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A DETAILED PROOF OF THE FUNDAMENTAL THEOREM OF STF MULTIPOLE EXPANSION IN LINEARIZED GRAVITY

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The linearized field equations of general relativity in harmonic coordinates are given by an inhomogeneous wave equation. In the region exterior to the matter field, the retarded solution of this wave equation can be expanded in terms of 10 Cartesian symmetric and tracefree (STF) multipoles in post-Minkowskian approximation. For such a multipole decomposition only three and rather weak assumptions are required:

(1) No-incoming-radiation condition.
(2) The matter source is spatially compact.
(3) A spherical expansion for the metric outside the matter source is possible.

During the last decades, the STF multipole expansion has been established as a powerful tool in several fields of gravitational physics: celestial mechanics, theory of gravitational waves and in the theory of light propagation and astrometry. But despite its formidable importance, an explicit proof of the fundamental theorem of STF multipole expansion has not been presented so far, while only some parts of it are distributed into several publications. In a technical but more didactical form, an explicit and detailed mathematical proof of each individual step of this important theorem of STF multipole expansion is represented.

Keywords: General relativity; linearized gravity; multipole expansion.

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1. Introduction

The field equations of gravity,[1,2] constitute a set of 10 coupled nonlinear partial differential equations which relate the metric tensor $g^{\alpha\beta}$ of curved spacetime to the stress-energy tensor of matter $T^{\alpha\beta}$. Due to the inherited mathematical difficulties of solving these field equations in closed form, exact and physically well interpretable solutions of general theory of relativity are on rare occasions.[3] The most well-known examples for the case of massive isolated sources are the metric of a spherically symmetric massive body derived by Schwarzschild,[4] the solution for a spherically symmetric and electrically charged body found by Reissner[5] and Nordström,[6] and the metric for rotating bodies obtained by Kerr.[7] However, for more realistic scenarios, like an accelerated body, an asymmetric body, or a N-body system, exact solutions are far out of reach or even do not exist. Therefore, approximative approaches of general relativity are essential for further progress in the theory of gravity. One of the most important approximative approaches is the theory of linearized gravity, where in harmonic gauge the coupled field equations of Einstein's theory are simplified to a set of decoupled inhomogeneous wave equations for each of the 10 components of the metric tensor[8]:

$$\Box_x \overline{h}^{\alpha\beta}(t,\boldsymbol{x}) = -\frac{16\pi G}{c^4} T^{\alpha\beta}(t,\boldsymbol{x}), \tag{1}$$

which is valid up to order $\mathcal{O}(G^2)$ and G is the gravitational constant. In Eq. (1), $\Box_x$ is the d'Alembert operator, $\overline{h}^{\alpha\beta} = \eta^{\alpha\beta} - \sqrt{-g}g^{\alpha\beta}$ is the metric perturbation ($g =$ determinant of $g^{\alpha\beta}$, $\eta^{\alpha\beta} = \mathrm{diag}(-1,+1,+1,+1)$ is the metric of flat spacetime) and c is the speed of light; the spacetime is assumed to be covered by harmonic coordinates $(ct,\boldsymbol{x})$.

The mathematical structure of linearized field equations (1) resembles the field equations of classical electrodynamics in Lorentz gauge, $\Box_x A^\mu = -\frac{4\pi}{c} j^\mu$, with A^μ being the four-potential and j^μ being the four-current, but with the addition that in classical electrodynamics the spacetime is Minkowskian, while the spacetime in linearized gravity is in fact curved. Especially, the Green functions of both field equations are formally the same, and the harmonic coordinates $(ct,\boldsymbol{x})$ can be treated as though they were Cartesian coordinates in the flat Minkowski space, cf. Ref. 9. Hence, like in classical electrodynamics, a solution of (1) is given by[8]

$$\overline{h}^{\alpha\beta}(t,\boldsymbol{x}) = \frac{4G}{c^4}\int_V d^3x' \frac{T^{\alpha\beta}(t_{\mathrm{ret}},\boldsymbol{x}')}{|\boldsymbol{x}'-\boldsymbol{x}|}, \tag{2}$$

where the integral runs over some finite spatial volume V of the extended matter field, $t_{\mathrm{ret}} = t - \frac{|\boldsymbol{x}'-\boldsymbol{x}|}{c}$ is the retarded time from a point inside the matter source with spatial coordinate $\boldsymbol{x}'$ to a field point with spatial coordinate $\boldsymbol{x}$. The so-called advanced solution, where t_{ret} in (2) is replaced by $t_{\mathrm{adv}} = t + \frac{|\boldsymbol{x}'-\boldsymbol{x}|}{c}$, is usually be regarded unphysical because it violates the causality condition and will not be considered here.

The multipole decomposition of (2) in terms of spherical harmonics is a highly effective approach to further analyze this solution. That the tool of multipole expansion has originally been applied a long time ago in classical electrodynamics[10] and later been transformed into the case of linearized gravity. In this respect, a bench mark was the investigation of Campbell *et al.*[11] who have worked out a multipole decomposition of the scalar (gravitational potential), vectorial (electrodynamical four-potential) and tensorial (linearized gravity) field outside the matter source in terms of spherical harmonics.

However, the use of so-called Cartesian symmetric and tracefree (STF) multipole moments[12–18] instead of spherical harmonics simplifies considerably the calculations in gravitational physics[19–22]: the mathematical relations and expressions in gravitational theory become simpler, the numerical algorithms can be performed more efficiently and the whole approach of gravitational theory becomes more elegant. By now, the STF multipole expansion, in post-Newtonian approximation ("slow-motion approximation", i.e. g_{00}, g_{ij} exact to order $\mathcal{O}(c^{-2})$, g_{0i} exact to order $\mathcal{O}(c^{-3})$) and post-Minkowskian approximation ("weak-field approximation", i.e. $g_{\alpha\beta}$ exact to order $\mathcal{O}(G)$), has been established as an important tool in linearized gravity and has found a wide range of applications: in celestial mechanics,[22–24] in the theory of gravitational waves,[25–27] and in the theory of light propagation in curved spacetime[28–31] which is a fundamental aspect of relativistic astrometry. Meanwhile, the STF multipole expansion in linearized gravity has a remarkable history and encompasses some decades of period of time. Let us mention some important contributions which are considered as cornerstones in the theory of multipole expansion; further historical facts can be found, for instance, in box 1 in Ref. 19, introductory sections in Refs. 20 and 32, and in Sec. 4.4 in Ref. 33.

First, the approach developed in Ref. 11 has been established in terms of STF tensors in a pioneering work by Thorne[19] in post-Newtonian approximation, where some initial steps of earlier investigations[12,13,34,35] have considerably been generalized. Especially, Thorne[19] has shown that the metric outside the matter source can be expanded in terms of 10 STF tensors as follows (Eqs. (8.4) in Ref. 19):

$$\overline{h}^{\alpha\beta}(t,\boldsymbol{x}) = \frac{4G}{c^4}\sum_{l=0}^{\infty}\frac{(-1)^l}{l!}\partial_L\left[\frac{\hat{F}_L^{\alpha\beta}(u)}{r}\right], \tag{3}$$

where $\partial_L = \frac{\partial^l}{\partial x^{a_1}\ldots\partial x^{a_l}}$ are l spatial derivatives, $r = |\boldsymbol{x}|$ is the spatial distance between the origin of coordinate system and the field point with spatial coordinate $\boldsymbol{x}$, $\hat{F}_L^{\alpha\beta}$ are 10 STF multipoles and $u = ct - r$. Moreover, Thorne[19] has shown, using energy–momentum conservation (Eqs. (8.6) and (8.7) in Ref. 19) and a sophisticated gauge transformation (Eqs. (8.9) in Ref. 19) which preserves the harmonic gauge, that outside the matter the metric can finally be expressed in terms of two independent multipoles in post-Newtonian approximation: mass multipoles $\hat{M}_L$ and spin multipoles $\hat{S}_L$ (Eqs. (8.13) in Ref. 19). However, the multipoles (Eqs. (5.32) in Ref. 19) were still formally divergent at spatial infinity.

Consequently, Blanchet and Damour[20] have further developed the approach in Ref. 19 and have demonstrated that Thorne's post-Newtonian multipoles are physically meaningful if one makes a rigorous use of the compact-support source of energy–momentum tensor. This important result has been achieved with the aid of the theory of distributions by means of which Blanchet and Damour[20] were able to extract the physically relevant and nondivergent part of Thorne's multipoles in post-Newtonian approximation.

Finally, Blanchet and Damour[21] have established a powerful theorem in post-Minkowskian approximation which states that outside of an isolated source the metric can be expanded in terms of 10 Cartesian STF multipoles (3) (Eqs. (B.2) and (B.3) in Ref. 21), defined by

$$\hat{F}_L^{\alpha\beta}(u) = \int_V d^3x' \hat{x}'_L \int_{-1}^{+1} dz \delta_l(z) T^{\alpha\beta}\left(\frac{u + zr'}{c}, \boldsymbol{x}'\right), \tag{4}$$

where $\hat{x}'_L = \underset{a_1,\ldots,a_l}{\mathrm{STF}} (x'_{a_1}, \ldots, x'_{a_l})$, and $r' = |\boldsymbol{x}'|$ is the distance between the origin of coordinate system and a point inside of the source with spatial coordinate $\boldsymbol{x}'$; the coefficient functions in (4) are given by

$$\delta_l(z) = \frac{(2l+1)!!}{2^{l+1} l!}(1 - z^2)^l, \tag{5}$$

which are normalized: $\int_{-1}^{+1} dz \delta_l(z) = 1$. As we will see, the expansion (3) is valid in regions $r > r_0$, where r_0 is the radius of the smallest possible sphere which encompasses completely the source of matter. The expansion (3) and (4) represents the fundamental theorem of STF multipole expansion in linearized gravity, e.g. Eqs. (B.2) and (B.3) in Ref. 21, Eqs. (5.3) and (5.4) in Ref. 36, Eqs. (56) and (57) in Ref. 26, or Eq. (25) in Ref. 27, and stands for a solution of linearized field equations (1) in post-Minkowskian approximation, hence it is even valid in case of ultra-relativistic motion of matter inside the source.

After all, using energy–momentum conservation (Eqs. (5.14) and (5.18) in Ref. 36) and applying a sophisticated gauge choice (Eq. (5.31) in Ref. 36) Damour and Iyer[36] have demonstrated, footing on the pioneering works of Thorne[19] and Blanchet and Damour,[20,21] that also in post-Minkowskian approximation the family of these 10 multipoles can be reduced to finally only two independent multipoles: mass multipoles $\hat{M}_L$ (Eq. (5.33) in Ref. 36) and spin multipoles $\hat{S}_L$ (Eq. (5.35) in Ref. 36):

$$\overline{h}^{\alpha\beta} = \overline{h}^{\alpha\beta}(\hat{M}_L, \hat{S}_L). \tag{6}$$

The demonstration, that the metric in (3) which depends on 10 multipoles $\hat{F}_L^{\alpha\beta}$ can be reduced to the form in (6) where the metric depends only on two multipoles $(\hat{M}_L, \hat{S}_L)$, is a rather ambitious assignment of a task and makes extensive use of irreducible Cartesian tensor techniques originally introduced in Refs. 16–18. Damour and Iyer[36] have also demonstrated that to order $\mathcal{O}(c^{-4})$ their post-Minkowskian multipoles coincide with the post-Newtonian multipoles of Ref. 21 (Eqs. (5.38) and (5.41) in Ref. 36). So, the investigation in Ref. 36 has been the final touch in the approach of STF multipole expansion to order $\mathcal{O}(G)$. This elaborated work of Damour and Iyer[36] will, however, not be on the scope of the present investigation. Instead, we will be focussed on Theorems 3 and 4, which is the heart and the core part of STF multipole expansion. An explicit proof of this important theorem is not so straightforward as one might believe and has not been presented in detail thus far; only some parts of it are published but scattered in several publications.[11,19–21] Here, in view of its formidable relevance in the theory of linearized gravity, we will outline a more detailed mathematical proof of each individual step of multipole expansion (3) in post-Minkowskian approximation.

The paper is organized as follows: In Sec. 2, a compendium of the exact field equations of gravity is provided. The linearized approximation of general relativity is given in Sec. 3. Section 4 is devoted to the main part of our investigation, where a detailed proof of the fundamental theorem 3 is represented and the required assumptions for its validity are defined. A summary is finally given in Sec. 5.

We shall use fairly standard notations of the STF tensor approach[13,19,20,22,36]:

- Lower case Latin indices $i, j, \ldots,$ take values 1, 2, 3.
- Lower case Greek indices $\mu, \nu, \ldots,$ take values 0, 1, 2, 3.
- $\delta_{ij} = \delta^{ij} = \text{diag}(+1, +1, +1)$ is Kronecker delta.
- $n! = n(n-1)(n-2) \cdots 2 \cdot 1$ is the faculty for positive integer; $0! = 1$.
- $n!! = n(n-2)(n-4) \cdots (2 \text{ or } 1)$ is the double faculty for positive integer; $0!! = 1$.
- $L = i_1 i_2 \ldots i_l$ and $Q = i_1 i_2 \ldots i_q$ are Cartesian multi-indices of a given tensor T, that means $T_L \equiv T_{i_1 i_2 \ldots i_l}$ and $T_Q \equiv T_{i_1 i_2 \ldots i_q}$, respectively.
- two identical multi-indices imply summation: $A_L B_L \equiv \sum_{i_1 \ldots i_l} A_{i_1 \ldots i_l} B_{i_1 \ldots i_l}$.
- The symmetric part of a Cartesian tensor T_L is, cf. Eq. (2.1) in Ref. 19:

$$T_{(L)} = T_{(i_1 \ldots i_l)} = \frac{1}{l!} \sum_{\sigma} A_{i_{\sigma(1)} \ldots i_{\sigma(l)}}, \tag{7}$$

  where σ is running over all permutations of $(1, 2, \ldots, l)$.
- The symmetric tracefree part of a Cartesian tensor T_L (notation: $\hat{T}_L \equiv \text{STF}_L T_L$) is, cf. Eq. (2.2) in Ref. 19:

$$\hat{T}_L = \sum_{k=0}^{[l/2]} a_{lk} \delta_{(i_1 i_2 \ldots} \delta_{i_{2k-1} i_{2k}} S_{i_{2k+1} \ldots i_l) a_1 a_1 \ldots a_k a_k}, \tag{8}$$

  where $[l/2]$ means the largest integer less than or equal to $l/2$, and $S_L \equiv T_{(L)}$ abbreviates the symmetric part of tensor T_L. For instance, $T_L^{\alpha\beta}$ means STF with respect to indices L but not with respect to indices α, β. The coefficient in (8) is given by

$$a_{lk} = (-1)^k \frac{l!}{(l-2k)!} \frac{(2l-2k-1)!!}{(2l-1)!!(2k)!!}. \tag{9}$$

  As instructive examples of (8) let us consider the cases $l = 2$ and $l = 3$:

$$\hat{T}_{ij} = T_{(ij)} - \frac{1}{3} \delta_{ij} T_{ss}, \tag{10}$$

$$\hat{T}_{ijk} = T_{(ijk)} - \frac{1}{5} (\delta_{ij} T_{(kss)} + \delta_{jk} T_{(iss)} + \delta_{ki} T_{(jss)}). \tag{11}$$

Further STF relations can be found in Refs. 19 and 20. We also will make use of Einstein's sum convention, that mans repeated indices are implicitly summed over.

2. Einstein's Field Equations

The gravitation is described by 10 coupled nonlinear partial differential equations for the metric tensor,[1,2] which can be written in the form:

$$R^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} R = \frac{8\pi G}{c^4} T^{\alpha\beta} \tag{12}$$

and which discover a fundamental relation between the metric of spacetime on the left-hand side and the matter field on the right-hand side. Essentially, Eq. (12) represents a relation among contravariant tensors, of which $R^{\alpha\beta}$ is the Ricci curvature tensor, $g^{\alpha\beta}$ is the metric tensor with signature $(-,+,+,+)$ and $T^{\alpha\beta}$ is the energy–momentum tensor of matter; $R = R^{\alpha}_{\alpha}$ is the Ricci scalar of curvature. The field equations (12) are valid in any coordinate system, that means the coordinates are arbitrary. For an asymptotically flat spacetime, it is useful to decompose the metric tensor as follows:

$$\sqrt{-g}\, g^{\alpha\beta} = \eta^{\alpha\beta} - \overline{h}^{\alpha\beta}, \tag{13}$$

where g is the determinant of metric tensor $g^{\alpha\beta}$, $\overline{h}^{\alpha\beta}$ is the metric perturbation which describes the deviation of the metric tensor of curved spacetime from the metric tensor of Minkowskian flat spacetime given by

$$\eta_{\alpha\beta} = \eta^{\alpha\beta} = \mathrm{diag}(-1, +1, +1, +1). \tag{14}$$

In harmonic gauge, also known as de Donder gauge

$$\partial_\beta \overline{h}^{\alpha\beta} = 0, \tag{15}$$

Einstein's field equations (12) read (Eq. (36.37) in Ref. 9 and Eq. (5.2b) in Ref. 19)

$$\Box_x \overline{h}^{\alpha\beta} = -\frac{16\pi G}{c^4} (\tau^{\alpha\beta} + t^{\alpha\beta}), \tag{16}$$

where $\Box_x = \eta^{\mu\nu} \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x^\nu}$ is the d'Alembert operator. These both tensors in (16) are given by (Eq. (5.3) in Ref. 19)

$$\tau^{\alpha\beta} = (-g) T^{\alpha\beta}, \tag{17}$$

$$t^{\alpha\beta} = (-g) t^{\alpha\beta}_{\mathrm{LL}} + \frac{c^4}{16\pi G} (\overline{h}^{\alpha\mu}{}_{,\nu}\, \overline{h}^{\beta\nu}{}_{,\mu} - \overline{h}^{\alpha\beta}{}_{,\mu\nu} \overline{h}^{\mu\nu}), \tag{18}$$

where $t^{\alpha\beta}_{\mathrm{LL}}$ is the Landau–Lifschitz pseudotensor of gravitational field, in explicit form given by Eq. (20.22) in Ref. 9 or by Eqs. (96.8) and (96.9) in Ref. 37. The field equations (16) are exact and the gravitational field is not necessarily weak, because the only assumptions made so far are the decomposition (13) and the choice of a harmonic coordinate system (15).

3. Linearized Theory of Gravity and STF Multipole Expansion

The tensors (17) and (18) can be expanded in terms of the coupling constant, cf. Eqs. (3.528) and (3.529) in Ref. 33:

$$\tau^{\alpha\beta} = T^{\alpha\beta} + \frac{8\pi G}{c^4}\tau_1^{\alpha\beta} + \mathcal{O}(G^2), \tag{19}$$

$$t^{\alpha\beta} = \frac{8\pi G}{c^4} t_1^{\alpha\beta} + \mathcal{O}(G^2). \tag{20}$$

As it can be deduced from Eq. (16) and expansions (19) and (20), in harmonic gauge and up to terms of the order $\mathcal{O}(G^2)$, the Einstein's field equations are simplified to d'Alembert's wave equation for each of the 10 components of the metric tensor[8]:

$$\Box_x \overline{h}^{\alpha\beta}(t,\boldsymbol{x}) = -\frac{16\pi G}{c^4} T^{\alpha\beta}(t,\boldsymbol{x}), \tag{21}$$

which is called linearized gravity, a term which refers to the fact that the approximative field equations (21) are linear partial differential equations, to be contrary to the nonlinear exact field equations of gravity (12); the harmonic coordinates are $(t,\boldsymbol{x})$.

Actually, there are formally infinitely many solutions of wave equation (21). These solutions of (21) consist of a general solution of the homogeneous wave equation $\Box_x \overline{h}^{\alpha\beta}_{\rm hom} = 0$ plus one particular solution of inhomogeneous wave equation (21): $\overline{h}^{\alpha\beta} = \overline{h}^{\alpha\beta}_{\rm hom} + \overline{h}^{\alpha\beta}_{\rm inhom}$. For an unique solution of (21) one has to impose initial and boundary conditions. In case of an infinite spacetime, there are no boundary conditions, and a well-posed problem (i.e. existence of one and only one unique solution) is given by the initial value problem at initial time t'' (Cauchy problem):

$$\overline{h}^{\alpha\beta}_{\rm hom}(t'',\boldsymbol{x}''), \quad \frac{\partial}{\partial ct''}\overline{h}^{\alpha\beta}_{\rm hom}(t'',\boldsymbol{x}''). \tag{22}$$

These initial conditions are valid in the entire three-dimensional space. According to Kirchhoff's rigorous integration of the wave equation,[38] an unique solution of (21) and (22) is given in terms of these initial conditions by an integral over an arbitrarily shaped but sufficiently smooth surface which contains completely the field point $\boldsymbol{x}$ and the spatially compact-matter field described by the energy–momentum tensor $T^{\alpha\beta}$; an explicit expression of Kirchhoff's solution can be found, for instance, in Eq. (13) in Ref. 39. Here, without loss of generality, the surrounding surface ∂S is assumed to be the surface of a sphere S. Then, the unique solution of (21) and (22) can be written as follows:

$$\overline{h}^{\alpha\beta}(t,\boldsymbol{x}) = \overline{h}^{\alpha\beta}_{\rm hom}(t,\boldsymbol{x}) + \overline{h}^{\alpha\beta}_{\rm inhom}(t,\boldsymbol{x}), \tag{23}$$

$$\overline{h}^{\alpha\beta}_{\rm hom}(t,\boldsymbol{x}) = \frac{1}{4\pi}\int_{\partial S} d\Omega'' \left[\frac{\partial}{\partial r''}\left(r''\overline{h}^{\alpha\beta}_{\rm hom}(t'',\boldsymbol{x}'')\right) + \frac{\partial}{\partial ct''}\left(r''\overline{h}^{\alpha\beta}_{\rm hom}(t'',\boldsymbol{x}'')\right)\right], \tag{24}$$

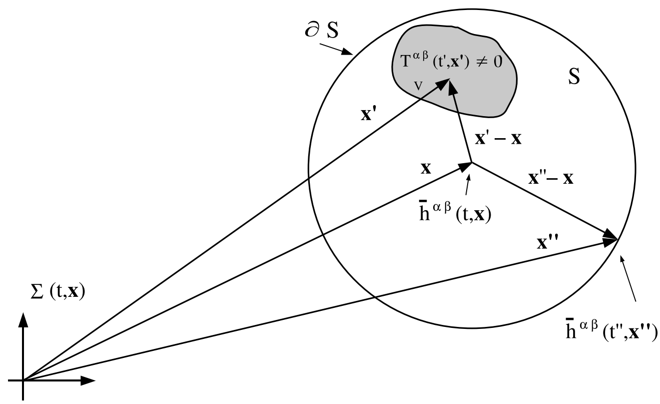

Fig. 1. Graphical representation of Kirchhoff's solution (23)–(25). The global reference frame with harmonic coordinates $(t, \boldsymbol{x})$ is denoted by $\Sigma(t, \boldsymbol{x})$. The vector $\boldsymbol{x}$ points from the origin of global coordinate system to the field point with spatial coordinate $\boldsymbol{x}$. The matter field is described by an energy–momentum tensor $T^{\alpha\beta}$. The matter field and the field point are enclosed by a virtual sphere S with surface ∂S. The center of the sphere is located at spatial coordinate $\boldsymbol{x}$ of the field point, so that the spatial distance between the field point and a point inside the matter field is given by $|\boldsymbol{x}' - \boldsymbol{x}|$, and the spatial distance between the field point and the surface ∂S is given by $|\boldsymbol{x}'' - \boldsymbol{x}|$. The matter field is assumed to be isolated, that means: (1) outside the region of some finite spatial volume V (gray colored) the matter field vanishes and (2) there is no gravitational radiation from outside through the surface ∂S of sphere S. The metric field $\overline{h}^{\alpha\beta}$ at field point $(t, \boldsymbol{x})$ and at surface point $(t'', \boldsymbol{x}'')$ has also been indicated.

$$\overline{h}^{\alpha\beta}_{\rm inhom}(t, \boldsymbol{x}) = \frac{4G}{c^4} \int_{-\infty}^{t} dt' \int_V d^3x' T^{\alpha\beta}(t', \boldsymbol{x}') \frac{\delta\left(t' - t + \dfrac{|\boldsymbol{x}' - \boldsymbol{x}|}{c}\right)}{|\boldsymbol{x}' - \boldsymbol{x}|}. \tag{25}$$

Here, $\delta(x)$ is Dirac's delta-distribution, normalized by $\int_{-\infty}^{+\infty} dx\, \delta(x) = 1$. For a graphical elucidation of Eqs. (23)–(25) see Fig. 1. According to (24), the solution of homogeneous wave equation is given by a surface integral over a sphere, while (25) is the particular solution of inhomogeneous wave equation which is called retarded solution. In (24) we use $r'' = |\boldsymbol{x}'' - \boldsymbol{x}|$, and for the retarded time between field point $\boldsymbol{x}$ and any point $\boldsymbol{x}''$ on the surface of sphere we use $t'' = t - \frac{r''}{c}$. The homogeneous solution (24) contains the initial conditions (22), that means $\overline{h}^{\alpha\beta}_{\rm hom}(t, \boldsymbol{x})$ in the whole spacetime is uniquely determined by its initial values (22) on surface ∂S; cf. Eq. (9) in Ref. 26. The surface integral is given in terms of spherical coordinates (r'', θ'', ϕ'') and the origin of the spherical coordinate system is located at the center of the sphere S, so that $d\Omega'' = \sin\theta'' d\theta'' d\phi''$. The integration in (24) runs over the surface with radius r''.

Physically, Kirchhoff's theorem[38] states that the homogeneous solution (24) is uniquely determined by source points which form a sphere with arbitrarily large radius r''. We will assume that the matter source $T^{\alpha\beta}$ in (21) is isolated, that means the source is spatially compact and does not receive any radiation from other sources far away; note, however, that the matter source itself can emit gravitational radiation. Accordingly, since the radius of sphere S can be arbitrarily large, we are allowed to take the limit up to spatial infinity and can replace the initial conditions (22) by the so-called no-incoming-radiation condition, cf. Eq. (10) in Ref. 26, and cf. Eqs. (2.5) and (2.6) in Ref. 40:

$$\lim_{\substack{r''\to+\infty \\ t''+\frac{r''}{c}=\text{const.}}} \left[\frac{\partial}{\partial r''}(r''\overline{h}^{\alpha\beta}_{\text{hom}}(\boldsymbol{x}'',t'')) + \frac{\partial}{\partial ct''}(r''\overline{h}^{\alpha\beta}_{\text{hom}}(\boldsymbol{x}'',t''))\right] = 0. \tag{26}$$

If we impose the no-incoming-radiation condition (26), then the unique solution of (21) is given by the retarded solution (25)[8]:

$$\overline{h}^{\alpha\beta}(t,\boldsymbol{x}) = \frac{4G}{c^4}\int_{-\infty}^{t} dt' \int_V d^3x' T^{\alpha\beta}(t',\boldsymbol{x}') \frac{\delta\left(t'-t+\dfrac{|\boldsymbol{x}'-\boldsymbol{x}|}{c}\right)}{|\boldsymbol{x}'-\boldsymbol{x}|}. \tag{27}$$

According to the fundamental theorem of STF multipole expansion, outside the matter field of an isolated source the retarded solution in (27) can be decomposed in terms of 10 STF multipoles: Eqs. (3) and (4). In what follows, we will present a detailed proof of key formulae of this STF multipole expansion.

4. Proof of STF Multipole Expansion

The inhomogeneous wave equation (21) is valid for any component of the tensors $\overline{h}^{\alpha\beta}$ and $T^{\alpha\beta}$, so we consider the inhomogeneous wave equation just for one of the field components:

$$\Box_x \overline{h}(t,\boldsymbol{x}) = -4\pi T(t,\boldsymbol{x}), \tag{28}$$

so that $\overline{h}$ stands either for $\overline{h}^{00}$, $\overline{h}^{0i}$ or $\overline{h}^{ij}$, while T stands either for $\frac{4G}{c^4}T^{00}$, $\frac{4G}{c^4}T^{0i}$ or $\frac{4G}{c^4}T^{ij}$, respectively. As it has been discussed above, if the source is isolated (i.e. source is spatially compact and no-incoming radiation) then there exists one and only one solution of (28), namely (cf. Eq. (27))

$$\overline{h}(t,\boldsymbol{x}) = \int_{-\infty}^{t} dt' \int_V d^3x' T(t',\boldsymbol{x}') G_R(t'-t,\boldsymbol{x}'-\boldsymbol{x}), \tag{29}$$

where the spatial integration runs over the volume V of the source and the retarded Green function is given by

$$G_R(t'-t,\boldsymbol{x}'-\boldsymbol{x}) = \frac{\delta\left(t'-t+\dfrac{|\boldsymbol{x}'-\boldsymbol{x}|}{c}\right)}{|\boldsymbol{x}'-\boldsymbol{x}|}. \tag{30}$$

The assumption that the source in (29) is spatially compact is formulated as follows (cf. text above Eq. (B1(a)) in Ref. 21, cf. text above Eq. (3.1) in Ref. 36 and cf. text in Sec. II A in Ref. 41):

$$T(t',\boldsymbol{x}') = 0, \quad \text{for } |\boldsymbol{x}'| > r_0, \tag{31}$$

where r_0 is the radius of some sphere which contains completely the source.

Consider the retarded Green function in (30), which can be expanded in a series of Legendre polynomials P_q, cf. Eqs. (D.1) and (D.2(a)) in Ref. 20:

$$G_R(t'-t,\boldsymbol{x}'-\boldsymbol{x}) = \frac{\Theta(t-t')\Theta(1-|\nu|)}{2rr'} \times \sum_{q=0}^{\infty} \frac{(2q+1)!!}{q!}\hat{n}_Q(\phi',\theta')\hat{n}_Q(\phi,\theta)P_q(\nu), \tag{32}$$

where $r = |\boldsymbol{x}|$, $r' = |\boldsymbol{x}'|$ and

$$\begin{aligned} \Theta(s) &= 0, \quad \text{for } s < 0, \\ \Theta(s) &= 1, \quad \text{for } s \geq 0, \end{aligned} \tag{33}$$

is the Heaviside step function, and

$$\nu = \frac{r^2 + r'^2 - c^2(t-t')^2}{2rr'}, \tag{34}$$

is the argument of the Legendre polynomial.

Proof 1. We will show the validity of Eq. (32). Some parts of this proof have been presented in Ref. 11 in terms of spherical harmonics, while here we present a proof in terms of STF tensors. The Legendre polynomials can be defined by (Rodrigues' formula, Eq. (12.65) in Ref. 42)

$$P_l(z) = \frac{1}{2^l l!}\frac{d^l}{dz^l}(z^2-1)^l \tag{35}$$

and the normalization is (Eq. (12.48) in Ref. 42)

$$\int_{-1}^{+1} dz P_n(z)P_m(z) = \frac{2}{2n+1}\delta_{\mathrm{nm}}. \tag{36}$$

Consider two directions given by two normalized vectors $\boldsymbol{n} = \frac{\boldsymbol{x}}{r}$, $\boldsymbol{n}' = \frac{\boldsymbol{x}'}{r'}$, with $r = |\boldsymbol{x}|$ and $r' = |\boldsymbol{x}'|$, and γ is the angle between $\boldsymbol{n}(\theta,\phi)$ and $\boldsymbol{n}'(\theta',\phi')$, that means $\cos\gamma = \boldsymbol{n}\cdot\boldsymbol{n}'$; this angle satisfies the trigonometric identity: $\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi')$ (Eq. (12.168) in Ref. 42). Then, let us consider a function $F(z,r,r')$ which depends on $z = \cos\gamma$. Further, we assume the function F to be an element of Hilbert space $V = L^2$ given by $V := L^2(z = [-1,+1];\mathcal{R})$, that means the function F is square-integrable over the surface of the unit sphere. Then, such a function F can be expanded in terms of Legendre polynomials (Eq. (12.49) in Ref. 42):

$$F(z,r,r') = \sum_{l=0}^{\infty} P_l(z)F_l(r,r'), \tag{37}$$

where the coefficients are given by (cf. Eq. (12.50) in Ref. 42; $x = \cos\gamma$)

$$F_l(r,r') = \frac{2l+1}{2}\int_{-1}^{+1} dx P_l(x)F(x,r,r'). \tag{38}$$

The Legendre polynomial addition theorem states (Eq. (8.189) or Eq. (12.170) in Ref. 42, for a detailed proof see Ref. 42 (Chapter 12.8)

$$P_l(z) = \frac{4\pi}{2l+1} \sum_{m=-l}^{l} Y_{lm}(\theta,\phi) Y^*_{lm}(\theta',\phi'). \tag{39}$$

By inserting (38) and (39) into (37), we obtain

$$F(z,r,r') = \frac{4\pi}{2} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} Y_{lm}(\theta,\phi) Y^*_{lm}(\theta',\phi') \int_{-1}^{+1} dx P_l(x) F(x,r,r'). \tag{40}$$

Now we use a relation between spherical harmonicals and STF-tensors (Eq. (2.11) in Ref. 19, or Eq. (2.19) in Ref. 22):

$$Y_{lm}(\theta,\phi) = \hat{Y}^{lm}_L \hat{n}_L(\theta,\phi), \tag{41}$$

where

$$\hat{n}_L(\theta,\phi) = \underset{i_1,i_2,\ldots,i_l}{\mathrm{STF}} \frac{x_{i_1}}{r} \frac{x_{i_2}}{r} \cdots \frac{x_{i_l}}{r} \tag{42}$$

and $n_x + i n_y = e^{i\phi}\sin\theta, n_z = \cos\theta$ (Eq. (2.10) in Ref. 19). The coefficients $\hat{Y}^{lm}_L$ (given by Eqs. (A.6(a))–(A.6(c)) in Ref. 20, or by Eq. (2.21) in Ref. 22) depend on l, m and on L, but they are independent of (θ,ϕ). Using (41) we verify

$$F(z,r,r') = \frac{4\pi}{2} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \hat{Y}^{lm}_L \hat{n}_L(\theta,\phi) Y^*_{lm}(\theta',\phi') \int_{-1}^{+1} dx P_l(x) F(x,r,r'). \tag{43}$$

By implementing the inversion of Eq. (41) (see Eq. (2.23) in Ref. 22)

$$\sum_{m=-l}^{l} \hat{Y}^{lm}_L Y^*_{lm}(\theta',\phi') = \frac{(2l+1)!!}{4\pi l!} \hat{n}_L(\theta',\phi'), \tag{44}$$

we obtain

$$F(z,r,r') = \frac{1}{2} \sum_{l=0}^{\infty} \frac{(2l+1)!!}{l!} \hat{n}_L(\theta,\phi) \hat{n}_L(\theta',\phi') \int_{-1}^{+1} dx F(x,r,r') P_l(x). \tag{45}$$

This expansion of a function of Hilbert space $V = L^2$ into a series of Legendre polynomials has been given by Eq. (A.26) in Ref. 20. According to Eq. (30), the function F as part of the integrand in Eq. (45) is given by

$$F(x,r,r') = \frac{\delta\left(t' - t + \dfrac{\sqrt{r^2 + r'^2 - 2rr'x}}{c}\right)}{\sqrt{r^2 + r'^2 - 2rr'x}}. \tag{46}$$

Note, that the Green function (46) is automatically retarded since $t \geq t'$. Using the formula

$$\delta(f(x)) = \sum_{i=1}^{n} \frac{\delta(x - \nu_i)}{|f'(\nu_i)|}, \tag{47}$$

where ν_i are the roots of f, i.e. $f(\nu_i)=0$ and $f'(\nu_i)=\frac{\partial f(x)}{\partial x}|_{x=\nu_i}$, and taking into account that the only root of $f(x)=t'-t+\frac{\sqrt{r^2+r'^2-2rr'x}}{c}$ is given by (cf. Eq. (34))

$$\nu=\frac{r^2+r'^2-c^2(t-t')^2}{2rr'}, \tag{48}$$

we can rewrite the function (46) as follows:

$$F(x,r,r')=\Theta(t-t')\Theta(1-|\nu|)\frac{c\delta(x-\nu)}{rr'}. \tag{49}$$

Here, by the Heaviside function $\Theta(t-t')$ we have taken into account the fact that the Green function (46) is retarded, i.e. $t>t'$. Moreover, since the root ν in (48) can take arbitrarily large numerical values, we have to consider the fact that $-1\leq x\leq +1$, which is taken into account by the Heaviside function $\Theta(1-|\nu|)$ in (49). By inserting relation (49) into Eq. (45) we can calculate the integral and get

$$\begin{aligned}F(\boldsymbol{n}\cdot\boldsymbol{n}',r,r') &= \frac{c}{2rr'}\Theta(t-t')\Theta(1-|\nu|)\\ &\quad\times\sum_{l=0}^{\infty}\frac{(2l+1)!!}{l!}\hat{n}_L(\theta,\phi)\hat{n}_L(\theta',\phi')P_l(\nu).\end{aligned} \tag{50}$$

This result is in agreement with Eq. (32). □

Furthermore, the source $T(t',\boldsymbol{x}')$ in (29) is expanded in spherical harmonics (cf. Eq. (B.4) in Ref. 21), which means in STF notation:

$$T(t',\boldsymbol{x}')=\sum_{l=0}^{\infty}\hat{n}_L(\phi',\theta')\hat{T}_L(t',r'), \tag{51}$$

where $\hat{T}_L$ are some STF tensorial functions, but their explicit structure is not relevant here because later the inversion of (51) will be used, see Eq. (126). Inserting the expansions (32) and (51) into (29) yields (in spherical coordinates we have $d^3x'=dr'r'^2\sin\theta' d\theta' d\phi'$ and $r=|\boldsymbol{x}|$, $r'=|\boldsymbol{x}'|$)

$$\begin{aligned}\overline{h}(t,\boldsymbol{x}) &= \frac{c}{2r}\int_{-\infty}^{t}dt'\int_V dr'r'\sum_{l=0}^{\infty}\hat{T}_L(t',r')\Theta(t-t')\Theta(1-|\nu|)\\ &\quad\times\sum_{q=0}^{\infty}\frac{(2q+1)!!}{q!}P_q(\nu)\hat{n}_Q(\phi,\theta)\int_0^{2\pi}d\theta'\sin\theta'\\ &\quad\times\int_0^{\pi}d\phi'\hat{n}_L(\phi',\theta')\hat{n}_Q(\phi',\theta').\end{aligned} \tag{52}$$

For the integration over the angles θ' and ϕ' we obtain (see Eq. (2.5) in Ref. 19)

$$\int_0^{2\pi}d\theta'\sin\theta'\int_0^{\pi}d\phi'\hat{n}_L(\phi',\theta')\hat{n}_Q(\phi',\theta')=\frac{4\pi l!}{(2l+1)!!}\delta_{lq}. \tag{53}$$

Thus, we arrive at (cf. Eq. (D.3) in Ref. 20)

$$\overline{h}(t,\boldsymbol{x}) = \frac{4\pi c}{2r}\sum_{l=0}^{\infty}\hat{n}_L(\phi,\theta)\int_{-\infty}^{t} dt' \times \int_V dr' r' \hat{T}_L(t',r') P_l(\nu)\Theta(t-t')\Theta(1-|\nu|). \tag{54}$$

Now we introduce the following four variables (cf. Eqs. (D.4(a)) and (D.4(b)) in Ref. 20) which are independent of each other:

$$u = ct - r, \quad u' = ct' - r', \tag{55}$$

$$v = ct + r, \quad v' = ct' + r'. \tag{56}$$

After coordinate transformation (55) and (56), the previous integration domain of (54), $\mathcal{D} = \{(t',r') : -\infty \le t' \le t \text{ and } 0 \le r' < r_0\}$, is given by (cf. comments above Eq. (D.5) in Ref. 20)

$$\mathcal{D} = \{(u',v') : u \le v' \le v \text{ and } u' \le u\}. \tag{57}$$

Proof 2. We will show that the integration domain $\mathcal{D}$ is given by (57). From the definition of the new variables (55) and (56) follows:

$$ct = \frac{u+v}{2}, \quad ct' = \frac{u'+v'}{2}, \tag{58}$$

$$r = \frac{v-u}{2}, \quad r' = \frac{v'-u'}{2}. \tag{59}$$

Let us consider the Heaviside function $\Theta(1-|\nu|)$, i.e. the relation:

$$\left|\frac{r^2 + {r'}^2 - c^2(t-t')^2}{2rr'}\right| \le 1. \tag{60}$$

This relation can also be written as follows:

$$(r+r')^2 \ge c^2(t-t')^2, \tag{61}$$

$$(r-r')^2 \le c^2(t-t')^2. \tag{62}$$

First, we consider condition (61). Due to Heaviside function $\Theta(t-t')$, i.e. $t \ge t'$, we can rewrite (61) as follows:

$$r + r' \ge c(t-t'). \tag{63}$$

By inserting (58) and (59) into (63) we find

$$v' \ge u. \tag{64}$$

Now let us consider condition (62), which can also be written as

$$c(t'-t) \le r - r' \le c(t-t'). \tag{65}$$

By inserting (58) and (59) into (63) we obtain from both conditions in (65):

$$v' \le v \quad \text{and} \quad u' \le u. \tag{66}$$

Finally, the conditions (64) and (66) can be summarized as follows:

$$\mathcal{D} = \{(u', v') : u \le v' \le v \text{ and } u' \le u\}, \tag{67}$$

which is just the integration domain (57). $\square$

Then, the integral (54) in these new variables (55) and (56) is given by (cf. Eq. (D.5) in Ref. 20)

$$\begin{aligned}\overline{h}(t, \boldsymbol{x}) &= \frac{4\pi}{4(v-u)} \sum_{l=0}^{\infty} \hat{n}_L(\phi, \theta) \iint_{\mathcal{D}} du' dv' (v' - u') \hat{T}_L \left(\frac{u' + v'}{2c}, \frac{v' - u'}{2} \right) \\ &\quad \times P_l \left(1 - 2 \frac{(u - u')(v - v')}{(v - u)(v' - u')} \right).\end{aligned} \tag{68}$$

Proof 3. We will show how to arrive at (68). First we note, by means of relations (58) and (59), that

$$dt' dx' = \begin{vmatrix} \dfrac{\partial t'}{\partial u'} & \dfrac{\partial t'}{\partial v'} \\ \dfrac{\partial x'}{\partial u'} & \dfrac{\partial x'}{\partial v'} \end{vmatrix} du' dv' = \frac{1}{2c} du' dv'.$$

Then, using $\frac{1}{2r} = \frac{1}{v-u}$ and $r' = \frac{v'-u'}{2}$ from (59), we obtain from (54) as intermediate step:

$$\overline{h}(t, \boldsymbol{x}) = \frac{4\pi}{4(v-u)} \sum_{l=0}^{\infty} \hat{n}_L(\phi, \theta) \iint_{\mathcal{D}} du' dv' (v' - u') \hat{T}_L(t', r') P_l(\nu), \tag{69}$$

where we also have implemented the integration domain $\mathcal{D}$ in virtue of (57). Now, for both arguments of the function $\hat{T}_L$ we use $t' = \frac{v'+u'}{2c}$ according to (58) and $r' = \frac{v'-u'}{2}$ according to (59) and obtain a further intermediate step:

$$\begin{aligned}\overline{h}(t, \boldsymbol{x}) &= \frac{4\pi}{4(v-u)} \sum_{l=0}^{\infty} \hat{n}_L(\phi, \theta) \iint_{\mathcal{D}} du' dv' (v' - u') \hat{T}_L \\ &\quad \times \left(\frac{u' + v'}{2c}, \frac{v' - u'}{2} \right) P_l(\nu).\end{aligned} \tag{70}$$

Finally, we have to reexpress the argument ν of Legendre polynomial P_l in terms of the new variables u, v, u', v'. First, from the definition of ν given by Eq. (34) and the new variables given by Eqs. (58) and (59) we get

$$\begin{aligned}\nu &= \frac{\left(\dfrac{v-u}{2}\right)^2 + \left(\dfrac{v'-u'}{2}\right)^2 - \left(\dfrac{u+v}{2} - \dfrac{u'+v'}{2}\right)^2}{2\left(\dfrac{v-u}{2}\right)\left(\dfrac{v'-u'}{2}\right)} \\ &= 1 - 2\frac{(u-u')(v-v')}{(v-u)(v'-u')},\end{aligned} \tag{71}$$

which can easily be checked. Thus, inserting (71) into (70) yields

$$\overline{h}(t,\boldsymbol{x}) = \frac{4\pi}{4(v-u)} \sum_{l=0}^{\infty} \hat{n}_L(\phi,\theta) \iint_{\mathcal{D}} du'dv'(v'-u')\hat{T}_L\left(\frac{u'+v'}{2c}, \frac{v'-u'}{2}\right)$$
$$\times P_l\left(1 - 2\frac{(u-u')(v-v')}{(v-u)(v'-u')}\right), \tag{72}$$

which is just in coincidence with expression (68). □

Then, we use the following relation for Legendre polynomial (cf. Eq. (D.6) in Ref. 20):

$$P_l\left(1 - 2\frac{(u-u')(v-v')}{(v-u)(v'-u')}\right) = \frac{(-1)^l}{l!}\frac{(v-u)^{l+1}}{(v'-u')^l}\frac{\partial^l}{\partial u^l}\left[\frac{(u-u')^l(u-v')^l}{(v-u)^{l+1}}\right]. \tag{73}$$

Proof 4. Blanchet and Damour[20] have found an elegant way to show the validity of (73) via Euler–Poisson–Darboux differential equation, see text below Eq. (D.6) in Ref. 20. Here, we will demonstrate (73) straightaway. According to Eq. (73), the Legendre polynomial under consideration is given by

$$P_l(\nu) = P_l\left(1 - 2\frac{(u-u')(v-v')}{(v-u)(v'-u')}\right). \tag{74}$$

Using Rodrigues' formula (35) we verify

$$P_l(\nu) = \frac{1}{2^l l!}\frac{d^l}{d\nu^l}(\nu^2-1)^l = \frac{2^l}{l!}\frac{d^l}{d\nu^l}\left(\frac{(u-u')^2(v-v')^2}{(v-u)^2(v'-u')^2} - \frac{(u-u')(v-v')}{(v-u)(v'-u')}\right)^l$$
$$= \frac{2^l}{l!}\frac{d^l}{d\nu^l}\left(\frac{(v-v')(v-u')}{(v'-u')^2}\frac{(u-u')(u-v')}{(v-u)^2}\right)^l. \tag{75}$$

Now we use the relation

$$\frac{d}{d\nu} = \left(\frac{du}{d\nu}\right)\frac{d}{du}, \tag{76}$$

while for any value of l we obtain

$$\frac{d^l}{d\nu^l} = \underbrace{\left[\left(\frac{du}{d\nu}\right)\frac{\partial}{\partial u}\right] \times \left[\left(\frac{du}{d\nu}\right)\frac{\partial}{\partial u}\right] \times \cdots \times \left[\left(\frac{du}{d\nu}\right)\frac{\partial}{\partial u}\right]}_{l} = \left[\left(\frac{du}{d\nu}\right)\frac{\partial}{\partial u}\right]^l. \tag{77}$$

Let us calculate the factor in (76). For that we have to reconvert

$$\nu = 1 - 2\frac{(u-u')(v-v')}{(v-u)(v'-u')}, \tag{78}$$

in terms of u, and get

$$u = \frac{u' + \dfrac{1-\nu}{2}\dfrac{v'-u'}{v-v'}v}{1 + \dfrac{1-\nu}{2}\dfrac{v'-u'}{v-v'}}. \tag{79}$$

With allowance for expression (79), we have

$$\left(\frac{du}{d\nu}\right) = \frac{1}{2}\frac{(v'-u')(v-u)^2}{(v'-v)(v-u')}. \tag{80}$$

Inserting operator (77) into (75), using (80), yields

$$P_l(\nu) = \frac{(-1)^l}{l!}\frac{1}{(v'-u')^l}\left[(v-u)^2\frac{\partial}{\partial u}\right]^l \left(\frac{(u-u')(u-v')}{(u-v)^2}\right)^l. \tag{81}$$

Now we apply the following relation, which is proven in Appendix A:

$$\left[(v-u)^2\frac{\partial}{\partial u}\right]^l \frac{(u-u')^l(u-v')^l}{(u-v)^{2l}} = (v-u)^{l+1}\frac{\partial^l}{\partial u^l}\frac{(u-u')^l(u-v')^l}{(v-u)^{l+1}}. \tag{82}$$

By inserting (82) into (81) we obtain

$$P_l(\nu) = \frac{(-1)^l}{l!}\frac{(v-u)^{l+1}}{(v'-u')^l}\frac{\partial^l}{\partial u^l}\frac{(u-u')^l(u-v')^l}{(v-u)^{l+1}}, \tag{83}$$

which represents the asserted relation (73). □

Now, by means of relation (73) and with the aid of (cf. Eq. (A.35(b)) in Ref. 20)

$$\frac{1}{l!}\hat{n}_L(\theta,\phi)(v-u)^l\frac{\partial^{2l}}{\partial u^l \partial v^l}\left(\frac{F(u)}{v-u}\right) = \frac{1}{2}\hat{\partial}_L\left(\frac{F(ct-r)}{r}\right), \tag{84}$$

Proof 5. We will show the validity of relation (84). The function F on the right-hand side in Eq. (84) does not depend explicitly on three-vector $\boldsymbol{x}$ but only on its absolute value $r = |\boldsymbol{x}|$. Therefore, it is meaningful to rewrite the differential operator $\hat{\partial}_L$ in a form where the vectorial dependence is projected out of the differential process. This can be achieved with virtue of the following relation (see also Eq. (A.30) in Ref. 20):

$$\hat{\partial}_L\left(\frac{F(ct-r)}{r}\right) = \hat{n}_L r^l \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^l \left(\frac{F(ct-r)}{r}\right). \tag{85}$$

For proofing (85) recall $\frac{\partial f(r)}{\partial x^{a_1}} = \frac{x^{a_1}}{r}\frac{\partial f(r)}{\partial r}$ and one verifies $\partial_L f(r) = n_L r^l (\frac{1}{r}\frac{\partial}{\partial r})^l f(r)$ plus terms containing at least one Kronecker delta which, however, vanish after STF operation; e.g. $\mathrm{STF}_{ab}\delta_{ab} = 0$, $\mathrm{STF}_{abc}\delta_{ab}x^c = 0$, etc.

By using the variables $u = ct - r$, $v = ct + r$, see Eqs. (55) and (56), we can rewrite (85) as follows:

$$\hat{\partial}_L\left(\frac{F(ct-r)}{r}\right) = 2\hat{n}_L(v-u)^l\left(\frac{1}{v-u}\left(\frac{\partial}{\partial v} - \frac{\partial}{\partial u}\right)\right)^l\left(\frac{F(u)}{v-u}\right), \tag{86}$$

where we have used the chain rule: $\frac{\partial}{\partial r} = (\frac{\partial v}{\partial r})\frac{\partial}{\partial v} + (\frac{\partial u}{\partial r})\frac{\partial}{\partial u} = \frac{\partial}{\partial v} - \frac{\partial}{\partial u}$. Now, we apply the following identity which can easily be proven with the aid of mathematical induction (i.e. show the validity of (87) for $l = 1$ and then prove that the validity of (87) for any one natural number l implies the validity of (87) for the next natural number $l + 1$):

$$\left(\frac{1}{v-u}\left(\frac{\partial}{\partial v} - \frac{\partial}{\partial u}\right)\right)^l \frac{F(u)}{v-u} = \frac{1}{l!}\frac{\partial^{2l}}{\partial u^l \partial v^l}\frac{F(u)}{v-u}. \tag{87}$$

By inserting (87) into (86) we obtain

$$\frac{1}{2}\hat{\partial}_L\left(\frac{F(ct-r)}{r}\right) = \hat{n}_L\frac{(v-u)^l}{l!}\frac{\partial^{2l}}{\partial v^l \partial u^l}\left(\frac{F(u)}{v-u}\right), \tag{88}$$

which is just relation (84). □

we can rewrite Eq. (68) using $\hat{\partial}_L$ as follows (cf. Eq. (D.7) in Ref. 20):

$$\overline{h}(t,\boldsymbol{x}) = \sum_{l=0}^{\infty}\frac{\pi}{2}\frac{1}{l!}\iint_{\mathcal{D}}\frac{du'dv'}{(v'-u')^{l-1}}\hat{T}_L\left(\frac{u'+v'}{2c},\frac{v'-u'}{2}\right)\hat{\partial}_L \times\left[\frac{(ct-r-u')^l(ct-r-v')^l}{r}\right]. \tag{89}$$

Proof 6. In order to obtain from Eq. (68) the expression in Eq. (89), the relation (73) is used, which yields

$$\overline{h}(t,\boldsymbol{x}) = \frac{4\pi}{4(v-u)}\sum_{l=0}^{\infty}\hat{n}_L(\phi,\theta)\iint_{\mathcal{D}}du'dv'(v'-u')\hat{T}_L\left(\frac{u'+v'}{2c},\frac{v'-u'}{2}\right) \times\frac{(-1)^l}{l!}\frac{(v-u)^{l+1}}{(v'-u')^l}\frac{\partial^l}{\partial u^l}\left[\frac{(u-u')^l(u-v')^l}{(v-u)^{l+1}}\right]. \tag{90}$$

For being able to apply relation (84), we have to rewrite the term

$$\frac{\partial^l}{\partial u^l}\left[\frac{(u-u')^l(u-v')^l}{(v-u)^{l+1}}\right]. \tag{91}$$

For doing that, we note the relation

$$\frac{\partial^l}{\partial v^l}\left[\frac{1}{v-u}\right] = \frac{(-1)^l l!}{(v-u)^{l+1}}. \tag{92}$$

By means of this relation we find for the term (91) the following expression:

$$\frac{\partial^l}{\partial u^l}\left[\frac{(u-u')^l(u-v')^l}{(v-u)^{l+1}}\right] = \frac{(-1)^l}{l!}\frac{\partial^{2l}}{\partial u^l \partial v^l}\left[\frac{(u-u')^l(u-v')^l}{v-u}\right]. \tag{93}$$

Inserting relation (93) into Eq. (90) yields

$$\overline{h}(t,\boldsymbol{x}) = \frac{4\pi}{4(v-u)} \sum_{l=0}^{\infty} \hat{n}_L(\phi,\theta) \iint_{\mathcal{D}} du' dv' (v'-u') \hat{T}_L \left(\frac{u'+v'}{2c}, \frac{v'-u'}{2}\right) \times \frac{1}{l!} \frac{(v-u)^{l+1}}{(v'-u')^l} \frac{1}{l!} \frac{\partial^{2l}}{\partial u^l \partial v^l} \left[\frac{(u-u')^l (u-v')^l}{v-u}\right], \tag{94}$$

where we have taken into account that $(-1)^l(-1)^l = 1$. Now we can apply relation (84) and obtain

$$\overline{h}(t,\boldsymbol{x}) = \sum_{l=0}^{\infty} \frac{\pi}{2} \frac{1}{l!} \iint_{\mathcal{D}} \frac{du' dv'}{(v'-u')^{l-1}} \hat{T}_L \left(\frac{u'+v'}{2c}, \frac{v'-u'}{2}\right) \hat{\partial}_L \times \left[\frac{(ct-r-u')^l (ct-r-v')^l}{r}\right], \tag{95}$$

which is just relation (89). $\square$

By means of the transformation

$$u' = s, \quad v' = s + 2y, \tag{96}$$

a straightforward calculation shows that (89) can be written as follows (cf. Eq. (D.9) in Ref. 20):

$$\overline{h}(t,\boldsymbol{x}) = \sum_{l=0}^{\infty} \frac{4\pi}{2^{l+1} l!} \int_{-\infty}^{ct-r} ds \int_{\frac{1}{2}(ct-r-s)}^{\frac{1}{2}(ct+r-s)} \frac{dy}{y^{l-1}} \hat{T}_L \left(\frac{s+y}{c}, y\right) \hat{\partial}_L \times \left[\frac{(ct-r-s)^l (ct-r-s-2y)^l}{r}\right]. \tag{97}$$

Furthermore, this expression can be written in the following form (cf. Eq. (D.8) in Ref. 20):

$$\overline{h}(t,\boldsymbol{x}) = -\sum_{l=0}^{\infty} \frac{4\pi}{2^{l+1} l!} \int_{-\infty}^{ct-r} ds \hat{\partial}_L \left[\int_a^{\frac{1}{2}(ct-r-s)} \frac{dy}{y^{l-1}} \hat{T}_L \times \left(\frac{s+y}{c}, y\right) \frac{(ct-r-s)^l (ct-r-s-2y)^l}{r} - \int_a^{\frac{1}{2}(ct+r-s)} \frac{dy}{y^{l-1}} \hat{T}_L \left(\frac{s+y}{c}, y\right) \frac{(ct+r-s)^l (ct+r-s-2y)^l}{r}\right]. \tag{98}$$

Here, we have commuted the operator $\hat{\partial}_L$ with the integrals, because all differentiations of the upper limits $\frac{1}{2}(ct - \epsilon r - s)$ with $\epsilon = \pm 1$ vanish, due to the factor $(ct - \epsilon r - s - 2y)^l$ inside the integrals.

Proof 7. We will show how to obtain (98) from (97). First, we separate the second integral in (97) into two parts as follows:

$$\begin{aligned}\overline{h}(t,\boldsymbol{x}) = &-\sum_{l=0}^{\infty}\frac{4\pi}{2^{l+1}l!}\int_{-\infty}^{ct-r}ds\int_{a}^{\frac{1}{2}(ct-r-s)}\frac{dy}{y^{l-1}}\hat{T}_L\left(\frac{s+y}{c},y\right)\hat{\partial}_L\\ &\times\left[\frac{(ct-r-s)^l(ct-r-s-2y)^l}{r}\right]\\ &+\sum_{l=0}^{\infty}\frac{4\pi}{2^{l+1}l!}\int_{-\infty}^{ct-r}ds\int_{a}^{\frac{1}{2}(ct+r-s)}\frac{dy}{y^{l-1}}\hat{T}_L\left(\frac{s+y}{c},y\right)\hat{\partial}_L\\ &\times\left[\frac{(ct-r-s)^l(ct-r-s-2y)^l}{r}\right],\end{aligned}\tag{99}$$

where a is an arbitrarily chosen constant which separates the region of integration variable y; in the first line in (99) the minus-sign in front of the integral takes into account that we have interchanged the upper and lower limits of integration. Now, let us recall the fundamental theorem of integral calculus:

$$\frac{d}{dx}\int_{a}^{x}dyf(y) = f(y)\Big|_{y=x} = f(x).\tag{100}$$

The differential operator $\hat{\partial}_L$ in (99) contains terms like $\frac{\partial}{\partial x^k} = (\frac{\partial r}{\partial x^k})\frac{\partial}{\partial r}$. Accordingly, in the first line of (99) we can take the differential operator $\hat{\partial}_L$ in front of the integral, because the differentiation of the upper limit $\frac{1}{2}(ct-r-s)$ would yield a term

$$(ct-r-s-2y)^l|_{y=\frac{1}{2}(ct-r-s)} = 0,\tag{101}$$

due to the term $(ct-r-s-2y)^l$ in the argument of the integral; the differentiation of the lower limit a gives zero because a is a constant. Thus, instead of (99) we can write

$$\begin{aligned}\overline{h}(t,\boldsymbol{x}) = &-\sum_{l=0}^{\infty}\frac{4\pi}{2^{l+1}l!}\int_{-\infty}^{ct-r}ds\hat{\partial}_L\int_{a}^{\frac{1}{2}(ct-r-s)}\frac{dy}{y^{l-1}}\hat{T}_L\left(\frac{s+y}{c},y\right)\\ &\times\left[\frac{(ct-r-s)^l(ct-r-s-2y)^l}{r}\right]\\ &+\sum_{l=0}^{\infty}\frac{4\pi}{2^{l+1}l!}\int_{-\infty}^{ct-r}ds\int_{a}^{\frac{1}{2}(ct+r-s)}\frac{dy}{y^{l-1}}\hat{T}_L\left(\frac{s+y}{c},y\right)\hat{\partial}_L\\ &\times\left[\frac{(ct-r-s)^l(ct-r-s-2y)^l}{r}\right].\end{aligned}\tag{102}$$

Now let us consider the second line in (102), especially the term:

$$\hat{\partial}_L \left[\frac{(ct - r - s)^l (ct - r - s - 2y)^l}{r} \right]$$
$$= \hat{\partial}_L \left[\frac{[(ct - r)^2 - 2(s + y)(ct - r) + s(s + 2y)]^l}{r} \right]. \tag{103}$$

We apply the following relation (for a proof of relation (104) see Appendix B, see also Eq. (A.36) in Ref. 20):

$$\hat{\partial}_L \left[\frac{(ct - r)^i}{r} \right] = \hat{\partial}_L \left[\frac{(ct + r)^i}{r} \right], \quad \text{if } i = 0, 1, \ldots, 2l, \tag{104}$$

and obtain for (103) the expression

$$\hat{\partial}_L \left[\frac{(ct - r - s)^l (ct - r - s - 2y)^l}{r} \right]$$
$$= \hat{\partial}_L \left[\frac{[(ct + r)^2 - 2(s + y)(ct + r) + s(s + 2y)]^l}{r} \right]$$
$$= \hat{\partial}_L \left[\frac{(ct + r - s)^l (ct + r - s - 2y)^l}{r} \right]. \tag{105}$$

Inserting (105) into (102) yields

$$\overline{h}(t, \boldsymbol{x}) = -\sum_{l=0}^{\infty} \frac{4\pi}{2^{l+1} l!} \int_{-\infty}^{ct-r} ds \hat{\partial}_L \int_a^{\frac{1}{2}(ct-r-s)} \frac{dy}{y^{l-1}} \hat{T}_L \left(\frac{s + y}{c}, y \right)$$
$$\times \left[\frac{(ct - r - s)^l (ct - r - s - 2y)^l}{r} \right]$$
$$+ \sum_{l=0}^{\infty} \frac{4\pi}{2^{l+1} l!} \int_{-\infty}^{ct-r} ds \int_a^{\frac{1}{2}(ct+r-s)} \frac{dy}{y^{l-1}} \hat{T}_L \left(\frac{s + y}{c}, y \right) \hat{\partial}_L$$
$$\times \left[\frac{(ct + r - s)^l (ct + r - s - 2y)^l}{r} \right]. \tag{106}$$

Now, also for the second line of (106) we can take the differential operator $\hat{\partial}_L$ in front of the integral, because the differentiation of the upper limit yields terms like

$$(ct + r - s - 2y)^l |_{y = \frac{1}{2}(ct+r-s)} = 0. \tag{107}$$

Accordingly, (106) can be written as follows:

$$\overline{h}(t,\boldsymbol{x}) = -\sum_{l=0}^{\infty} \frac{4\pi}{2^{l+1}l!} \int_{-\infty}^{ct-r} ds\hat{\partial}_L \int_a^{\frac{1}{2}(ct-r-s)} \frac{dy}{y^{l-1}} \hat{T}_L\left(\frac{s+y}{c}, y\right) \times \left[\frac{(ct-r-s)^l(ct-r-s-2y)^l}{r}\right] + \sum_{l=0}^{\infty} \frac{4\pi}{2^{l+1}l!} \int_{-\infty}^{ct-r} ds\hat{\partial}_L \int_a^{\frac{1}{2}(ct+r-s)} \frac{dy}{y^{l-1}} \hat{T}_L\left(\frac{s+y}{c}, y\right) \times \left[\frac{(ct+r-s)^l(ct+r-s-2y)^l}{r}\right], \tag{108}$$

which is nothing else but related to (98). □

It is readily seen that the expression (98) can also be written as (cf. Eq. (6.4) in Ref. 20)

$$\overline{h}(t,\boldsymbol{x}) = -4\pi \sum_{l=0}^{\infty} \int_{-\infty}^{ct-r} ds\hat{\partial}_L \times \left[\frac{R_a\left(\frac{1}{2}(ct-r-s), s\right) - R_a\left(\frac{1}{2}(ct+r-s), s\right)}{r}\right], \tag{109}$$

with the function (cf. Eq. (B.6) in Ref. 21)

$$R_a(r,s) = r^l \int_a^r dy \frac{(r-y)^l}{l!} \left(\frac{2}{y}\right)^{l-1} \hat{T}_L\left(\frac{s+y}{c}, y\right). \tag{110}$$

Furthermore, by commuting the differential operator $\hat{\partial}_L$ with the integral of the first term in (109) (for the proof one can use the very same arguments as presented in detail in Proof 7), the expression (109) can be written as follows (see Eq. (6.8) in Ref. 20 or see Eq. (B.5) in Ref. 21):

$$\overline{h}(t,\boldsymbol{x}) = -\sum_{l=0}^{\infty} \hat{\partial}_L \left[\frac{4\pi}{r} \int_{-\infty}^{ct-r} ds R_a\left(\frac{ct-r-s}{2}, s\right)\right] - \sum_{l=0}^{\infty} \int_{-\infty}^{ct-r} ds\hat{\partial}_L \left[\frac{4\pi}{r} R_a\left(\frac{ct+r-s}{2}, s\right)\right]. \tag{111}$$

The solution (111) is independent of the choice of a. In case of a field point outside the source $r > r_0$, the first argument of the second term in (111) will satisfy $\frac{1}{2}(ct+r-s) > r_0$. Hence, if we choose $a = r_0$ it becomes evident from Eq. (110) that the second term in (111) will vanish when $r > r_0$, because the source is spatially compact (cf. Eq. (31); note that the sequence of transformations (55), (56) and (96) yields $y = |\boldsymbol{x}'|$):

$$\hat{T}_L\left(\frac{s+y}{c}, y\right) = 0, \quad \text{for } y > r_0. \tag{112}$$

This argumentation immediately yields (Eq. (B.7) in Ref. 21)

$$\overline{h}(t, \boldsymbol{x}) = \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \hat{\partial}_L \left[\frac{\hat{F}_L(u)}{r}\right], \tag{113}$$

with (cf. Eq. (B.8) in Ref. 21)

$$\begin{aligned} \hat{F}_L(u) = {} & -4\pi(-1)^l \int_{-\infty}^{u} ds \left(\frac{u-s}{2}\right)^l \int_{r_0}^{(u-s)/2} dy \\ & \times \left(\frac{u-s}{2} - y\right)^l \left(\frac{2}{y}\right)^{l-1} \hat{T}_L\left(\frac{s+y}{c}, y\right). \end{aligned} \tag{114}$$

By a change of variables $s = u + (z-1)y$, the expression (114) can be transformed into (cf. Eq. (B.9) in Ref. 21):

$$\hat{F}_L(u) = \frac{4\pi}{2^{l+1}} \int_{-1}^{+1} dz (1-z^2)^l \int_0^{r_0} dy y^{l+2} \hat{T}_L\left(\frac{u+zy}{c}, y\right). \tag{115}$$

Proof 8. We will show the validity of Eq. (115). Consider the expression given by Eq. (114):

$$\begin{aligned} \hat{F}_L(u) = {} & -4\pi(-1)^l \int_{-\infty}^{u} ds \left(\frac{u-s}{2}\right)^l \int_{r_0}^{(u-s)/2} dy \\ & \times \left(\frac{u-s}{2} - y\right)^l \left(\frac{2}{y}\right)^{l-1} \hat{T}_L\left(\frac{s+y}{c}, y\right). \end{aligned} \tag{116}$$

The transformation reads

$$s = u + (z-1)y \tag{117}$$

and the differentials

$$ds dy = \begin{vmatrix} \dfrac{\partial s}{\partial z} & \dfrac{\partial s}{\partial y} \\[2ex] \dfrac{\partial y}{\partial z} & \dfrac{\partial y}{\partial y} \end{vmatrix} dz dy = y dz dy.$$

Thus, one obtains

$$\hat{F}_L(u) = -\frac{4\pi}{2^{l+1}} \int_{-\infty}^{u} dz (1-z^2)^l \int_{r_0}^{(u-s)/2} dy y^{l+2} \hat{T}_L\left(\frac{u+zy}{c}, y\right). \tag{118}$$

Now we have to transform the integration limits. First, we take into account that (cf. Eqs. (31) and (112))

$$\hat{T}_L\left(\frac{u+zy}{c}, y\right) = 0, \quad \text{for } y > r_0. \tag{119}$$

Consequently, we conclude

$$y_{\min} = \frac{u-s}{2}, \quad y_{\max} = r_0 \tag{120}$$

and write (118) as follows:

$$\hat{F}_L(u) = \frac{4\pi}{2^{l+1}} \int_{z_{\min}}^{z_{\max}} dz (1-z^2)^l \int_{(u-s)/2}^{r_0} dy y^{l+2} \hat{T}_L\left(\frac{u+zy}{c}, y\right). \tag{121}$$

From (117) we conclude

$$z_{\min} = \frac{s_{\min}-u}{y_{\max}} + 1, \quad z_{\max} = \frac{s_{\max}-u}{y_{\min}} + 1. \tag{122}$$

From (120) and taking into account the upper limit in (118), i.e. $s \leq u$, we immediately get

$$s_{\min} = u - 2r_0, \quad s_{\max} = u. \tag{123}$$

Then, by inserting (120) and (123) into (122), we obtain the limits:

$$z_{\min} = -1, \quad z_{\max} = +1. \tag{124}$$

Accordingly, the integral (121) reads

$$\hat{F}_L(u) = \frac{4\pi}{2^{l+1}} \int_{-1}^{+1} dz (1-z^2)^l \int_0^{r_0} dy y^{l+2} \hat{T}_L\left(\frac{u+zy}{c}, y\right), \tag{125}$$

which is just in coincidence with Eq. (115). □

Finally, we use the inversion of Eq. (51) (see Eq. (A.9(b)) in Ref. 20 or Eq. (B.10) in Ref. 21)

$$\hat{T}_L(t, y) = \frac{(2l+1)!!}{4\pi l!} \int_0^{2\pi} \sin\theta d\theta \int_0^{\pi} d\phi \hat{n}_L(\theta, \phi) T(t, y, \theta, \phi). \tag{126}$$

Inserting (126) into (115), yields for (113) the following expression (cf. Eq. (B.2) in Ref. 21)

$$\overline{h}(t, \boldsymbol{x}) = \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \partial_L \left[\frac{\hat{F}_L(u)}{r}\right], \tag{127}$$

where $r = |\boldsymbol{x}|$ is the spatial distance between the origin of coordinate system and the field point. By a transformation from spherical coordinates $\boldsymbol{y} = (y, \theta, \phi)$ to

Cartesian-like coordinates (cf. Eq. (1.1) in Ref. 19) $\boldsymbol{x}' = (x'_1, x'_2, x'_3)$, the STF multipole moments of the source are given by

$$\hat{F}_L(u) = \int_V d^3x' \hat{x}'_L \int_{-1}^{+1} dz \delta_l(z) T\left(\frac{u + zr'}{c}, \boldsymbol{x}'\right), \tag{128}$$

where the spatial integral runs over the volume V of the source, $r' = |\boldsymbol{x}'|$ is the spatial distance between the origin of coordinate system and a point inside the source with spatial coordinate $\boldsymbol{x}'$, and $u = ct - r$, cf. Eq. (55). In order to derive the form of Eq. (127), we also have used the relation $\hat{\partial}_L \hat{F}_L(u) = \partial_L \hat{F}_L(u)$ since $\hat{F}_L$ are STF multipoles, that means the trace over any pair of indices in $\hat{F}_L$ vanishes: e.g. for $l = 2$ we would have $\hat{\partial}_{i_1 i_2} = \frac{\partial^2}{\partial x^{i_1} \partial x^{i_2}} - \frac{\delta_{i_1 i_2}}{3} \frac{\partial^2}{\partial r^2}$ and due to $\delta_{i_1 i_2} \hat{F}_{i_1 i_2} = 0$, we have $\hat{\partial}_{i_1 i_2} \hat{F}_{i_1 i_2} = \partial_{i_1 i_2} \hat{F}_{i_1 i_2}$ and so on.

The functions in (128) are given by

$$\delta_l(z) = \frac{(2l+1)!!}{2^{l+1} l!} (1 - z^2)^l. \tag{129}$$

In view that $\overline{h}$ in (127) stands either for $\overline{h}^{00}$, $\overline{h}^{0i}$ or $\overline{h}^{ij}$, while T in (128) stands either for $\frac{4G}{c^4} T^{00}$, $\frac{4G}{c^4} T^{0i}$ or $\frac{4G}{c^4} T^{ij}$, respectively, we can rewrite Eqs. (127) and (128) in terms of their explicit tensorial structure:

$$\overline{h}^{\alpha\beta}(t, \boldsymbol{x}) = \frac{4G}{c^4} \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \partial_L \left[\frac{\hat{F}_L^{\alpha\beta}(u)}{r}\right], \tag{130}$$

where the STF multipoles are given by

$$\hat{F}_L^{\alpha\beta}(u) = \int_V d^3x' \hat{x}'_L \int_{-1}^{+1} dz \delta_l(z) T^{\alpha\beta}\left(\frac{u + zr'}{c}, \boldsymbol{x}'\right). \tag{131}$$

The equations (130) and (131) represent the fundamental theorem of STF multipole expansion in post-Minkowskian approximation, as previously emphasized by Eqs. (3) and (4) in the introductory section. In virtue of Eq. (31) (compact support source) it is obvious that the multipole expansion (130) and (131) is valid for regions $r > r_0$, where r_0 is the radius of the smallest possible sphere which encloses completely the matter source. Finally, it should be noted that a straightforward application of theorem (127) and (128) for the case of electrodynamics leads immediately to the STF expansion given by Eqs. (4.2) and (4.3) in Ref. 36.

5. Summary

In linearized gravity, the Einstein's field equations are given by an inhomogeneous partial differential equation (1) for each of the 10 components of the metric tensor. In the region exterior to the source, the retarded solution (2) can be expanded in terms of 10 Cartesian STF multipoles in post-Minkowskian approximation: Eqs. (3) and (4) (= Eqs. (130) and (131)). These 10 multipoles in (4) are not independent of each other, because using energy–momentum conservation (four-relations) and gauge transformation (four-relations) they can be reduced to finally two independent STF multipoles: mass multipoles and spin multipoles, $\hat{M}_L$ and $\hat{S}_L$, respectively, in post-Newtonian approximation demonstrated by Thorne[19] and Blanchet and Damour,[20,21] while in post-Minkowskian approximation this fact has been established by Damour and Iyer.[36]

Meanwhile, the STF multipole expansion has become an important tool in linearized gravity and has demonstrated its efficiency for a wide spectrum of applications: in celestial mechanics,[22–24] in the theory of gravitational waves,[25–27] and in high precision astrometry where a particularly important aspect thereof is the theory of light propagation in curved spacetime.[28–31]

Theorems 3 and 4 is the fundamental and the heart part of STF multipole expansion; see Eqs. (B.2) and (B.3) in Ref. 21, Eqs. (5.3) and (5.4) in Ref. 36, Eqs. (56) and (57) in Ref. 26, or Eq. (25) in Ref. 27. But despite its formidable importance, an explicit proof of Eqs. (3) and (4) has not been presented so far, while some parts of the mathematical proof are distributed into several publications.[11,19–21] In this investigation, a detailed proof of the STF multipole decomposition has been represented in the form of a more didactical manuscript. Only three and rather weak assumptions are required for the validity of the STF multipole expansion:

(1) No-incoming-radiation condition, Eq. (26).
(2) The source is spatially compact, Eq. (31).
(3) A spherical expansion of the metric outside the source is possible, Eq. (51).

We hope that our investigation elucidates fundamental aspects of the main theorem of STF multipole expansion (3), where the multipoles in post-Minkowskian approximation are defined by (4).

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Appendix A. Proof of Eq. (82)

Relation (82) contains only derivatives with respect to variable u, and since u and v are independent variables, here we can treat v as a constant. Accordingly, we introduce a new variable $x = u - v$ with $\frac{\partial}{\partial x} = \frac{\partial}{\partial u}$, and rewrite relation (82) as follows:

$$\left[x^2 \frac{\partial}{\partial x}\right]^l \frac{(x+a)^l (x+b)^l}{x^{2l}} = x^{l+1} \frac{\partial^l}{\partial x^l} \frac{(x+a)^l (x+b)^l}{x^{l+1}}, \tag{A.1}$$

where $a = v - u'$ and $b = v - v'$, and the independent variable u' and v' are also considered as constant quantities. In order to show the validity of relation (A.1), we apply the binomial theorem:

$$(x+a)^l = \sum_{p=0}^{l} \binom{l}{p} x^{l-p} a^p, \quad (x+b)^l = \sum_{q=0}^{l} \binom{l}{q} x^{l-q} b^q, \tag{A.2}$$

where the binomial coefficients are defined by

$$\binom{l}{p} = \frac{l!}{(l-p)!p!}, \quad \binom{l}{q} = \frac{l!}{(l-q)!q!}. \tag{A.3}$$

Inserting (A.2) into (A.1) yields

$$\begin{aligned} &\sum_{p,q=0}^{l} \binom{l}{p} \binom{l}{q} a^p b^q \left[x^2 \frac{\partial}{\partial x}\right]^l \frac{x^{l-p} x^{l-q}}{x^{2l}} \\ &\quad = \sum_{p,q=0}^{l} \binom{l}{p} \binom{l}{q} a^p b^q x^{l+1} \frac{\partial^l}{\partial x^l} \frac{x^{l-p} x^{l-q}}{x^{l+1}}. \end{aligned} \tag{A.4}$$

Let us consider each individual term in (A.4). One can easily show the validity of the following both relations by means of mathematical induction:

$$\left[x^2 \frac{\partial}{\partial x}\right]^l \frac{x^{l-p} x^{l-q}}{x^{2l}} = (-1)^l x^{-(p+q-l)} \prod_{k=0}^{l-1} (p+q-k), \tag{A.5}$$

$$x^{l+1} \frac{\partial^l}{\partial x^l} \frac{x^{l-p} x^{l-q}}{x^{l+1}} = (-1)^l x^{-(p+q-l)} \prod_{k=0}^{l-1} (p+q-k). \tag{A.6}$$

Accordingly, we can conclude the following identity for each individual term in (A.4):

$$\left[x^2 \frac{\partial}{\partial x}\right]^l \frac{x^{l-p} x^{l-q}}{x^{2l}} = x^{l+1} \frac{\partial^l}{\partial x^l} \frac{x^{l-p} x^{l-q}}{x^{l+1}}. \tag{A.7}$$

That means, each individual term on the left-hand side in (A.4) coincides with the corresponding term on the right-hand side in (A.4). Thus, we have shown the validity of relation (A.4) and, therefore, the validity of relation (A.1) and (82), respectively.

Appendix B. Proof of Eq. (104)

Let us consider both expressions in (104), which we write as follows (for a proof of relation (B.1) see Eqs. (85)–(88), while the proof of (B.2) is very similar, see also relations (A.35(b)) and (A.36(c)) in Ref. 20):

$$\hat{\partial}_L \frac{(ct-r)^n}{r} = \frac{2}{l!}\hat{n}_L (v-u)^l \frac{\partial^{2l}}{\partial u^l \partial v^l}\frac{u^n}{v-u}, \tag{B.1}$$

$$\hat{\partial}_L \frac{(ct+r)^n}{r} = \frac{2}{l!}\hat{n}_L (v-u)^l \frac{\partial^{2l}}{\partial u^l \partial v^l}\frac{v^n}{v-u}, \tag{B.2}$$

where $u = ct - r$ and $v = ct + r$. By subtraction of (B.1) from (B.2) one obtains

$$\hat{\partial}_L \frac{(ct+r)^n}{r} - \hat{\partial}_L \frac{(ct-r)^n}{r} = \frac{2}{l!}\hat{n}_L (v-u)^l \frac{\partial^{2l}}{\partial u^l \partial v^l}\frac{v^n - u^n}{v-u}. \tag{B.3}$$

Now we recall the generalized version of third binomial theorem

$$\frac{v^n - u^n}{v-u} = \sum_{j=0}^{n-1} v^{n-j-1} u^j. \tag{B.4}$$

Due to $2l \geq n$, the $(2l)$th derivative of the polynomial in (B.4) yields zero:

$$\frac{\partial^{2l}}{\partial u^l \partial v^l}\frac{v^n - u^n}{v-u} = \frac{\partial^{2l}}{\partial u^l \partial v^l}\sum_{j=0}^{n-1} v^{n-j-1} u^j = 0. \tag{B.5}$$

Thus, inserting (B.5) into (B.3) yields

$$\hat{\partial}_L \frac{(ct+r)^n}{r} - \hat{\partial}_L \frac{(ct-r)^n}{r} = 0, \tag{B.6}$$

which is just relation (104).

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