	5.79×10^{-5}

quasars still having an accuracy of 1 μas and valid for an observer situated within a few million kilometers of the Earth's orbit reads

$$\delta\boldsymbol{\sigma}_{\rm Q}^A = \frac{1+\gamma}{2}\,\frac{G}{c^2}\,\boldsymbol{\alpha}_A'\,\frac{\dot{\mathcal{U}}_A}{c}\,. \tag{9}$$

The simplified formula of quadrupole light deflection (9) is still a complicated expression. In order to avoid evaluation of this term for each object in the data reduction, a simple criterion is needed allowing one to decide whether or not it is necessary to compute the quadrupole light deflection for a source. The absolute value of light deflection due to the quadrupole field of objects A can be estimated as [17]

$$\left|\delta\boldsymbol{\sigma}_{\rm Q}^A\right| \leqslant \frac{9}{4}\,\frac{1+\gamma}{2}\,\frac{GM_A}{c^2}\,\left|J_2^A\right|\,\frac{P_A^2}{d_A^3}\left(1+\frac{\boldsymbol{\sigma}\cdot\boldsymbol{r}_1^A}{r_1^A}\right). \tag{10}$$

A comparison of (10) with the absolute value of the monopole deflection given by (4) gives

$$\left|\,\delta\boldsymbol{\sigma}_{\rm Q}^A\right| \leqslant \frac{9}{8}\,\left|J_2^A\right|\,\frac{P_A^2}{d_A^2}\left|\,\delta\boldsymbol{\sigma}_{\rm pN}^A\right|. \tag{11}$$

This estimate relates the quadrupole light deflection for stars and quasars to the corresponding monopole deflection. The latter is relatively large, defined by a simple formula and usually computed for each source and each deflecting body. In this case the estimate (11) can be computed at the cost of two multiplications (note that d_A is known since it is used for $\delta\boldsymbol{\sigma}_{\rm pN}^A$). In the case when $\left|\,\delta\boldsymbol{\sigma}_{\rm pN}\,\right|$ is not readily available, one can use [17]

$$\left|\delta\boldsymbol{\sigma}_{\rm Q}^A\right| \leqslant 2\,(1+\gamma)\,\frac{GM_A}{c^2}\,\left|J_2^A\right|\,\frac{P_A^2}{d_A^3} \tag{12}$$

$$\leqslant 2\,(1+\gamma)\,\frac{GM_A}{c^2}\,\left|J_2^A\right|\,\frac{1}{P_A}\,, \tag{13}$$

where we use $d_A \geqslant P_A$ and (A.1) is estimated by $|2+3\cos\alpha-\cos^3\alpha| \leqslant 4$ for α being the angle between vectors $\boldsymbol{\sigma}$ and $\boldsymbol{r}_1^A$. Estimate (13) coincides with equation (41) of [6].

4. The quadrupole light deflection for solar system objects

The quadrupole light deflection for solar system objects $\delta\boldsymbol{k}_{\mathrm{Q}}$ is defined by equations (36)–(47) and (69) of [5] and can be written as [8, 17]

$$\delta\boldsymbol{k}_{\mathrm{Q}} = \sum_A \delta\boldsymbol{k}_{\mathrm{Q}}^A, \qquad \delta\boldsymbol{k}_{\mathrm{Q}}^A = \frac{1+\gamma}{2}\frac{G}{c^2}\left[\boldsymbol{\alpha}''_A\frac{\mathcal{A}_A}{c} + \boldsymbol{\beta}''_A\frac{\mathcal{B}_A}{c} + \boldsymbol{\gamma}''_A\frac{\mathcal{C}_A}{c} + \boldsymbol{\delta}''_A\frac{\mathcal{D}_A}{c}\right]. \tag{14}$$

The scalar functions and vectorial coefficients are given in equations (A.10)–(A.17) in appendix A.2. In [17] it has been shown that the last three terms in (14) can be estimated by

$$\begin{aligned} T_2^A &= \frac{1+\gamma}{2}\frac{G}{c^2}\left|\boldsymbol{\beta}''_A\frac{\mathcal{B}_A}{c} + \boldsymbol{\gamma}''_A\frac{\mathcal{C}_A}{c} + \boldsymbol{\delta}''_A\frac{\mathcal{D}_A}{c}\right| \\ &\leqslant \left(\frac{9}{2}\frac{1}{P_A^2 r_1^{A\,\min}} + \frac{1}{P_A\left(r_1^{A\,\min}\right)^2} + \frac{19}{2}\frac{1}{\left(r_1^{A\,\min}\right)^3}\right)\frac{GM_A}{c^2}\left|J_2^A\right| P_A^2. \end{aligned} \tag{15}$$

Using the parameters given in table 1 we obtain the numerical estimates of T_2^A given in table 2. Moreover, numerical simulations have confirmed the correctness of the values for T_2^A given in table 2. In view of these numerical values, the simplest possible form of quadrupole light deflection (14) for solar system objects with an accuracy of 1 μas and valid for an observer situated within a few million kilometers of the Earth's orbit is given by

$$\delta\boldsymbol{k}_{\mathrm{Q}}^A = \frac{1+\gamma}{2}\frac{G}{c^2}\,\boldsymbol{\alpha}''_A\frac{\mathcal{A}_A}{c}. \tag{16}$$

Again, in order to avoid unnecessary computations, one needs an efficient way to estimate the magnitude of $\delta\boldsymbol{k}_{\mathrm{Q}}^A$. This can be done using the following inequality [17]:

$$\left|\delta\boldsymbol{k}_{\mathrm{Q}}^A\right| \leqslant \frac{3(1+\gamma)}{2}\frac{GM_A}{c^2}\frac{1}{d_A^2}\frac{1}{r_1^A}\left|J_2^A\right| P_A^2\frac{\left|\boldsymbol{r}_0^A\times\boldsymbol{r}_1^A\right|}{r_0^A r_1^A + \boldsymbol{r}_0^A\cdot\boldsymbol{r}_1^A}. \tag{17}$$

A comparison with (6) yields

$$\left|\delta\boldsymbol{k}_{\mathrm{Q}}^A\right| \leqslant \frac{3}{2}\frac{P_A^2}{d_A^2}\left|J_2^A\right|\left|\delta\boldsymbol{k}_{\mathrm{pN}}^A\right|. \tag{18}$$

This estimate allows one to estimate the magnitude of the quadrupole light deflection in observations of solar system objects using the monopole deflection. As was mentioned above, the monopole deflection is significantly larger and should be usually calculated for each source and each gravitating body (at least for those bodies, for which the quadrupole deflection could be sufficiently large). If $|\delta\boldsymbol{k}_{\mathrm{pN}}|$ has been calculated, the magnitude of $\delta\boldsymbol{k}_{\mathrm{Q}}^A$ can be estimated at cost of three multiplications (we note that d_A is required to compute $|\delta\boldsymbol{k}_{\mathrm{pN}}|$ and can be considered as known). If $|\delta\boldsymbol{k}_{\mathrm{pN}}|$ is not readily available, we can use [17]

$$\left|\delta\boldsymbol{k}_{\mathrm{Q}}^A\right| \leqslant 2(1+\gamma)\frac{GM_A}{c^2}\left|J_2^A\right|\frac{P_A^2}{d_A^3} \tag{19}$$

$$\leqslant 2(1+\gamma)\frac{GM_A}{c^2}\left|J_2^A\right|\frac{1}{P_A}, \tag{20}$$

where in the last estimate we have used $P_A \leqslant d_A$.

Table 3. Numerical values of estimate (26).

Parameter	Sun	Jupiter	Saturn	Uranus	Neptune
$3\left\vert J_2^A\right\vert \frac{G\,M_A}{c^2}$ (mm)	0.89	62.16	20.68	0.68	0.81

5. Shapiro effect for solar system objects

According to equation (1), the propagation time $c\tau = c\,(t_1 - t_0)$ is given by [6]

$$c\tau = R + c\sum_i \delta\tau_i + \mathcal{O}(c^{-4}), \qquad c\delta\tau_i = -\boldsymbol{k}\cdot\Delta\boldsymbol{x}_i(t_1). \tag{21}$$

The formula for the Shapiro delay due to one mass monopole with mass M_A is well known:

$$c\delta\tau^A_{\rm pN} = (1+\gamma)\,\frac{GM_A}{c^2}\,\log\frac{r_0^A + r_1^A + R}{r_0^A + r_1^A - R}. \tag{22}$$

This monopole Shapiro delay becomes unboundedly large for the growing distance R between the points of emission and observations (although it is growing logarithmically with R). The quadrupole Shapiro effect $c\delta\tau_{\rm Q} = -\boldsymbol{k}\cdot\Delta\boldsymbol{x}_{\rm Q}(t_1)$ is given by [6]

$$c\delta\tau_{\rm Q} = \sum_A c\delta\tau^A_{\rm Q}, \qquad c\delta\tau^A_{\rm Q} = \frac{1+\gamma}{2}\,\frac{G}{c^2}\,(\delta_A\mathcal{V}_A + \gamma_A\mathcal{F}_A + \beta_A\mathcal{E}_A)\,, \tag{23}$$

where the scalar functions and scalar coefficients are given in equations (A.18)–(A.23) in appendix A.3. This expression for the quadrupole Shapiro delay cannot be reasonably simplified. However, one can give a strict upper bound for the quadrupole effect in the Shapiro delay. One can demonstrate [17] that

$$\frac{G}{c^2}\,|\delta_A\mathcal{V}_A| \leqslant \frac{GM_A}{c^2}\,\left\vert J_2^A\right\vert\,\frac{P_A^2}{d_A^2}, \tag{24}$$

$$\frac{G}{c^2}\,|\gamma_A\mathcal{F}_A + \beta_A\mathcal{E}_A| \leqslant \frac{GM_A}{c^2}\,\left\vert J_2^A\right\vert\left(\frac{P_A^2}{\left(r_0^A\right)^2} + \frac{P_A^2}{\left(r_1^A\right)^2}\right). \tag{25}$$

Now since $d_A \geqslant P_A$, $r_0^A \geqslant P_A$ and $r_1^A \geqslant P_A$, we conclude that

$$\left\vert c\delta\tau^A_{\rm Q}\right\vert \leqslant 3\left\vert J_2^A\right\vert\frac{GM_A}{c^2}, \tag{26}$$

which represents a strict upper bound of quadrupole Shapiro delay and slightly improves the estimate given in equation (47) in [6]. This estimate implies that the quadrupole Shapiro delay has an upper bound that depends only on physical parameters of the massive body. Table 3 gives maximal possible quadrupole effects in the Shapiro delay for *any* positions of the source and observer.

6. Efficiency of the upper estimates

As explained above, the principal merit of the simple upper estimates for the quadrupole light deflection (equations (11)–(13) for stars and quasars and equations (18)–(20) for solar system objects) is the possibility of using them, at very low computational cost, as criteria to decide if the quadrupole deflection should be calculated or not for a given configuration and a given numerical accuracy. In this section we investigate the numerical efficiency of the criteria

Table 4. Statistical properties of the ratios r_i for two distributions of sources (see the text for further explanations).

	Random			Grazing		
Ratio	Minimum	Mean	Maximum	Minimum	Mean	Maximum
r_1	0	$\frac{40}{81}$	1	0	$\frac{16}{27}$	$\frac{8}{9}$
r_2	0	$\frac{1}{3}$	1	} 0	$\frac{2}{3}$	1
r_3	0	$\frac{1}{3}\left(P_A/r_1^A\right)^2$	1			
r_4	0	$\frac{10}{27}$	1	0	$\frac{4}{9}$	$\frac{2}{3}$
r_5	0	$\frac{1}{3}$	1	} 0	$\frac{2}{3}$	1
r_6	0	$\frac{1}{3}\left(P_A/r_1^A\right)^2$	1			

for two situations: (1) purely random homogeneous distribution of sources and the position of observer with respect to the deflecting body, and (2) light rays grazing the surface of the deflecting body, but with directions still randomly distributed with respect to the body (note that the body is not spherically symmetric and the orientation does play a role). For both of these situations we compute minimal, maximal and mean values of the ratio between the quadrupole deflection and its upper estimate. The higher the mean value the more efficient the corresponding estimate as a criterion.

For stars and quasars, starting from (11)–(13), we consider the following ratios:

$$r_1=\frac{\left|\delta\boldsymbol{\sigma}_{\rm Q}^A\right|}{\frac{9}{8}\frac{P_A^2}{d_A^2}\left|J_2^A\right|\left|\delta\boldsymbol{\sigma}_{\rm pN}^A\right|},\qquad r_2=\frac{\left|\delta\boldsymbol{\sigma}_{\rm Q}^A\right|}{4\frac{GM_A}{c^2}\left|J_2^A\right|\frac{P_A^2}{d_A^3}},\qquad r_3=\frac{\left|\delta\boldsymbol{\sigma}_{\rm Q}^A\right|}{4\frac{GM_A}{c^2}\left|J_2^A\right|\frac{1}{P_A}},\tag{27}$$

where $\delta\boldsymbol{\sigma}_{\rm Q}^A$ and $\delta\boldsymbol{\sigma}_{\rm pN}^A$ are determined by equations (9) and (4), respectively. For solar system objects, starting from equations (18)–(20), we consider the ratios

$$r_4=\frac{\left|\delta\boldsymbol{k}_{\rm Q}^A\right|}{\frac{3}{2}\frac{P_A^2}{d_A^2}\left|J_2^A\right|\left|\delta\boldsymbol{k}_{\rm pN}^A\right|},\qquad r_5=\frac{\left|\delta\boldsymbol{k}_{\rm Q}^A\right|}{4\frac{GM_A}{c^2}\left|J_2^A\right|\frac{P_A^2}{d_A^3}},\qquad r_6=\frac{\left|\delta\boldsymbol{k}_{\rm Q}^A\right|}{4\frac{GM_A}{c^2}\left|J_2^A\right|\frac{1}{P_A}},\tag{28}$$

where $\delta\boldsymbol{k}_{\rm Q}^A$ and $\delta\boldsymbol{k}_{\rm pN}^A$ are determined by equations (16) and (6), respectively.

For all six ratios it is easy to compute the minimal and maximal values analytically. Besides that, for both distributions of sources and observers, it is possible to compute analytically the mathematical expectations (i.e. the mean values) of each of six ratios r_i. These values are given in table 4. The analytical calculations have also been confirmed by direct numerical simulations in which the ratios r_i were computed for correspondingly distributed sources and positions of the observer and statistically analysed.

The minimal values of all r_i are zero. The maximal values of the ratios are 1 except for r_1 and r_4 for grazing rays. In the latter case the maximal values are less than 1. This reflects the fact that for grazing rays the numerical coefficients in (11) and (18) can be improved (but only for grazing rays and not for an arbitrary situation). The fact that no maximal values are greater than 1 confirms the validity of the estimates.

Note that the mean values of r_3 and r_6 for the random distribution depend on the ratio P_A/r_1^A of the equatorial radius P_A of the deflecting body and the distance between the body and observer r_1^A. Although this ratio can be as large as 1, it is small in typical applications where the observer is situated far from the body compared to the size of the latter. For Gaia the numerical values for the mean values of r_3 and r_6 can be computed using P_A and $r_1^{A\,\rm min}$ from table 1. For example, for Jupiter $\frac{1}{3}\left(P_A/r_1^{A\,\rm min}\right)^2=0.49\times10^{-8}$.

The efficiency of the criteria is characterized by the mean values of the ratios. Considering that the random distribution of sources is a much more realistic situation than the grazing rays we can conclude that the trivial estimates (13) and (20) leading to r_3 and r_6 are extremely inefficient: the value of quadrupole deflection is typically many orders of magnitude lower than 'predicted' by those estimates. Criteria (12) and (19) are already better: the quadrupole deflection is typically only three times lower than 'predicted' (the mean value of both r_2 and r_5 for random sources is $1/3$). It is clear, however, that the most efficient criteria are given by (11) and (18). For stars and quasars the value of the quadrupole deflection 'predicted' by (11) is only two times larger than the real value. It means that only 50% of the computations based on (11) lead to values lower than the desired numerical cut-off value and could be saved. Estimates (11) and (18) will be used for the Gaia data processing.

7. Summary

In this paper we have developed efficient numerical algorithms allowing one to compute the quadrupole light deflection with minimal computational efforts. These algorithms will be used for data processing of the ESA astrometric survey mission Gaia and can be useful in other cases. In this work we assume that the observer is situated within a few million kilometers from the Earth's orbit. This is clearly the most practical case. Other situations can be analysed along the lines of our reasoning. The main results which are valid with an accuracy of at least 1μas are as follows.

(1) Quadrupole light deflection for stars and quasars can be computed as (9).
(2) Equations (11)–(13) can be used as an *a priori* criterion if the quadrupole light deflection (9) has to be computed for a given source.
(3) Quadrupole light deflection for solar system sources can be computed as (16).
(4) Equations (18)–(20) can be used as an *a priori* criterion if the quadrupole light deflection (16) has to be computed for a given solar system object.

The efficiency of the upper estimates has been investigated numerically and analytically, and the results are shown in table 4. They demonstrate high efficiency and correctness of the upper estimates, both for randomly distributed sources and sources which generate grazing rays. According to these investigation, the most efficient upper estimate of quadrupole light deflection is (11) for stars and quasars and (18) for solar system objects. The correctness of simplified quadrupole formulas has also been shown by numerical simulations.

Additionally, we give a strict upper bound (26) for the quadrupole effect in the Shapiro delay. This upper bound can be used to decide whether or not the quadrupole Shapiro delay should be taken into account if high-accuracy ranging measurements are to be modelled, e.g. in the framework of missions like BepiColombo [15] or Juno.

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Appendix A. Explicit formulas for the quadrupole terms

We give the full expressions of coefficients and scalar functions of the quadrupole light deflection and quadrupole Shapiro effect in the post-Newtonian order, because so far they have not been presented in a refereed journal.

A.1. Light deflection for stars and quasars

The functions in (7) read

$$\frac{\dot{\mathcal{U}}_A}{c} = \frac{1}{d_A^3}\left(2 + 3\frac{\boldsymbol{\sigma}\cdot\boldsymbol{r}_1^A}{r_1^A} - \frac{\left(\boldsymbol{\sigma}\cdot\boldsymbol{r}_1^A\right)^3}{\left(r_1^A\right)^3}\right), \tag{A.1}$$

$$\frac{\dot{\mathcal{E}}_A}{c} = \frac{\left(r_1^A\right)^2 - 3\left(\boldsymbol{\sigma}\cdot\boldsymbol{r}_1^A\right)^2}{\left(r_1^A\right)^5}, \tag{A.2}$$

$$\frac{\dot{\mathcal{F}}_A}{c} = -3\; d_A\;\frac{\boldsymbol{\sigma}\cdot\boldsymbol{r}_1^A}{\left(r_1^A\right)^5}, \tag{A.3}$$

$$\frac{\dot{\mathcal{V}}_A}{c} = -\frac{1}{\left(r_1^A\right)^3}, \tag{A.4}$$

$$\alpha_A^{'\,k} = -M_{ij}^A\sigma^i\sigma^j\frac{d_A^k}{d_A} + 2M_{kj}^A\frac{d_A^j}{d_A} - 2M_{ij}^A\sigma^i\sigma^k\frac{d_A^j}{d_A} - 4M_{ij}^A\frac{d_A^i d_A^j d_A^k}{d_A^3}, \tag{A.5}$$

$$\beta_A^{'\,k} = 2M_{ij}^A\sigma^i\frac{d_A^j d_A^k}{d_A^2}\;, \tag{A.6}$$

$$\gamma_A^{'\,k} = M_{ij}^A\frac{d_A^i d_A^j d_A^k}{d_A^3} - M_{ij}^A\sigma^i\sigma^j\frac{d_A^k}{d_A}, \tag{A.7}$$

$$\delta_A^{'\,k} = -2M_{ij}^A\sigma^i\sigma^j\sigma^k + 2M_{kj}^A\sigma^j - 4M_{ij}^A\sigma^i\frac{d_A^j d_A^k}{d_A^2}. \tag{A.8}$$

Here M_{ij}^A is the symmetric and trace-free quadrupole moment of body A. For an axial symmetric body (this approximation is sufficient for the solar system and the accuracy of 1μas) one has

$$M_{ij}^A = \tfrac{1}{3}M_A P_A^2 J_2^A \mathcal{R}_A \mathrm{diag}\,(1, 1, -2)\,\mathcal{R}_A^T, \tag{A.9}$$

where $\mathcal{R}_A$ is the rotational matrix giving the orientation of the figure axis of body A (see equations (48)– (53) of [5]).

A.2. Light deflection for solar system objects

The functions in (14) read

$$\frac{\mathcal{A}_A}{c} = \frac{1}{d_A}\frac{1}{R}\left(\frac{1}{r_0^A}\frac{r_0^A + \boldsymbol{k}\cdot\boldsymbol{r}_0^A}{r_0^A - \boldsymbol{k}\cdot\boldsymbol{r}_0^A} - \frac{1}{r_1^A}\frac{r_1^A + \boldsymbol{k}\cdot\boldsymbol{r}_1^A}{r_1^A - \boldsymbol{k}\cdot\boldsymbol{r}_1^A}\right) + \frac{1}{d_A^3}\left(2 + 3\frac{\boldsymbol{k}\cdot\boldsymbol{r}_1^A}{r_1^A} - \frac{\left(\boldsymbol{k}\cdot\boldsymbol{r}_1^A\right)^3}{\left(r_1^A\right)^3}\right), \tag{A.10}$$

$$\frac{\mathcal{B}_A}{c} = \frac{1}{R}\left(\frac{\boldsymbol{k}\cdot\boldsymbol{r}_0^A}{\left(r_0^A\right)^3} - \frac{\boldsymbol{k}\cdot\boldsymbol{r}_1^A}{\left(r_1^A\right)^3}\right) + \frac{\left(r_1^A\right)^2 - 3\left(\boldsymbol{k}\cdot\boldsymbol{r}_1^A\right)^2}{\left(r_1^A\right)^5}, \tag{A.11}$$

$$\frac{\mathcal{C}_A}{c} = \frac{d_A}{R}\left(\frac{1}{\left(r_0^A\right)^3} - \frac{1}{\left(r_1^A\right)^3}\right) - 3d_A\frac{\boldsymbol{k}\cdot\boldsymbol{r}_1^A}{\left(r_1^A\right)^5}, \tag{A.12}$$

$$\frac{\mathcal{D}_A}{c} = -\frac{1}{d_A^2}\frac{1}{R}\left(\frac{\boldsymbol{k}\cdot\boldsymbol{r}_0^A}{r_0^A} - \frac{\boldsymbol{k}\cdot\boldsymbol{r}_1^A}{r_1^A}\right) - \frac{1}{\left(r_1^A\right)^3}, \tag{A.13}$$

$$\alpha_A^{\prime\prime k} = -M_{ij}^A k^i k^j \frac{d_A^k}{d_A} + 2M_{kj}^A \frac{d_A^j}{d_A} - 2M_{ij}^A k^i k^k \frac{d_A^j}{d_A} - 4M_{ij}^A \frac{d_A^i d_A^j d_A^k}{d_A^3}, \tag{A.14}$$

$$\beta_A^{\prime\prime k} = 2M_{ij}^A k^i \frac{d_A^j d_A^k}{d_A^2}, \tag{A.15}$$

$$\gamma_A^{\prime\prime k} = M_{ij}^A \frac{d_A^i d_A^j d_A^k}{d_A^3} - M_{ij}^A k^i k^j \frac{d_A^k}{d_A}, \tag{A.16}$$

$$\delta_A^{\prime\prime k} = -2M_{ij}^A k^i k^j k^k + 2M_{kj}^A k^j - 4M_{ij}^A k^i \frac{d_A^j d_A^k}{d_A^2}. \tag{A.17}$$

A.3. Shapiro delay

The functions in (23) read

$$\mathcal{E}_A = \frac{\boldsymbol{k}\cdot\boldsymbol{r}_0^A}{\left(r_0^A\right)^3} - \frac{\boldsymbol{k}\cdot\boldsymbol{r}_1^A}{\left(r_1^A\right)^3}, \tag{A.18}$$

$$\mathcal{F}_A = d_A\left(\frac{1}{\left(r_0^A\right)^3} - \frac{1}{\left(r_1^A\right)^3}\right), \tag{A.19}$$

$$\mathcal{V}_A = -\frac{1}{d_A^2}\left(\frac{\boldsymbol{k}\cdot\boldsymbol{r}_0^A}{r_0^A} - \frac{\boldsymbol{k}\cdot\boldsymbol{r}_1^A}{r_1^A}\right), \tag{A.20}$$

$$\beta_A = M_{ij}^A k_i k_j - M_{ij}^A \frac{d_A^i}{d_A}\frac{d_A^j}{d_A}, \tag{A.21}$$

$$\gamma_A = 2M_{ij}^A k^i \frac{d_A^j}{d_A}, \tag{A.22}$$

$$\delta_A = M_{ij}^A k_i k_j + 2M_{ij}^A \frac{d_A^i}{d_A}\frac{d_A^j}{d_A}. \tag{A.23}$$

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