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# Renormalization of the two-photon vacuum polarization and the self energy vacuum polarization for a tightly bound electron

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**Abstract.** The renormalization method of Bogoljubov-Parasiuk-Hepp-Zimmermann (BPHZ) is used in order to derive the renormalized energy shift due to the gauge invariant Källén-Sabry diagram of the two-photon vacuum polarization (VPVP) as well as the self energy vacuum polarization S(VP)E beyond the Uehling approximation. It is outlined, that no outer renormalization is required for the two-photon vacuum polarization and that only the inner renormalization has to be accomplished. It is shown that the so-called nongauge invariant spurious term is absent for a wide class of vacuum polarization (VP) diagrams if one applies the widely used spherical expansion of bound and free-electron propagator. This simplifies significantly calculations in bound state quantum electrodynamics. As one result of our paper the use of the BPHZ-approach in bound state QED is established.

**PACS.** 31.30.Jv Relativistic and quantum electrodynamic effects in atoms and molecules

## 1 Introduction

Highly charged ions provide an ideal scenario to demonstrate the validity of QED in strong electric and magnetic fields by measurements of the Lamb shift at utmost precision. In this respect the recent experimental progress made in measurements of the Lamb shift in hydrogen-like ions [1,2] indicate that calculations of all QED corrections of order $\alpha^2$ ($\alpha \approx 1/137.036$ is the fine structure constant), but exact in the coupling constant $Z\alpha$ become relevant. The present status of theoretical prediction for the Lamb shift have been presented in [3,4]. Even the most difficult gauge invariant set, the second-order electron self energy correction, have been recently calculated for the hydrogen-like ions uranium and bismuth [5]. The renormalization of these diagrams has been carried out in reference [6] and within a different approach in reference [7]. At present two of these two-photon diagrams are known in the Uehling approximation only, namely the two-photon vacuum polarization VPVP and the effective self energy S(VP)E (see Fig. 1). Consequently, mainly the VPVP correction remains an additional source of theoretical uncertainty in the Lamb shift predictions of hydrogen-like ions [3,4]. The main contribution originates from the Uehling part of the VPVP diagram ($-0.6 \pm 0.12$ eV for uranium and $-0.34 \pm 0.07$ eV for lead). To provide a result, complete in order $\alpha^2$, for the Lamb shift is indeed a challenge for theory. As a step towards this goal we derive the renormalized expression for the two-photon vacuum polarization. At the same time we derive a renormalized expression for the higher-order energy shift of the effective self energy diagram S(VP)E. This diagram is of special interest not only since its contribution to the Lamb shift is unknown for all elements in nature but also due to the fact that it contains the one-loop vacuum polarization (VP) as a subdiagram. It is well-known that in general the evaluation of VP is connected with a so-called nongauge invariant spurious term. We shall show that this spurious term is absent for a wide class of diagrams which contain the VP diagram. This result simplifies essentially evaluations of such diagrams in bound state QED.

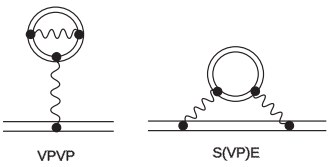


**Fig. 1.** The two-photon vacuum polarization (VPVP) and the effective self energy S(VP)E.

The paper is organized as follows. In Section 2 we explain the BPHZ-renormalization approach and it's application to bound state QED. The renormalization of the VPVP diagram is discussed in Section 3. It is shown that no outer renormalization is required and the renormalized expression is specified in Section 4. In Section 5 we shall discuss some basic properties of the fourth-rank vacuum polarization tensor. In Section 6 the renormalized expression of higher-order of the S(VP)E diagram is derived. Finally it will be demonstrated that the nongauge invariant term is absent if one applies the partial wave decomposition of the electron propagator. The renormalized expression of S(VP)E diagram is specified in Section 7.

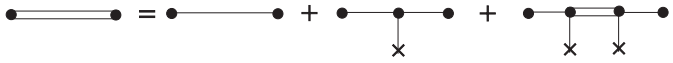

**Fig. 2.** Potential expansion of the bound-electron propagator.

## 2 BPHZ-renormalization scheme for QED

The general renormalization scheme of QED is formulated for graphs with free-electron propagators. Therefore, in order to isolate the ultraviolet divergencies we first have to perform a potential expansion [10–12] of the bound electron-propagator

$$S_F(\mathbf{r}_1,\mathbf{r}_2,\omega) = S_F^0(\mathbf{r}_1-\mathbf{r}_2,\omega) + \int \mathrm{d}^3\mathbf{r}\; S_F^0(\mathbf{r}_1-\mathbf{r},\omega)\gamma_0 V(\mathbf{r}) S_F^0(\mathbf{r}-\mathbf{r}_2,\omega) + \int \mathrm{d}^3\mathbf{r}\int \mathrm{d}^3\mathbf{r}' S_F^0(\mathbf{r}_1-\mathbf{r},\omega)\gamma_0 V(\mathbf{r}) \times S_F(\mathbf{r},\mathbf{r}',\omega)\gamma_0 V(\mathbf{r}') S_F^0(\mathbf{r}'-\mathbf{r}_2,\omega) \quad (1)$$

that is represented graphically in Figure 2. After this it is possible to apply the standard renormalization prescription for the free QED. We use the BPHZ-renormalization-method [13,14]. For any given divergent loop-integral $\hat{F}_\Gamma$ corresponding to a Feynman diagram $\Gamma$ this approach allows for a systematic isolation of the divergent parts and to derive the finite contribution $\hat{F}_\Gamma^{'}$ of the loop-integral under consideration. This procedure is formally expressed in terms of Bogoljubov's $\mathcal{R}$-operation acting onto the integrand $\hat{I}_\Gamma$ of the divergent loop-integral $\hat{F}_\Gamma$. This operation determines the integrand $\hat{R}_\Gamma$, which yields the finite integral $\hat{F}_\Gamma^{'}$:

$$\hat{F}_\Gamma = \int \mathrm{d}^4q_1\;...\;\mathrm{d}^4q_n\,\hat{I}_\Gamma \;\longrightarrow\; \hat{F}_\Gamma^{'} = \int \mathrm{d}^4q_1\;...\;\mathrm{d}^4q_n\,\hat{R}_\Gamma. \quad (2)$$

Depending on the superficial degree of divergence $\omega(\Gamma)$ of the diagram $\Gamma$ the action of Bogoljubov's $\mathcal{R}$-operation is defined recursively *via*

$$\hat{R}_\Gamma = \left\{ \begin{array}{rl} (1-t^\Gamma)\hat{\overline{R}}_\Gamma & \text{if} \quad \omega(\Gamma)\geq 0 \\ \hat{\overline{R}}_\Gamma & \text{if} \quad \omega(\Gamma)<0 \end{array} \right\} \quad (3)$$

where Bogoliubov's recursion formula is given by

$$\hat{\overline{R}}_\Gamma = \hat{I}_\Gamma + \sum_{\{\gamma_1,\ldots,\gamma_k\}} \hat{I}_{\Gamma/\{\gamma_1,\ldots,\gamma_k\}} \prod_{\tau=1}^{k}(-t^{\gamma_\tau})\hat{\overline{R}}_{\gamma_\tau}. \quad (4)$$

Here $\omega(\gamma) = 4 - E_\mathrm{B} - (3/2)E_\mathrm{F}$ defines the superficial degree of divergence for QED where $E_\mathrm{B}$ and $E_\mathrm{F}$ denote the number of external bosonic and fermionic legs, respectively, assigned to an arbitrary diagram or subdiagram $\gamma$. The sum in equation (4) runs over all combinations of disjunct superficially divergent subdiagrams $\gamma_i$ of the diagram $\Gamma$, and $\gamma_\tau$ denote the superficially divergent subdiagrams of the subdiagrams $\gamma_i$. The mathematical expressions for the subdiagrams are obtained by means of the usual Feynman rules. The symbol $\hat{I}_{\Gamma/\gamma_1,\ldots,\gamma_k}$ stands for the integrand of the diagram $\Gamma$ where the subdiagrams $\gamma_1,...,\gamma_k$ are merged to a point. The operator $t^\gamma$ denotes the Taylor expansion with respect to the independent external electron momenta $p_i$ and photon momenta $k_j$ up to order $\omega(\gamma)$ of the diagram $\gamma$. According to the renormalization group equation of QED the reference point of this Taylor expansion can be chosen arbitrary. However, a special choice of a renormalization scheme with fixed reference point implies renormalization conditions which are kept fixed. Here we employ on-shell-renormalization conditions and utilize the gauge invariance of the coupling between electron-positron field and the radiation field. This implies the following conditions for the renormalized electron self energy, the renormalized vertex function and for the second- and fourth-rank vacuum polarization tensors of the photon, respectively, in perturbation theory of order $(\alpha)^n$ $(n=1,2,3,...)$:

$$\hat{\Sigma}^{(n)\,\mathrm{ren}}(p)\Big|_{\not{p}=m} = 0, \quad \hat{\Lambda}_\mu^{(n)\,\mathrm{ren}}(p_1,p_2)\Big|_{\not{p}_1=\not{p}_2=m} = \mathrm{i}e\gamma_\mu,$$

$$\hat{\Pi}_{\mu\nu}^{(n)\,\mathrm{ren}}(k)\Big|_{k=0} = 0, \quad \hat{\Pi}_{\mu\nu\rho\sigma}^{(n)\,\mathrm{ren}}(k_1,k_2,k_3)\Big|_{k_1=k_2=k_3=0} = 0, \quad (5)$$

where $m$ denotes the physical electron mass and $k_i = 0$ means actually $k_i = (0,0,0,0)$. Therefore from the renormalization-conditions (5) one can define the Taylor-Operator $t^\gamma$ up to first order according to the prescription.

The operator $t^\gamma$ expands the integrand belonging to the diagram $\gamma$

(a) with respect to all independent $N_\mathrm{F}$ external electron momenta $\not{p}_i$ around $m$
(b) with respect to all independent $N_\mathrm{B}$ external photon momenta $k_j$ around 0.

In our context here it will be sufficient to perform the Taylor expansion up to first order in external momenta. In mathematical terms we can write

$$t^\gamma\hat{I}_\gamma(p_i,k_j) = \hat{I}_\gamma(p_i,k_j)\Big|_{\not{p}_i=m,k_j=0} + \sum_{i=1}^{N_\mathrm{F}}\left(\frac{\partial}{\partial p_i^\mu}\hat{I}_\gamma(p_i,k_j)\Big|_{\not{p}_i=m,k_j=0}\right)\left(p_i^\mu - m\frac{\gamma^\mu}{4}\right) + \sum_{j=1}^{N_\mathrm{B}}\left(\frac{\partial}{\partial k_j^\mu}\hat{I}_\gamma(p_i,k_j)\Big|_{\not{p}_i=m,k_j=0}\right)k_j^\mu + \ldots \quad (6)$$

## 3 Renormalization of VPVP

The contribution of the Källén-Sabry-diagrams with the free electron propagator are well-known [8,9]. Accordingly we subtract these terms from the two-photon vacuum polarization diagram and concentrate on the higher-order contribution of VPVP which is graphically shown in the Figure 3. As the major result in this chapter we derive the corresponding renormalized energy shift.

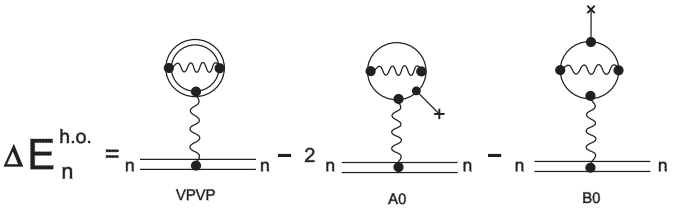


**Fig. 3.** Unrenormalized energy shift due to higher-order contribution in $(Z\alpha)$ of the VPVP diagram.

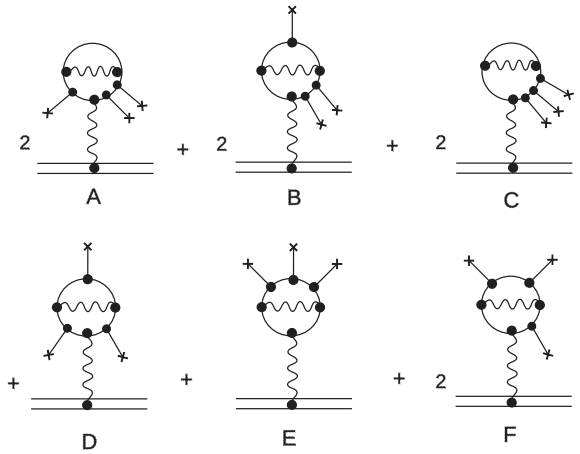


**Fig. 4.** Lowest-order terms of the potential expansion contributing to the energy shift of Figure 3 with $\omega = 0$.

### 3.1 The outer renormalization

According to the BPHZ-renormalization scheme as defined above the case where the Taylor-Operator $t^{\Gamma}$ acts on the whole diagram $\Gamma$ may be called outer renormalization. From equation (3) it becomes evident that outer renormalization is always required for a superficial degree of divergence $\omega(\Gamma) \geq 0$. In the case of the VPVP diagram we have the following situation.

In the potential expansion of the energy shift in Figure 3 diagrams with at least four external Coulomb-photon legs occur. According to the Furry's theorem diagrams with an odd number of external photon or bosonic potential legs vanish. The superficial degree of divergence $\omega$ is zero for diagrams with four external bosonic legs and is negative for all diagrams of higher-order in $(Z\alpha)^n$ with $n>4$. In view of equation (3) outer renormalization is required if and only if the number of outer bosonic legs is equal to four. These diagrams have to be considered with special care.

At this point we note that in the case of the one-loop vacuum polarization VP one must consider the diagram with four outer bosonic legs with special care as well (see for example [16]). Employing any regularization scheme in the one-loop case VP the corresponding counter term vanishes. Moreover, if the spherical expansion for the free and bound electron propagator is used no spurious non-gauge invariant term occurs in the 1-loop case VP [17]. For detailed discussions we refer to the literature ([16–18] and [19]). With this in mind one could expect that, after any regularization is performed, the counter terms required for the outer renormalization may vanish in the two-loop-case VPVP as well. In any case an explicit proof is desirable.

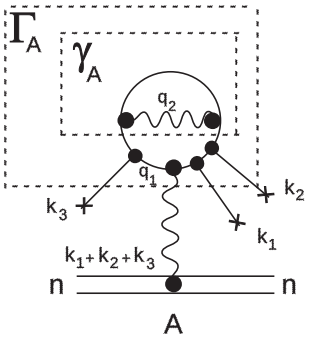


**Fig. 5.** Diagram $\Gamma_{\rm A}$ with $\omega(\Gamma_{\rm A}) = 0$. $\Gamma_{\rm A}$ contains a divergent subdiagram $\gamma_{\rm A}$ (= one-loop self energy). $n$ denotes a bound electron state.

We turn to the investigation of the couterterms of the outer renormalization for the VPVP diagram. Performing the potential expansion of the energy shift shown in Figure 3 leads to six diagrams with $\omega = 0$ (see Fig. 4). To give an explicit example we consider graph $\Gamma_{\rm A}$ depicted in Figure 5. This diagram includes the subdiagram $\gamma_{\rm A}$. It's contribution to the energy shift of an arbitrary bound electron state $\varphi_n$ reads

$$\begin{aligned}\Delta E_n^{\Gamma_A} = & e^2 \int \mathrm{d}^3\mathbf{r}\; \overline{\varphi}_n(\mathbf{r}) \\ & \times \int \frac{\mathrm{d}^3\mathbf{k}_1}{(2\pi)^3} \int \frac{\mathrm{d}^3\mathbf{k}_2}{(2\pi)^3} \int \frac{\mathrm{d}^3\mathbf{k}_3}{(2\pi)^3}\; \mathrm{e}^{\mathrm{i}(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)\mathbf{r}} \\ & \times V(\mathbf{k}_1)\, V(\mathbf{k}_2)\, V(\mathbf{k}_3)\, \frac{1}{0^2 - (\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)^2 + \mathrm{i}\epsilon} \\ & \times \gamma_0\; \hat{F}_{\Gamma_{\rm A}}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)\; \varphi_n(\mathbf{r}), \end{aligned} \tag{7}$$

where $V(\mathbf{k}_j)$ $(j = 1,2,3)$ is the Fourier-transform of the Coulomb potential of the nucleus:

$$V(\mathbf{k}_j) = \int \mathrm{d}^3\mathbf{r}\; \mathrm{e}^{-\mathrm{i}\mathbf{k}_j\mathbf{r}}\; V(\mathbf{r}). \tag{8}$$

The explicit mathematical expression for the Feynman diagram is given by (with $k_i = (0, \mathbf{k}_i),\ i = 1,2,3$)

$$\hat{F}_{\Gamma_{\rm A}}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) = \int \frac{\mathrm{d}^4q_1}{(2\pi)^4} \frac{\mathrm{d}^4q_2}{(2\pi)^4}\; I_{\Gamma_{\rm A}}(q_1,q_2,\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3),$$

$$\begin{aligned} & I_{\Gamma_{\rm A}}(q_1,q_2,\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) = \\ & \quad \mathrm{i}\,\mathrm{Tr}\Bigg[\gamma_0 \frac{1}{\not q_1 + \not k_1 + \not k_2 + \not k_3 - m + \mathrm{i}\epsilon} \\ & \quad \times \gamma_0 \frac{1}{\not q_1 + \not k_2 + \not k_3 - m + \mathrm{i}\epsilon} \gamma_0 \frac{1}{\not q_1 + \not k_3 - m + \mathrm{i}\epsilon} \\ & \quad \times \left(-\mathrm{i}e^2 \frac{1}{q_2^2 + \mathrm{i}\epsilon} \gamma_\alpha \frac{1}{\not q_1 - \not q_2 + \not k_3 - m + \mathrm{i}\epsilon} \gamma^\alpha\right) \\ & \quad \times \frac{1}{\not q_1 + \not k_3 - m + \mathrm{i}\epsilon} \gamma_0 \frac{1}{\not q_1 - m + \mathrm{i}\epsilon}\Bigg]. \end{aligned} \tag{9}$$

In round brackets the mathematical expression of integrand referring to the subdiagram $\gamma_{\rm A}$ (one-loop-self

energy) has been inserted. Applying the $\mathcal{R}$-operation equations (3, 4) yields

$$\hat{R}_{\Gamma_\mathrm{A}} = \hat{R}_{\Gamma_\mathrm{A}}^{\mathrm{inner}} + \hat{R}_{\Gamma_\mathrm{A}}^{\mathrm{outer}} \tag{10}$$

with

$$\hat{R}_{\Gamma_\mathrm{A}}^{\mathrm{inner}} = \hat{I}_{\Gamma_\mathrm{A}} - \hat{I}_{\Gamma_\mathrm{A}/\gamma_\mathrm{A}} \left( t^{\gamma_\mathrm{A}} \hat{I}_{\gamma_\mathrm{A}} \right),$$

$$\hat{R}_{\Gamma_\mathrm{A}}^{\mathrm{outer}} = -t^{\Gamma_\mathrm{A}} \hat{I}_{\Gamma_\mathrm{A}} + t^{\Gamma_\mathrm{A}} \left( \hat{I}_{\Gamma_\mathrm{A}/\gamma_\mathrm{A}} \left( t^{\gamma_\mathrm{A}} \hat{I}_{\gamma_\mathrm{A}} \right) \right). \tag{11}$$

The Taylor-Operator $t^{\Gamma_\mathrm{A}}$ acts up to zeroth order while the other one $t^{\gamma_\mathrm{A}}$ acts up to first order since $\omega(\Gamma_\mathrm{A}) = 0$ and $\omega(\gamma_\mathrm{A}) = 1$. Accordingly, the terms for the outer renormalization are

$$\int \frac{\mathrm{d}^4 q_1}{(2\pi)^4} \frac{\mathrm{d}^4 q_2}{(2\pi)^4}\, t^{\Gamma_\mathrm{A}} \left[ \hat{I}_{\Gamma_\mathrm{A}}(q_1, q_2, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) \right] = \int \frac{\mathrm{d}^4 q_1}{(2\pi)^4} \frac{\mathrm{d}^4 q_2}{(2\pi)^4}\, \hat{I}_{\Gamma_\mathrm{A}}(q_1, q_2, \mathbf{0}, \mathbf{0}, \mathbf{0}) \tag{12}$$

and

$$\begin{aligned} \int \frac{\mathrm{d}^4 q_1}{(2\pi)^4} \frac{\mathrm{d}^4 q_2}{(2\pi)^4}\, t^{\Gamma_\mathrm{A}} \Big[ & \hat{I}_{\Gamma_\mathrm{A}/\gamma_\mathrm{A}}(q_1, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) \\ & \times \left( t^{\gamma_\mathrm{A}} \hat{I}_{\gamma_\mathrm{A}}(q_1, q_2, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) \right) \Big] \\ & = \Sigma^{(1)} \int \frac{\mathrm{d}^4 q_1}{(2\pi)^4}\, \hat{I}_{\Gamma_\mathrm{A}}(q_1, \mathbf{0}, \mathbf{0}, \mathbf{0}) \\ + \Sigma^{(1)'} & \int \frac{\mathrm{d}^4 q_1}{(2\pi)^4}\, \underline{(\not{q}_1 - m)\, \hat{I}_{\Gamma_\mathrm{A}/\gamma_\mathrm{A}}}(q_1, q_2, \mathbf{0}, \mathbf{0}, \mathbf{0}), \end{aligned} \tag{13}$$

respectively. The denotation $\underline{(\not{q}_1 - m)\, \hat{I}_{\Gamma_\mathrm{A}/\gamma_\mathrm{A}}}$ implies insertion of the Dirac-structure $(\not{q}_1 - m)$ at the same place in the diagram $\Gamma_\mathrm{A}/\gamma_\mathrm{A}$ where there was the subdiagram $\gamma_\mathrm{A}$ before. In equation (13) we used the usual definition of the one-loop-counter terms $\Sigma^{(1)}$ and $\Sigma^{(1)'}$ of free QED:

$$\Sigma^{(1)} = -\mathrm{i}e^2 \int \frac{\mathrm{d}^4 q_2}{(2\pi)^4} \gamma_\alpha \frac{1}{\not{q}_1 - \not{q}_2 - m + \mathrm{i}\epsilon} \gamma^\alpha \frac{1}{q_2^2 + \mathrm{i}\epsilon} \Big|_{\not{q}_1 = m} \tag{14}$$

and

$$\gamma_\rho \Sigma^{(1)'} = -\mathrm{i}e^2 \frac{\partial}{\partial q_1^\rho} \int \frac{\mathrm{d}^4 q_2}{(2\pi)^4} \gamma_\alpha \frac{1}{\not{q}_1 - \not{q}_2 - m + \mathrm{i}\epsilon} \times \gamma^\alpha \frac{1}{q_2^2 + \mathrm{i}\epsilon} \Big|_{\not{q}_1 = m}. \tag{15}$$

Similar steps have to be performed in all other diagrams of Figure 4. With the aid of the one-loop Ward-identity

$$\frac{\partial \hat{\Sigma}^{(1)\mathrm{ren}}(q_1)}{\partial q_1^0} = \hat{\Lambda}_0^{(1)\mathrm{ren}}(q_1, q_1), \quad \frac{\partial \hat{\Sigma}^{(1)}(q_1)}{\partial q_1^0} = \hat{\Lambda}_0^{(1)}(q_1, q_1), \quad \Sigma^{(1)'} = -\Lambda^{(1)} \tag{16}$$

together with the identity

$$\underline{(\not{q}_1 - m)\, \hat{I}_{\Gamma_\mathrm{A}/\gamma_\mathrm{A}}} = \underline{\gamma_0 \hat{I}_{\Gamma_\mathrm{B}/\gamma_\mathrm{B}}} \tag{17}$$

and the relations between renormalized and unrenormalized vertex and self-energy operator

$$\hat{\Lambda}_0^{(1)}(\not{q}_1 - \not{q}_2, \not{q}_1 - \not{q}_2) = \gamma_0 \Lambda^{(1)} + \hat{\Lambda}_0^{(1)\mathrm{ren}}(\not{q}_1 - \not{q}_2, \not{q}_1 - \not{q}_2),$$

$$\hat{\Sigma}^{(1)}(\not{q}_1) = \Sigma^{(1)} + (\not{q}_1 - m)\, \Sigma^{(1)'} + \hat{\Sigma}^{(1)\mathrm{ren}}(\not{q}_1), \tag{18}$$

all counter terms referring to the outer renormalization can be collected. One ends up with the following expression for the sum of all counter terms generated by the operator $\hat{R}_\Gamma^{\mathrm{outer}}$ corresponding to the six diagrams of Figure 4:

$$\begin{aligned} \int \frac{\mathrm{d}^4 q_1}{(2\pi)^4} \int \frac{\mathrm{d}^4 q_2}{(2\pi)^4} \Big( & 2\hat{R}_{\Gamma_\mathrm{A}}^{\mathrm{outer}} + 2\hat{R}_{\Gamma_\mathrm{B}}^{\mathrm{outer}} + 2\hat{R}_{\Gamma_\mathrm{C}}^{\mathrm{outer}} \\ & + \hat{R}_{\Gamma_\mathrm{D}}^{\mathrm{outer}} + \hat{R}_{\Gamma_\mathrm{E}}^{\mathrm{outer}} + \hat{R}_{\Gamma_\mathrm{F}}^{\mathrm{outer}} \Big) = \mathrm{Tr} \int \frac{\mathrm{d}^4 q_1}{(2\pi)^4} \\ & \times \frac{\partial}{\partial q_1^0} \Bigg[ \frac{1}{\not{q}_1 - m + \mathrm{i}\epsilon} \hat{\Lambda}_0^{(1)\mathrm{ren}}(\not{q}_1, \not{q}_1) \frac{1}{\not{q}_1 - m + \mathrm{i}\epsilon} \gamma_0 \\ & \times \frac{1}{\not{q}_1 - m + \mathrm{i}\epsilon} \gamma_0 + \frac{1}{\not{q}_1 - m + \mathrm{i}\epsilon} \hat{\Sigma}^{(1)\mathrm{ren}}(\not{q}_1) \\ & \times \frac{1}{\not{q}_1 - m + \mathrm{i}\epsilon} \gamma_0 \frac{1}{\not{q}_1 - m + \mathrm{i}\epsilon} \gamma_0 \frac{1}{\not{q}_1 - m + \mathrm{i}\epsilon} \gamma_0 \Bigg]. \end{aligned} \tag{19}$$

The last integral is understood as being regularized. The renormalized self energy as well as the renormalized vertex correction do not possess any poles. Therefore the total integrand has simple poles only at $q_1^0 = \pm\sqrt{\mathbf{q}_1^2 + (m - \mathrm{i}\epsilon)^2}$. Accordingly, a Wick-rotation $q_1^0 \to \mathrm{i}q_1^0$ to Euclidean space can be carried out and all surface terms vanishes after regularization. This implies that all counter terms generated by $\hat{R}_\Gamma^{\mathrm{outer}}$ cancel each other. Therefore all singularities associated with the outer renormalization must cancel each other in the unrenormalized expression of VPVP already.

### 3.2 The inner renormalization

According to the previous subsection only an inner renormalization associated with the action of $\hat{R}_\Gamma^{\mathrm{inner}}$ is required. For this purpose we perform the potential expansion of the bound-electron propagator until three outer bosonic potential legs. An appropriate potential expansion of the energy correction of higher-order in equation (3) is given in Figure 6. For the last two diagrams C2 and C3 an inner renormalization is not adequate. For the other six diagrams A1, A2, A3, B1, B2 and C1 a simple inner renormalization is necessary. There are the inner divergencies of the one-loop-self energy as well as the one-loop-vertex correction. The renormalization of this one-loop-expressions has to be carried out using equation (18). Accordingly, one should perform the replacements of Figure 7 in order to receive the full renormalized expression for VPVP.

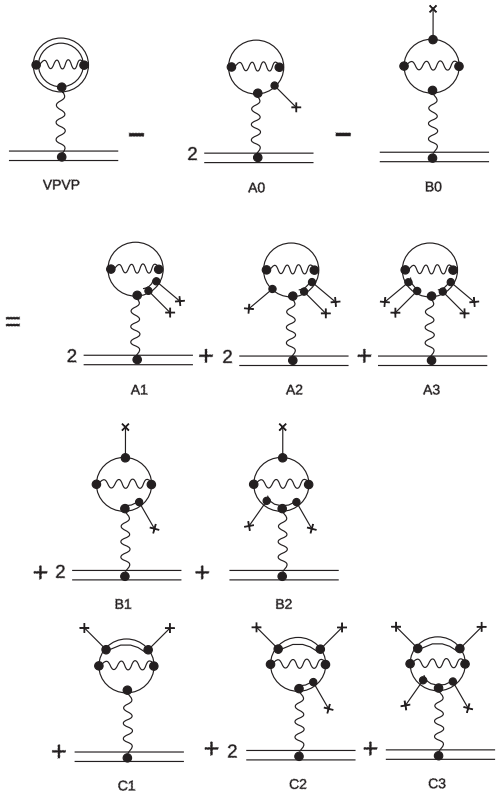


**Fig. 6.** The potential expansion of the unrenormalized expression (Fig. 3).

If one uses the Ward-identity equation (16), Furry's theorem and the potential expansion shown in Figure 2, it is not difficult to realize that all the terms which are proportional to $\Sigma^{(1)'}$ are cancelled against the terms which are proportional to $\Lambda^{(1)}$. Especially, it holds the following relations for these counter terms:

$$2\,\Lambda^{(1)}\,\mathrm{CT(B1)} + 2\,\Sigma^{(1)'}\,\mathrm{CT(A1)} = 0,$$
$$\Lambda^{(1)}\,\mathrm{CT(B2)} + \Lambda^{(1)}\,\mathrm{CT(C1)} + 2\,\Sigma^{(1)'}\,\mathrm{CT(A2)} + \Sigma^{(1)'}\,\mathrm{CT(A3)} = 0. \quad (20)$$

Finally, if one collects all terms proportional to $\Sigma^{(1)}$ and retransforms the potential expansion one ends up with the renormalized expression for the total energy shift of the VPVP, see Figure 8.

## 4 The renormalized expression of VPVP

Now we proceed to write down the mathematical expression of the renormalized vacuum polarization correction. The energy shift in higher-order reads:

$$\Delta E_n^{\mathrm{VPVP\ ren\ h.o.}} = \int \mathrm{d}^3\mathbf{r}_1\, \overline{\varphi}_n(\mathbf{r}_1)\, \gamma_0\, \varphi_n(\mathbf{r}_1) \times \Big[\hat{U}^{\mathrm{VPVP}}(\mathbf{r}_1) - 2\hat{U}^{\mathrm{A0}}(\mathbf{r}_1) - \hat{U}^{\mathrm{B0}}(\mathbf{r}_1) - \Sigma^{(1)}\Big(\hat{U}^{\mathrm{CT1}}(\mathbf{r}_1) - 2\hat{U}^{\mathrm{CT2}}(\mathbf{r}_1)\Big)\Big]\cdot \quad (21)$$

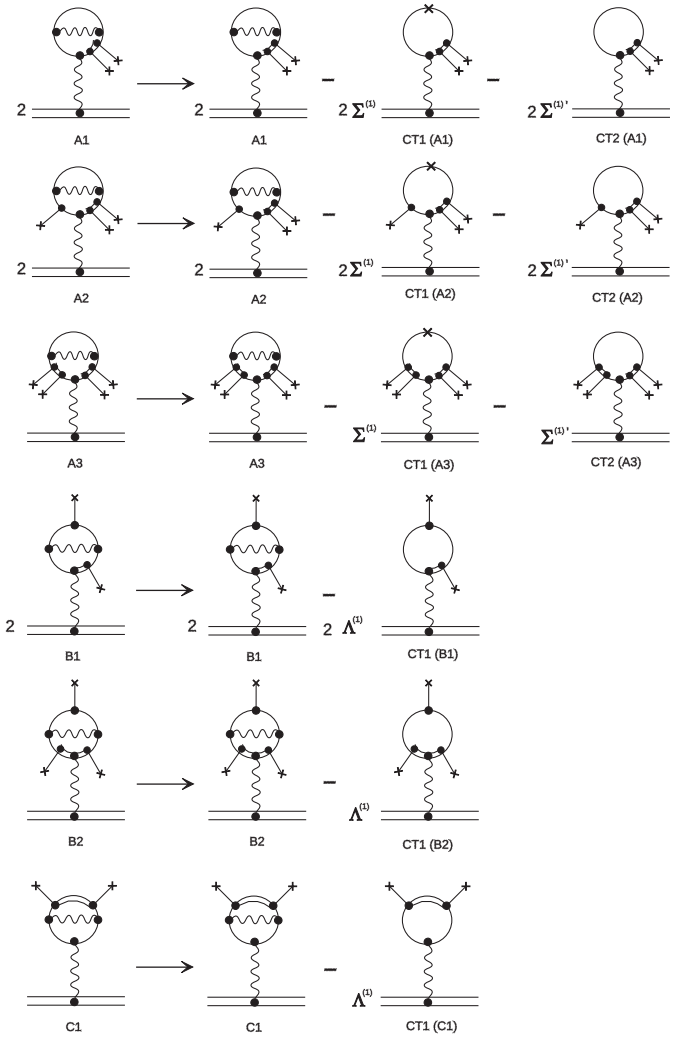


**Fig. 7.** The inner renormalization according to equation (18).

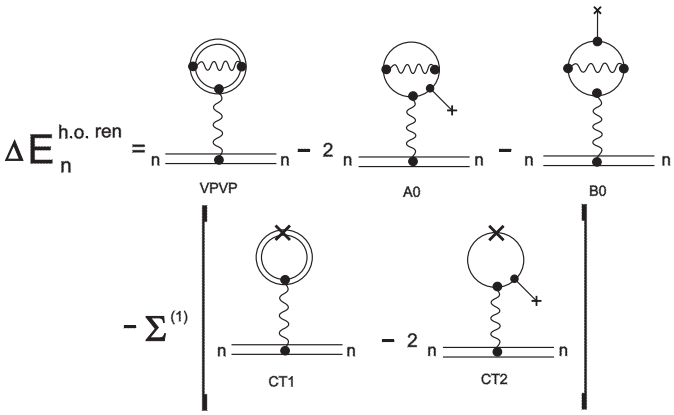


**Fig. 8.** This is the final expression for the renormalization of the VPVP diagram if one subtracts the Källén-Sabry-terms.

It is depicted in Figure 8. The potentials are given by (note that $\Sigma^{(1)}$ is proportional to $e^2$)

$$\hat{U}^{\mathrm{VPVP}}(\mathbf{r}_1) = e^4 \int \mathrm{d}^3\mathbf{r}_2 \int \mathrm{d}^3\mathbf{r}_3 \int \mathrm{d}^3\mathbf{r}_4 \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}E_1}{2\pi} \times \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}E_2}{2\pi} D^{00}(E_1, \mathbf{r}_1 - \mathbf{r}_2) D^{\alpha\beta}(E_2, \mathbf{r}_3 - \mathbf{r}_4) \times \mathrm{Tr}\left[\gamma_0 S_F(E_1, \mathbf{r}_2, \mathbf{r}_3)\gamma_\alpha S_F(E_1 - E_2, \mathbf{r}_3, \mathbf{r}_4) \times \gamma_\beta S_F(E_1, \mathbf{r}_4, \mathbf{r}_2)\right]. \quad (22)$$

$$\hat{U}^{\mathrm{A0}}(\mathbf{r}_1) = e^4 \int \mathrm{d}^3\mathbf{r}_2\, \mathrm{d}^3\mathbf{r}_3\, \mathrm{d}^3\mathbf{r}_4\, \mathrm{d}^3\mathbf{r}_5 \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}E_1}{2\pi} \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}E_2}{2\pi} \times D^{00}(\mathbf{r}_1 - \mathbf{r}_2, 0) D^{\alpha\beta}(\mathbf{r}_3 - \mathbf{r}_4, E_2) V(\mathbf{r}_5) \times \mathrm{Tr}\Big[S_F^0(\mathbf{r}_2 - \mathbf{r}_3, E_1)\gamma_\alpha S_F^0(\mathbf{r}_3 - \mathbf{r}_4, E_1 - E_2) \times \gamma_\beta S_F^0(\mathbf{r}_4 - \mathbf{r}_5, E_1) S_F^0(\mathbf{r}_5 - \mathbf{r}_2, E_1)\Big], \quad (23)$$

$$\hat{U}^{\mathrm{B0}}(\mathbf{r}_1) = e^4 \int \mathrm{d}^3\mathbf{r}_2\, \mathrm{d}^3\mathbf{r}_3\, \mathrm{d}^3\mathbf{r}_4\, \mathrm{d}^3\mathbf{r}_5 \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}E_1}{2\pi} \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}E_2}{2\pi} \times D^{00}(\mathbf{r}_1 - \mathbf{r}_2, 0) D^{\alpha\beta}(\mathbf{r}_3 - \mathbf{r}_4, E_2) V(\mathbf{r}_5) \times \mathrm{Tr}\Big[S_F^0(\mathbf{r}_2 - \mathbf{r}_3, E_1)\gamma_\alpha S_F^0(\mathbf{r}_3 - \mathbf{r}_5, E_1 - E_2) \times S_F^0(\mathbf{r}_5 - \mathbf{r}_4, E_1 - E_2)\gamma_\beta S_F^0(\mathbf{r}_4 - \mathbf{r}_2, E_1)\Big], \quad (24)$$

$$\hat{U}^{\mathrm{CT1}}(\mathbf{r}_1) = e^2 \int \mathrm{d}^3\mathbf{r}_2\, \mathrm{d}^3\mathbf{r}_3 \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}E}{2\pi} D^{00}(\mathbf{r}_1 - \mathbf{r}_2, 0) \times \mathrm{Tr}\left[S_F(\mathbf{r}_2, \mathbf{r}_3, E)\gamma_0 S_F(\mathbf{r}_3, \mathbf{r}_2, E)\right], \quad (25)$$

$$\hat{U}^{\mathrm{CT2}}(\mathbf{r}_1) = e^2 \int \mathrm{d}^3\mathbf{r}_2\, \mathrm{d}^3\mathbf{r}_3\, \mathrm{d}^3\mathbf{r}_4 \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}E}{2\pi} D^{00}(\mathbf{r}_1 - \mathbf{r}_2, 0) \times V(\mathbf{r}_3)\mathrm{Tr}\left[\gamma_0 S_F^0(\mathbf{r}_2 - \mathbf{r}_3, E) S_F^0(\mathbf{r}_3 - \mathbf{r}_4, E) \times S_F^0(\mathbf{r}_4 - \mathbf{r}_2, E)\right]. \quad (26)$$

In the expressions above $S_F$ denotes the bound-electron propagator and $S_F^0$ denotes the free-electron propagator, respectively, and $D^{\mu\nu}$ is the photon propagator (see Appendix).

## 5 The fourth-rank vacuum polarization tensor

In order to derive a renormalized expression for the self energy vacuum polarization S(VP)E it is useful to mention some properties of the fourth-rank vacuum polarization tensor. Especially we reexamine some features of this tensor which are related to the gauge invariance of the

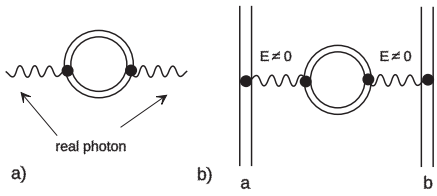


**Fig. 9.** The diagrams of Delbrück scattering (a) and energy splitting (b).

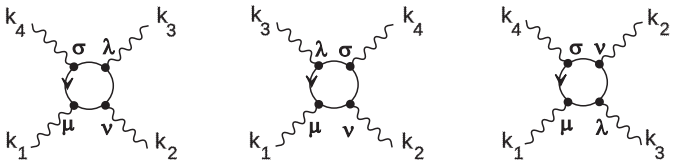


**Fig. 10.** The diagrams of light-light scattering.

vacuum polarization diagram VP. The fourth-rank vacuum polarization tensor is defined as amplitude of the light-light scattering which is shown in Figure 10. The expression of the unrenormalized tensor reads:

$$\hat{\Pi}^{\mathrm{unren}}_{\mu,\nu,\sigma,\lambda}(k_1, k_2, k_3, m) = 2\Big[\hat{T}^{\mathrm{unren}}_{\mu,\nu,\lambda,\sigma}(k_1, k_2, k_3, m) + \hat{T}^{\mathrm{unren}}_{\mu,\nu,\sigma,\lambda}(k_1, k_2, -k_1 - k_2 - 2k_3, m) + \hat{T}^{\mathrm{unren}}_{\mu,\lambda,\nu,\sigma}(k_1, k_3, k_2, m)\Big], \quad (27)$$

where the single three tensors $\hat{T}^{\mathrm{unren}}$ are corresponding to the expressions belonging to the diagrams depicted in Figure 10 and they are defined by

$$\hat{T}^{\mathrm{unren}}_{\mu,\nu,\lambda,\sigma}(k_1, k_2, k_3, m) = \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \mathrm{Tr}\left(\gamma_\mu \frac{1}{\not{q} + \not{k}_1 - m + \mathrm{i}\epsilon} \times \gamma_\nu \frac{1}{\not{q} + \not{k}_1 + \not{k}_2 - m + \mathrm{i}\epsilon} \gamma_\lambda \frac{1}{\not{q} + \not{k}_1 + \not{k}_2 + \not{k}_3 - m + \mathrm{i}\epsilon} \times \gamma_\sigma \frac{1}{\not{q} - m + \mathrm{i}\epsilon}\right). \quad (28)$$

For our purposes it is necessary to consider the general case where the external momenta $k_i$ are not on mass shell ($k_i^2 \neq 0$). The general structure of the solution for this tensor is complicated and was derived at the first time by Karplus, Neumann in [20]. For our intension here it is only important that the single tensors $\hat{T}^{\mathrm{unren}}_{\mu,\nu,\sigma,\lambda}$ are logarithmically divergent but their sum $\hat{\Pi}^{\mathrm{unren}}_{\mu,\nu,\sigma,\lambda}$ is convergent. This surprising behaviour of $\hat{\Pi}^{\mathrm{unren}}_{\mu,\nu,\sigma,\lambda}$ has been proven a long time ago [21,22], see also [18]. Therefore a renormalization of $\hat{\Pi}^{\mathrm{unren}}_{\mu,\nu,\sigma,\lambda}$ seems to be not necessary, which, however, is not true. The vacuum polarization tensor $\hat{\Pi}^{\mathrm{unren}}_{\mu,\nu,\sigma,\lambda}$ is not gauge invariant and therefore a renormalization is necessary. Following the BPHZ-renormalization method described in Section 2, we obtain the renormalized expression

$$\hat{\Pi}^{\mathrm{ren}}_{\mu,\nu,\sigma,\lambda}(k_1, k_2, k_3, m) = \hat{\Pi}^{\mathrm{unren}}_{\mu,\nu,\sigma,\lambda}(k_1, k_2, k_3, m) - \hat{\Pi}^{\mathrm{unren}}_{\mu,\nu,\sigma,\lambda}(0, 0, 0, m). \quad (29)$$

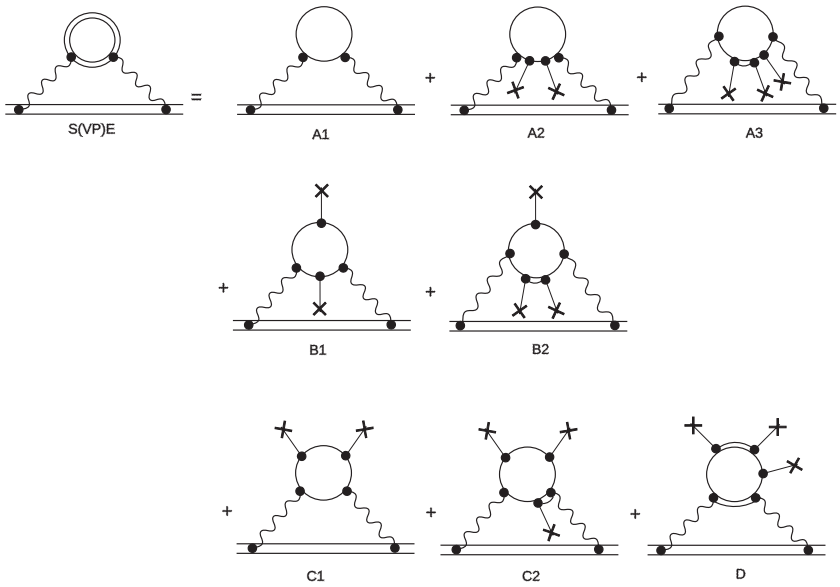


**Fig. 11.** The potential expansion of the S(VP)E diagram. The diagrams A3, B2, C2 and D are convergent. The diagrams A2, B1 and C1 are separately divergent but their sum is convergent but not gauge invariant. Diagram A1 is the so-called Uehling contribution of the S(VP)E.

Gauge invariance implies (see for example [14,18,22,23] and references therein)

$$\begin{aligned} k_1^\mu\, \hat{\Pi}^{\rm ren}_{\mu,\nu,\sigma,\lambda}(k_1,k_2,k_3,m) &= 0,\\ k_2^\nu\, \hat{\Pi}^{\rm ren}_{\mu,\nu,\sigma,\lambda}(k_1,k_2,k_3,m) &= 0,\\ k_3^\sigma\, \hat{\Pi}^{\rm ren}_{\mu,\nu,\sigma,\lambda}(k_1,k_2,k_3,m) &= 0. \end{aligned} \tag{30}$$

In other words the counter term $\hat{\Pi}^{\rm unren}_{\mu,\nu,\sigma,\lambda}(0,0,0,m)$ ensure the gauge invariance of the fourth-rank vacuum polarization tensor. We note that the regularized tensor

$$\begin{aligned} \hat{\Pi}^{\rm reg}_{\mu,\nu,\sigma,\lambda}(k_1,k_2,k_3,m,M) = {}& \hat{\Pi}^{\rm unren}_{\mu,\nu,\sigma,\lambda}(k_1,k_2,k_3,m)\\ & - \hat{\Pi}^{\rm unren}_{\mu,\nu,\sigma,\lambda}(k_1,k_2,k_3,M), \end{aligned} \tag{31}$$

is also gauge invariant ($M$ denotes a large electron mass of the Pauli-Villars-regularization). A renormalization of the regularized tensor in equation (31) is not necessary because the counter term $\hat{\Pi}^{\rm unren}_{\mu,\nu,\sigma,\lambda}(0,0,0,m)$ would vanish if one performs such a regularization. Sometimes the subtracted term $\hat{\Pi}^{\rm unren}_{\mu,\nu,\sigma,\lambda}(k_1,k_2,k_3,M)$ in equation (31), which ensures the gauge invariance like the counter term in equation (29), is called spurious term. Similarly we can consider the counter term $\hat{\Pi}^{\rm unren}_{\mu,\nu,\sigma,\lambda}(0,0,0,m)$ in equation (29) as a spurious term.

There are regularization methods, in which the counter term $\hat{\Pi}^{\rm unren}_{\mu,\nu,\sigma,\lambda}(0,0,0,m)$ in equation (29) or the subtracted term $\hat{\Pi}^{\rm unren}_{\mu,\nu,\sigma,\lambda}(k_1,k_2,k_3,M)$ in equation (31) will vanish [17,19]. It means that these terms are not necessary in such cases in order to ensure gauge invariance. Because of this it is important to note that both approaches alone, equation (29) as well as equation (31), ensure the gauge invariance of the fourth-rank vacuum polarization tensor. Therefore is does not play any role which term will vanish. However, because of the simpler mathematical structure of the spurious term of equation (29) in comparison with the spurious term of equation (31) it turns out to be much easier to use the approach of equation (29) instead of equation (31). As we will see in the next section the spherical expansion of electron propagator is such a special condition where the counter term in equation (29) will vanish.

## 6 Renormalization of the S(VP)E diagram and disappearance of the spurious term

In order to isolate the ultraviolet divergencies of the S(VP)E diagram we first have to perform a potential expansion of the bound electron-propagator according to equation (1) as we have seen already in the case of the VPVP. For the diagram of the S(VP)E such an appropriate potential expansion is shown in Figure 11. The diagrams A3, B2, C2 and D are convergent. The sum of diagrams A2, B1 and C1 are nothing else but the fourth-rank vacuum polarization tensor. Therefore they are separately divergent but their sum is convergent but not gauge invariant as we have discussed in the previous section. It should be mention here that only for hydrogen an evaluation of the diagrams A2, B1 and C1 was performed (see [24] and references therein). At present, for all other elements of the periodic system their contribution is unknown. Diagram A1 is the so-called Uehling contribution of the S(VP)E. This Uehling contribution was evaluated for low- and high-$Z$ atomic systems few years ago [25] and gave the leading contribution at least for high-$Z$-systems like uranium and lead.

Now we proceed to prove the disappearance of counter terms of diagrams A2, B1 and C1 if one uses the spherical expansion of the free-electron propagator (see Appendix). Their corresponding energy shift is given by:

$$E_n^{(\mathrm{A2})+(\mathrm{B1})+(\mathrm{C1})} = e^4 \int_{-\infty}^{\infty} \frac{\mathrm{d}E_1}{2\pi} \int \frac{\mathrm{d}^3\mathbf{k}_1}{(2\pi)^3} \int \frac{\mathrm{d}^3\mathbf{k}_3}{(2\pi)^3} V(\mathbf{k}_3)$$
$$\times \int \frac{\mathrm{d}^3\mathbf{k}_4}{(2\pi)^3} V(\mathbf{k}_4) \frac{1}{E_1^2 - {\mathbf{k}_1}^2 + \mathrm{i}\epsilon} \frac{1}{E_1^2 - (\mathbf{k}_1 - \mathbf{k}_3 - \mathbf{k}_4)^2 + \mathrm{i}\epsilon}$$
$$\times \frac{1}{E_n - E_1 - E_m(1 - \mathrm{i}\epsilon)} \sum_m \int \mathrm{d}^3\mathbf{r}_1\, \varphi_n(\mathbf{r}_1) \alpha^\mu \mathrm{e}^{\mathrm{i}\mathbf{k}_1\mathbf{r}_1} \varphi_m(\mathbf{r}_1)$$
$$\times \int \mathrm{d}^3\mathbf{r}_2\, \varphi_m(\mathbf{r}_2) \alpha^\nu \mathrm{e}^{-\mathrm{i}(\mathbf{k}_1 + \mathbf{k}_3 + \mathbf{k}_4)\mathbf{r}_2} \varphi_n(\mathbf{r}_2)$$
$$\times \hat{\Pi}_{\mu,\nu,0,0}(E_1, \mathbf{k}_1; E_1, \mathbf{k}_1 + \mathbf{k}_3 + \mathbf{k}_4; 0, \mathbf{k}_3; 0, \mathbf{k}_4; m). \quad (32)$$

Using the free-electron propagator

$$S_F^0(E, \mathbf{r}_1 - \mathbf{r}_2) = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \sum_{i=1}^{4} \frac{\varphi_i(\mathbf{k}) \mathrm{e}^{\mathrm{i}\mathbf{k}\,\mathbf{r}_1} \varphi_i^\dagger(\mathbf{k}) \mathrm{e}^{-\mathrm{i}\mathbf{k}\,\mathbf{r}_2}}{E - E_k(1 - \mathrm{i}\epsilon)}, \quad (33)$$

where the functions $\varphi_i(\mathbf{k})$ are the solutions of the free Dirac equation in momentum space:

$$(\not{k} - m)\, \varphi_i(\mathbf{k}) = 0, \quad \varphi_{i_1}^\dagger(\mathbf{k}) \varphi_{i_2}(\mathbf{k}) = \delta_{i_1,i_2}, \quad (34)$$

(here the indices $i$ indicate the four linear independent solutions of the free Dirac equation) one may derive the relation

$$\frac{1}{E_2\gamma_0 - \mathbf{k}_2\boldsymbol{\gamma} - m + \mathrm{i}\epsilon} = \sum_{i=1}^{4} \frac{\varphi_i(\mathbf{k}_2) \varphi_i^\dagger(\mathbf{k}_2)}{E_2 - E_{k_2,i}(1 - \mathrm{i}\epsilon)} \cdot \quad (35)$$

The last three equations lead immediately to the following expression for the counter term of the fourth-rank vacuum polarization tensor in equation (32):

$$\hat{\Pi}_{\mu,\nu,0,0}(0,0,0,0,m) = \frac{1}{2} \int_{-\infty}^{\infty} \frac{\mathrm{d}E_2}{2\pi} \int \mathrm{d}^3(\mathbf{r}_1 - \mathbf{r}_2)$$
$$\times \mathrm{Tr}\left[ \frac{\partial^2}{\partial E_2^2} \left( \alpha_\mu S_F^0(E_2, \mathbf{r}_1 - \mathbf{r}_2) \alpha_\nu S_F^0(E_2, \mathbf{r}_2 - \mathbf{r}_1) \right) \right] \cdot \quad (36)$$

Using the identity ($\hat{H}_0^{\mathrm{Dirac}}$ is the free Dirac operator without Coulomb potential)

$$\alpha_j = \left( \hat{H}_0^{\mathrm{Dirac}} - E \right) r_j - r_j \left( \hat{H}_0^{\mathrm{Dirac}} - E \right), \quad j = 1, 2, 3 \quad (37)$$

and the relation

$$\left( \hat{H}_0^{\mathrm{Dirac}} - E \right) S_F^0(E, \mathbf{r}_1 - \mathbf{r}_2) = \delta(\mathbf{r}_1 - \mathbf{r}_2), \quad (38)$$

and their conjugate version, respectively, it is not difficult to see that the spatial components in equation (36) do not contribute. Finally, with $\alpha_0 = \mathbb{1}$ we get

$$\hat{\Pi}_{\mu=0,\nu=0,0,0}(0,0,0,0,m) =$$
$$\frac{1}{6} \int_{-\infty}^{\infty} \frac{\mathrm{d}E_2}{2\pi} \mathrm{Tr}\left[ \frac{\partial^3}{\partial E_2^3} \left( S_F^0(E_2, \mathbf{r}_1 - \mathbf{r}_1) \right) \right] \cdot \quad (39)$$

If one inserts the spherical expansion of the free-electron propagator (see Appendix) one ends up with

$$\hat{\Pi}_{\mu=0,\nu=0,0,0}(0,0,0,0,m) = \frac{1}{6} \sum_\kappa \frac{|\kappa|}{2\pi} \int_{-\infty}^{\infty} \frac{\mathrm{d}E_2}{2\pi} \frac{\partial^3}{\partial E_2^3}$$
$$\times \left[ G_{0,\kappa}^{11}(E_2, r_1, r_1) + G_{0,\kappa}^{22}(E_2, r_1, r_1) \right] \cdot \quad (40)$$

The same expression was derived and investigated already in [17] where it has been shown that this expression vanishes if the summation over $\kappa$ is restricted to a finite number of terms. This can be seen by inserting the explicit expression of the spherical expansion where one gets after a Wick-rotation the following expression:

$$\hat{\Pi}_{\mu=0,\nu=0,0,0}(0,0,0,0,m) = \frac{1}{6} \sum_\kappa \frac{|\kappa|}{2\pi} \int_{-\infty}^{\infty} \frac{\mathrm{d}E_2}{2\pi} \frac{\partial^3}{\partial E_2^3}$$
$$\times \mathrm{Re}\left( \mathrm{i} \left[ G_{0,\kappa}^{11}(\mathrm{i}E_2) + G_{0,\kappa}^{22}(\mathrm{i}E_2) \right] \right) = -\frac{1}{6} \sum_\kappa \frac{|\kappa|}{2\pi} \frac{\partial^2}{\partial E_2^2}$$
$$\times \left[ (\mathrm{i}E_2 + 1) \sqrt{1 + E_2^2}\, j_{|\kappa+1/2|-1/2} \left( \mathrm{i}\sqrt{1 + E_2^2}\, r_1 \right) \right.$$
$$\times h_{|\kappa+1/2|-1/2}^{(1)} \left( \mathrm{i}\sqrt{1 + E_2^2}\, r_1 \right) + (\mathrm{i}E_2 - 1) \sqrt{1 + E_2^2}$$
$$\times j_{|\kappa-1/2|-1/2} \left( \mathrm{i}\sqrt{1 + E_2^2}\, r_1 \right) h_{|\kappa-1/2|-1/2}^{(1)}$$
$$\left. \times \left( \mathrm{i}\sqrt{1 + E_2^2}\, r_1 \right) \right]_{E_2=-\infty}^{E_2=\infty} \cdot \quad (41)$$

Here $j_{|\kappa\pm1/2|-1/2}(z)$ and $h_{|\kappa\pm1/2|-1/2}^{(1)}(z)$ are the spherical Bessel function and spherical Hankel function of first kind, respectively. In the last line we used the Gauss law in Euclidean space. It can readily be seen that the last expression vanishes if one performs both derivations. So it has been proven that the counter term is zero if one uses the spherical expansion of the electron propagator where one has to sum over $\kappa$ in the last step. This result simplifies essentially the evaluation of arbitrary diagrams of bound-state-QED which include the one-loop-vacuum polarization.

## 7 The renormalized expression of S(VP)E

As we have seen in the previous section the counter term vanishes and therefore it is only necessary to subtract the

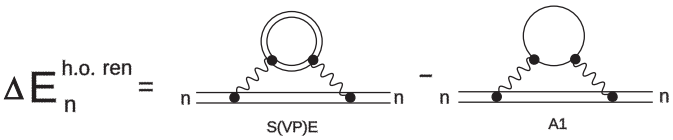


**Fig. 12.** The renormalized expression of energy shift of higher-order of the self energy vacuum polarization.

known Uehling contribution of the S(VP)E diagram. The renormalized expression of the energy correction of higher order is shown in Figure 12 and the corresponding energy shift reads:

$$\Delta E_n^{\mathrm{S(VP)E}} = e^4 \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}E_1}{2\pi} \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}E_2}{2\pi} \int \frac{\mathrm{d}^3\mathbf{r}_1}{(2\pi)^3} \int \frac{\mathrm{d}^3\mathbf{r}_2}{(2\pi)^3}$$
$$\times \int \frac{\mathrm{d}^3\mathbf{r}_3}{(2\pi)^3} \int \frac{\mathrm{d}^3\mathbf{r}_4}{(2\pi)^3} \varphi_n^\dagger(\mathbf{r}_1)\alpha_\mu S_F(E_n - E_1, \mathbf{r}_1, \mathbf{r}_2)\alpha_\sigma \varphi_n(\mathbf{r}_2)$$
$$\times D^{\mu\nu}(E_1, \mathbf{r}_1, \mathbf{r}_3) D^{\rho\sigma}(E_1, \mathbf{r}_4, \mathbf{r}_2)$$
$$\times \mathrm{Tr}\left[\alpha_\nu S_F(E_1 + E_2, \mathbf{r}_3, \mathbf{r}_4)\alpha_\rho S_F(E_2, \mathbf{r}_4, \mathbf{r}_3)\right] \cdot \qquad (42)$$

The mathematical expression for the subtracted Uehling-diagram can be obtained from equation (42) by the replacement of the bound-electron propagator under the trace by the free-electron propagator.

## 8 Summary

In this article, using the Bogoljubov-Parasiuk-Hepp-Zimmermann (BPHZ)-renormalization method, we derived renormalized expressions for the energy shift of higher-order for the last two unknown second-order diagrams of bound state QED, the two-photon vacuum polarization VPVP and self energy vacuum polarization S(VP)E, respectively. It was shown in some detail that counter terms of the outer renormalization for the VPVP diagram are cancelled against each other. This result simplifies significantly numerical evaluations of this diagram.

In a second part of this paper it has been proven for a wide class of diagrams for bound-state-QED which contains the one-loop vacuum polarization VP that the counter term of the fourth-rank vacuum polarization tensor vanishes since the sum over the quantum number $\kappa$ has been performed until a finite $\kappa_{\max}$ and as the last step in the spherical expansion of the electron propagator. It has been discussed in some detail that gauge invariance is not broken in such cases. The disappearance of the counter term simplify essentially the evaluation of such diagrams which contain the diagram VP. The investigations presented here are to be considered as generalization of corresponding investigations presented in [17,19]. We also mention that the numerical evaluation of VPVP and S(VP)E for hydrogen-like ions has to be consider as important step in the future to reduce the uncertainty in Lamb shift predictions and still is under consideration. Simultaneously, the results of this paper establish the use of the BPHZ-approach in bound state QED.

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## Appendix

The bound-electron propagator is given by

$$S_F(E, \mathbf{r}_1, \mathbf{r}_2) = \sum_{n,\kappa,\mu} \frac{\varphi_{n,\kappa,\mu}(\mathbf{r}_1)\overline{\varphi}_{n,\kappa,\mu}(\mathbf{r}_2)}{E - E_{n,\kappa}(1 - \mathrm{i}\epsilon)}, \qquad (43)$$

where $\varphi$ are solutions of the Dirac equation with Coulomb potential:

$$(-\mathrm{i}\boldsymbol{\alpha}\nabla + \beta + V(\mathbf{r}))\,\varphi(\mathbf{r}) = E\varphi(\mathbf{r}). \qquad (44)$$

The free-electron propagator of equation (33) can be rewritten in a similar form as the bound-electron propagator:

$$S_F^0(E, \mathbf{r}_1 - \mathbf{r}_2) = \int\limits_0^\infty \mathrm{d}p \sum_{\kappa,\mu} \sum_{s=1}^{2} \frac{\psi_{s,\kappa,\mu}(\mathbf{r}_1)\overline{\psi}_{s,\kappa,\mu}(\mathbf{r}_2)}{E - E_{s,p}(1 - \mathrm{i}\epsilon)}, \qquad (45)$$

where the index $s$ signifies the positive and negative energy states of a free electron

$$E_{s,p} = \pm\sqrt{m^2 + \mathbf{p}^2}, \quad (s = 1, 2), \qquad (46)$$

and $\Psi$ are the solutions of the free Dirac equation:

$$(-\mathrm{i}\boldsymbol{\alpha}\nabla + \beta)\,\Psi(\mathbf{r}) = E\,\Psi(\mathbf{r}). \qquad (47)$$

The free-electron propagator in spherical expansion can be written as [26]:

$$S_F^0(E, \mathbf{r}_1 - \mathbf{r}_2) =$$
$$\sum_\kappa \begin{pmatrix} G_{0,\kappa}^{11}(E, r_1, r_2)\ \pi_\kappa^{11} & G_{0,\kappa}^{12}(E, r_1, r_2)\ \pi_\kappa^{12} \\ G_{0,\kappa}^{21}(E, r_1, r_2)\ \pi_\kappa^{21} & G_{0,\kappa}^{22}(E, r_1, r_2)\ \pi_\kappa^{22} \end{pmatrix}, \qquad (48)$$

with the radial components (for $r_1 > r_2$)

$$G_{0,\kappa}^{11}(E, r_1, r_2) = -(\mathrm{i}E + 1)\sqrt{E^2+1}\, j_{|\kappa+1/2|-1/2} \times \left(\mathrm{i}r_1\sqrt{E^2+1}\right) h^{(1)}_{|\kappa+1/2|-1/2}\left(\mathrm{i}r_2\sqrt{E^2+1}\right),$$
$$G_{0,\kappa}^{12}(E, r_1, r_2) = -\mathrm{i}\left(E^2+1\right)\mathrm{sign}(\kappa)\, j_{|\kappa+1/2|-1/2} \times \left(\mathrm{i}r_1\sqrt{E^2+1}\right) h^{(1)}_{|\kappa-1/2|-1/2}\left(\mathrm{i}r_2\sqrt{E^2+1}\right),$$
$$G_{0,\kappa}^{21}(E, r_1, r_2) = -\mathrm{i}\left(E^2+1\right)\mathrm{sign}(\kappa)\, j_{|\kappa-1/2|-1/2} \times \left(\mathrm{i}r_1\sqrt{E^2}+1\right) h^{(1)}_{|\kappa+1/2|-1/2}\left(\mathrm{i}r_2\sqrt{E^2+1}\right),$$
$$G_{0,\kappa}^{22}(E, r_1, r_2) = -(\mathrm{i}E - 1)\sqrt{E^2+1}\, j_{|\kappa-1/2|-1/2} \times \left(\mathrm{i}r_1\sqrt{E^2+1}\right) h^{(1)}_{|\kappa-1/2|-1/2}\left(\mathrm{i}r_2\sqrt{E^2+1}\right). \qquad (49)$$

For the case $r_2 > r_1$ one simply can employ the symmetry relations:

$$\begin{aligned} G^{11}_{0,\kappa}(E,r_1,r_2) &= G^{11}_{0,\kappa}(E,r_2,r_1),\\ G^{12}_{0,\kappa}(E,r_1,r_2) &= G^{21}_{0,\kappa}(E,r_2,r_1),\\ G^{21}_{0,\kappa}(E,r_1,r_2) &= G^{12}_{0,\kappa}(E,r_2,r_1),\\ G^{22}_{0,\kappa}(E,r_1,r_2) &= G^{22}_{0,\kappa}(E,r_2,r_1). \end{aligned} \tag{50}$$

In equation (48) $\pi^{ij}_\kappa$ denote the spin-angular functions which are given by:

$$\begin{aligned} \pi^{11}_\kappa &= \sum_\mu \chi^\mu_\kappa(\theta_1,\varphi_1)\ \chi^{\mu\,\dagger}_\kappa(\theta_2,\varphi_2)\\ \pi^{12}_\kappa &= \sum_\mu \chi^\mu_\kappa(\theta_1,\varphi_1)\ \chi^{\mu\,\dagger}_{-\kappa}(\theta_2,\varphi_2)\\ \pi^{21}_\kappa &= \sum_\mu \chi^\mu_{-\kappa}(\theta_1,\varphi_1)\ \chi^{\mu\,\dagger}_\kappa(\theta_2,\varphi_2)\\ \pi^{22}_\kappa &= \sum_\mu \chi^\mu_{-\kappa}(\theta_1,\varphi_1)\ \chi^{\mu\,\dagger}_{-\kappa}(\theta_2,\varphi_2), \end{aligned} \tag{51}$$

and $\chi^\mu_\kappa$ is defined as:

$$\chi^\mu_\kappa(\theta,\phi) = \begin{pmatrix} -\dfrac{\kappa}{|\kappa|}\sqrt{\dfrac{\kappa+\frac{1}{2}-\mu}{2\kappa+1}}\ Y_{|\kappa+\frac{1}{2}|-\frac{1}{2},\mu-\frac{1}{2}}(\theta,\phi)\\[2ex] \sqrt{\dfrac{\kappa+\frac{1}{2}+\mu}{2\kappa+1}}\ Y_{|\kappa+\frac{1}{2}|-\frac{1}{2},\mu+\frac{1}{2}}(\theta,\phi) \end{pmatrix}, \tag{52}$$

where $Y_{|\kappa+\frac{1}{2}|-\frac{1}{2},\mu\pm\frac{1}{2}}(\theta,\phi)$ are the spherical harmonics. The photon propagator reads in Feynman gauge:

$$\begin{aligned} D_{\mu\nu}(t_1-t_2,\mathbf{r}_1-\mathbf{r}_2) = -g_{\mu\nu}\int\limits_{-\infty}^{\infty}\frac{\mathrm{d}E}{2\pi}\\ \times\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\frac{\mathrm{e}^{-\mathrm{i}[E(t_1-t_2)-\mathbf{k}(\mathbf{r}_1-\mathbf{r}_2)]}}{E^2-\mathbf{k}^2+\mathrm{i}\epsilon}\,. \end{aligned} \tag{53}$$

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