Efficient computation of the quadrupole light deflection in the Gaia relativity model

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Efficient computation of the quadrupole light deflection for both stars/quasars and solar system objects within the framework of the baseline Gaia relativity model (GREM) is discussed. Two refinements have been achieved with the goal to improve the performance of the model:

– The quadrupole deflection formulas for both cases are simplified as much as possible considering the Gaia nominal orbit (only approximate minimal distances between Gaia and the giant planets were used here), physical parameters of the giant planets and the level of up to 0.1 μas for individual systematic effects to be considered in the model. The recommended formulas are given by Eq. (31) for stars/quasars and by Eq. (77) for solar system objects.

– Simple expressions for the upper estimate of the quadrupole light deflection have been found allowing, with a few additional arithmetical operations, to judge a priori if the quadrupole light deflection should be computed or not for a given source and for a given requested accuracy. The recommended criteria are given by Eq. (37) for stars/quasars and by Eq. (90) for solar system objects.

The obtained expressions have been incorporated into the current reference C implementation of the GREM. Numerical experiments with the C implementation show that the derived formulas are both correct and efficient.

I.   THE QUADRUPOLE LIGHT DEFLECTION FOR STARS AND QUASARS

The impact of quadrupole field of massive bodies on the light deflection and the corresponding data reduction have been examined in [2, 3]. The quadrupole light deflection for star and quasars is obtained by inserting Eqs. (44) - (47) into Eq. (62) of [3], and looks as follows (γ is the PPN parameter, and the form and notations used in this report are taken from Section 6.1 of [4]):

$$\delta\boldsymbol{\sigma}_Q(t_o) \;=\; \frac{1+\gamma}{2} \sum_A \frac{G}{c^2} \left[\boldsymbol{\alpha}'_A \frac{\dot{\mathcal{U}}_A(t_o)}{c} + \boldsymbol{\beta}'_A \frac{\dot{\mathcal{E}}_A(t_o)}{c} + \boldsymbol{\gamma}'_A \frac{\dot{\mathcal{F}}_A(t_o)}{c} + \boldsymbol{\delta}'_A \frac{\dot{\mathcal{V}}_A(t_o)}{c} \right] , \quad (1)$$

with

$$\frac{\dot{\mathcal{U}}_A(t_o)}{c} \;=\; \frac{d_A}{r_{oA}^3} \frac{2\,r_{oA} - \boldsymbol{\sigma}\boldsymbol{r}_{oA}}{(r_{oA} - \boldsymbol{\sigma}\boldsymbol{r}_{oA})^2} \;=\; \frac{1}{d_A^3} \left[2 + 3\frac{\boldsymbol{\sigma}\,\boldsymbol{r_{oA}}}{r_{oA}} - \frac{(\boldsymbol{\sigma}\,\boldsymbol{r_{oA}})^3}{r_{oA}^3} \right] , \quad (2)$$

$$\frac{\dot{\mathcal{E}}_A(t_o)}{c} \;=\; \frac{r_{oA}^2 - 3\,(\boldsymbol{\sigma}\boldsymbol{r}_{oA})^2}{r_{oA}^5} \;, \quad (3)$$

$$\frac{\dot{\mathcal{F}}_A(t_o)}{c} \;=\; -3\,d_A\,\frac{\boldsymbol{\sigma}\boldsymbol{r}_{oA}}{r_{oA}^5} \;, \quad (4)$$

$$\frac{\dot{\mathcal{V}}_A(t_o)}{c} \;=\; -\frac{1}{r_{oA}^3} \;, \quad (5)$$

and the time-independent vectorial coefficients are

$$\alpha'^{\,k}_A \;=\; -\hat{M}^A_{ij}\,\sigma^i\,\sigma^j\,\frac{d_A^k}{d_A} \;+\; 2\,\hat{M}^A_{kj}\,\frac{d_A^j}{d_A} \;-\; 2\,\hat{M}^A_{ij}\,\sigma^i\,\sigma^k\,\frac{d_A^j}{d_A} \;-\; 4\,\hat{M}^A_{ij}\,\frac{d_A^i\,d_A^j\,d_A^k}{d_A^3} \;, \quad (6)$$

$$\beta'^{\,k}_A \;=\; 2\hat{M}^A_{ij}\,\sigma^i\,\frac{d_A^j\,d_A^k}{d_A^2} \;, \quad (7)$$

$$\gamma'^{\,k}_A \;=\; \hat{M}^A_{ij}\,\frac{d_A^i\,d_A^j\,d_A^k}{d_A^3} \;-\; \hat{M}^A_{ij}\,\sigma^i\,\sigma^j\,\frac{d_A^k}{d_A} \;, \quad (8)$$

$$\delta'^{\,k}_A \;=\; -2\,\hat{M}^A_{ij}\,\sigma^i\,\sigma^j\,\sigma^k \;+\; 2\,\hat{M}^A_{kj}\,\sigma^j \;-\; 4\,\hat{M}^A_{ij}\,\sigma^i\,\frac{d_A^j\,d_A^k}{d_A^2} \;. \quad (9)$$

The quadrupole formula (1) is valid for sources at infinite distance from the observer. The sum over A in (1) runs, in principle, over all bodies inside the solar system, but only the giant planets contribute within the accuracy of 1 μas. The vector

$$\boldsymbol{r}_A \;=\; \boldsymbol{x}_p(t) - \boldsymbol{x}_A \quad (10)$$

is time-dependent and directed from body A toward the position of photon at time t, where $\boldsymbol{x}_A$ is the BCRS position of body A, while the vector

$$\boldsymbol{r}_{oA} \;=\; \boldsymbol{x}_p(t_o) - \boldsymbol{x}_A = \boldsymbol{x}_o(t_o) - \boldsymbol{x}_A \tag{11}$$

is directed from body A toward the position of photon at observation time t_o which, of course, coincides with the position of observer. The vector

$$\boldsymbol{d}_A \;=\; \boldsymbol{\sigma} \times (\boldsymbol{r}_{oA} \times \boldsymbol{\sigma}) \tag{12}$$

is time-independent and directed from object A towards the trajectory ($\boldsymbol{\sigma}\,\boldsymbol{d}_A = 0$), $d_A = |\boldsymbol{d}_A|$, $r_{oA} = |\boldsymbol{r}_{oA}|$, and G is the gravitational constant. The symmetric and tracefree quadrupole moment of an object A are defined as

$$\hat{M}^A_{ij} \;=\; \int\limits_A d^3 x \, \rho_A(x) \left(r^i\, r^j - \frac{1}{3}\, \delta^{ij}\, r^2 \right), \quad r^i = x^i - x^i_A\,, \tag{13}$$

with the mass density ρ_A, and the integral is taken over the volume of body A; the Kronecker symbol $\delta^{ij} = 1$ for $i = j$ and zero otherwise.

II. APPROXIMATION OF THE QUADRUPOLE LIGHT DEFLECTION FORMULA FOR STARS AND QUASARS

The numerical evaluation of exact quadrupole formula (1) for all of the aimed observation of 10^9 stars by GAIA mission is time-consuming. This, however, can be improved if (1) can be substituted by a simpler formula. In this section we will derive an approximation of (1) sufficient for the envisaged accuracy of at least 1 μas. From (1) one obtains the estimate

$$| \delta\boldsymbol{\sigma}_Q(t_o)| \;\leq\; \frac{1+\gamma}{2} \sum_A \frac{G}{c^2} \left[|\boldsymbol{\alpha}'_A| \frac{|\dot{\mathcal{U}}_A(t_o)|}{c} + |\boldsymbol{\beta}'_A| \frac{|\dot{\mathcal{E}}_A(t_o)|}{c} + |\boldsymbol{\gamma}'_A| \frac{|\dot{\mathcal{F}}_A(t_o)|}{c} + |\boldsymbol{\delta}'_A| \frac{|\dot{\mathcal{V}}_A(t_o)|}{c} \right], \tag{14}$$

where t_o is the time moment of observation by the GAIA satellite. Note that since $\delta\boldsymbol{\sigma}_Q$ is perpendicular to $\boldsymbol{\sigma}$ the absolute value $|\delta\boldsymbol{\sigma}_Q(t_o)|$ directly gives, in the adopted post-Newtonian approximation, the change of the calculated or observed direction to star or quasar due to the quadrupole light deflection.

A. Estimate of the vectorial coefficients

The vectorial coefficients of the last three individual terms in (14) can be estimated as follows. For an axial symmetric body (this approximation is sufficient for the giant planets and goal accuracy of 1 μas) one has

$$\hat{M}^A_{ij} = M_A \, J^A_2 \, P^2_A \, \frac{1}{3} \, \hat{R} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \hat{R}^T\,, \tag{15}$$

where $\hat{R}$ is the rotational matrix giving the orientation of the symmetry (rotational) axis of a planet in the BCRS, M_A is the mass of body A, J_2^A is the coefficient of the second zonal harmonic of the gravitational field, P_A is the minimal radius of a sphere containing the body A and whose center coincides with the center of mass of A (for the giant planets P_A is just the equatorial radius). Below we are only looking for the maximal possible absolute value of the coefficients $|\boldsymbol{\beta}'_A|$, $|\boldsymbol{\gamma}'_A|$, $|\boldsymbol{\delta}'_A|$. Therefore, $\hat{R}$ can be omitted.

Inserting (15) into (6)–(9) yields

$$\boldsymbol{\alpha}'_A = -M_A J_2^A P_A^2 \frac{1}{d_A} \left[\left(1 - (\boldsymbol{\sigma}\boldsymbol{e}_3)^2 - 4\frac{(\boldsymbol{d}_A\boldsymbol{e}_3)^2}{d_A^2} \right) \boldsymbol{d}_A + 2(\boldsymbol{d}_A\boldsymbol{e}_3)\boldsymbol{e}_3 - 2(\boldsymbol{\sigma}\boldsymbol{e}_3)(\boldsymbol{d}_A\boldsymbol{e}_3)\boldsymbol{\sigma} \right] , \tag{16}$$

$$\boldsymbol{\beta}'_A = -2 M_A J_2^A P_A^2 \frac{1}{d_A^2} (\boldsymbol{\sigma}\boldsymbol{e}_3)(\boldsymbol{d}_A\boldsymbol{e}_3)\boldsymbol{d}_A , \tag{17}$$

$$\boldsymbol{\gamma}'_A = -M_A J_2^A P_A^2 \frac{1}{d_A^3} \left[(\boldsymbol{d}_A\boldsymbol{e}_3)^2 \boldsymbol{d}_A - (\boldsymbol{\sigma}\boldsymbol{e}_3)^2 d_A^2 \boldsymbol{d}_A \right] , \tag{18}$$

$$\boldsymbol{\delta}'_A = 2 M_A J_2^A P_A^2 \left[(\boldsymbol{\sigma}\boldsymbol{e}_3)^2 \boldsymbol{\sigma} + \frac{2}{d_A^2} (\boldsymbol{\sigma}\boldsymbol{e}_3)(\boldsymbol{d}_A\boldsymbol{e}_3)\boldsymbol{d}_A - (\boldsymbol{\sigma}\boldsymbol{e}_3)\boldsymbol{e}_3 \right] , \tag{19}$$

where $\boldsymbol{e}_3$ is the unit direction along the axis of symmetry (rotation). Here, $(\boldsymbol{\sigma}\boldsymbol{e}_3)$ and $(\boldsymbol{d}_A\boldsymbol{e}_3)$ are the projections of the vectors $\boldsymbol{\sigma}$ and $\boldsymbol{d}_A$, respectively, on the axis of symmetry. With the aid of (17)–(19) we can explicitly determine the maximal absolute values of the last three individual terms in (14):

$$|\boldsymbol{\beta}'_A| \leq 2 M_A J_2^A P_A^2 \left[|(\boldsymbol{\sigma}\boldsymbol{e}_3)| \frac{|\boldsymbol{d}_A\boldsymbol{e}_3|}{d_A} \right] \leq M_A J_2^A P_A^2 , \tag{20}$$

$$|\boldsymbol{\gamma}'_A| = M_A J_2^A P_A^2 \left| \frac{(\boldsymbol{d}_A\boldsymbol{e}_3)^2}{d_A^2} - (\boldsymbol{\sigma}\boldsymbol{e}_3)^2 \right| \leq M_A J_2^A P_A^2 , \tag{21}$$

$$|\boldsymbol{\delta}'_A| \leq 2 M_A J_2^A P_A^2 \left[(\boldsymbol{\sigma}\boldsymbol{e}_3)^2 + 2\frac{|(\boldsymbol{\sigma}\boldsymbol{e}_3)(\boldsymbol{d}_A\boldsymbol{e}_3)|}{d_A} + (\boldsymbol{\sigma}\boldsymbol{e}_3) \right] \leq 6 M_A J_2^A P_A^2 , \tag{22}$$

where for the estimates (20) and (22) we have taken into account that

$$|\boldsymbol{\sigma}\boldsymbol{e}_3| \frac{|\boldsymbol{d}_A\boldsymbol{e}_3|}{d_A} \leq \frac{1}{2} , \tag{23}$$

valid due to $\boldsymbol{\sigma}\boldsymbol{d}_A = 0$.

B. Estimate of the scalar functions

Furthermore, from (3)–(5) we deduce the estimates

$$\frac{|\dot{\mathcal{E}}_A(t_o)|}{c} \leq \frac{[r_{oA}(t_o)]^2 + 3[r_{oA}(t_o)]^2}{[r_{oA}(t_o)]^5} = 4\frac{1}{[r_{oA}(t_o)]^3} , \tag{24}$$

Parameter	Jupiter	Saturn	Uranus	Neptune
GM_A/c^2 [m]	1.40987	0.42215	0.064473	0.076067
J_2^A [10^{-3}]	14.697	16.331	3.516	3.538
P_A [10^6 m]	71.492	60.268	25.559	24.764
$r_{oA}^{\min}$ [10^{12} m]	0.59	1.20	2.59	4.31
$GM_A J_2^A P_A^2/c^2$ [10^{15} m^3]	0.106	0.025	0.000148	0.000165

TABLE I: Numerical parameters of the giant planets taken from [1, 6].

$$\frac{|\dot{\mathcal{F}}_A(t_o)|}{c} \;\leq\; d_A \, \frac{3\, r_{oA}(t_o)}{[r_{oA}(t_o)]^5} \leq 3\, \frac{1}{[r_{oA}(t_o)]^3} \;, \tag{25}$$

$$\frac{|\dot{\mathcal{V}}_A(t_o)|}{c} \;=\; \frac{1}{[r_{oA}(t_o)]^3} \;. \tag{26}$$

By inserting these estimates (20)–(22) and (24)–(26) into (14) we get

$$\frac{G}{c^2}\, |\boldsymbol{\beta}'_A| \, \frac{|\dot{\mathcal{E}}_A(t_o)|}{c} \;\leq\; 4\, \frac{G}{c^2}\, M_A\, J_2^A\, P_A^2\, \frac{1}{(r_{oA}^{\min})^3} \;, \tag{27}$$

$$\frac{G}{c^2}\, |\boldsymbol{\gamma}'_A| \, \frac{|\dot{\mathcal{F}}_A(t_o)|}{c} \;\leq\; 3\, \frac{G}{c^2}\, M_A\, J_2^A\, P_A^2\, \frac{1}{(r_{oA}^{\min})^3} \;, \tag{28}$$

$$\frac{G}{c^2}\, |\boldsymbol{\delta}'_A| \, \frac{|\dot{\mathcal{V}}_A(t_o)|}{c} \;\leq\; 6\, \frac{G}{c^2}\, M_A\, J_2^A\, P_A^2\, \frac{1}{(r_{oA}^{\min})^3} \;. \tag{29}$$

The quantity $r_{oA}^{\min}$ represents the minimal distance between the object A and the observer (GAIA satellite).

C. Collection of all terms

Table I summarizes physical parameters of the giant planets. In the Table and in the following discussion we use values of $r_{oA}^{\min}$ computed under assumtion that the observer (Gaia) is within a few million kilometers from the Earth's orbit. From the values given in Table I and (27)–(29) we deduce (γ can be safely set to unity for these estimates)

$$\frac{G}{c^2} \left[|\boldsymbol{\beta}'_A| \, \frac{|\dot{\mathcal{E}}_A(t_o)|}{c} + |\boldsymbol{\gamma}'_A| \, \frac{|\dot{\mathcal{F}}_A(t_o)|}{c} + |\boldsymbol{\delta}'_A| \, \frac{|\dot{\mathcal{V}}_A(t_o)|}{c} \right] \;\leq\; 1.61 \times 10^{-9} \; \mu\text{as} \quad \text{for Jupiter}\,,$$
$$\leq\; 4.52 \times 10^{-11} \; \mu\text{as} \quad \text{for Saturn}\,,$$
$$\leq\; 2.64 \times 10^{-14} \; \mu\text{as} \quad \text{for Uranus}\,,$$
$$\leq\; 7.66 \times 10^{-15} \; \mu\text{as} \quad \text{for Neptune}\,. \tag{30}$$

Obviously, by comparing the estimates given in (30) with the envisaged accuracy of 1μas we can conclude that these last three terms in (1), i.e. $\boldsymbol{\beta}'_A \dot{\mathcal{E}}_A$ and $\boldsymbol{\gamma}'_A \dot{\mathcal{F}}_A$ and $\boldsymbol{\delta}'_A \dot{\mathcal{V}}_A$, can safely be neglected. Accordingly, the simplified formula for the quadrupole light deflection for stars and quasars is

$$\delta\boldsymbol{\sigma}_Q(t_o) \;=\; \sum_A \frac{1+\gamma}{2} \frac{G}{c^2} \boldsymbol{\alpha}'_A \frac{\dot{\mathcal{U}}_A(t_o)}{c} \;, \tag{31}$$

with $\dot{\mathcal{U}}_A$ given by (2) and $\boldsymbol{\alpha}'_A$ given by (6).

III.   AN UPPER ESTIMATE OF THE QUADRUPOLE LIGHT DEFLECTION FOR STARS AND QUASARS

Due to the complexity of (31) it is advantageous to find a simple criterion which allows one, with a few additional arithmetical operations, to judge if the quadrupole light deflection should be computed for a given source and for a given accuracy. Therefore, we first evaluate the absolute value of the vectorial coefficient (16),

$$|\boldsymbol{\alpha}'_A| \;=\; M_A \, J_2^A \, P_A^2 \left(1 - (\boldsymbol{\sigma}\,\boldsymbol{e}_3)^2\right) \;, \tag{32}$$

which yields for the absolute value of light deflection angle caused by the quadrupole field of objects A,

$$|\delta\boldsymbol{\sigma}_Q(t_o)| \;=\; \frac{1+\gamma}{2} \sum_A \frac{GM_A}{c^2} \, J_2^A \, P_A^2 \left(1 - (\boldsymbol{\sigma}\,\boldsymbol{e}_3)^2\right) \frac{1}{d_A^3} \left(3\frac{\boldsymbol{\sigma}\,\boldsymbol{r}_{oA}}{r_{oA}} - \frac{(\boldsymbol{\sigma}\,\boldsymbol{r}_{oA})^3}{r_{oA}^3} + 2\right). \tag{33}$$

From Eq. (63) in [3], we deduce the corresponding absolute value of deflection angle caused by the spherically symmetric part of objects A,

$$|\delta\boldsymbol{\sigma}_{pN}(t_o)| \;=\; \frac{1+\gamma}{2} \sum_A \frac{2\,GM_A}{c^2} \frac{1}{d_A} \left(1 + \frac{\boldsymbol{\sigma}\,\boldsymbol{r}_{oA}}{r_{oA}}\right) \;. \tag{34}$$

A direct comparison of (33) with (34) and taking into account the fact

$$3\frac{\boldsymbol{\sigma}\,\boldsymbol{r}_{oA}}{r_{oA}} - \frac{(\boldsymbol{\sigma}\,\boldsymbol{r}_{oA})^3}{r_{oA}^3} + 2 \leq \frac{9}{4}\left(1 + \frac{\boldsymbol{\sigma}\,\boldsymbol{r}_{oA}}{r_{oA}}\right) \;, \tag{35}$$

we obtain the criterion

$$|\delta\boldsymbol{\sigma}_Q(t_o)| \;\leq\; \frac{9}{8}\frac{P_A^2}{d_A^2}\,J_2^A\left(1 - (\boldsymbol{\sigma}\boldsymbol{e}_3)^2\right)|\delta\boldsymbol{\sigma}_{pN}(t_o)| \;. \tag{36}$$

Due to $1 \geq (\boldsymbol{\sigma}\boldsymbol{e}_3)^2$, the estimate (36) can be further approximated by

$$|\delta\boldsymbol{\sigma}_Q(t_o)| \;\leq\; \frac{9}{8}\,J_2^A\,\frac{P_A^2}{d_A^2}\,|\delta\boldsymbol{\sigma}_{pN}(t_o)| \;. \tag{37}$$

This criterion relates the quadrupole light deflection for stars and quasars to the simpler case of spherically symmetric part. It is recommended for GAIA to use (37) as a criterion if the quadrupole light deflection has to be computed for a given star or quasar. For completeness let us quote the well-known estimate of the monopole light deflection

$$| \delta \boldsymbol{\sigma}_{pN}(t_o)| \ \leq \ \frac{2 \, (1 + \gamma) \, GM_A}{c^2 \, d_A} \ . \tag{38}$$

Eq. (33) can be used to estimate $| \delta \boldsymbol{\sigma}_Q(t_o)|$ directly:

$$| \delta \boldsymbol{\sigma}_Q(t_o)| \ \leq \ \frac{2 \, (1 + \gamma) \, GM_A}{c^2} \, \frac{P_A^2}{d_A^3} \, J_2^A \leq \frac{2 \, (1 + \gamma) \, GM_A}{c^2 \, d_A} \, J_2^A. \tag{39}$$

Estimate (39) coincides with the results of [2].

IV.   THE QUADRUPOLE LIGHT DEFLECTION FOR SOLAR SYSTEM OBJECTS

While (1) is valid for stars and quasars, the formulas considered in this Section describe light deflection due to quadrupole field of massive bodies for sources in the solar system. This issue has also been examined in [3]. Substituting Eqs. (36)–(47) into Eq. (69) of [3], the quadrupole light deflection for sources in the solar system can be written as follows (the notations are again taken from Section 6.1 of [4])

$$\delta \boldsymbol{k}_Q(t_o) \ = \ \frac{1 + \gamma}{2} \, \sum_A \frac{G}{c^2} \left[\boldsymbol{\alpha}''_A \, \frac{\mathcal{A}_A(t_o)}{c} + \boldsymbol{\beta}''_A \, \frac{\mathcal{B}_A(t_o)}{c} + \boldsymbol{\gamma}''_A \, \frac{\mathcal{C}_A(t_o)}{c} + \boldsymbol{\delta}''_A \, \frac{\mathcal{D}_A(t_o)}{c} \right] \, , \tag{40}$$

with

$$\frac{\mathcal{A}_A(t_o)}{c} \ = \ \frac{1}{d_A} \frac{1}{R} \left(\frac{1}{r_{eA}} \frac{r_{eA} + \boldsymbol{k} \, \boldsymbol{r}_{eA}}{r_{eA} - \boldsymbol{k} \, \boldsymbol{r}_{eA}} - \frac{1}{r_{oA}} \frac{r_{oA} + \boldsymbol{k} \, \boldsymbol{r}_{oA}}{r_{oA} - \boldsymbol{k} \, \boldsymbol{r}_{oA}} \right) + \frac{d_A}{r_{oA}^3} \frac{2 \, r_{oA} - \boldsymbol{k} \, \boldsymbol{r}_{oA}}{(r_{oA} - \boldsymbol{k} \, \boldsymbol{r}_{oA})^2} \, , \tag{41}$$

$$\frac{\mathcal{B}_A(t_o)}{c} \ = \ \frac{1}{R} \left(\frac{\boldsymbol{k} \, \boldsymbol{r}_{eA}}{r_{eA}^3} - \frac{\boldsymbol{k} \, \boldsymbol{r}_{oA}}{r_{oA}^3} \right) + \frac{r_{oA}^2 - 3 \, (\boldsymbol{k} \, \boldsymbol{r}_{oA})^2}{r_{oA}^5} \, , \tag{42}$$

$$\frac{\mathcal{C}_A(t_o)}{c} \ = \ \frac{d_A}{R} \left(\frac{1}{r_{eA}^3} - \frac{1}{r_{oA}^3} \right) - 3 \, d_A \, \frac{\boldsymbol{k} \, \boldsymbol{r}_{oA}}{r_{oA}^5} \, , \tag{43}$$

$$\frac{\mathcal{D}_A(t_o)}{c} \ = \ -\frac{1}{d_A^2} \frac{1}{R} \left(\frac{\boldsymbol{k} \, \boldsymbol{r}_{eA}}{r_{eA}} - \frac{\boldsymbol{k} \, \boldsymbol{r}_{oA}}{r_{oA}} \right) - \frac{1}{r_{oA}^3} \, , \tag{44}$$

and the time-independent vectorial coefficients in spatial component notation are

$$\alpha_A''^{\, k} \ = \ -\hat{M}_{ij}^A \, k^i \, k^j \, \frac{d_A^k}{d_A} \ + 2 \, \hat{M}_{kj}^A \, \frac{d_A^j}{d_A} - 2 \, \hat{M}_{ij}^A \, k^i \, k^k \, \frac{d_A^j}{d_A} - 4 \, \hat{M}_{ij}^A \, \frac{d_A^i \, d_A^j \, d_A^k}{d_A^3} \, , \tag{45}$$

$$\beta_A''^k = 2\hat{M}_{ij}^A k^i \frac{d_A^j d_A^k}{d_A^2} , \tag{46}$$

$$\gamma_A''^k = \hat{M}_{ij}^A \frac{d_A^i d_A^j d_A^k}{d_A^3} - \hat{M}_{ij}^A k^i k^j \frac{d_A^k}{d_A} , \tag{47}$$

$$\delta_A''^k = -2\hat{M}_{ij}^A k^i k^j k^k + 2\hat{M}_{kj}^A k^j - 4\hat{M}_{ij}^A k^i \frac{d_A^j d_A^k}{d_A^2} . \tag{48}$$

Thereby, the following vectors have been used. The vector

$$\boldsymbol{r}_{eA} = \boldsymbol{x}_s(t_e) - \boldsymbol{x}_A , \tag{49}$$

is directed from body A toward the position of the source $\boldsymbol{x}_s$ at the moment of emission t_e of the signal which is observed at $\boldsymbol{x}_o(t_o)$, while the vector

$$\boldsymbol{R} = \boldsymbol{x}_o(t_o) - \boldsymbol{x}_s(t_e) = \boldsymbol{r}_{oA} - \boldsymbol{r}_{eA} , \tag{50}$$

is directed from source toward the observer. Furthermore, the unit direction from source to observer is

$$\boldsymbol{k} = \frac{\boldsymbol{R}}{R} , \tag{51}$$

and $R = |\boldsymbol{R}|$, $r_{eA} = |\boldsymbol{r}_{eA}|$. In the following we will investigate how (40) can be simplified for a goal accuracy of at least 1 μas and taking into account that in the case of Gaia the observer is situated within a few million kilometers from the Earth's orbit.

V.   APPROXIMATION OF QUADRUPOLE LIGHT DEFLECTION FOR SOLAR SYSTEM OBJECTS

To determine the magnitude of the individual terms in (40) we first notice that at observation time t_o,

$$|\delta\boldsymbol{k}_Q(t_o)| \leq \frac{1+\gamma}{2} \sum_A \frac{G}{c^2} \left[|\boldsymbol{\alpha}_A''| \frac{|\mathcal{A}_A(t_o)|}{c} + |\boldsymbol{\beta}_A''| \frac{|\mathcal{B}_A(t_o)|}{c} + |\boldsymbol{\gamma}_A''| \frac{|\mathcal{C}_A(t_o)|}{c} + |\boldsymbol{\delta}_A''| \frac{|\mathcal{D}_A(t_o)|}{c} \right] . \tag{52}$$

Here again since $\delta\boldsymbol{k}_Q$ is perpendicular to $\boldsymbol{k}$ the absolute value $|\delta\boldsymbol{k}_Q(t_o)|$ directly gives, in the adopted post-Newtonian approximation, the change of the calculated or observed direction to a solar system object due to the quadrupole light deflection.

Then again, in order to estimate the maximal value of vectorial coefficients we make use of the diagonalized form of quadrupole moment given in (15), which yields for the vectorial coefficients (45)–(48)

$$\boldsymbol{\alpha}_A^{''} = -M_A\, J_2^A\, P_A^2\, \frac{1}{d_A}$$
$$\times \left[\left(1 - (\boldsymbol{k}\,\boldsymbol{e}_3)^2 - 4\frac{(\boldsymbol{d}_A\,\boldsymbol{e}_3)^2}{d_A^2}\right)\boldsymbol{d}_A + 2(\boldsymbol{d}_A\,\boldsymbol{e}_3)\,\boldsymbol{e}_3 - 2(\boldsymbol{k}\,\boldsymbol{e}_3)(\boldsymbol{d}_A\,\boldsymbol{e}_3)\,\boldsymbol{k} + \frac{2}{3}(\boldsymbol{k}\,\boldsymbol{d}_A)\,\boldsymbol{k}\right],$$

$$\tag{53}$$

$$\boldsymbol{\beta}_A^{''} = -2\,M_A\, J_2^A\, P_A^2\, \frac{1}{d_A^2}\left[(\boldsymbol{k}\,\boldsymbol{e}_3)\,(\boldsymbol{d}_A\,\boldsymbol{e}_3)\,\boldsymbol{d}_A - \frac{1}{3}(\boldsymbol{k}\,\boldsymbol{d}_A)\,\boldsymbol{d}_A\right], \tag{54}$$

$$\boldsymbol{\gamma}_A^{''} = -M_A\, J_2^A\, P_A^2\, \frac{1}{d_A^3}\left[(\boldsymbol{d}_A\,\boldsymbol{e}_3)^2\,\boldsymbol{d}_A - (\boldsymbol{k}\,\boldsymbol{e}_3)^2 d_A^2\,\boldsymbol{d}_A\right], \tag{55}$$

$$\boldsymbol{\delta}_A^{''} = 2\,M_A\, J_2^A\, P_A^2\, \left[(\boldsymbol{k}\,\boldsymbol{e}_3)^2\,\boldsymbol{k} + \frac{2}{d_A^2}(\boldsymbol{k}\,\boldsymbol{e}_3)(\boldsymbol{d}_A\,\boldsymbol{e}_3)\,\boldsymbol{d}_A - (\boldsymbol{k}\,\boldsymbol{e}_3)\,\boldsymbol{e}_3 - \frac{2}{3}\frac{1}{d_A^2}(\boldsymbol{k}\,\boldsymbol{d}_A)\,\boldsymbol{d}_A\right], \tag{56}$$

where $\boldsymbol{k}\,\boldsymbol{e}_3$ and $\boldsymbol{d}_A\,\boldsymbol{e}_3$ are the projections of the vectors $\boldsymbol{k}$ and $\boldsymbol{d}_A$, respectively, on the axis of symmetry.

A.   Estimate of vectorial coefficients

From (54)–(56) we deduce the following absolute values for the last three vectorial coefficients,

$$|\boldsymbol{\beta}_A^{''}| \le 2\,M_A\, J_2^A\, P_A^2\, \left[|\boldsymbol{k}\,\boldsymbol{e}_3|\frac{|\boldsymbol{d}_A\,\boldsymbol{e}_3|}{d_A} + \frac{1}{3}\frac{|\boldsymbol{k}\,\boldsymbol{d}_A|}{d_A}\right] \le M_A\, J_2^A\, P_A^2, \tag{57}$$

$$|\boldsymbol{\gamma}_A^{''}| = M_A\, J_2^A\, P_A^2\, \left|\frac{|\boldsymbol{d}_A\,\boldsymbol{e}_3|^2}{d_A^2} - |\boldsymbol{k}\,\boldsymbol{e}_3|^2\right| \le M_A\, J_2^A\, P_A^2, \tag{58}$$

$$|\boldsymbol{\delta}_A^{''}| = 2\,M_A\, J_2^A\, P_A^2\, \left[\frac{4}{3}\frac{1}{d_A^2}\left(3\,(\boldsymbol{k}\,\boldsymbol{e}_3)^3 - (\boldsymbol{k}\,\boldsymbol{e}_3)\right)(\boldsymbol{k}\,\boldsymbol{d}_A)(\boldsymbol{d}_A\,\boldsymbol{e}_3)\right.$$
$$\left. - \frac{4}{3}\frac{1}{d_A^2}(\boldsymbol{k}\,\boldsymbol{d}_A)^2(\boldsymbol{k}\,\boldsymbol{e}_3)^2 - (\boldsymbol{k}\,\boldsymbol{e}_3)^4 + (\boldsymbol{k}\,\boldsymbol{e}_3)^2 + \frac{4}{9}\frac{1}{d_A^2}(\boldsymbol{k}\,\boldsymbol{d}_A)^2\right]^{1/2}$$
$$\le M_A\, J_2^A\, P_A^2, \tag{59}$$

where for the estimates (57) and (59) we have taken into account that

$$|\boldsymbol{k}\,\boldsymbol{e}_3|\frac{|\boldsymbol{d}_A\,\boldsymbol{e}_3|}{d_A} \le \frac{1}{2}, \tag{60}$$

$$(\boldsymbol{k}\,\boldsymbol{e}_3)^2 - (\boldsymbol{k}\,\boldsymbol{e}_3)^4 \le \frac{1}{4}. \tag{61}$$

The first estimate uses the fact that $\boldsymbol{k}\,\boldsymbol{d}_A = \mathcal{O}\left(c^{-2}\right)$. In the following we estimate the magnitude of the scalar functions (42)–(44).

1. Estimate of $\mathcal{B}_A$

The coefficient given in (42) can be written as follows,

$$\frac{\mathcal{B}_A(t_o)}{c} = \frac{1}{\sqrt{r_{eA}^2 + r_{oA}^2 - 2\,\boldsymbol{r}_{eA}\,\boldsymbol{r}_{oA}}} \left(\frac{\boldsymbol{k}\,\boldsymbol{r}_{eA}}{r_{eA}^3} - \frac{\boldsymbol{k}\,\boldsymbol{r}_{oA}}{r_{oA}^3} \right) + \frac{r_{oA}^2 - 3(\boldsymbol{k}\,\boldsymbol{r}_{oA})^2}{r_{oA}^5} \; . \tag{62}$$

Inserting the definition of vector $\boldsymbol{k}$ yields

$$\frac{\mathcal{B}_A(t_o)}{c} = \frac{1}{r_{eA}^2 + r_{oA}^2 - 2\,\boldsymbol{r}_{eA}\,\boldsymbol{r}_{oA}} \left(\frac{r_{eA}}{r_{oA}^2} \cos\alpha + \frac{r_{oA}}{r_{eA}^2} \cos\alpha - \frac{1}{r_{eA}} - \frac{1}{r_{oA}} \right)$$
$$+ \frac{r_{oA}^2 - 3(\boldsymbol{k}\,\boldsymbol{r}_{oA})^2}{r_{oA}^5} \; , \tag{63}$$

where $\cos\alpha = (\boldsymbol{r}_{oA}\,\boldsymbol{r}_{eA})/(|\boldsymbol{r}_{oA}|\,|\boldsymbol{r}_{eA}|)$. By means of the inequality (with $x = r_{eA}, y = r_{oA}$)

$$\left| \frac{1}{x^2 + y^2 - 2\,x\,y\,\cos\alpha} \left(\frac{x}{y^2} \cos\alpha + \frac{y}{x^2} \cos\alpha - \frac{1}{x} - \frac{1}{y} \right) \right| \le \frac{x + y}{x^2\,y^2} \tag{64}$$

valid for any $x \ge 0$ and $y \ge 0$, we obtain the estimate

$$\frac{|\mathcal{B}_A(t_o)|}{c} \le \frac{r_{eA} + r_{oA}}{r_{eA}^2\,r_{oA}^2} + \frac{4}{r_{oA}^3} \le \frac{1}{d_A\,r_{oA}^2} + \frac{1}{d_A^2\,r_{oA}} + \frac{4}{r_{oA}^3} \; . \tag{65}$$

2. Estimate of $\mathcal{C}_A$

The coefficient given in (43) can be written as follows,

$$\frac{\mathcal{C}_A(t_o)}{c} = \frac{d_A}{r_{eA}^3\,r_{oA}^3} \frac{r_{oA}^3 - r_{eA}^3}{\sqrt{r_{eA}^2 + r_{oA}^2 - 2\,\boldsymbol{r}_{eA}\boldsymbol{r}_{oA}}} - 3\,d_A\,\frac{\boldsymbol{k}\,\boldsymbol{r}_{oA}}{r_{oA}^5} \; . \tag{66}$$

Since

$$\frac{1}{\sqrt{r_{eA}^2 + r_{oA}^2 - 2\,\boldsymbol{r}_{eA}\boldsymbol{r}_{oA}}} \le \frac{1}{\sqrt{(r_{eA} - r_{oA})^2}} \; , \tag{67}$$

we find for the absolute value

$$\frac{|\mathcal{C}_A(t_o)|}{c} \le \frac{d_A}{r_{eA}^3\,r_{oA}^3} \frac{|r_{eA}^3 - r_{oA}^3|}{|r_{eA} - r_{oA}|} + 3\,\frac{d_A}{r_{oA}^4} \; . \tag{68}$$

By means of the inequality

$$\frac{|x^3 - y^3|}{|x - y|} \leq \frac{3}{2}\left(x^2 + y^2\right) \tag{69}$$

that is valid for any x and y, we obtain the estimate

$$\frac{|\mathcal{C}_A(t_o)|}{c} \leq \frac{3}{2} d_A \frac{r_{eA}^2 + r_{oA}^2}{r_{eA}^3 \, r_{oA}^3} + 3 \frac{d_A}{r_{oA}^4} \leq \frac{3}{2} \frac{1}{r_{oA}^3} + \frac{3}{2} \frac{1}{d_A^2 \, r_{oA}} + 3 \frac{d_A}{r_{oA}^4} \; . \tag{70}$$

3.   Estimate of $\mathcal{D}_A$

The coefficient given in (44) can be written as follows,

$$\frac{\mathcal{D}_A(t_o)}{c} = -\frac{1}{d_A^2} \frac{1}{\sqrt{r_{eA}^2 + r_{oA}^2 - 2\, \boldsymbol{r}_{eA} \, \boldsymbol{r}_{oA}}} \left(\frac{\boldsymbol{k} \, \boldsymbol{r}_{eA}}{r_{eA}} - \frac{\boldsymbol{k} \, \boldsymbol{r}_{oA}}{r_{oA}}\right) - \frac{1}{r_{oA}^3} \; . \tag{71}$$

Inserting the definition of vector $\boldsymbol{k}$ yields

$$\frac{\mathcal{D}_A(t_o)}{c} = -\frac{1}{d_A^2} \frac{r_{eA} \cos \alpha + r_{oA} \cos \alpha - r_{eA} - r_{oA}}{r_{eA}^2 + r_{oA}^2 - 2\, \boldsymbol{r}_{eA} \, \boldsymbol{r}_{oA}} - \frac{1}{r_{oA}^3} \; . \tag{72}$$

With the aid of the inequality

$$\left|\frac{x \cos \alpha + y \cos \alpha - x - y}{x^2 + y^2 - 2\, x\, y \, \cos \alpha}\right| \leq \frac{2}{x + y} \tag{73}$$

valid for $x \geq 0$ and $y \geq 0$, we obtain the estimate

$$\frac{|\mathcal{D}_A(t_o)|}{c} \leq 2 \, \frac{1}{d_A^2} \, \frac{1}{r_{oA}} + \frac{1}{r_{oA}^3} \; . \tag{74}$$

B.   Collection of all terms

Altogether, by inserting the estimates of vectorial coefficients, (57)–(59), and the scalar coefficients (65), (70), (74) into (52) yields

$$\begin{aligned}
\frac{G}{c^2} &\left[|\boldsymbol{\beta}''_A| \frac{|\mathcal{B}_A(t_o)|}{c} + |\boldsymbol{\gamma}''_A| \frac{|\mathcal{C}_A(t_o)|}{c} + |\boldsymbol{\delta}''_A| \frac{|\mathcal{D}_A(t_o)|}{c}\right] \\
&\leq \frac{G}{c^2} M_A J_2^A P_A^2 \left[\frac{9}{2} \frac{1}{d_A^2 \, r_{oA}} + \frac{1}{d_A \, r_{oA}^2} + \frac{13}{2} \frac{1}{r_{oA}^3} + 3 \frac{d_A}{r_{oA}^4}\right] \\
&\leq \frac{G}{c^2} M_A J_2^A P_A^2 \left[\frac{9}{2} \frac{1}{P_A^2 \, r_{oA}^{\min}} + \frac{1}{P_A \, (r_{oA}^{\min})^2} + \frac{19}{2} \frac{1}{(r_{oA}^{\min})^3}\right] ,
\end{aligned} \tag{75}$$

where we have used that $P_A \le d_A \le r_{oA}$. Note, in the last line of (75) the first term in the brackets is at least by a factor of $\simeq 10^4$ larger than the other two terms. Using the parameters given in Table I we obtain for the giant planets (γ can be safely set to unity for these estimates)

$$\frac{G}{c^2}\left[|\boldsymbol{\beta}''_A|\,\frac{|\mathcal{B}_A(t_o)|}{c} + |\boldsymbol{\gamma}''_A|\,\frac{|\mathcal{C}_A(t_o)|}{c} + |\boldsymbol{\delta}''_A|\,\frac{|\mathcal{D}_A(t_o)|}{c}\right] \quad \begin{aligned} &\le\ 3.26 \times 10^{-2}\mu\text{as} \quad \text{for Jupiter} \\[4pt] &\le\ 5.32 \times 10^{-3}\mu\text{as} \quad \text{for Saturn} \\ &\le\ 8.11 \times 10^{-4}\mu\text{as} \quad \text{for Uranus} \\ &\le\ 5.79 \times 10^{-5}\mu\text{as} \quad \text{for Neptune} \end{aligned}$$

$$(76)$$

In view of these estimates, for the envisaged accuracy of 1 μas the quadrupole light deflection (40) for sources in the solar system can be approximated by

$$\delta\boldsymbol{k}_Q(t_o) \;=\; \frac{1+\gamma}{2}\sum_A \frac{G}{c^2}\left[\boldsymbol{\alpha}''_A\,\frac{\mathcal{A}_A(t_o)}{c}\right] \tag{77}$$

with $\mathcal{A}_A$ given by (41) and $\boldsymbol{\alpha}''_A$ given by (45).

VI.   AN UPPER ESTIMATE OF THE QUADRUPOLE LIGHT DEFLECTION FOR SOLAR SYSTEM OBJECTS

The absolute value of the vectorial coefficient (53) is given by

$$|\boldsymbol{\alpha}''_A| \;=\; M_A\,J_2^A\,P_A^2\left(1 - (\boldsymbol{\sigma}\,\boldsymbol{e}_3)^2\right)\ , \tag{78}$$

so that an estimate of the absolute value (77) is

$$|\,\delta\boldsymbol{k}_Q(t_o)| \;=\; \sum_A \frac{G}{c^2}\,M_A\,J_2^A\,P_A^2\left(1 - (\boldsymbol{\sigma}\,\boldsymbol{e}_3)^2\right)\frac{|\mathcal{A}_A(t_o)|}{c}\ , \tag{79}$$

where the scalar coefficient A_A is given in (41). Due to

$$(\boldsymbol{k}\ \times\ \boldsymbol{r}_{eA})^2 \;=\; (\boldsymbol{k}\ \times\ \boldsymbol{r}_{oA})^2 \;=\; d_A^2 \;+\; \mathcal{O}\!\left(\frac{1}{c^2}\right)\ , \tag{80}$$

we obtain

$$\frac{\mathcal{A}_A(t_o)}{c} \;=\; \frac{1}{d_A^3}\,\frac{1}{R}\left(\frac{1}{r_{eA}}\left(r_{eA} + \boldsymbol{k}\boldsymbol{r}_{eA}\right)^2 - \frac{1}{r_{oA}}\left(r_{oA} + \boldsymbol{k}\boldsymbol{r}_{oA}\right)^2\right)$$

$$+\,\frac{1}{r_{oA}^3}\,\frac{1}{d_A^3}\left(2\,r_{oA} - \boldsymbol{k}\boldsymbol{r}_{oA}\right)\left(r_{oA} + \boldsymbol{k}\boldsymbol{r}_{oA}\right)^2\ . \tag{81}$$

Inserting the explicit form of vector $\boldsymbol{k}$ defined in (51) and (50), respectively, and collecting all terms together we obtain

$$\frac{\mathcal{A}_A(t_o)}{c} = \frac{1}{d_A^3}\frac{1}{R^3}\left(1 - \cos\alpha\right)^2\left(2\,r_{eA}^3 + r_{oA}^2\,r_{eA} + 2\,r_{eA}^2\,r_{oA} + r_{eA}^3\,\cos\alpha\right). \quad (82)$$

By means of the inequality (see Appendix A for a proof)

$$\begin{aligned}
&(1 - \cos\alpha)^2\,\frac{2x^3 + x\,y^2 + 2x^2\,y + x^3\,\cos\alpha}{(x^2 + y^2 - 2\,x\,y\,\cos\alpha)^{3/2}} \\
&\leq 3\,\frac{x}{(x^2 + y^2 - 2xy\,\cos\alpha)^{1/2}}\,\frac{\sin^2\alpha}{1 + \cos\alpha},
\end{aligned} \quad (83)$$

valid for any $x \geq 0$ and $y \geq 0$ and with the aid of

$$\frac{r_{eA}}{R}\,\sin\alpha = \frac{d_A}{r_{oA}}, \quad (84)$$

we obtain the estimate,

$$\frac{\mathcal{A}_A(t_o)}{c} \leq 3\,\frac{1}{r_{oA}}\frac{1}{d_A^2}\frac{\sin\alpha}{1 + \cos\alpha}, \quad (85)$$

and, therefore, with the aid of (77) and (78) we achieve

$$|\delta\boldsymbol{k}_Q(t_o)| \leq \frac{3\,(1+\gamma)}{2}\,\frac{GM_A}{c^2}\,\frac{1}{d_A^2}\,\frac{1}{r_{oA}}\,J_2^A\,P_A^2\left(1 - (\boldsymbol{\sigma}\,\boldsymbol{e}_3)^2\right)\frac{\sin\alpha}{1 + \cos\alpha}. \quad (86)$$

This result can be related to the spherically symmetric part, given in Eq. (70) in [3] by

$$\delta\boldsymbol{k}_{pN}(t_o) = -\sum_A (1+\gamma)\,\frac{GM_A}{c^2}\,\frac{\boldsymbol{R}\times(\boldsymbol{r}_{eA}\times\boldsymbol{r}_{oA})}{R\,r_{oA}^2\,r_{eA}\,(1 + \cos\alpha)}, \quad (87)$$

and it's absolute value, respectively,

$$|\,\delta\boldsymbol{k}_{pN}(t_o)| = \sum_A (1+\gamma)\,\frac{GM_A}{c^2}\,\frac{1}{r_{oA}}\,\frac{\sin\alpha}{1 + \cos\alpha}. \quad (88)$$

A direct comparison between (86) and (88) yields the criterion

$$|\delta\boldsymbol{k}_Q(t_o)| \leq \frac{3}{2}\,\frac{P_A^2}{d_A^2}\,J_2^A\left(1 - (\boldsymbol{\sigma}\boldsymbol{e}_3)^2\right)|\,\delta\boldsymbol{k}_{pN}(t_o)|. \quad (89)$$

Due to $1 \geq (\boldsymbol{\sigma}\boldsymbol{e}_3)^2$, the estimate (89) can be further approximated by

$$|\,\delta \boldsymbol{k}_Q(t_o)|\ \leq\ \frac{3}{2}\,J_2^A\,\frac{P_A^2}{d_A^2}\,|\,\delta \boldsymbol{k}_{pN}(t_o)|\ . \tag{90}$$

This criterion relates the quadrupole light deflection of sources in the solar system to the simpler case of spherically symmetric part. For GAIA it is recommended to use (90) as a criterion if the quadrupole light deflection has to be calculated for a given solar system object. The estimate of the monopole light deflection for solar system objects

$$|\,\delta \boldsymbol{k}_{pN}(t_o)|\ \leq\ \frac{2\,(1+\gamma)\,GM_A}{c^2\,d_A}\ \leq\ \frac{2\,(1+\gamma)\,GM_A}{c^2\,P_A}\ , \tag{91}$$

which is easy to prove from (88) can be used in the case when $|\,\delta \boldsymbol{k}_{pN}(t_o)|$ is not available. From (79) and (82) one can directly see that (for a proof see Appendix B)

$$|\,\delta \boldsymbol{k}_Q(t_o)|\ \leq\ 2\,(1+\gamma)\,\frac{GM_A}{c^2}\,\frac{P_A^2}{d_A^3}\,J_2 \leq 2\,(1+\gamma)\,\frac{GM_A}{c^2\,P_A}\,J_2\ . \tag{92}$$

VII. NUMERICAL TESTS

The obtained simplified formulas given by Eq. (31) for stars/quasars and by Eq. (77) for solar system objects and the a priori criteria given for these objects by Eq. (37) and (90), respectively, have been incorporated into the current reference C implementation of GREM as described by Klioner & Blankenburg [4] and Klioner [5].

Numerical experiments with the C implementation have confirmed the correctness and the efficiency of the estimates and criteria. In the experiments we used about 10^8 objects (both randomly distributed over the sky and specially generated to give grazing rays to the giant planets). The results can be summarized as follows:

I. Stars and quasars.

- The maximal difference between the full quadrupole deflection formula (1) and the simplified one (31) amounts to 1.1×10^{-10} μas in good agreements with (30). Therefore, the actual values of the neglected terms are in case of Gaia about 10–15 times less than given by (30).

- The upper estimate (37) holds and is attainable for randomly distributed sources.

- The mean value of the ratio between the actual value of the quadrupole light deflection and its upper estimate (37) amounts to 0.48 for randomly distributed sources that indicates the high numerical efficiency of the estimate.

II. Solar system objects.

- The maximal difference between the full quadrupole deflection formula (40) and the simplified one (77) amounts to 0.0017 μas in good agreements with (76). Therefore, the actual values of the neglected terms are again about 10–15 times less than given by (76).

– The upper estimate (90) holds and is attainable for randomly distributed sources.

– The mean value of the ratio between the actual value of the quadrupole light deflection and its upper estimate (90) amounts to 0.40 for randomly distributed sources that again indicates the high numerical efficiency of the estimate.

Implementation of the criteria (37) and (90) has allowed to significantly reduce the number of "false alarms" (cases for which the full quadrupole deflection has been computed and turned out to be much smaller than the requested goal accuracy). The "false alarms" were caused by the use of a more primitive ad hoc criteria implemented in the C code of GREM previously. This, in turn, slightly increases the performance of the GREM implementation.

VIII. SUMMARY

Let us summarize the results of this report.

1. The quadrupole light deflection (1) for stars and quasars can be approximated by (31) for the Gaia nominal orbit and for the accuracy of 1 μas.

2. Eq. (37) can be used as an a priori criterion if the quadrupole light deflection (1) has to be computed for a given source.

3. The quadrupole light deflection (40) for solar system sources can be approximated by (77) for the Gaia nominal orbit and for the accuracy of 1 μas.

4. Eq. (90) can be used as an a priori criterion if the quadrupole light deflection (77) has to be computed for a given solar system object.

[1] IERS Conventions (2003). Dennis D. McCarthy and Gérard Petit. (IERS Technical Note 32) Frankfurt am Main: Verlag des Bundesamts für Kartographie und Geodäsie, 2004, 127 pp.
[2] Klioner, S.A., 1991, Sov. Astron., **35**, 523
[3] Klioner, S.A., 2003, AJ, **125** 1580
[4] Klioner, S. A., Blankenburg, R. 2003, Technical report on the implementation of the GAIA relativistic model, available on Livelink
[5] Klioner, S. A., 2003, Technical report on the implementation of the GAIA relativistic model, Amendment for version 1.0g, available on Livelink
[6] P.R. Weissman, L.-A. McFadden, T.V. Johnson, Encyclopedia of the Solar System, (San Diego: Academic) eds. 1999.

APPENDIX A: PROOF OF EQ. (83)

In this Appendix we prove the inequality (83). The latter can be rewritten as

$$(1 - \cos \alpha)^2 \, (2x^3 + x \, y^2 + 2x^2 \, y + x^3 \, \cos \alpha) \; \leq \; 3 \, x \, (x^2 + y^2 - 2xy \, \cos \alpha) \, (1 - \cos \alpha) \, . \tag{A1}$$

Denoting $z = x/y$ we obtain the relation

$$f \; \equiv \; -1 - 2 \, z^2 + 2 \, z - \cos \alpha + 4 \, z \, \cos \alpha - \cos^2 \alpha - z^2 \, \cos \alpha \; \leq \; 0 \, . \tag{A2}$$

To prove the inequality (A2) it is sufficient to investigate the values of f at extrema and at the boundaries given by $z \geq 0$ and $0 \leq \alpha \leq \pi$. To determine the extrema of this function we set the first derivatives zero,

$$f_z \; = \; -4 \, z + 2 + 4 \cos \alpha - 2 \, z \cos \alpha \; = \; 0 \, , \tag{A3}$$
$$f_\alpha \; = \; \sin \alpha \, (1 - 4 \, z + 2 \cos \alpha + z^2) \; = \; 0 \, . \tag{A4}$$

The only solutions of the coupled system (A3), (A4) are

$$P_1 \; = \; (z = 0, \alpha = \frac{2}{3} \pi) \, , \quad P_2 \; = \; (z = 1, \alpha = 0) \, . \tag{A5}$$

The values of f at these points are

$$f(P_1) \; = \; -\frac{3}{4} \, , \quad f(P_2) \; = \; 0 \, . \tag{A6}$$

At the boundaries we get

$$f(z = 0) \; = \; -1 - \cos \alpha - \cos^2 \alpha \; < 0 \, , \tag{A7}$$
$$f(z \to \infty) \; = \; -z^2 (2 + \cos \alpha) \; < 0 \, , \tag{A8}$$
$$f(\alpha = 0) \; = \; -3(1 - z)^2 \; \leq 0 \, , \tag{A9}$$
$$f(\alpha = \pi) \; = \; -z(z + 2) \; \leq \; 0 \, . \tag{A10}$$

From the results (A5)–(A10) we conclude the validity of the inequality (A2) and (A1).

APPENDIX B: PROOF OF EQ. (92)

From (79) it is clear that (92) is true if for $\mathcal{A}_A$ defined by (82) one has

$$\frac{\mathcal{A}_A(t_o)}{c} \leq \frac{4}{d_A^3} \ . \tag{B1}$$

To prove this it is sufficient to demonstrate that

$$\frac{1}{R^3} \left(1 - \cos \alpha\right)^2 \left(2\, r_{eA}^3 + r_{oA}^2\, r_{eA} + 2\, r_{eA}^2\, r_{oA} + r_{eA}^3\, \cos \alpha\right) \ \leq \ 4 \ , \tag{B2}$$

or introducing again $z = r_{eA}/r_{oA}$

$$g \equiv \frac{(1 - \cos \alpha)^2\, z\, \left((1 + z)^2 + z^2\, (1 + \cos \alpha)\right)}{4\, (1 + z^2 - 2z \cos \alpha)^{3/2}} \ \leq \ 1 \ . \tag{B3}$$

The derivatives of g with respect to α and z vanish simultaneously only for $\alpha = 0$ which is one of the boundaries. At the boundaries we get

$$g(z = 0) \ = \ 0 \ , \tag{B4}$$

$$\lim_{z \to \infty} g \ = \ \frac{1}{4}\, (1 - \cos \alpha)^2\, (2 + \cos \alpha) \leq 1 \ , \tag{B5}$$

$$g(\alpha = 0) \ = \ 0 \ , \tag{B6}$$

$$g(\alpha = \pi) \ = \ \frac{z}{1 + z} \leq 1 \ . \tag{B7}$$

From this we conclude that (B2) and, therefore, (B1) and (92) are valid.

APPENDIX C: PROOF OF SEVERAL FURTHER INEQUALITIES USING MAPLE

A Maple worksheet is attached where several additional inequalities are proved. The worksheet, although rather trivial, is available from the authors upon request.

Proof of inequalities (29) and (66):

```
> restart;
> with(linalg):
```
Warning, the protected names norm and trace have been redefined and unprotected

The goal is to prove that the product of projections of two perpendicular unit vectors v1 and v2 on an arbitrary third unit vector v3 is not greater than 1/2. Since the scalar products are independent of the orientation of the coordinates we choose the orientation where the problem looks as simple as possible. Without loss of generality the first vector can be chosen to coincide with axis z:

```
> v1:=vector([0,
              0,
              1]);
```

$$v1 := [0, 0, 1]$$

Again without loss of generality we can consider that v3 lies in the x-z plane and its y component vanishes. This v3 is a unit vector it can be represented as

```
> v3:=vector([cos(delta),
              0,
              sin(delta)]);
```

$$v3 := [\cos(\delta), 0, \sin(\delta)]$$

Now the orientation of the coordinate is completely fixed and the second unit vector v2 is than

```
> v2:=vector([cos(alpha2)*cos(delta2),
              sin(alpha2)*cos(delta2),
              sin(delta2)]);
```

$$v2 := [\cos(\alpha 2)\cos(\delta 2), \sin(\alpha 2)\cos(\delta 2), \sin(\delta 2)]$$

The orthogonality condition of v1 and v2 gives

```
> ortho:=dotprod(v1,v2,'orthogonal')=0;
```

$$ortho := \sin(\delta 2) = 0$$

which means that delta2=0 and the representation of v2 can be simplified as

```
> v2:=simplify(subs(delta2=0,evalm(v2)));
```

$$v2 := [\cos(\alpha 2), \sin(\alpha 2), 0]$$

Now, the function the maximum of which should be found reads

```
> f:=dotprod(v1,v3,'orthogonal')*dotprod(v2,v3,'orthogonal');
```
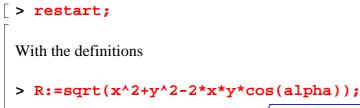
$$f := \sin(\delta)\cos(\alpha 2)\cos(\delta)$$

and since

```
> f=combine(coeff(f,cos(alpha2)))*cos(alpha2);
```

$$\sin(\delta)\cos(\alpha 2)\cos(\delta) = \frac{1}{2}\sin(2\,\delta)\cos(\alpha 2)$$

and angles delta and alpha2 are independent it becomes evident that f does not exceed 1/2. QED.

⊟ Proof of inequality (70):

```
> restart;
```

With the definitions

```
> R:=sqrt(x^2+y^2-2*x*y*cos(alpha));
```

$$R := \sqrt{x^2 + y^2 - 2\,x\,y\cos(\alpha)}$$

```
> LHS:=(1/R^2)*(x/y^2*cos(alpha)+y/x^2*cos(alpha)-1/x-1/y);
```

$$LHS := \frac{\dfrac{x\cos(\alpha)}{y^2} + \dfrac{y\cos(\alpha)}{x^2} - \dfrac{1}{x} - \dfrac{1}{y}}{x^2 + y^2 - 2\,x\,y\cos(\alpha)}$$

```
> RHS:=(x+y)/(x^2*y^2);
```

$$RHS := \frac{x + y}{x^2\,y^2}$$

the inequality (68) states that f <= 0 with

```
> f:=LHS-RHS;
```

$$f := \frac{\dfrac{x\cos(\alpha)}{y^2} + \dfrac{y\cos(\alpha)}{x^2} - \dfrac{1}{x} - \dfrac{1}{y}}{x^2 + y^2 - 2\,x\,y\cos(\alpha)} - \frac{x + y}{x^2\,y^2}$$

Consider the factorized form

```
> f:=factor(f);
```

$$f := \frac{(x + y)\,(y^2 + x\,y + x^2)\,(\cos(\alpha) - 1)}{(x^2 + y^2 - 2\,x\,y\cos(\alpha))\,x^2\,y^2}$$

Since both x and y are non-negative, all the factors in f are non-negative expect for cos(alpha)-1 that is non-positive. Therefore f<=0. QED.

Proof of inequality (75):

```
> restart;
```

Definitions:

```
> LHS:=(x^3-y^3)/(x-y);
```

$$LHS := \frac{x^3 - y^3}{x - y}$$

which for any x and y can be simplified to be

```
> LHS:=simplify(LHS);
```

$$LHS := x^2 + y\,x + y^2$$

```
> RHS:=(3/2)*(x^2+y^2);
```

$$RHS := \frac{3\,x^2}{2} + \frac{3\,y^2}{2}$$

The inequality (73) states that g <= 0

```
> g:=LHS-RHS;
```

$$g := -\frac{1}{2}\,x^2 + y\,x - \frac{1}{2}\,y^2$$

The factorized simplified form of g

```
> g:=factor(g);
```

$$g := -\frac{(x - y)^2}{2}$$

makes it evident that g is non-positive. QED.

Proof of inequality (79):

```
> restart;
```

Definitions:

```
> R:=sqrt(x^2+y^2-2*x*y*cos(alpha));
```

$$R := \sqrt{x^2 + y^2 - 2\,x\,y\cos(\alpha)}$$

```
> LHS:=(x*cos(alpha)+y*cos(alpha)-x-y)/R^2;
```

$$LHS := \frac{x \cos(\alpha) + y \cos(\alpha) - x - y}{x^2 + y^2 - 2\,x\,y\,\cos(\alpha)}$$

> **`RHS:=2/(x+y);`**

$$RHS := \frac{2}{x+y}$$

The left-hand side can be factored

> **`LHS:=factor(LHS);`**

$$LHS := \frac{(\cos(\alpha) - 1)(x+y)}{x^2 + y^2 - 2\,x\,y\,\cos(\alpha)}$$

which make it evident that LHS is non-positive (considering that both x and y are non-negative). Therefore the absolute value of LHS is -LHS:

> **`ABSLHS:=-LHS;`**

$$ABSLHS := -\frac{(\cos(\alpha) - 1)(x+y)}{x^2 + y^2 - 2\,x\,y\,\cos(\alpha)}$$

The inequality (77) states that h <= 0 with

> **`h:=ABSLHS-RHS;`**

$$h := -\frac{(\cos(\alpha) - 1)(x+y)}{x^2 + y^2 - 2\,x\,y\,\cos(\alpha)} - \frac{2}{x+y}$$

The factored form of function h:

> **`h:=factor(h);`**

$$h := -\frac{(x-y)^2\,(1 + \cos(\alpha))}{(x^2 + y^2 - 2\,x\,y\,\cos(\alpha))(x+y)}$$

makes it evident that h is non-positive. QED.

>