Analytical solution for light propagation in Schwarzschild field having an accuracy of 1 μas

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Numerical integration of the differential equations of light propagation in the Schwarzschild metric shows that in some extreme situations relevant for practical observations (e.g. for Gaia) the well-known standard post-Newtonian formula for the boundary problem has an error up to 16 μas. The aim of this note is to identify the reason for this error and to derive an extended formula accurate at the level of 1 μas as needed e.g. for Gaia.

The analytical parametrized post-post-Newtonian solution for light propagation derived by Klioner & Zschocke [1] gives the solution for the boundary problem with all analytical terms of order $\mathcal{O}(c^{-4})$ taken into account. Giving analytical upper estimates of each term we investigate which post-post-Newtonian terms may play a role for an observer in the solar system at the level of 1 μas. We conclude that only one post-post-Newtonian term remains important for this numerical accuracy and derive a simplified analytical solution for the boundary problem for light propagation containing all the terms that are indeed relevant at the level of 1 μas. The derived analytical solution has been verified using the results of a high-accuracy numerical integration of differential equations of light propagation and found to be correct at the level well below 1 μas for arbitrary observer situated within the solar system.

Contents

I. INTRODUCTION

It is well known that adequate relativistic modelling is indispensable for the success of microarcsecond space astrometry. One of the most important relativistic effects for astrometric observations in the solar system is the gravitational light deflection. The largest contribution in the light deflection comes from the spherically symmetric (Schwarzschild) parts of the gravitational fields of each solar system body. Although the planned astrometric satellites Gaia, SIM, etc. will not observe very close to the Sun, they can observe very close to the giant planets also producing significant light deflection. This poses the problem of modelling this light deflection with a numerical accuracy of better than 1 μas.

The exact differential equation of motion for a light ray in the Schwarzschild field can be solved numerically as well as analytically. However, the exact analytical solution is given in terms of elliptic integrals, implying numerical efforts comparable with direct numerical integration, so that approximate analytical solutions are usually used. In fact, the standard parametrized post-Newtonian (PPN) solution is sufficient in many cases and has been widely applied. So far, there was no doubt that the post-Newtonian order of approximation is sufficient for astrometric missions even up to microarcsecond level of accuracy, besides astrometric observations close to the edge of the Sun. However, a direct comparison reveals a deviation between the standard post-Newtonian approach and the exact numerical solution of the geodetic equations. In particular, we have found a difference of up to 16 μas in light deflection for solar system objects observed close to giant planets. This error has triggered detailed numerical and analytical investigations of the problem.

Usually, in the framework of general relativity or the PPN formalism analytical orders of smallness of various terms are considered. Here the role of small parameter is played by c^{-1} where c is the light velocity. Standard post-Newtonian and post-post-Newtonian solutions are derived by retaining terms of relevant analytical orders of magnitude. On the other hand, for practical calculations only numerical magnitudes of various terms are relevant. In this note we attempt to close this gap and combine the analytical post-post-Newtonian solution derived in Klioner & Zschocke [1] with estimates of numerical magnitudes of various terms. In this way we will derive a compact analytical solution for the boundary problem for light propagation where all terms are indeed relevant at the level of 1 μas. The derived analytical solution is then verified using high-accuracy numerical integration of the differential equations of light propagation and found to be correct at the level well below 1 μas.

We use fairly standard notations:

- G is the Newtonian constant of gravitation;
- c is the velocity of light;
- β and γ are the parameters of the Parametrized Post-Newtonian (PPN) formalism which characterize possible deviation of the physical reality from general relativity theory ($\beta = \gamma = 1$ in general relativity);
- lower case Latin indices i, j, ... take values 1, 2, 3;
- lower case Greek indices μ, ν, ... take values 0, 1, 2, 3;
- repeated indices imply the Einstein's summation irrespective of their positions (e.g. $a^i\, b^i = a^1\, b^1 + a^2\, b^2 + a^3\, b^3$ and $a^\alpha\, b^\alpha = a^0\, b^0 + a^1\, b^1 + a^2\, b^2 + a^3\, b^3$);

- a dot over any quantity designates the total derivative with respect to the coordinate time of the corresponding reference system: e.g. $\dot{a} = \frac{da}{dt}$;
- the 3-dimensional coordinate quantities ("3-vectors") referred to the spatial axes of the corresponding reference system are set in boldface: $\boldsymbol{a} = a^i$;
- the absolute value (Euclidean norm) of a "3-vector" $\boldsymbol{a}$ is denoted as $|\boldsymbol{a}|$ or, simply, a and can be computed as $a = |\boldsymbol{a}| = (a^1\, a^1 + a^2\, a^2 + a^3\, a^3)^{1/2}$;
- the scalar product of any two "3-vectors" $\boldsymbol{a}$ and $\boldsymbol{b}$ with respect to the Euclidean metric δ_{ij} is denoted by $\boldsymbol{a}\, \cdot\, \boldsymbol{b}$ and can be computed as $\boldsymbol{a}\, \cdot\, \boldsymbol{b} = \delta_{ij}\, a^i\, b^j = a^i\, b^i$;
- the vector product of any two "3-vectors" $\boldsymbol{a}$ and $\boldsymbol{b}$ is designated by $\boldsymbol{a} \times \boldsymbol{b}$ and can be computed as $(\boldsymbol{a} \times \boldsymbol{b})^i = \varepsilon_{ijk}\, a^j\, b^k$, where $\varepsilon_{ijk} = (i-j)(j-k)(k-i)/2$ is the fully antisymmetric Levi-Civita symbol;
- for any two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ the angle between them is designated as $\delta(\boldsymbol{a}, \boldsymbol{b})$.

The paper is organized as follows. In Section II we present the exact differential equations. High-accuracy numerical integration of these equations is discussion in Section III. In Section IV we discuss the standard post-Newtonian approximation and demonstrate the problem with the standard post-Newtonian solution by direct comparison between numerical results and the PPN solution. In Section V, the formulas for the boundary problem in post-post-Newtonian approximation are considered. A detailed estimation of all relevant terms is given, and simplified expressions are derived. We demonstrate by explicit numerical examples the applicability of this analytical approach for the GAIA astrometric mission. In Section VI we consider the important case of objects situated infinitely far from the observer as a limit of the boundary problem. The results are summarized in Section VII. In the Appendices detailed derivations for a number of analytical formulas are given.

II. SCHWARZSCHILD METRIC AND NULL GEODESICS IN HARMONIC COORDINATES

For the reasons given above we need a tool to calculate the real numerical accuracy of some analytical formulas for the light propagation. To this end, we consider the exact Schwarzschild metric and its null geodesics in harmonic gauge. Those exact differential equations for the null geodesics will be solved numerically with high accuracy (see below) and that numerical solution provides the required reference.

A. Metric tensor

As it has been already discussed in Klioner & Zschocke [1] in harmonic gauge

$$\frac{\partial\left(\sqrt{-g}\, g^{\alpha\beta}\right)}{\partial x^\beta} = 0 \qquad (1)$$

the components of the covariant metric tensor of the Schwarzschild solution are given by

$$g_{00} = - \frac{1-a}{1+a},$$
$$g_{0i} = 0,$$
$$g_{ij} = (1+a)^2 \, \delta_{ij} \, + \, \frac{a^2}{x^2} \frac{1+a}{1-a} \, x^i \, x^j \tag{2}$$

where

$$a = \frac{m}{x}, \tag{3}$$

$m = \frac{GM}{c^2}$ is the Schwarzschild radius of a body with mass M. The contravariant components read

$$g^{00} = \frac{1+a}{1-a} \, ,$$
$$g^{0i} = 0 \, ,$$
$$g^{ij} = \frac{1}{(1+a)^2} \, \delta_{ij} \, - \, \frac{a^2}{x^2} \, \frac{1}{(1+a)^2} \, x^i \, x^j \, . \tag{4}$$

Considering that the determinant of the metric can be computed as

$$g = -(1+a)^4, \tag{5}$$

one can easily check that this metric satisfies the harmonic conditions (1).

B. Christoffel symbols

The Christoffel symbols of second kind are defined as

$$\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2} \, g^{\mu\nu} \, \left(\frac{\partial g_{\nu\alpha}}{\partial x^{\beta}} \, + \, \frac{\partial g_{\nu\beta}}{\partial x^{\alpha}} \, - \, \frac{\partial g_{\alpha\beta}}{\partial x^{\nu}} \right) \, . \tag{6}$$

Using (2) and (4) one gets

$$\Gamma^{0}_{0i} = \frac{a}{x^2} \, \frac{1}{1-a^2} \, x^i \, ,$$
$$\Gamma^{i}_{00} = \frac{a}{x^2} \, \frac{1-a}{(1+a)^3} \, x^i \, ,$$
$$\Gamma^{i}_{jk} = \frac{a}{x^2} \, x^i \, \delta_{jk} \, - \, \frac{a}{x^2} \, \frac{1}{1+a} \, \left(x^j \, \delta_{ik} \, + \, x^k \, \delta_{ij} \right) \, - \, \frac{a^2}{x^4} \, \frac{2-a}{1-a^2} \, x^i \, x^j \, x^k \, , \tag{7}$$

and all other Christoffel symbols vanish.

C. Isotropic condition

As it has been pointed out in Section II.C of Klioner & Zschocke [1] the condition of isotropy

$$g_{\alpha\beta} \, \frac{d\,x^{\alpha}}{d\,\lambda} \, \frac{d\,x^{\beta}}{d\,\lambda} = 0 \, , \tag{8}$$

leads to the following integral of the equations of light propagation

$$s = \frac{1-a}{1+a}\,\left(1-a^2+\frac{a^2}{x^2}(\boldsymbol{x}\cdot\boldsymbol{\mu})^2\right)^{-1/2}\,, \tag{9}$$

where μ^i is the coordinate direction of propagation ($\boldsymbol{\mu}\cdot\boldsymbol{\mu}=1$), $\boldsymbol{x}$ is the position of the photon and s is the absolute value of the coordinate light velocity normalized by c: $s=|\dot{\boldsymbol{x}}|/c$.

D. Equation of isotropic geodesics

Reparametrizing the geodetic equations

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\alpha\beta}\,\frac{dx^\alpha}{d\lambda}\,\frac{dx^\beta}{d\lambda} = 0\,, \tag{10}$$

by coordinate time $t=x^0$ (see e.g. Section II.D of Klioner & Zschocke [1]) and using the Christoffel symbols computed above one gets the differential equations for the light propagation in metric (2):

$$\ddot{\boldsymbol{x}} = \frac{a}{x^2}\left[-c^2\frac{1-a}{(1+a)^3} - \dot{\boldsymbol{x}}\cdot\dot{\boldsymbol{x}} + a\frac{2-a}{1-a^2}\left(\frac{\boldsymbol{x}\cdot\dot{\boldsymbol{x}}}{x}\right)^2\right]\boldsymbol{x} + 2\frac{a}{x^2}\,\frac{2-a}{1-a^2}(\boldsymbol{x}\cdot\dot{\boldsymbol{x}})\,\dot{\boldsymbol{x}}\,. \tag{11}$$

Eq. (9) for the isotropic condition together with $\dot{\boldsymbol{x}}\cdot\dot{\boldsymbol{x}} = c^2\,s^2$ could be used to avoid the term containing $\dot{\boldsymbol{x}}\cdot\dot{\boldsymbol{x}}$, but it does not simplify the equations.

III. NUMERICAL INTEGRATION OF THE EQUATIONS OF LIGHT PROPAGATION

Our goal is to integrate Eq. (11) numerically to get a solution for the trajectory of a light ray with an accuracy much higher than the goal accuracy of 1μas $\approx 4.8\times10^{-12}$. For this numerical integration a simple FORTRAN 95 code using quadrupole (128 bit) arithmetic has been written. Numerical integrator ODEX [2] has been adapted to the quadrupole precision. ODEX is an extrapolation algorithm based on the explicit midpoint rule. It has automatic order selection, local accuracy control and dense output. Using forth and back integration to estimate the accuracy, each numerical integration is automatically checked to achieve a numerical accuracy of at least 10^{-24} in the components of both position and velocity of the photon at each moment of time.

The numerical integration is first used to solve the initial value problem for differential equations (11). Eq. (9) should be used to choose the initial conditions. The problem of light propagation has thus only 5 degrees of freedom: 3 degrees of freedom correspond to the position of the photon and two other degrees of freedom correspond to the unit direction of light propagation. The absolute value of the coordinate light velocity can be computed from (9). Fixing initial position of the photon $\boldsymbol{x}(t_0)$ and initial direction of propagation $\boldsymbol{\mu}$ one gets the initial velocity of the photon as function of $\boldsymbol{\mu}$ and s computed for given $\boldsymbol{\mu}$ and $\boldsymbol{x}$:

$$\begin{aligned}\boldsymbol{x}(t_0) &= \boldsymbol{x}_0\,,\\ \dot{\boldsymbol{x}}(t_0) &= c\,s\,\boldsymbol{\mu}\,.\end{aligned} \tag{12}$$

The numerical integration yields the position $\boldsymbol{x}$ and velocity $\dot{\boldsymbol{x}}$ of a photon as function of time t. The dense output of ODEX allows one to obtain the position and velocity of the photon on a selected grid of moments of time. Eq. (9) holds for any moment of time as soon as it is satisfied by the initial conditions. Therefore, (9) can be also used to estimate the accuracy of numerical integration at each moment of integration.

For the purposes of this work we need to have an accurate solution of two-value boundary problem. That is, a solution of Eq. (11) with boundary conditions

$$\begin{aligned}\boldsymbol{x}(t_0) &= \boldsymbol{x}_0,\\ \boldsymbol{x}(t) &= \boldsymbol{x}\,,\end{aligned} \tag{13}$$

where $\boldsymbol{x}_0$ and $\boldsymbol{x}$ are two given constants, t_0 is assumed to be fixed and t is unknown and should be determined by solving (11). Instead of using some numerical methods to solve this boundary problem directly, we generate solutions of a family of boundary problems from our solution of initial value problem (12). Each intermediate result computed by ODEX during the integration with initial conditions (12) gives us a high-accuracy solution of the corresponding two-value boundary problem (13): t and $\boldsymbol{x}$ are just taken from the intermediate steps of our numerical integration.

In the following discussion we will compare predictions of various analytical models for the unit direction of light propagation $\boldsymbol{n}(t)$ for a given moment of time t. The reference value for these comparisons can be derived directly from the numerical integration as

$$\boldsymbol{n}(t) = \frac{\dot{\boldsymbol{x}}(t)}{|\dot{\boldsymbol{x}}(t)|}. \tag{14}$$

The accuracy of this numerically computed $\boldsymbol{n}$ in our numerical integrations is guaranteed to be of the order of 10^{-24} radiant and can be considered as exact for our purposes.

IV. STANDARD POST-NEWTONIAN APPROACH

In this Section we will recall the standard post-Newtonian approach and will compare the results for the light deflection with the accurate numerical solution of the geodetic equations described in the previous Section.

A. Equations of post-Newtonian approach

The well-known equations of light propagation in first post-Newtonian approximation with PPN parameters have been discussed by many authors. The differential equations for the light rays are given by the post-Newtonian terms of Eq. (22) of Klioner & Zschocke [1]:

$$\ddot{\boldsymbol{x}} = -\,\left(c^2 + \gamma\,\dot{x}^k\,\dot{x}^k\right)\,\frac{a\,\boldsymbol{x}}{x^2} + 2\,(1+\gamma)\,\frac{a\,\dot{\boldsymbol{x}}\,(\dot{x}^k\,x^k)}{x^2} + \mathcal{O}(c^{-2})\,. \tag{15}$$

The analytical solution of (15) can be written in the form

$$\boldsymbol{x}(t) = \boldsymbol{x}_{\rm pN} + \mathcal{O}(c^{-4})\,, \tag{16}$$

$$\boldsymbol{x}_{\rm pN} = \boldsymbol{x}_0 + c\,(t-t_0)\,\boldsymbol{\sigma} + \Delta\boldsymbol{x}(t)\,, \tag{17}$$

where

$$\Delta\boldsymbol{x}(t) = -(1+\gamma)m\left(\boldsymbol{\sigma}\times(\boldsymbol{x}_0\times\boldsymbol{\sigma})\left(\frac{1}{x-\boldsymbol{\sigma}\cdot\boldsymbol{x}}-\frac{1}{x_0-\boldsymbol{\sigma}\cdot\boldsymbol{x}_0}\right)+\boldsymbol{\sigma}\,\log\frac{x+\boldsymbol{\sigma}\cdot\boldsymbol{x}}{x_0+\boldsymbol{\sigma}\cdot\boldsymbol{x}_0}\right). \tag{18}$$

Solution (16)–(18) satisfies the following initial conditions:

$$\boldsymbol{x}(t_0) = \boldsymbol{x}_0\,,$$
$$\lim_{t\to-\infty}\dot{\boldsymbol{x}}(t) = c\,\boldsymbol{\sigma}\,. \tag{19}$$

From Eqs. (16)–(18) it is easy to derive the following expression for the unit tangent vector at observer's position (note, in boundary problem we consider $\boldsymbol{x}_{\rm pN}$ as the exact position $\boldsymbol{x}$, according to Eq. (16)):

$$\boldsymbol{n}_{\rm pN} = \boldsymbol{k} - (1+\gamma)\,m\,\frac{\boldsymbol{k}\times(\boldsymbol{x}_0\times\boldsymbol{x})}{x\,(x\,x_0+\boldsymbol{x}\cdot\boldsymbol{x}_0)}\,, \tag{20}$$

where $\boldsymbol{R} = \boldsymbol{x} - \boldsymbol{x}_0$, $\boldsymbol{k} = \boldsymbol{R}/R$. By means of Eq. (47) given below we obtain that for the angle $\delta(\boldsymbol{n}_{\rm pN},\boldsymbol{k})$ between $\boldsymbol{n}_{\rm pN}$ and $\boldsymbol{k}$ one has (for $\gamma = 1$)

$$\delta(\boldsymbol{n}_{\rm pN},\boldsymbol{k}) \le \frac{4m}{d}\,\frac{x_0}{x+x_0}\,, \tag{21}$$

where

$$\boldsymbol{d} = \boldsymbol{k}\times(\boldsymbol{x}_0\times\boldsymbol{k}) = \boldsymbol{k}\times(\boldsymbol{x}\times\boldsymbol{k})\,. \tag{22}$$

In the limit of a source at infinity one gets

$$\lim_{x_0\to\infty}\delta(\boldsymbol{n}_{\rm pN},\boldsymbol{k}) \le \frac{4m}{d}\,. \tag{23}$$

B. Comparison between the post-Newtonian approximation and numerical solution

In order to determine the accuracy of the standard post-Newtonian approach we have to compare the post-Newtonian predictions of the light deflection with the results of the numerical solution of geodetic equations. Here, we compare the difference between the unit tangent vector $\boldsymbol{n}_{\rm pN}$ defined by (20) and the vector $\boldsymbol{n}$ calculated from the numerical integration using (14).

Having performed extensive tests, we have found that, in the real solar system, the error of $\boldsymbol{n}_{\rm pN}$ for observations made by an observer situated in the vicinity of the Earth attains 16 μas. These results are illustrated by Table I and Fig. 1. Table I contains the parameters we have used in our numerical simulations as well as the maximal deviation between $\boldsymbol{n}_{\rm pN}$ and $\boldsymbol{n}$ in each set of simulations. We have performed simulations with different bodies of the solar systems, assuming that the minimal impact distance d is equal to the radius of the corresponding body, and the maximal distance x between the gravitation body and the observer is given by the maximal distance between the gravitational body and the Earth. The simulation shows that the error of $\boldsymbol{n}_{\rm pN}$ is generally increasing for larger x and decreasing for larger d. The dependence of the error of $\boldsymbol{n}_{\rm pN}$ for fixed d and x and increasing distance between the gravitating body and the source at x_0 is given on Fig. 1 for the case of Jupiter, d being taken to be minimal and x to be maximal as given in Table I. Moreover, the error of $\boldsymbol{n}_{\rm pN}$ is found to be proportional to m^2 which leads us to the necessity to deal with the post-post-Newtonian approximation for the light propagation.

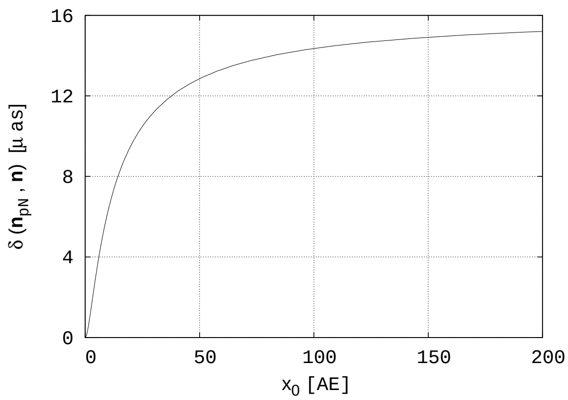

FIG. 1: The angle between $\boldsymbol{n}_{\rm pN}$ and $\boldsymbol{n}$ for Jupiter. The vector $\boldsymbol{n}_{\rm pN}$ is evaluated by means of the standard Newtonian formula (20), while $\boldsymbol{n}$ is taken from the numerical integration as described in Section III. Impact parameter d is taken to be the radius of Jupiter and the distance x between Jupiter and the observer is 6 AU.

	Sun	Sun at 45°	Jupiter	Saturn	Uranus	Neptune
$m = GM/c^2$ [m]	1476.6	1476.6	1.40987	0.42215	0.064473	0.076067
$d_{\rm min}$ [10^6 m]	696.0	105781.7	71.492	60.268	25.559	24.764
$x_{\rm max}$[AU]	1	1	6	11	21	31
$\delta_{\rm max}$[μas]	3187.8	6.32×10^{-4}	16.13	4.42	2.58	5.84

TABLE I: Numerical parameters of the Sun and giant planets are taken from [4, 5]. $d_{\rm min}$ is the minimal value of the impact parameter d that was used in the simulations. For each body $d_{\rm min}$ are equal its radius. For the Sun at 45° the impact parameter is computed as $d = \sin 45° \times 1\,\text{AU}$. $x_{\rm max}$ is the maximal absolute value of the position of observer x that was used in the simulations. $\delta_{\rm max}$ is the maximal angle between $\boldsymbol{n}_{\rm pN}$ and $\boldsymbol{n}$ found in the numerical tests.

V. POST-POST-NEWTONIAN SOLUTION OF BOUNDARY PROBLEM

In [1] an explicit analytical solution of the parametrized post-post-Newtonian equations of light propagation in the gravitational field of one spherically symmetric static body has been derived. The solution $\boldsymbol{x}_{\rm ppN}$ is given by Eqs. (26)–(35) of [1]. Boundary problem (13) has been considered in [1]. In this Section, we derive analytical upper estimates of all the terms in the post-post-Newtonian solution of the boundary problem and find which terms are responsible for numerical errors of the post-Newtonian solution described in the previous Section. In this way we derive the simplest possible formulas that agree with exact solution at a given numerical level.

A. Analytical estimates of the individual terms in $c\tau$

The propagation time between $\boldsymbol{x}_0$ and $\boldsymbol{x}$ is given by Eq. (50) of Klioner & Zschocke [1]:

$$\begin{aligned}
\text{N}\quad & c\,\tau = R \\
\text{pN}\quad & \qquad +(1+\gamma)\,m\,\log\frac{x+x_0+R}{x+x_0-R} \\
\Delta\text{pN}\quad & \qquad +\frac{1}{2}\,(1+\gamma)^2\,m^2\,\frac{R}{|\boldsymbol{x}\times\boldsymbol{x}_0|^2}\,\left((x-x_0)^2-R^2\right) \\
\text{ppN}\quad & \qquad +\frac{1}{8}\,\alpha\,\epsilon\,\frac{m^2}{R}\,\left(\frac{x_0^2-x^2-R^2}{x^2}+\frac{x^2-x_0^2-R^2}{x_0^2}\right) \\
\text{ppN}\quad & \qquad +\frac{1}{4}\alpha\,\left(8(1+\gamma)-4\beta+3\epsilon\right)\,m^2\,\frac{R}{|\boldsymbol{x}\times\boldsymbol{x}_0|}\,\delta(\boldsymbol{x},\boldsymbol{x}_0) \\
& \qquad +\mathcal{O}(c^{-6})\,.
\end{aligned} \tag{24}$$

Here and below we classify the nature of the individual terms by labels N (Newtonian), pN (post-Newtonian), ppN (post-post-Newtonian) and ΔpN (terms that are formally of post-post-Newtonian order, but may numerically become significantly larger than other post-post-Newtonian terms, see below). Using $|\boldsymbol{x}\times\boldsymbol{x}_0| = R\,d$ where d is the impact parameter defined by (22), and assuming general-relativistic values of all parameters $\alpha=\beta=\gamma=\epsilon=1$ one gets the following estimates of the sums of the terms labelled "ppN" and "ΔpN":

$$|c\,\delta\tau_{\Delta\text{pN}}| \le 2\,\frac{m^2}{d^2}\,R\,\frac{4\,x\,x_0}{(x+x_0)^2}\;\le\;2\,\frac{m^2}{d^2}\,R\,, \tag{25}$$

$$c\,\delta\tau = |c\,\delta\tau_{\text{ppN}}| \le \frac{15}{4}\,\pi\,\frac{m^2}{d}\,. \tag{26}$$

Estimates (25)–(26) are proved in Appendix A. Note that here and below all estimates we give are reachable for some values of parameters and, in this sense, cannot be improved. From these estimates we can conclude that among the post-post-Newtonian terms $c\,\delta\tau_{\Delta\text{pN}}$ can become significantly larger compared to the other post-post-Newtonian terms.

A series of additional Monte-Carlo tests using randomly chosen boundary conditions have been performed to verify the given estimates of the post-post-Newtonian terms. The results of these simulations are described in Table III.

The effect of $c\delta\tau$ for the Sun is less than 3.8 cm for arbitrary boundary conditions. Therefore, the formula for the time of light propagation between two given points can be simplified by taking only the relevant term:

$$\begin{aligned}
c\,\tau = R\;+\;(1+\gamma)\,m\,\log\frac{x+x_0+R}{x+x_0-R} \\
-\frac{1}{2}\,(1+\gamma)^2\,m^2\,\frac{R}{|\boldsymbol{x}\times\boldsymbol{x}_0|^2}\,\left(R^2-(x-x_0)^2\right)+\mathcal{O}\left(\frac{m^2}{d}\right)+\mathcal{O}(m^3).
\end{aligned} \tag{27}$$

This expression can be written in an elegant form

$$c\,\tau = R + (1+\gamma)\,m\,\log\frac{x+x_0+R+(1+\gamma)\,m}{x+x_0-R+(1+\gamma)\,m} + \mathcal{O}\left(\frac{m^2}{d}\right) + \mathcal{O}(m^3) \tag{28}$$

that has been already derived by Moyer [3] in an inconsistent way (see Section 8.3.1.1 and Eq. (8-54) of Moyer [3]). As a criterion if the additional post-post-Newtonian term is required for a given situation, one can use Eq. (25) giving the upper boundary of the additional term.

B. Analytical estimates of the individual terms in transformation from $\boldsymbol{k}$ to $\boldsymbol{\sigma}$

Transformation between $\boldsymbol{k}$ and $\boldsymbol{\sigma}$ is given by Eq. (51) of Klioner & Zschocke [1]:

$$\begin{aligned}
&\text{N}\Big| && \boldsymbol{\sigma} = \boldsymbol{k} \\
&\text{pN}\Big| && + (1+\gamma)\,m\,\frac{x-x_0+R}{|\boldsymbol{x}\times\boldsymbol{x}_0|^2}\,\boldsymbol{k}\times(\boldsymbol{x}_0\times\boldsymbol{x}) \\
&\Delta\text{pN}\Big| && + \frac{1}{2}\,(1+\gamma)^2\,m^2\,\boldsymbol{k}\times(\boldsymbol{x}_0\times\boldsymbol{x})\,\frac{1}{|\boldsymbol{x}\times\boldsymbol{x}_0|^4}\,(x+x_0)\,(x-x_0-R)\,(x-x_0+R)^2 \\
&\text{scaling}\Big| && - \frac{(1+\gamma)^2}{2}\,m^2\,\frac{(x-x_0+R)^2}{|\boldsymbol{x}\times\boldsymbol{x}_0|^2}\,\boldsymbol{k} \\
&\text{ppN}\Big| && + m^2\,\boldsymbol{k}\times(\boldsymbol{x}_0\times\boldsymbol{x})\,\Bigg[-\frac{1}{4}\,\alpha\,\epsilon\,\frac{1}{R^2}\left(\frac{1}{x^2}-\frac{1}{x_0^2}\right) \\
&\text{ppN}\Big| && + \frac{1}{8}\,\left(8(1+\gamma-\alpha\,\gamma)(1+\gamma) - 4\,\alpha\,\beta + 3\alpha\,\epsilon\right)\frac{1}{|\boldsymbol{x}\times\boldsymbol{x}_0|^3} \\
&\text{ppN}\Big| && \qquad \times\left(2R^2\,(\pi-\delta(\boldsymbol{k},\boldsymbol{x})) + \left(x^2-x_0^2-R^2\right)\delta(\boldsymbol{x},\boldsymbol{x}_0)\right)\Bigg] \\
& && + \mathcal{O}(c^{-6})\,.
\end{aligned} \tag{29}$$

Let us estimate the magnitude of the individual terms in Eq. (29) in the angle $\delta(\boldsymbol{\sigma},\boldsymbol{k})$ between $\boldsymbol{\sigma}$ and $\boldsymbol{k}$. This angle can be computed from vector product $\boldsymbol{k}\times\boldsymbol{\sigma}$, and, therefore, the term in (29) proportional to $\boldsymbol{k}$ and labelled as “scaling” plays no role. Here and below terms proportional to $\boldsymbol{k}$ do not influence the directions in the given order of magnitude, but are only necessary to keep the involved vectors to have unit length. The total effects of the terms of the other groups on $\boldsymbol{k}\times\boldsymbol{\sigma}$ can be estimated using

$$|\,\boldsymbol{k}\times[\boldsymbol{k}\times(\boldsymbol{x}_0\times\boldsymbol{x})]\,| = |\,\boldsymbol{k}\times(\boldsymbol{x}_0\times\boldsymbol{x})\,| = R\,d\,, \tag{30}$$

and general-relativistic values of the parameters $\alpha=\beta=\gamma=\epsilon=1$ (see Appendix B):

$$|\boldsymbol{\rho}_{\rm pN}| \le \frac{4m}{d}\left[\begin{array}{ll} 1, & x_0\le x, \\ \dfrac{x}{x+x_0}, & x_0>x \end{array}\right. \le \frac{4m}{d}\,, \tag{31}$$

$$|\boldsymbol{\rho}_{\Delta\rm pN}| \le 16\,\frac{m^2}{d^3}\left[\begin{array}{ll} \frac{4}{27}\,(x+x_0), & \frac{1}{2}\,x\le x_0\le x, \\ \dfrac{x^2\,x_0}{(x+x_0)^2}, & x_0<\frac{1}{2}\,x \text{ or } x_0>x\,, \end{array}\right. \tag{32}$$

$$\rho = |\boldsymbol{\rho}_{\rm ppN}| \le \frac{15}{4}\,\pi\,\frac{m^2}{d^2}\,. \tag{33}$$

Note that $\boldsymbol{\rho}_{\rm pN}$ and $\boldsymbol{\rho}_{\Delta\rm pN}$ themselves as well as their estimates are not continuous for $\boldsymbol{x}\to\boldsymbol{x}_0$ since in this limit an infinitely small change of $\boldsymbol{x}$ leads to big changes in $\boldsymbol{k}$. Discontinuity of the same origin appears for many other terms. The limit $\boldsymbol{x}\to\boldsymbol{x}_0$ and the corresponding discontinuity have, clearly, no physical importance.

We see that among terms of order m^2 only $|\boldsymbol{\rho}_{\Delta\rm pN}|$ cannot be estimated as $\text{const}\times m^2/d^2$. The sum of the three other terms can be estimated as given by (33). The values of ρ for solar system bodies are given in Table II. In most cases these terms can be neglected at the level of 1 μas. Indeed, it is easy to see that ρ can be comparable with 1 μas and even exceed this limit only for observations within 5 angular radii from the Sun. Again, Monte-Carlo simulations have been performed to check the actual maximal magnitude of these terms. The results are given in Table III. Accordingly, we obtain a simplified formula for the transformation from $\boldsymbol{k}$ to $\boldsymbol{\sigma}$ keeping only the post-post-Newtonian term that can become larger than 1 μas also far from the Sun:

$$\begin{aligned}\boldsymbol{\sigma} = \boldsymbol{k} \,+\, (1+\gamma)\,m\,\frac{x-x_0+R}{|\boldsymbol{x}\times\boldsymbol{x}_0|^2}\,\boldsymbol{k}\times(\boldsymbol{x}_0\times\boldsymbol{x}) \\ + \frac{(1+\gamma)^2}{2}\,m^2\,(x+x_0)\,\frac{(x-x_0+R)^2\,(x-x_0-R)}{|\boldsymbol{x}\times\boldsymbol{x}_0|^4}\,\boldsymbol{k}\times(\boldsymbol{x}_0\times\boldsymbol{x}) + \mathcal{O}\left(\frac{m^2}{d^2}\right) + \mathcal{O}(m^3)\,.\end{aligned} \tag{34}$$

This can be also written as

$$\boldsymbol{\sigma} = \boldsymbol{k} + \boldsymbol{d}\,S\,\left(1-S\,\frac{1}{2}\,(x+x_0)\left(1+\frac{x_0-x}{R}\right)\right) + \mathcal{O}\left(\frac{m^2}{d^2}\right) + \mathcal{O}(m^3)\,, \tag{35}$$

$$S = (1+\gamma)\,\frac{m}{d^2}\,\left(1-\frac{x_0-x}{R}\right)\,, \tag{36}$$

where $\boldsymbol{d}$ is defined by (22). Eq. (32) can be used as a criterion if the additional post-post-Newtonian term in (34) or (35) is necessary for a given accuracy and configuration.

C. Analytical estimates of individual terms in transformation from $\boldsymbol{\sigma}$ to $\boldsymbol{n}$

Transformation between $\boldsymbol{n}$ and $\boldsymbol{\sigma}$ is given by Eq. (55) of Klioner & Zschocke [1]:

$$
\begin{aligned}
&\text{N}\Big| \quad \boldsymbol{n} = \boldsymbol{\sigma} \\
&\text{pN}\Big| \quad -(1+\gamma)\, m\, \boldsymbol{k}\times(\boldsymbol{x}_0\times\boldsymbol{x})\frac{R}{|\,\boldsymbol{x}\times\boldsymbol{x}_0\,|^2}\left(1+\frac{\boldsymbol{k}\cdot\boldsymbol{x}}{x}\right) \\
&\text{scaling}\Big| \quad +\frac{1}{4}\,(1+\gamma)^2\, m^2\,\frac{\boldsymbol{k}}{|\,\boldsymbol{x}\times\boldsymbol{x}_0\,|^2}\,\frac{R}{x}\left(1+\frac{\boldsymbol{k}\cdot\boldsymbol{x}}{x}\right)(3x-x_0-R)\,(x-x_0+R) \\
&\Delta\text{pN}\Big| \quad +m^2\,\boldsymbol{k}\times(\boldsymbol{x}_0\times\boldsymbol{x})\Bigg[(1+\gamma)^2\,\frac{R}{|\,\boldsymbol{x}\times\boldsymbol{x}_0\,|^2}\left(1+\frac{\boldsymbol{k}\cdot\boldsymbol{x}}{x}\right)\frac{R\,(R^2-(x-x_0)^2)}{2\,|\,\boldsymbol{x}\times\boldsymbol{x}_0\,|^2} \\
&\text{ppN}\Big| \quad\quad +(1+\gamma)^2\,\frac{R}{|\,\boldsymbol{x}\times\boldsymbol{x}_0\,|^2}\left(1+\frac{\boldsymbol{k}\cdot\boldsymbol{x}}{x}\right)\frac{1}{x} \\
&\text{ppN}\Big| \quad\quad -\frac{1}{2}\,\alpha\,\epsilon\,\frac{\boldsymbol{k}\cdot\boldsymbol{x}}{R\,x^4} \\
&\text{ppN}\Big| \quad\quad -\frac{1}{4}\,(8\,(1+\gamma-\alpha\,\gamma)(1+\gamma)\,-4\,\alpha\,\beta\,+3\,\alpha\,\epsilon)\,\frac{\boldsymbol{k}\cdot\boldsymbol{x}}{x^2}\,\frac{R}{|\,\boldsymbol{x}\times\boldsymbol{x}_0\,|^2} \\
&\text{ppN}\Big| \quad\quad -\frac{1}{4}\,(8(1+\gamma-\alpha\,\gamma)(1+\gamma)\,-4\,\alpha\,\beta\,+3\,\alpha\,\epsilon)\,\frac{R^2}{|\,\boldsymbol{x}\times\boldsymbol{x}_0\,|^3}\,(\pi-\delta(\boldsymbol{k},\boldsymbol{x}))\Bigg] \\
&\quad +\mathcal{O}(c^{-6})\,. \qquad (37)
\end{aligned}
$$

Let us estimate the magnitude of the individual terms in Eq. (37) in the angle $\delta(\boldsymbol{\sigma},\boldsymbol{n})$ between $\boldsymbol{n}$ and $\boldsymbol{\sigma}$. This angle can be computed from vector product $\boldsymbol{\sigma}\times\boldsymbol{n}$, and, therefore, the term in (37) proportional to $\boldsymbol{k}$ plays no role since $\boldsymbol{\sigma}\times\boldsymbol{k}=\mathcal{O}(m)$. To estimate the effects of the other terms in (37) we take into account that

$$|\,\boldsymbol{\sigma}\times\big(\boldsymbol{k}\times(\boldsymbol{x}_0\times\boldsymbol{x})\big)\,| = R\,d+\mathcal{O}(m), \qquad (38)$$

and assume again $\alpha=\beta=\gamma=\epsilon=1$. We get (see Appendix C)

$$|\boldsymbol{\varphi}_{\rm pN}| = 2\,m\,\Big|\boldsymbol{\sigma}\times[\boldsymbol{k}\times(\boldsymbol{x}_0\times\boldsymbol{x})]\Big|\frac{R}{|\,\boldsymbol{x}\times\boldsymbol{x}_0\,|^2}\left(1+\frac{\boldsymbol{k}\cdot\boldsymbol{x}}{x}\right)\le 4\,\frac{m}{d}\,, \qquad (39)$$

$$
\begin{aligned}
|\boldsymbol{\varphi}_{\Delta\rm pN}| &= 4\,m^2\,\Big|\boldsymbol{\sigma}\times[\boldsymbol{k}\times(\boldsymbol{x}_0\times\boldsymbol{x})]\Big|\left(1+\frac{\boldsymbol{k}\cdot\boldsymbol{x}}{x}\right)\frac{R^2}{|\,\boldsymbol{x}\times\boldsymbol{x}_0\,|^4}\,\frac{R^2-(x-x_0)^2}{2} \\
&\le 4\,\frac{m^2}{d^2}\,\frac{R}{d}\,\frac{4x\,x_0}{(x+x_0)^2}\le 4\,\frac{m^2}{d^2}\,\frac{R}{d}, \qquad (40)
\end{aligned}
$$

$$\varphi = |\boldsymbol{\varphi}_{\rm ppN}| \le \frac{15}{4}\,\pi\,\frac{m^2}{d^2}. \qquad (41)$$

Again here $\boldsymbol{\varphi}_{ppN}$ is the sum of all the terms in (37) labelled as "ppN". These terms can attain 1 μas only if one observes within 5 angular radii from the Sun. The values of φ for solar system bodies are given in Table II. In most cases these terms can be neglected at the level of 1 μas. Again, Monte-Carlo simulations have been performed to check the actual maximal magnitude of these terms. The results can be found in Table III. Accordingly, we obtain a simplified formula for the transformation from $\boldsymbol{\sigma}$ to $\boldsymbol{n}$ keeping only the post-post-Newtonian term that can become significantly larger than the others:

$$
\begin{aligned}
\boldsymbol{n} = \boldsymbol{\sigma} \; &- \; (1+\gamma)\, m\, \frac{\boldsymbol{d}}{d^2} \left(1 + \frac{\boldsymbol{k}\cdot\boldsymbol{x}}{x}\right) \\
&+ (1+\gamma)^2\, m^2\, \frac{\boldsymbol{d}}{d^4} \left(1 + \frac{\boldsymbol{k}\cdot\boldsymbol{x}}{x}\right) \frac{R^2 - (x - x_0)^2}{2\,R} + \mathcal{O}\left(\frac{m^2}{d^2}\right) + \mathcal{O}(m^3)\,, \qquad (42)
\end{aligned}
$$

where $\boldsymbol{d}$ is defined by (22). The same formula can be written as

$$
\boldsymbol{n} = \boldsymbol{\sigma} + \boldsymbol{d}\, T \left(1 + T\, x\, \frac{R + x_0 - x}{R + x_0 + x}\right) + \mathcal{O}\left(\frac{m^2}{d^2}\right) + \mathcal{O}(m^3)\,, \qquad (43)
$$

$$
T = -(1+\gamma)\, \frac{m}{d^2} \left(1 + \frac{\boldsymbol{k}\cdot\boldsymbol{x}}{x}\right)\,. \qquad (44)
$$

D. Analytical estimates of individual terms in transformation from $\boldsymbol{k}$ to $\boldsymbol{n}$

Transformation between $\boldsymbol{n}$ and $\boldsymbol{\sigma}$ is given by Eq. (56)–(57) of Klioner & Zschocke [1]:

$$
\begin{aligned}
\text{N}\;\Big| \quad & \boldsymbol{n} = \boldsymbol{k} \\
\text{pN}\;\Big| \quad & -(1+\gamma)\, m\, \frac{\boldsymbol{k}\times(\boldsymbol{x}_0\times\boldsymbol{x})}{x\,(x\,x_0 + \boldsymbol{x}\cdot\boldsymbol{x}_0)} \\
\Delta\text{pN}\;\Big| \quad & -(1+\gamma)\, m\, \frac{\boldsymbol{k}\times(\boldsymbol{x}_0\times\boldsymbol{x})}{x\,(x\,x_0 + \boldsymbol{x}\cdot\boldsymbol{x}_0)}\, F \\
\text{scaling}\;\Big| \quad & -\frac{1}{8}\,(1+\gamma)^2\, \frac{m^2}{x^2}\, \boldsymbol{k}\, \frac{\left((x - x_0)^2 - R^2\right)^2}{|\boldsymbol{x}\times\boldsymbol{x}_0|^2} \\
\text{ppN}\;\Big| \quad & + m^2\, \boldsymbol{k}\times(\boldsymbol{x}_0\times\boldsymbol{x}) \Bigg[\frac{1}{2}\,(1+\gamma)^2\, \frac{R^2 - (x - x_0)^2}{x^2\,|\boldsymbol{x}\times\boldsymbol{x}_0|^2} \\
\text{ppN}\;\Big| \quad & + \frac{1}{4}\,\alpha\,\epsilon\, \frac{1}{R} \left(\frac{1}{R\,x_0^2} - \frac{1}{R\,x^2} - 2\,\frac{\boldsymbol{k}\cdot\boldsymbol{x}}{x^4}\right) \\
\text{ppN}\;\Big| \quad & - \frac{1}{4}\,\left(8(1+\gamma-\alpha\gamma)(1+\gamma) - 4\alpha\beta + 3\,\alpha\,\epsilon \right)\, R\, \frac{\boldsymbol{k}\cdot\boldsymbol{x}}{x^2\,|\,\boldsymbol{x}\times\boldsymbol{x}_0\,|^2} \\
\text{ppN}\;\Big| \quad & + \frac{1}{8}\,(8(1+\gamma-\alpha\,\gamma)(1+\gamma) - 4\,\alpha\,\beta + 3\alpha\,\epsilon)\, \frac{x^2 - x_0^2 - R^2}{|\boldsymbol{x}\times\boldsymbol{x}_0|^3}\, \delta(\boldsymbol{x},\boldsymbol{x}_0) \Bigg] \\
& + \mathcal{O}\left(c^{-6}\right) \qquad (45)
\end{aligned}
$$

where

$$
F = -(1+\gamma)\, m\, \frac{x + x_0}{x\,x_0 + \boldsymbol{x}\cdot\boldsymbol{x}_0}\,. \qquad (46)
$$

As in other cases our goal is to estimate the effect of the individual terms in Eq. (45) on the angle $\delta(\boldsymbol{k},\boldsymbol{n})$ between $\boldsymbol{k}$ and $\boldsymbol{n}$. This angle can be computed from vector product $\boldsymbol{k}\times\boldsymbol{n}$. The term in (45) proportional to $\boldsymbol{k}$ obviously plays no role here and can be ignored. For the other terms taking into account Eq. (30) and considering the general-relativistic values $\alpha=\beta=\gamma=\epsilon=1$ one gets (see Appendix D)

$$|\boldsymbol{\omega}_{\rm pN}| = 2\,m\,\frac{1}{x}\,\frac{|\boldsymbol{k}\times(\boldsymbol{x}_0\times\boldsymbol{x})|}{x\,x_0+\boldsymbol{x}\cdot\boldsymbol{x}_0}\;\le\;4\,\frac{m}{d}\,\frac{x_0}{x\,+\,x_0}\;\le\;4\,\frac{m}{d}\,, \tag{47}$$

$$\begin{aligned}|\boldsymbol{\omega}_{\Delta\rm pN}| &= 2\,m\,\frac{1}{x}\,\frac{|\boldsymbol{k}\times(\boldsymbol{x}_0\times\boldsymbol{x})|}{x\,x_0+\boldsymbol{x}\cdot\boldsymbol{x}_0}\,|F| \\ &\le 16\,\frac{m^2}{d^3}\,\frac{R\,x\,x_0^2}{(x+x_0)^3}\;\le\;16\,\frac{m^2}{d^3}\,\frac{x\,x_0^2}{(x+x_0)^2}\le 16\frac{m^2}{d^2}\,\frac{x}{d}\,,\end{aligned} \tag{48}$$

or, alternatively,

$$|\boldsymbol{\omega}_{\Delta\rm pN}| \le \frac{64}{27}\,\frac{m^2}{d^2}\,\frac{R}{d}\,. \tag{49}$$

We give four possible estimates of $|\boldsymbol{\omega}_{\Delta\rm pN}|$. These estimates can be useful in different situations. Note that the last estimate in (48) and the estimate in (49) cannot be related to each other and reflect different properties of $|\boldsymbol{\omega}_{\Delta\rm pN}|$ as function of multiple variables.

The sum of all the terms in (45) labelled as "ppN" is denoted as $\boldsymbol{\omega}_{ppN}$ and can be estimated as

$$\omega = |\boldsymbol{\omega}_{\rm ppN}| \le \frac{15}{4}\,\pi\,\frac{m^2}{d^2}\,. \tag{50}$$

Again these terms can attain 1 μas only for observations within 5 angular radii from the Sun. The values of ω for solar system bodies are given in Table II. One can see that these terms can be neglected at the level of 1 μas in most cases. The corresponding results of our Monte-Carlo simulations can be found in Table III. Accordingly, we obtain a simplified formula for the transformation from $\boldsymbol{k}$ to $\boldsymbol{n}$ keeping only the terms which cannot be estimated as m^2/d^2:

$$\boldsymbol{n} = \boldsymbol{k} - (1+\gamma)\,m\,\frac{1}{x}\,\frac{\boldsymbol{k}\times(\boldsymbol{x}_0\times\boldsymbol{x})}{x\,x_0+\boldsymbol{x}\cdot\boldsymbol{x}_0}\,(1+F)+\mathcal{O}\left(\frac{m^2}{d^2}\right)+\mathcal{O}(m^3)\,, \tag{51}$$

where F is given by (46). This can be also written as

$$\boldsymbol{n} = \boldsymbol{k}+\boldsymbol{d}\,P\,\left(1+P\,x\,\frac{x_0+x}{R}\right)+\mathcal{O}\left(\frac{m^2}{d^2}\right)+\mathcal{O}(m^3)\,, \tag{52}$$

$$P = -(1+\gamma)\,\frac{m}{d^2}\,\left(\frac{x_0-x}{R}+\frac{\boldsymbol{k}\cdot\boldsymbol{x}}{x}\right)\,, \tag{53}$$

where $\boldsymbol{d}$ is given by Eq. (22). Let us also note that the post-post-Newtonian term in (51) and (52) is maximal for sources at infinity:

$$|\boldsymbol{\omega}_{\Delta\rm pN}| \le \lim_{x_0\to\infty}|\boldsymbol{\omega}_{\Delta\rm pN}| = \lim_{x_0\to\infty}(1+\gamma)\,m\,\frac{1}{x}\,\frac{|\boldsymbol{k}\times(\boldsymbol{x}_0\times\boldsymbol{x})|}{x\,x_0+\boldsymbol{x}\cdot\boldsymbol{x}_0}\,|F| = (1+\gamma)^2\,(1-\cos\Phi)^2\,\frac{m^2}{d^2}\,\frac{x}{d}\,, \tag{54}$$

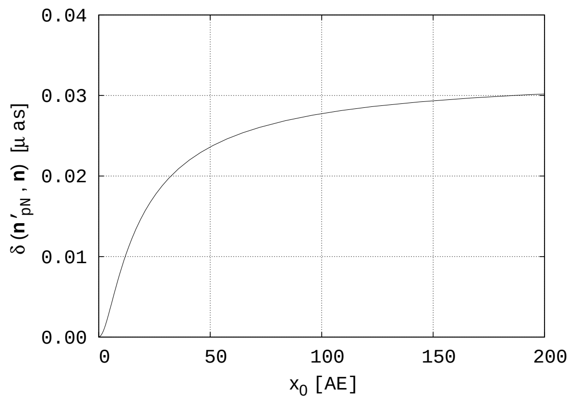

FIG. 2: The angle between $\boldsymbol{n}'_{\rm pN}$ and $\boldsymbol{n}$ for Jupiter. Vector $\boldsymbol{n}'_{\rm pN}$ is evaluated with the aid of (51), while $\boldsymbol{n}$ is the high-accuracy numerical solution of exact geodetic equations as described in Section III. Impact parameter d is taken to be the radius of Jupiter and the distance x between Jupiter and the observer is 6 AU as on Fig. 1. This figure demonstrates the numerical validity of (52) at the level well below 1 μas.

	Sun	Sun at 45°	Jupiter	Saturn	Uranus	Neptune
$c\,\delta\,\tau$ [10^{-6} m]	36906.0	242.9	0.328	0.036	0.002	0.003
ρ, φ, ψ, ω [10^{-3} μas]	10937.4	0.474	0.945	0.120	0.016	0.023

TABLE II: Numerical values of the analytical upper estimates of the post-post-Newtonian terms of order of $\mathcal{O}(m^2/d)$ in Eq. (24) of order $\mathcal{O}(m^2/d^2)$ in Eqs. (29), (37), (45), and (58). The analytical estimates are given by Eqs. (26), (33), (41), (50), and (61), respectively. One can see that at the level of 10 cm in the distance and 1 μas in angles these terms are irrelevant except for observations within 5 angular radii from the Sun.

where $\Phi = \delta(\boldsymbol{x}_0, \boldsymbol{x})$ is the angle between vectors $\boldsymbol{x}_0$ and $\boldsymbol{x}$. Several useful estimates of these terms are given by (48)–(49). These estimates can be used as a criterion which allows one to decide if the post-post-Newtonian correction is important for a particular situation.

Using estimate (54) and the parameters of the solar system bodies given in Table I one gets the maximal values of the post-post-Newtonian correction shown in Table IV. For grazing rays one can apply $\cos\Phi \simeq -1$, while for the Sun at 45° one can apply $\cos\Phi \simeq -1/\sqrt{2}$. Comparing these values with those in the last line of Table I one sees that the post-post-Newtonian correction matches the error of the standard post-Newtonian formula. The deviation for a grazing ray to the Sun is a few μas and originates from the post-post-Newtonian terms neglected in Eq. (52). Vector $\boldsymbol{n}$ computed by (52) can be denoted as $\boldsymbol{n}'_{\rm pN}$ (a post-Newtonian formula enhanced by one post-post-Newtonian term that can become large). The numerical validity of $\boldsymbol{n}'_{\rm pN}$ can be confirmed by direct comparisons of $\boldsymbol{n}'_{\rm pN}$ and vector $\boldsymbol{n}$ computed by numerical integration of geodetic equations as discussed in Section IV B. The results for Jupiter are given on Fig. 2 (cf. Fig. 1).

	Sun	Sun at 45°	Jupiter	Saturn	Uranus	Neptune
$c\,\delta\,\tau$ $[10^{-6}$ m]	36846.9	178.71	0.327	0.0348	0.00192	0.00274
ρ $[10^{-3}\times\mu\mathrm{as}]$	10747.6	0.349	0.942	0.119	0.0154	0.0228
φ $[10^{-3}\times\mu\mathrm{as}]$	10713.2	0.112	0.942	0.119	0.0154	0.0228
ψ $[10^{-3}\times\mu\mathrm{as}]$	10713.2	0.112	0.942	0.119	0.0154	0.0228
ω $[10^{-3}\times\mu\mathrm{as}]$	10682.8	0.239	0.834	0.096	0.0109	0.0134

TABLE III: Maximal values of the sum of the terms of order of $\mathcal{O}(m^2/d)$ in Eq. (24) and of order $\mathcal{O}(m^2/d^2)$ in Eqs. (29), (37), (45) and (58) obtained from numerical simulations. Two simulations have been performed. For the first simulation 10^8 starting points $\boldsymbol{x}_0$ were taken within 50 AU from the relevant massive body. For the second simulation 10^8 starting points were taken at a random distance but with a constraint that in each case the straight line between the starting and final points is tangent to the surface of the body under consideration. Final point $\boldsymbol{x}$ is always chosen on the orbit of the Earth. The position on the Earth orbit is taken randomly. For each of the 2×10^8 points the corresponding terms were evaluated numerically and the maximal value is given in the Table. The fact that these values are always below the corresponding analytical estimations given in Table II can be considered as additional confirmation of the estimates (26), (33), (41), (50), and (61). Note that the values of ω for all cases and all the values for "Sun at 45°" are systematically smaller than the estimates given in Table II. This behaviour is well understood and expected in the described set-up of the Monte-Carlo simulations.

	Sun	Sun at 45°	Jupiter	Saturn	Uranus	Neptune
$\max\lvert\boldsymbol{\omega}_{\Delta\mathrm{pN}}\rvert$ [μas]	3192.8	0.663×10^{-3}	16.11	4.42	2.58	5.83

TABLE IV: Maximal numerical values (54) of the post-post-Newtonian correction in Eqs. (51) or (52) for the solar system bodies given in Table I. A comparison of these values with the last column of Table I allows one to conclude that the post-post-Newtonian term in (51) is responsible for the errors of the the standard post-Newtonian formula (20).

VI. TRANSFORMATION FROM $\boldsymbol{k}$ TO $\boldsymbol{n}$ FOR STARS AND QUASARS

In principle, the formulas for the boundary problem given above are valid also for stars and quasars. However, for sufficiently large x_0 the formulas could be simplified. It is the purpose of this Section to derive necessary formulas for this case.

A. Transformation from $\boldsymbol{k}$ to $\boldsymbol{\sigma}$

First, let us show that for stars and quasars the approximation

$$\boldsymbol{\sigma}=\boldsymbol{k} \tag{55}$$

is valid for an accuracy of 1 μas. Using estimates (31) and (32) for the two terms in Eq.

x_0 [pc]	Sun	Sun at 45°	Jupiter	Saturn	Uranus	Neptune
1	8.506	0.056	0.473	0.309	0.212	0.382
10	0.851	5.586×10^{-3}	0.047	0.031	0.021	0.038
100	0.085	0.559×10^{-3}	4.740×10^{-3}	3.086×10^{-3}	2.122×10^{-3}	3.819×10^{-3}

TABLE V: Numerical values of estimate (56) in μas for the light deflection due to the solar system bodies with various values of x_0.

(34) one can see that for $x_0 \gg x$ the angle $\delta(\boldsymbol{\sigma}, \boldsymbol{k})$ can be estimated as

$$\delta(\boldsymbol{\sigma}, \boldsymbol{k}) \leq 4\,\frac{m}{d}\,\frac{x}{x+x_0}\,\left(1+4\,\frac{m}{d}\,\frac{x}{d}\,\frac{x_0}{x+x_0}\right). \tag{56}$$

Clearly, $\delta(\boldsymbol{\sigma}, \boldsymbol{k})$ goes to zero for $x_0 \to \infty$. Numerical values of this estimate are given in Table V for x_0 equal to 1, 10 and 100 pc. Angle $\delta(\boldsymbol{\sigma}, \boldsymbol{k})$ is smaller for stars at larger distances. However, for hypothetical objects with $x_0 < 1$ pc the difference between $\boldsymbol{\sigma}$ and $\boldsymbol{k}$ must be explicitly taken into account.

B. Transformation from $\boldsymbol{\sigma}$ to $\boldsymbol{n}$

As soon as we accept the equality of $\boldsymbol{\sigma}$ and $\boldsymbol{k}$ for our case the only relevant step is the transformation between $\boldsymbol{\sigma}$ and $\boldsymbol{n}$. This transformation in the post-post-Newtonian approximation is given by Eqs. (53)–(54) of Klioner & Zschocke [1]. Introducing impact vector computed using $\boldsymbol{\sigma}$ and the position of the observer $\boldsymbol{x}$

$$\boldsymbol{d}_\sigma = \boldsymbol{\sigma} \times (\boldsymbol{x} \times \boldsymbol{\sigma}) \tag{57}$$

we can re-write Eqs. (53)–(54) of Klioner & Zschocke [1] as

$$
\begin{aligned}
&\text{N}\quad && \boldsymbol{n} = \boldsymbol{\sigma}\\
&\text{pN}\quad && -(1+\gamma)\, m\, \frac{\boldsymbol{d}_\sigma}{d_\sigma^2}\, \left(1+\frac{\boldsymbol{\sigma}\cdot\boldsymbol{x}}{x}\right)\\
&\Delta\text{pN}\quad && +(1+\gamma)^2\, m^2 \frac{\boldsymbol{d}_\sigma}{d_\sigma^3}\, \frac{x}{d_\sigma}\left(1+\frac{\boldsymbol{\sigma}\cdot\boldsymbol{x}}{x}\right)^2\\
&\text{scaling}\quad && -\frac{1}{2}\, m^2 (1+\gamma)^2\, \frac{\boldsymbol{\sigma}}{d_\sigma^2}\left(1+\frac{\boldsymbol{\sigma}\cdot\boldsymbol{x}}{x}\right)^2\\
&\text{ppN}\quad && -\frac{1}{2}\, m^2 \alpha\, \epsilon\, \frac{\boldsymbol{\sigma}\cdot\boldsymbol{x}}{x^4}\, \boldsymbol{d}_\sigma\\
&\text{ppN}\quad && +(1+\gamma)^2\, m^2 \frac{\boldsymbol{d}_\sigma}{d_\sigma^2}\, \frac{1}{x}\, \left(1+\frac{\boldsymbol{\sigma}\cdot\boldsymbol{x}}{x}\right)\\
&\text{ppN}\quad && -\frac{1}{4}\, \left(8\,(1+\gamma-\alpha\,\gamma)\,(1+\gamma)-4\,\alpha\,\beta+3\,\alpha\,\epsilon\right)\, m^2\, \frac{\boldsymbol{d}_\sigma}{d_\sigma^2}\, \frac{\boldsymbol{\sigma}\cdot\boldsymbol{x}}{x^2}\\
&\text{ppN}\quad && -\frac{1}{4}\, \left(8\,(1+\gamma-\alpha\,\gamma)\,(1+\gamma)-4\,\alpha\,\beta+3\,\alpha\,\epsilon\right)\, m^2\, \frac{\boldsymbol{d}_\sigma}{d_\sigma^3}\, \left(\pi-\delta(\boldsymbol{\sigma},\boldsymbol{x})\right)\\
& && +\mathcal{O}(m^3),
\end{aligned}
\tag{58}
$$

where $d_\sigma = |\boldsymbol{d}_\sigma| = |\boldsymbol{\sigma}\times\boldsymbol{x}|$. Now we need to estimate the effect of the individual terms in Eq. (58) on the angle $\delta(\boldsymbol{\sigma},\boldsymbol{n})$ between $\boldsymbol{\sigma}$ and $\boldsymbol{n}$. This angle can be computed from vector product $\boldsymbol{\sigma}\times\boldsymbol{n}$. The term in (58) proportional to $\boldsymbol{\sigma}$ obviously plays no role and can be ignored. For the other terms taking into account that $|\boldsymbol{\sigma}\times\boldsymbol{d}_\sigma| = d_\sigma$ and considering the general-relativistic values $\alpha=\beta=\gamma=\epsilon=1$ we get

$$
|\boldsymbol{\psi}_{\text{pN}}| = 2\, m\, \frac{|\boldsymbol{\sigma}\times\boldsymbol{d}_\sigma|}{d_\sigma^2}\, \left(1+\frac{\boldsymbol{\sigma}\cdot\boldsymbol{x}}{x}\right) \le 4\, \frac{m}{d_\sigma}\,, \tag{59}
$$

$$
|\boldsymbol{\psi}_{\Delta\text{pN}}| = 4m^2\, \frac{|\boldsymbol{\sigma}\times\boldsymbol{d}_\sigma|}{d_\sigma^3}\, \frac{x}{d_\sigma}\left(1+\frac{\boldsymbol{\sigma}\cdot\boldsymbol{x}}{x}\right)^2 \le 16\, \frac{m^2}{d_\sigma^2}\, \frac{x}{d_\sigma}\,, \tag{60}
$$

$$
\psi = |\boldsymbol{\psi}_{\text{ppN}}| \le \frac{15}{4}\, \pi\, \frac{m^2}{d_\sigma^2}\,, \tag{61}
$$

where $\boldsymbol{\psi}_{\text{ppN}}$ is the sum of all terms of order m^2/d_σ^2 in (58). Estimate (61) obviously agrees with estimate (41) for φ. Numerical values of this estimate can be found in Table II. The estimates show that these terms can be neglected at the level of 1 μas except for the observations within 5 angular radii from the Sun. Omitting these terms one gets an expression valid at the level of 1 μas in all other cases:

$$
\boldsymbol{n} = \boldsymbol{\sigma} + \boldsymbol{d}_\sigma\, Q\, (1+Q\,x) + \mathcal{O}\left(\frac{m^2}{d_\sigma^2}\right) + \mathcal{O}(m^3)\,, \tag{62}
$$

$$
Q = -(1+\gamma)\, \frac{m}{d_\sigma^2}\, \left(1+\frac{\boldsymbol{\sigma}\cdot\boldsymbol{x}}{x}\right)\,. \tag{63}
$$

Note that for $x_0\to\infty$ this coincides with (52)–(53) and with (43)–(44). This formula together with $\boldsymbol{\sigma}=\boldsymbol{k}$ can be applied for sources at distances larger than 1 pc to attain the accuracy of 1 μas. Alternatively Eqs. (52)–(53) can be used for the same purpose giving slightly better accuracy for very close stars. However, distance information (parallax) is necessary to use (52)–(53).

VII. SUMMARY AND CONCLUDING REMARKS

In this report the numerical accuracy of the post-Newtonian and post-post-Newtonian formulas for light propagation in the parametrized Schwarzschild field has been investigated. Analytical formulas have been compared with high-accuracy numerical integrations of the geodetic equations. In this way we demonstrate that the standard post-Newtonian formulas for the boundary problem (light propagation between two given points) cannot be used at the accuracy level of 1 μas for observations performed by an observer situated within the solar system. The error of the standard formula may attain ~ 16 μas. Detailed analysis has shown that the error is of post-post-Newtonian order $\mathcal{O}(m^2)$. On the other hand, the post-post-Newtonian terms are often thought to be of order m^2/d^2 and can be estimated to be much smaller than 1 μas in this case. To clarify this contradiction we have investigated the post-post-Newtonian solution for the light propagation derived in [1]. For each individual term in relevant formulas upper estimates have been found. It turns out that in each case one post-post-Newtonian term may become much larger than the other ones and cannot be estimated as const $\times\, m^2/d^2$. These terms depend only on γ and do not come from the post-post-Newtonian terms of the corresponding metric. The formulas for transformations between directions $\boldsymbol{\sigma}$, $\boldsymbol{n}$ and $\boldsymbol{k}$ containing both post-Newtonian terms and post-post-Newtonian ones that can be relevant at the level of 10 cm for the Shapiro delay and 1 μas for the directions have been derived. The formulas are given by Eqs. (28), (35)–(36), (43)–(44), (52)–(53), and (62)–(63). These formulas should be considered as formulas that guarantee this numerical accuracy.

The derived analytical solution shows that no "native" post-post-Newtonian terms are relevant for the accuracy of 1 μas in the conditions of this note (no observations closer than five angular radii of the Sun). "Native" refers here to the terms coming from the post-post-Newtonian terms in the metric tensor. It is, therefore, not the post-Newtonian solution itself, but the standard analytical way to convert the solution of the initial value problem into the solution for the boundary problem that is responsible for the numerical error of 16 μas mentioned above.

Let us finally note that the post-post-Newtonian term in (52)–(53) is closely related to the standard gravitational lens formula. Here we only note that all the formulas given in Klioner & Zschocke [1] and in this paper are not valid for $d = 0$ (d always appear in the denominators of the relevant formulas). On the other hand, the standard post-Newtonian lens equation successfully treats this case, known as the Einstein ring solution. The relation between the lens approximation and the standard post-Newtonian expansion is a different topic which will be considered in a subsequent paper.

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APPENDIX A: ESTIMATES OF TERMS IN THE SHAPIRO DELAY

In order to get (25) we write the corresponding term as

$$|c\,\delta\,\tau_{\Delta\mathrm{pN}}| = \left| 2\,m^2 \frac{R}{|\boldsymbol{x}\times\boldsymbol{x}_0|^2}\left((x-x_0)^2 - R^2\right)\right| = 2\,\frac{m^2}{d^2}\,R\,\frac{2z(1-\cos\Phi)}{1+z^2-2z\,\cos\Phi}\,, \tag{A1}$$

where $\Phi = \delta(\boldsymbol{x},\boldsymbol{x}_0)$ is the angle between $\boldsymbol{x}$ and $\boldsymbol{x}_0$, and $z = x_0/x$. It is easy to see that for $0 \le \Phi \le \pi$ and $z \ge 0$

$$f_1 = \frac{2z(1-\cos\Phi)}{1+z^2-2z\,\cos\Phi} \le \frac{4z}{(1+z)^2} \le 1. \tag{A2}$$

This immediately gives (25). Here and below we always give estimates that cannot be improved in the sense that they are reachable for certain values of the parameters.

For (26) we write

$$\begin{aligned} c\,\delta\tau = |c\delta\tau_{\mathrm{ppN}}| &= \left|\frac{1}{8}\frac{m^2}{R}\left(\frac{x_0^2-x^2-R^2}{x^2}+\frac{x^2-x_0^2-R^2}{x_0^2}\right)+\frac{15}{4}m^2\,\frac{R}{|\boldsymbol{x}\times\boldsymbol{x}_0|}\,\delta(\boldsymbol{x},\boldsymbol{x}_0)\right| \\ &= \frac{1}{4}\,\frac{m^2}{d}\,\left|\sin\Phi\,\frac{z^2\,\cos\Phi - 2\,z + \cos\Phi}{1+z^2-2\,z\,\cos\Phi} + 15\,\Phi\right|\,. \end{aligned} \tag{A3}$$

Here and below $\Phi = \delta(\boldsymbol{x},\boldsymbol{x}_0)$ is the angle between $\boldsymbol{x}$ and $\boldsymbol{x}_0$, and $z = x_0/x$. One can show that for $0 \le \Phi \le \pi$ and $z \ge 0$

$$f_2 = \left|\sin\Phi\,\frac{z^2\,\cos\Phi - 2\,z + \cos\Phi}{1+z^2-2\,z\,\cos\Phi} + 15\,\Phi\right| \le 15\,\pi. \tag{A4}$$

and this immediately gives (26).

APPENDIX B: ESTIMATES OF TERMS IN THE TRANSFORMATION BETWEEN $\boldsymbol{\sigma}$ AND $\boldsymbol{k}$

For Eq. (31) we note that

$$\begin{aligned} |\boldsymbol{\rho}_{\mathrm{pN}}| = 2\,m\,\frac{x-x_0+R}{|\boldsymbol{x}\times\boldsymbol{x}_0|^2}\,|\boldsymbol{k}\times(\boldsymbol{x}_0\times\boldsymbol{x})| &= \frac{2m}{d}\,\frac{x-x_0+R}{R} \\ &= \frac{2m}{d}\left(\frac{1-z}{\sqrt{1+z^2-2z\cos\Phi}}+1\right). \end{aligned} \tag{B1}$$

Again $\Phi = \delta(\boldsymbol{x},\boldsymbol{x}_0)$ is the angle between $\boldsymbol{x}$ and $\boldsymbol{x}_0$, and $z = x_0/x$. One can show that for $0 \le \Phi \le \pi$ and $z \ge 0$

$$f_3 = \frac{1-z}{\sqrt{1+z^2-2z\cos\Phi}} + 1 \le \left[\begin{array}{ll} 2, & z \le 1 \\ \dfrac{2}{1+z}, & z > 1 \end{array}\right. \le 2 \tag{B2}$$

and this leads to (31). The discontinuity of $\boldsymbol{\rho}_{\mathrm{pN}}$ and its estimate at $z \to 1$ are discussed in the main text after Eq. (32).

The term $\boldsymbol{\rho}_{\Delta\mathrm{pN}}$ can be written as

$$|\boldsymbol{\rho}_{\Delta\mathrm{pN}}| = 2\,\frac{m^2}{d^3}\,(x+x_0)\,\left|\,\frac{(x-x_0+R)\,[(x-x_0)^2-R^2\,]}{R^3}\,\right|$$
$$= 4\,\frac{m^2}{d^2}\,\frac{x}{d}\,z\,(1+z)\,\frac{1-\cos\Phi}{1+z^2-2z\,\cos\Phi}\,\left(1+\frac{1-z}{\sqrt{1+z^2-2z\,\cos\Phi}}\right). \tag{B3}$$

For $0 \le \Phi \le \pi$ and $z \ge 0$ one has

$$f_4 = z\,(1+z)\,\frac{1-\cos\Phi}{1+z^2-2z\,\cos\Phi}\,\left(1+\frac{1-z}{\sqrt{1+z^2-2z\,\cos\Phi}}\right) \le \left[\begin{array}{ll} \frac{16}{27}\,(1+z), & \frac{1}{2} \le z \le 1, \\ 4\,\dfrac{z}{(1+z)^2}, & z < \frac{1}{2} \text{ or } z > 1\,. \end{array}\right. \tag{B4}$$

This gives Eq. (32). The function itself and its estimate (B4) are again not continuous for $z = 1$ (implying $x = x_0$). This is discussed after Eq. (32).

In order to get (33) we write

$$\rho = |\boldsymbol{\rho}_{\mathrm{ppN}}| = m^2\,\,|\boldsymbol{k}\times(\boldsymbol{x}_0\times\boldsymbol{x})|\,\left|\,-\frac{1}{4}\,\frac{1}{R^2}\,\left(\frac{1}{x^2}-\frac{1}{x_0^2}\right)+\frac{15}{8}\,\frac{1}{|\boldsymbol{x}\times\boldsymbol{x}_0|^3}\right.$$
$$\left.\times\left(2R^2\,(\pi-\delta(\boldsymbol{k},\boldsymbol{x}))+\left(x^2-x_0^2-R^2\right)\delta(\boldsymbol{x},\boldsymbol{x}_0)\right)\right|$$
$$= \frac{1}{4}\,\frac{m^2}{d^2}\,\left|\,-\frac{z(z^2-1)\sin^3\Phi}{\left(1+z^2-2\,z\,\cos\Phi\right)^2}-15\,\arccos\frac{1-z\,\cos\Phi}{\sqrt{1+z^2-2\,z\,\cos\Phi}}\right.$$
$$\left.+\,15\,\frac{z\,(\cos\Phi-z)\,\Phi}{1+z^2-2\,z\,\cos\Phi}+15\,\pi\right|. \tag{B5}$$

One can show that for $0 \le \Phi \le \pi$ and $z \ge 0$

$$f_5 = \left|\,-\frac{z(z^2-1)\sin^3\Phi}{\left(1+z^2-2\,z\,\cos\Phi\right)^2}-15\,\arccos\frac{1-z\,\cos\Phi}{\sqrt{1+z^2-2\,z\,\cos\Phi}}\right.$$
$$\left.+15\,\frac{z\,(\cos\Phi-z)\,\Phi}{1+z^2-2\,z\,\cos\Phi}+15\,\pi\right| \le 15\pi. \tag{B6}$$

This immediately leads to (33).

APPENDIX C: ESTIMATES OF TERMS IN THE TRANSFORMATION BETWEEN $\boldsymbol{n}$ AND $\boldsymbol{\sigma}$

Estimate (39) for $\boldsymbol{\varphi}_{\rm pN}$ is trivial. For estimate (40) of $\boldsymbol{\varphi}_{\Delta\rm pN}$ we write

$$\begin{aligned}
|\boldsymbol{\varphi}_{\Delta\rm pN}| &= 4\,m^2 \left| \boldsymbol{\sigma}\times[\boldsymbol{k}\times(\boldsymbol{x}_0\times\boldsymbol{x})] \right| \left(1+\frac{\boldsymbol{k}\cdot\boldsymbol{x}}{x}\right) \frac{R^2}{|\,\boldsymbol{x}\times\boldsymbol{x}_0\,|^4}\,\frac{R^2-(x-x_0)^2}{2} \\
&= 4\,\frac{m^2}{d^2}\,\frac{R}{d}\left(1+\frac{\boldsymbol{k}\cdot\boldsymbol{x}}{x}\right)\frac{R^2-(x-x_0)^2}{2\,R^2} \\
&= 4\,\frac{m^2}{d^2}\,\frac{R}{d}\left(\frac{1-z\,\cos\Phi}{\sqrt{1+z^2-2z\,\cos\Phi}}+1\right)\frac{z\,(1-\cos\Phi)}{1+z^2-2z\,\cos\Phi},
\end{aligned} \tag{C1}$$

where again $\Phi=\delta(\boldsymbol{x},\boldsymbol{x}_0)$ is the angle between $\boldsymbol{x}$ and $\boldsymbol{x}_0$, and $z=x_0/x$. It is easy to see that for $0\le\Phi\le\pi$ and $z\ge 0$

$$f_6=\left(\frac{1-z\,\cos\Phi}{\sqrt{1+z^2-2z\,\cos\Phi}}+1\right)\frac{z\,(1-\cos\Phi)}{1+z^2-2z\,\cos\Phi}\le\frac{4z}{(1+z)^2}\le 1. \tag{C2}$$

This immediately leads to (40). For Eq. (41) we write

$$\begin{aligned}
\varphi=|\boldsymbol{\varphi}_{\rm ppN}| &= m^2|\boldsymbol{k}\times(\boldsymbol{x}_0\times\boldsymbol{x})|\,\Bigg|4\,\frac{R}{|\,\boldsymbol{x}\times\boldsymbol{x}_0\,|^2}\left(1+\frac{\boldsymbol{k}\cdot\boldsymbol{x}}{x}\right)\frac{1}{x}-\frac{1}{2}\,\frac{\boldsymbol{k}\cdot\boldsymbol{x}}{R\,x^4} \\
&\qquad -\frac{15}{4}\,\frac{\boldsymbol{k}\cdot\boldsymbol{x}}{x^2}\,\frac{R}{|\,\boldsymbol{x}\times\boldsymbol{x}_0\,|^2}-\frac{15}{4}\,\frac{R^2}{|\,\boldsymbol{x}\times\boldsymbol{x}_0\,|^3}\,(\pi-\delta(\boldsymbol{k},\boldsymbol{x}))\Bigg| \\
&= \frac{1}{4}\,\frac{m^2}{d^2}\left|16\,\frac{d}{x}+\frac{d}{x}\,\frac{\boldsymbol{k}\cdot\boldsymbol{x}}{x}-2\left(\frac{d}{x}\right)^3\frac{\boldsymbol{k}\cdot\boldsymbol{x}}{x}-15\,(\pi-\delta(\boldsymbol{k},\boldsymbol{x}))\right| \\
&= \frac{1}{4}\,\frac{m^2}{d^2}\left|16\,\sin\Psi+\sin\Psi\,\cos\Psi-2\sin^3\Psi\,\cos\Psi-15\,\pi+15\,\Psi\right|,
\end{aligned} \tag{C3}$$

where $\Psi=\delta(\boldsymbol{k},\boldsymbol{x})$ is the angle between vectors $\boldsymbol{k}$ and $\boldsymbol{x}$. Here we used that $\boldsymbol{k}\cdot\boldsymbol{x}=x\,\cos\Psi$ and $d=|\boldsymbol{k}\times\boldsymbol{x}|=x\,\sin\Psi$. For $0\le\Psi\le\pi$ we have

$$f_7=\left|16\,\sin\Psi+\sin\Psi\,\cos\Psi-2\sin^3\Psi\,\cos\Psi-15\,\pi+15\,\Psi\right|\le 15\pi \tag{C4}$$

and this proves Eq. (41).

APPENDIX D: ESTIMATES OF TERMS IN THE TRANSFORMATION BETWEEN $\boldsymbol{n}$ AND $\boldsymbol{k}$

In order to get (47) we write

$$|\boldsymbol{\omega}_{\rm pN}|=2\,m\,\frac{1}{x}\,\frac{|\boldsymbol{k}\times(\boldsymbol{x}_0\times\boldsymbol{x})|}{x\,x_0+\boldsymbol{x}\cdot\boldsymbol{x}_0}=2\,\frac{m}{d}\,\frac{x\,x_0-\boldsymbol{x}\cdot\boldsymbol{x}_0}{x\,R}=2\,\frac{m}{d}\,\frac{z\,(1-\cos\Phi)}{\sqrt{1+z^2-2z\,\cos\Phi}}\,. \tag{D1}$$

Here again $\Phi = \delta(\boldsymbol{x}, \boldsymbol{x}_0)$ is the angle between $\boldsymbol{x}$ and $\boldsymbol{x}_0$, and $z = x_0/x$. One can show that for $0 \le \Phi \le \pi$ and $z \ge 0$

$$f_8 = \frac{z\,(1-\cos\Phi)}{\sqrt{1+z^2-2z\,\cos\Phi}} \le \frac{2z}{1+z}, \tag{D2}$$

that immediately gives (47). To derive (48) and (49) we write

$$\begin{aligned} |\boldsymbol{\omega}_{\Delta\mathrm{pN}}| &= 2\,m\,\frac{1}{x}\,\frac{|\boldsymbol{k}\times(\boldsymbol{x}_0\times\boldsymbol{x})|}{x\,x_0+\boldsymbol{x}\cdot\boldsymbol{x}_0}\,|F| = 4\,m^2\,\frac{|\,\boldsymbol{k}\times(\boldsymbol{x}_0\times\boldsymbol{x})\,|}{(x\,x_0+\boldsymbol{x}\cdot\boldsymbol{x}_0)^2}\,\frac{x+x_0}{x} \\ &= 4\,\frac{m^2}{d^2}\,\frac{R}{d}\,\frac{z^2\,(1+z)\,(1-\cos\Phi)^2}{(1+z^2-2z\,\cos\Phi)^2}. \end{aligned} \tag{D3}$$

For $0 \le \Phi \le \pi$ and $z \ge 0$ one gets

$$f_9 = \frac{z^2\,(1+z)\,(1-\cos\Phi)^2}{(1+z^2-2z\,\cos\Phi)^2} \le \frac{4z^2}{(1+z)^3}. \tag{D4}$$

This gives the first estimate in (48). Trivial inequalities $R \le x+x_0$, $\frac{z^2}{(1+z)^2} \le 1$ and $\frac{z^2}{(1+z)^3} \le \frac{4}{27}$ give the second and third estimates in (48) and estimate (49), respectively.

For (50) we write

$$\begin{aligned} \omega = |\boldsymbol{\omega}_{\mathrm{ppN}}| &= m^2|\boldsymbol{k}\times(\boldsymbol{x}_0\times\boldsymbol{x})|\,\Bigg|\,2\,\frac{R^2-(x-x_0)^2}{x^2\,|\boldsymbol{x}\times\boldsymbol{x}_0|^2} + \frac{1}{4\,R}\left(\frac{1}{R\,x_0^2} - \frac{1}{R\,x^2} - 2\,\frac{\boldsymbol{k}\cdot\boldsymbol{x}}{x^4}\right) \\ &\quad - \frac{15}{4}\,R\,\frac{\boldsymbol{k}\cdot\boldsymbol{x}}{x^2\,|\,\boldsymbol{x}\times\boldsymbol{x}_0\,|^2} + \frac{15}{8}\,\frac{x^2-x_0^2-R^2}{|\boldsymbol{x}\times\boldsymbol{x}_0|^3}\,\delta(\boldsymbol{x},\boldsymbol{x}_0)\Bigg| \\ &= \frac{1}{4}\,\frac{m^2}{d^2}\,\Bigg|\frac{z\,(16z - z\,\cos\Phi - 15)\,\sin\Phi}{1+z^2-2z\,\cos\Phi} + \frac{z(1-3z^2+2z^3\cos\Phi)\,\sin^3\Phi}{(1+z^2-2z\,\cos\Phi)^2} \\ &\quad + \frac{15z\,(\cos\Phi - z)\,\Phi}{1+z^2-2z\,\cos\Phi}\Bigg|. \end{aligned} \tag{D5}$$

For $0 \le \Phi \le \pi$ and $z \ge 0$ one can demonstrate that

$$\begin{aligned} f_{10} &= \left|\frac{z\,(16z - z\,\cos\Phi - 15)\,\sin\Phi}{1+z^2-2z\,\cos\Phi} + \frac{z(1-3z^2+2z^3\cos\Phi)\,\sin^3\Phi}{(1+z^2-2z\,\cos\Phi)^2} + \frac{15z\,(\cos\Phi - z)\,\Phi}{1+z^2-2z\,\cos\Phi}\right| \\ &\le 15\,\pi \end{aligned} \tag{D6}$$

and this leads to (50).

APPENDIX E: ESTIMATES OF TERMS IN THE TRANSFORMATION BETWEEN $\boldsymbol{n}$ AND $\boldsymbol{\sigma}$ FOR STARS AND QUASARS

Estimates (59)–(60) are trivial. For (61) we write

$$\begin{aligned}
\psi = |\boldsymbol{\psi}_{\rm ppN}| &= m^2\,|\boldsymbol{d}_\sigma| \left| -\frac{1}{2}\,\frac{\boldsymbol{\sigma}\cdot\boldsymbol{x}}{x^4} + 4\frac{1}{d_\sigma^2\,x}\left(1+\frac{\boldsymbol{\sigma}\cdot\boldsymbol{x}}{x}\right) - \frac{15}{4}\,\frac{\boldsymbol{\sigma}\cdot\boldsymbol{x}}{d_\sigma^2\,x^2} - \frac{15}{4}\,\frac{\pi-\delta(\boldsymbol{\sigma},\boldsymbol{x})}{d_\sigma^3} \right| \\
&= \frac{1}{4}\,\frac{m^2}{d_\sigma^2} \left| -2\,\frac{\boldsymbol{\sigma}\cdot\boldsymbol{x}}{x}\,\frac{d_\sigma^3}{x^3} + \frac{d_\sigma}{x}\,\frac{\boldsymbol{\sigma}\cdot\boldsymbol{x}}{x} + 16\,\frac{d_\sigma}{x} - 15\,\pi + 15\,\delta(\boldsymbol{\sigma},\boldsymbol{x}) \right| \\
&= \frac{1}{4}\,\frac{m^2}{d_\sigma^2} \left| 16\,\sin\Psi_\sigma + \cos\Psi_\sigma\,\sin\Psi_\sigma - 2\,\sin^3\Psi_\sigma\,\cos\Psi_\sigma - 15\,\pi + 15\,\Psi_\sigma \right|, \qquad \text{(E1)}
\end{aligned}$$

where $\Psi_\sigma = \delta(\boldsymbol{\sigma},\boldsymbol{x})$ is the angle between vectors $\boldsymbol{\sigma}$ and $\boldsymbol{x}$. Here we use $d_\sigma = |\boldsymbol{\sigma}\times\boldsymbol{x}| = x\,\sin\Psi_\sigma$ and $\boldsymbol{\sigma}\cdot\boldsymbol{x} = x\,\cos\Psi_\sigma$. Therefore, for $0 \le \Psi_\sigma \le \pi$ one can use estimate (C4) for f_7 to prove (61).