Radiative Corrections in Highly Charged Ions and Tests of QED in Strong Electric and Magnetic Fields

I. A. Goidenko[1], L. N. Labzowsky[1], A.V. Nefiodov[2], U. D. Jentschura[3], G. Plunien[3], S. Zschocke[3] and G. Soff[3]

[1] St. Petersburg State University, St. Petersburg, Russia
[2] Petersburg Nuclear Physics Institute, St. Petersburg, Russia
[3] Technische Universität Dresden, Dresden, Germany

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Abstract

To provide predictions of the Lamb shift of highly charged ions on the level of accuracy of about 10^{-6} has been achieved after exact results for the contributions of all two-photon self-energy diagrams have been performed. We report on the present status of our Lamb-shift calculations including all QED-corrections of first- and second-order in the finestructure constant α and all relevant nuclear effects. An excellent agreement between the most recent experimental data for Lamb shift of the ls-ground state in hydrogenlike uranium can be stated. This can serve as a sensitive test of QED in the strongest electric fields accessible in nature. In a second part of this article we report about an all-order numerical evaluation of the one-photon selfenergy at low nuclear charge ($Z = 1, \ldots, 5$). A sensitive comparison of our numerical approach with analytical approach to the one-photon selfenergy confirms the consistency of these two different approaches (numerical and analytic) to very high precision.

A precise knowledge concerning the validity of quantum electrodynamic (QED) in strong external fields is crucial for the detection of new physics beyond QED. The possible test of QED in strong external electric and magnetic fields has initiated intensive theoretical and experimental activities. Measurements of the Lamb shift and the hyperfine structure splitting in one–electron systems at utmost precision are considered as ideal scenarios for this purpose. In this paper we will report on the present status and recent calculations with respect to QED corrections to the Lamb shift for the one–electron systems uranium and lead as well as for hydrogen.

At the SIS/ESR facility at GSI in Darmstadt one is aiming at an accuracy of about 1eV in measurements of the ground-state Lamb shift for hydrogen–like uranium [1,2]. Accordingly, to provide predictions for the ground-state energy in H-like uranium with relative accuracy of about 10^{-5} or even 10^{-6} – this ultimate limit is set by nuclear polarization effects and uncertainties in the parameters – necessitates the evaluation of all QED corrections of order α^2 but to all orders $Z\alpha$ in the interaction with the Coulomb field of the nucleus. We performed a complete calculation for the lead system as well. This system seems more favorable for precision tests of QED in strong fields since nuclear uncertainties are expected to be about one order of magnitude smaller.

The complete set of second-order radiative corrections are displayed in Fig. 1. These diagrams are naturally divided into separately gauge invariant subsets: the two-photon self–energy SESE (a),(b),(c), the second-order vacuum polarization VPVP (d),(e),(f), the mixed self-energy vacuum polarization SEVP (g),(h) and the effective self-energy S(VP)E (i). The abbreviation SE stands for self energy and VP denotes vacuum polarization. Most of these corrections have been already calculated numerically for $^{238}U^{91+}$ and $^{208}Pb^{81+}$ ions [3,4].

The contributions of the two-photon self energy SESE (a), SESE (b) and SESE (c) were considered to represent the major theoretical uncertainty.

The diagram SESE (a) consists of an irreducible (*irred.*) and a reducible (*red.*) part. The contribution SESE (a) (irred.) can be renormalized and evaluated separately. It has been calculated first in Ref. [5] for high nuclear charge numbers Z and recently for arbitrary values of Z in Refs. [6,7]. Although the results obtained are in fair agreement for high Z a discrepancy between those in Refs. [6] and

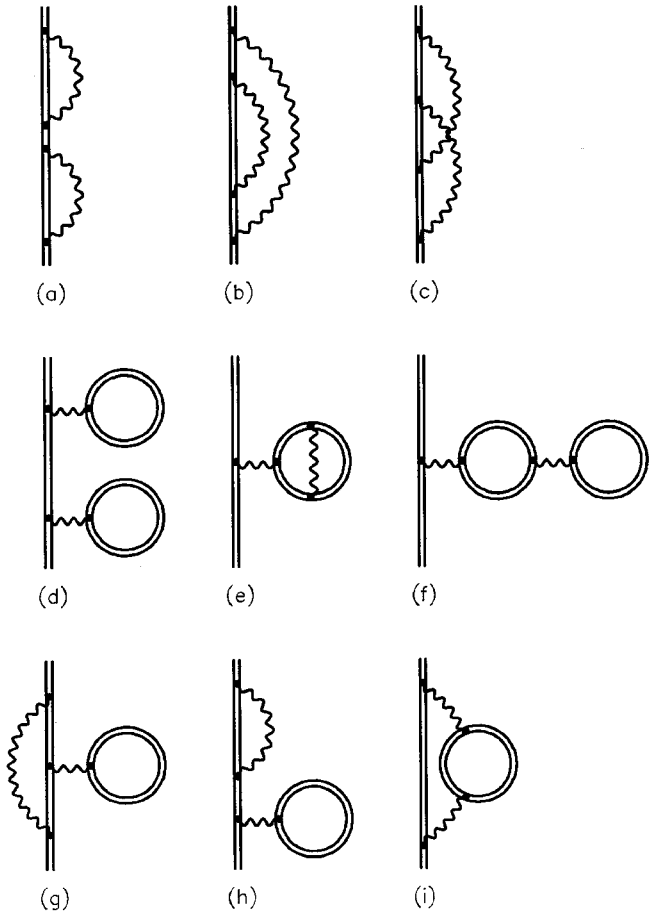

Fig. 1. Feynman diagrams corresponding to the second-order radiative corrections in hydrogen-like ions. The double solid line denotes the bound electron and the wavy line indicates the photon.

[7] has been observed in the case of low and intermediate Z values. Here we report on the solution of this challenging problem and present exact results for the energy shift of the 1s-ground state in H-like uranium due to the SESE diagrams. For our complete evaluation of SESE we adopt the renormalization scheme, which is based on (double) partial wave expansions. To evaluate summations over the Dirac spectrum we utilize the B-Spline method. For the first time a rigorous calculation of all second-order self-energy diagrams for the $1s_{1/2}$-state in H-like uranium and lead is now available [8]. In Table I the results are compiled for uranium together with all relevant QED-corrections of order α and α^2 and nuclear effects. The corrections VPVP (e) and S(VP)E (i) are known only in Uehling approximation. The inaccuracies assigned to these rather small corrections are estimated as the average of the inaccuracies of the Uehling approximation deduced from exact results for the corrections VPVP (f) and SEVP (g),(h).

In Table II we collected the results for lead. The inaccuracy of the Uehling approximation for the VPVP (e) and S(VP)E (i) corrections is neglected. The zero value presented for the nuclear polarization is due to the cancellation of the usual nuclear polarization with the mixed nuclear polarization (NP)-vacuum polarization correction. The latter effect arises when the nucleus interacts with a virtual electron–positron pair. For lead, due to the collective monopole vibrations, specific for this nucleus, the mixed NP–VP effect becomes rather large. Therefore, nuclear polarization effects which otherwise limit very precise Lamb shift predictions are almost completely negligible for ^{208}Pb, making this ion especially suitable for most precise theoretical predictions.

With the present result at hand the most serious QED uncertainty in present Lamb-shift predictions has been removed.

Let us now turn to another interesting atomic system, the case of hydrogen. Precise values for the magnetic moment of the proton are available [9]. Predictions of the hyperfine structure splitting in this system are effected by the insufficiently known size and the internal structure of the proton. However, it is possible to deduce an accurate value for the proton radius from measurements of the Lamb shift in muonic hydrogen.

Accurate QED calculations in hydrogen are of crucial importance for the determination of fundamental constants like the Rydberg constant. Such calculations serves also as a basis for a detailed comparison between the numerical and the analytic approaches to the self-energy problem.

But as in the case for the two-photon self-energy correction for one–electron systems with high nuclear charge the numerical evaluation of one-loop self-energy corrections to all orders in the binding Coulomb field represented a long-standing problem for hydrogenlike systems with low nuclear charge. We have performed an all-order numerical evaluation of the one-photon self energy at low nuclear charge [10]. As we mentioned already for systems with a high nuclear charge number, extensive numerical evaluations have been performed over the past decades, and the available data are very precise (see also [11,12]). Due to convergence problems and numerical cancellations, the numerical calculations have been confined to high nuclear charge. For low Z, the preferred method of evaluation has been the semi-analytic expansion in the strength $Z\alpha$ of the binding Coulomb field [by the term semi-analytic we refer to an expansion in the quantities $(Z\alpha)$ and $\ln(Z\alpha)^{-2}$]. For low Z, the best theoretical predictions for the Lamb shift have been obtained by a combination of analytic results (for low Z) and numerical results (for high Z) through extrapolation of the available numerical data [13]. However, this treatment is still not completely satisfactory because of the unknown higher-order terms in the semi-analytic expansion. The extrapolation method exploys an incomplete approximation to the unknown higher-order terms. It therefore contains a component of uncertainty that is difficult to reliably assess. In order to quantify the uncertainty from the as yet unknown higher-order terms, we observe that termination of the power series at the order of $\alpha(Z\alpha)^6$ leads to an error of 27 kHz as compared to the all-order numerical result for hydrogen listed in Table III. After the inclusion of a result recently obtained in [14] for the logarithmic term of order $\alpha(Z\alpha)^7 \ln(Z\alpha)^{-2}$ the difference between the truncated semi-analytic expansion and the numerical result is still 13 kHz. This has to be compared to the best available experimental values for the 1S–2S transition frequency in hydrogen which has an uncertainty of 46 Hz [15]. Addition-

Table I. *Lamb shift contribution for the ground state of $^{238}U^{91+}$ ion (in eV). The finite nuclear–size correction is calculated for a Fermi distribution with $\langle r^2\rangle^{1/2} = 5.860 \pm 0.002$ (fm).*

Binding energy E_B (point nucleus)	–132279.66 eV	
Finite nuclear size	198.82	±0.10
Self Energy (order α)	355.05	
Vacuum Polarization (order α)	–88.60	
SESE (a) (irred.)	–0.97	
SESE (a) (red.) (b) (c)	1.28	±0.15
VPVP (d)	–0.22	
VPVP (e) (Uehling approx.)	–0.60	±0.10
VPVP (f)	–0.15	
SP (g),(h)	1.12	
S(VP)E (i) (Uehling approx.)	0.13	
relativistic recoil	0.16	
Nuclear polarization	–0.20	±0.10
Lamb Shift (theory)	465.82	±0.45
Lamb Shift (experiment)	469	±16

Table II. *Lamb shift contribution for the ground state of $^{208}Pb^{81+}$ ion (in eV). The notations are the same as in Table I. The finite nuclear size correction is calculated for a Fermi distribution with $\langle r^2\rangle^{1/2} = 5.505 \pm 0.001$ fm.*

Binding energy E_B (point nucleus)	–101581.37 eV	
Finite nuclear size	67.25	
Self Energy (order α)	226.33	
Vacuum Polarization (order α)	–48.41	
SESE (a) (irred.)	–0.51	
SESE (a) (red.) (b) (c)	0.73	±0.09
VPVP (d)	–0.09	
VPVP (e) (Uehling approx.)	–0.34	
VPVP (f)	–0.07	
SEVP (g),(h)	0.53	
S(VP)E (i) (Uehling approx.)	0.07	
relativistic recoil	0.10	
Nuclear polarization	0.00	
Lamb Shift (theory)	245.59	±0.09
Lamb Shft (experiment)	290.0	±75

Table III. *Nonperturbative self-energy remainder function for low-Z hydrogenlike systems [10].*

Z	$G_{SE}(Z\alpha)$
1	–30.290 24(2)
2	–29.770 967(5)
3	–29.299 170(2)
4	–28.859 222(1)
5	–28.443 472 3(8)

ally, it should be pointed out that further experimental progress is likely in the near future.

The energy shift ΔE_{SE} due to the electron self energy is

$$\Delta E_{SE} = \frac{\alpha}{\pi} \frac{(Z\alpha)^4}{n^3} m_e c_0^2 F(Z\alpha) . \quad (1)$$

For the function $F(Z\alpha)$, the semi-analytic expansion is given by (the coefficients are list e.g. in [16])

$$F(Z\alpha) = A_{41} \ln(Z\alpha)^{-2} + A_{40} + (Z\alpha) A_{50} + (Z\alpha)^2 \left[A_{62} \ln^2(Z\alpha)^{-2} + A_{61} \ln(Z\alpha)^{-2} + G_{SE}(Z\alpha) \right] .$$

Numerical data for the nonperturbative self-energy remainder function $G_{SE}(Z\alpha)$ of the 1S ground state are given in Table III.

A sensitive comparison of numerical and analytic approaches to the self energy can be made by extrapolating the nonperturbative self-energy remainder function $G_{SE}(Z\alpha)$ to the point $Z\alpha = 0$. We find that our numerical data is consistent with the value

$$A_{60} \equiv G_{SE}(0) = -30.924\,15(1) \quad (2)$$

[17]. The numerical value of $A_{60} = -30.924\,15(1)$ has been obtained by K. Pachucki after reevaluation of certain poorly convergent one-dimensional numerical integrals in his calculation [17]. This result is important because it confirms the consistency of the two completely different approaches to the self energy (numerical and analytic) to very high precision. Hydrogenlike systems (both for low and high Z) belong to the most accurately understood physical systems today.

Acknowledgements

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