Formal proof of some inequalities used in the analysis of the post-post-Newtonian light propagation theory

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GAIA-CA-TN-LO-SZ-003-1

January 28, 2010

A rigorous analytical solution of light propagation in Schwarzschild metric in post-post Newtonian approximation has been presented in [1]. In [2] it has been claimed that the sum of all those terms which are of order $\mathcal{O}\left(\dfrac{m^2}{d^2}\right)$ and $\mathcal{O}\left(\dfrac{m^2}{d_\sigma^2}\right)$ is not greater than $\dfrac{15}{4}\,\pi\,\dfrac{m^2}{d^2}$ and $\dfrac{15}{4}\,\pi\,\dfrac{m^2}{d_\sigma^2}$, respectively. All these terms can be neglected on microarcsecond level of accuracy, leading to considerably simplified analytical transformations of light propagation. In this report, we give formal mathematical proofs for the inequalities used in the appendices of [2].

Contents

I. INTRODUCTION

In [1], the rigorous analytical solution of light propagation in Schwarzschild metric has been presented in post-post Newtonian approximation. Especially, the analytical expressions for Shapiro delay, the three transformations between $\boldsymbol{k}$ and $\boldsymbol{\sigma}$, $\boldsymbol{\sigma}$ and $\boldsymbol{n}$, $\boldsymbol{k}$ and $\boldsymbol{n}$, and the transformation between $\boldsymbol{n}$ and $\boldsymbol{\sigma}$ for stars and quasars have been given explicitly. A detailed investigation in [2] has shown that many of the terms, occurring in these five transformations, are of order $\mathcal{O}\left(\frac{m^2}{d^2}\right)$ and $\mathcal{O}\left(\frac{m^2}{d_\sigma^2}\right)$. At the microarcsecond level of accuracy all these terms can be neglected, leading to considerably simplified analytical transformations. Furthermore, in [2] it has been claimed that the sum of all these terms is not greater than $\frac{15}{4}\,\pi\,\frac{m^2}{d^2}$ and $\frac{15}{4}\,\pi\,\frac{m^2}{d_\sigma^2}$, respectively. This has been demonstrated without formal proof of several inequalities involving functions $f_1, f_2, ..., f_{10}$ as defined in the appendices of [2]. The goal of this report is to close this gap and to give formal proofs of these inequalities. Throughout this report, for all estimations we use that $0 \leq \Phi \leq \pi$, $0 \leq \Psi \leq \pi$ and $z \geq 0$; the angles Φ, Ψ and variable z were defined in [2].

II. ESTIMATE OF FUNCTION f_1

In this section we want to proof the inequality (A2) from Appendix A of [2]:

$$f_1 = \frac{2\,z\,\left(1-\cos\Phi\right)}{1+z^2-2\,z\,\cos\Phi} \leq \frac{4\,z}{\left(1+z\right)^2} \leq 1\,. \tag{1}$$

The proof of the first inequality in (1) is straightforward and can be written as inequality

$$-\left(1-z\right)^2\,\left(1+\cos\Phi\right) \leq 0\,, \tag{2}$$

which is obviously valid. The second inequality in (1) is equivalent to

$$4\,z-\left(1+z\right)^2 = -\left(1-z\right)^2 \leq 0\,. \tag{3}$$

From the inequalities (2) and (3) we conclude the validity of (1).

III. ESTIMATE OF FUNCTION f_2

In this section we want to proof the inequality (A4) from Appendix A of [2]:

$$f_2 = \left|\sin\Phi\,\frac{z^2\,\cos\Phi-2\,z+\cos\Phi}{1+z^2-2\,z\,\cos\Phi}+15\,\Phi\right| \leq 15\,\pi\,. \tag{4}$$

To proof the validity of (4) we consider the extremal conditions $f_{2,\Phi}=0$ and $f_{2,z}=0$, which yield

$$\begin{aligned} 0 &= 7-z^3\,\cos^3\Phi-z\,\cos^3\Phi+z^4\,\cos^2\Phi+32\,z^2\,\cos^2\Phi \\ &\quad +\cos^2\Phi-31\,z^3\,\cos\Phi-31\,z\,\cos\Phi+7\,z^4+16\,z^2\,, \end{aligned} \tag{5}$$

$$0 = \sin^3\Phi\,\left(z^2-1\right). \tag{6}$$

Inserting the solutions of Eq. (6), given by $\Phi = 0$, $\Phi = \pi$, $z = 1$, into Eq. (5) yields the equations

$$0 = (1 - z)^4 \,, \tag{7}$$
$$0 = (1 + z)^4 \,, \tag{8}$$
$$0 = (\cos\Phi - 1)^2 \,(\cos\Phi - 15) \,. \tag{9}$$

The solution of Eq. (7) is $z = 1$, while Eq. (8) has no solution, and the only solution of Eq. (9) is $\Phi = 0$. Thus, the extremal point $P_{\mathrm{e}} : (\Phi = 0, z = 1)$ which is only one special point of one of the boundaries of function f_2. The boundaries are given by

$$f_2\Big|_{z=0} = \sin\Phi\ \cos\Phi + 15\,\Phi \leq 15\,\pi \,, \tag{10}$$
$$f_2\Big|_{z=\infty} = \sin\Phi\ \cos\Phi + 15\,\Phi \leq 15\,\pi \,, \tag{11}$$
$$f_2\Big|_{\Phi=0} = 0 \,, \tag{12}$$
$$f_2\Big|_{\Phi=\pi} = 15\,\pi \,. \tag{13}$$

From Eqs. (10) - (13) we conclude the validity of inequality (4).

IV. ESTIMATE OF FUNCTION f_3

In this section we want to proof the inequality (B2) from Appendix B of [2]:

$$f_3 = \frac{1 - z}{\sqrt{1 + z^2 - 2z\cos\Phi}} + 1 \leq \left[\begin{array}{ll} 2\,, & z \leq 1 \\ \dfrac{2}{1+z}\,, & z > 1 \end{array}\right. \leq 2\,. \tag{14}$$

Let us first consider the case $z \leq 1$ where (14) is equivalent to the inequality

$$1 + z^2 - 2\,z \leq 1 + z^2 - 2\,z\ \cos\Phi \,, \tag{15}$$

which is obviously valid. Let us now consider the case $z > 1$, where we have to show

$$\frac{1 - z}{\sqrt{1 + z^2 - 2\,z\ \cos\Phi}} \leq \frac{1 - z}{1 + z} \,, \tag{16}$$

or (note, that $1 - z$ is negative)

$$\sqrt{1 + z^2 - 2\,z\ \cos\Phi} \leq 1 + z \,, \tag{17}$$

which is obviously valid. Thus, by means of (15) and (17), we have shown the validity of (14).

V. ESTIMATE OF FUNCTION f_4

In this section we want to proof the inequality (B4) from Appendix B of [2]:

$$f_4 = z\,(1+z)\,\frac{1-\cos\Phi}{1+z^2-2z\,\cos\Phi}\left(1+\frac{1-z}{\sqrt{1+z^2-2z\,\cos\Phi}}\right)$$
$$\leq \left[\begin{array}{ll} \frac{16}{27}\,(1+z), & \frac{1}{2}\leq z\leq 1, \\ 4\,\dfrac{z}{(1+z)^2}, & z<\frac{1}{2} \text{ or } z>1\,. \end{array}\right. \tag{18}$$

A. $1/2 \leq z \leq 1$

Let us first consider the case $1/2 \leq z \leq 1$, where (18) reduces to the inequality

$$\frac{16}{27}-\frac{z\,(1-w)}{1+z^2-2\,w\,z}\geq\frac{z\,(1-w)\,(1-z)}{\left(1+z^2-2\,w\,z\right)^{3/2}}\,, \tag{19}$$

where $w = \cos\Phi$. Note, while the r.h.s. of (19) is obviously positive, the l.h.s. of (19) is also positive, because the inequality

$$\frac{16}{27}-\frac{z\,(1-w)}{1+z^2-2\,w\,z}\geq 0 \tag{20}$$

leads to $16+16\,z^2-5\,w\,z-27\,z\geq 16\,(1-z)^2\geq 0$. Therefore, by squaring both sides of (19), we obtain the equivalent inequality

$$\left(8-25\,w\,z+9\,z+8\,z^2\right)\left(w\,z+4-9\,z+4\,z^2\right)^2\geq 0\,. \tag{21}$$

Since the quadratic term in (21) is by definition larger than zero, we have only to show that

$$h_1 = 8-25\,w\,z+9\,z+8\,z^2\geq 0\,. \tag{22}$$

The extremal conditions $h_{1,w} = 0$ and $h_{1,z} = 0$ yield

$$-25\,z = 0\,, \tag{23}$$
$$-25\,w+9+16\,z = 0\,. \tag{24}$$

The solution of (23) is $z = 0$, however the region under consideration is $1/2 \leq z \leq 1$, that means there is no extremal point. The boundaries of function h_1 are

$$h_1\Big|_{z=1/2} = \frac{29}{2}-\frac{25}{2}\,w>0\,, \tag{25}$$
$$h_1\Big|_{z=1} = 25\,(1-w)\geq 0\,, \tag{26}$$
$$h_1\Big|_{w=-1} = 8\,z^2+34\,z+8>0\,, \tag{27}$$
$$h_1\Big|_{w=1} = 8\,(z-1)^2\geq 0\,. \tag{28}$$

From Eqs. (25) - (28) we conclude the validity of inequality (22) and (19).

B. $z > 1$

The case $z > 1$ has actually been shown already, because by means of inequality (1) we obtain the estimate

$$f_4 \leq \frac{2\,z}{(1+z)} \left(1 + \frac{1-z}{\sqrt{1+z^2-2\,z\,\cos\Phi}} \right), \tag{29}$$

and with the aid of inequality (14) we just obtain the estimate (18) for $z > 1$.

C. $z < 1/2$

Let us now consider the case $z < 1/2$, where the inequality (18) reduces to

$$\frac{(1+z)^3\,(1-w)}{1+z^2-2\,w\,z} + \frac{(1+z)^3\,(1-w)\,(1-z)}{(1+z^2-2\,w\,z)^{3/2}} \leq 4\,. \tag{30}$$

We simplify (30) as follows:

$$z^2 - z^2\,w - 4\,w\,z + 3 + w \geq \frac{(1+z)^3\,(1-w)}{\sqrt{1+z^2-2\,w\,z}}\,. \tag{31}$$

Squaring both sides of (31), which obviously are positive, yields the relation

$$\begin{aligned} &\left(-z^5 - 8\,z^4 - 14\,z^3 + 8\,z^2 - z\right) w^2 + \left(4\,z^5 + 16\,z^4 + 8\,z^3 + 16\,z^2 - 12\,z\right) w \\ &-3\,z^5 - 4\,z^4 - 10\,z^3 - 3\,z + 4 \geq 0\,. \end{aligned} \tag{32}$$

In order to proof the validity of (32), we recall that $-1 \leq w \leq 1$, that means the inequality (32) is valid, if the following inequality is satisfied:

$$\begin{aligned} h_2 = &\left| -z^5 - 8\,z^4 - 14\,z^3 + 8\,z^2 - z \right| + \left| 4\,z^5 + 16\,z^4 + 8\,z^3 + 16\,z^2 - 12\,z \right| \\ &+ \left| -3\,z^5 - 4\,z^4 - 10\,z^3 - 3\,z \right| \leq 4\,. \end{aligned} \tag{33}$$

To proof inequality (33), we first note that

$$\left| 4\,z^5 + 16\,z^4 + 8\,z^3 + 16\,z^2 - 12\,z \right| \leq 4\,z^5 + \left| 16\,z^4 + 8\,z^3 + 16\,z^2 - 12\,z \right|\,. \tag{34}$$

Second, we note the obvious inequalities

$$-z^5 - 8\,z^4 - 14\,z^3 + 8\,z^2 - z = -z\left(z^2 + 4\,z - 1\right)^2 \leq 0\,, \tag{35}$$

$$16\,z^4 + 8\,z^3 + 16\,z^2 - 12\,z = 4\,z\,(2\,z-1)\left(2\,z^2 + 2\,z + 3\right) \leq 0\,, \tag{36}$$

$$-3\,z^5 - 4\,z^4 - 10\,z^3 - 3\,z \leq 0\,. \tag{37}$$

Accordingly, by means of relation (34) and inequalities (35) - (37), we obtain

$$\begin{aligned} h_2 \leq &\; z\left(z^2 + 4\,z - 1\right)^2 + 4\,z\,(1-2\,z)\left(2\,z^2 + 2\,z + 3\right) \\ &+ \left(3\,z^5 + 4\,z^4 + 10\,z^3 + 3\,z\right) + 4\,z^5 \leq 4\,. \end{aligned} \tag{38}$$

Accordingly, we have to show the inequality

$$h_3 = z \left| 2\,z^4 - z^3 + 4\,z^2 - 6\,z + 4 \right| \leq 1\,. \tag{39}$$

The extremal condition $h_{3,z} = 0$ yields

$$5\,z^4 - 2\,z^3 + 6\,z^2 - 6\,z + 2 = 0\,. \tag{40}$$

Eq. (40) has no real solution due to $5\,z^4 - 2\,z^3 + 6\,z^2 - 6\,z + 2 \geq 2\,(1-z)^3 > 0$ for $0 \leq z \leq 1/2$. Thus, there are no extremal points of h_3 in the region under consideration. The boundaries of function h_3 are given by

$$h_3 \Big|_{z=0} = 0\,, \tag{41}$$

$$h_3 \Big|_{z=1/2} = 1\,. \tag{42}$$

From (41) and (42) we conclude the validity of inequality (39), and by means of which we conclude the validity of inequality (30).

VI. ESTIMATE OF FUNCTION f_5

In this section we want to proof the inequality (B6) from Appendix B of [2]:

$$\begin{aligned} f_5 = \Bigg| &- \frac{z(z^2-1)\sin^3\Phi}{\left(1+z^2-2\,z\,\cos\Phi\right)^2} - 15\,\arccos\frac{1-z\,\cos\Phi}{\sqrt{1+z^2-2\,z\,\cos\Phi}} \\ &+15\,\frac{z\,(\cos\Phi - z)\,\Phi}{1+z^2-2\,z\,\cos\Phi} + 15\,\pi \Bigg| \leq 15\,\pi\,. \end{aligned} \tag{43}$$

The extremal conditions $f_{5,\Phi} = 0$ and $f_{5,z} = 0$ yield

$$\begin{aligned} 0 = \sin\Phi\;z\;(z-1)\;(z+1)\;\Big(&-3\,z^2\,\sin\Phi\,\cos\Phi + 15\,z^2\,\Phi + 2\,z\,\sin\Phi\cos^2\Phi \\ &+4\,z\,\sin\Phi - 30\,z\,\Phi\,\cos\Phi - 3\,\sin\Phi\,\cos\Phi + 15\,\Phi\Big)\,, \end{aligned} \tag{44}$$

$$0 = C_0 + C_1\,z + C_2\,z^2 + C_3\,z^3 + C_4\,z^4\,, \tag{45}$$

with

$$C_0 = C_4 = -\sin^3\Phi + 15\,\sin\Phi - 15\,\Phi\,\cos\Phi\,, \tag{46}$$

$$C_1 = C_3 = 30\,\Phi + 30\,\Phi\,\cos^2\Phi - 2\,\sin^3\Phi\,\cos\Phi - 60\,\sin\Phi\,\cos\Phi\,, \tag{47}$$

$$C_2 = -90\,\Phi\,\cos\Phi + 30\,\sin\Phi + 60\,\sin\Phi\,\cos^2\Phi + 6\,\sin^3\Phi\,. \tag{48}$$

The complete set of solutions of extremal condition (44) reads

$$z = 0\,, \tag{49}$$

$$z = 1\,, \tag{50}$$

$$\Phi = \pi\,, \tag{51}$$

$$\Phi = 0\,. \tag{52}$$

Note, that the expression in the parentheses of Eq. (44) vanishes only at $\Phi = 0$ (see Appendix A) which, however, represents no additional solution because it is already considered in Eq. (52). Inserting the solutions (49) - (51) into (45) considerably simplifies the extremal condition $f_{5,z} = 0$ and yields the relations

$$0 = -\sin^3\Phi + 15\sin\Phi - 15\,\Phi\cos\Phi\,, \tag{53}$$

$$0 = (1-\cos\Phi)^2\,\left(15\,\Phi + \sin\Phi\,(1+\cos\Phi) + 15\sin\Phi\right)\,, \tag{54}$$

$$0 = (z+1)^4\,, \tag{55}$$

while inserting the solution (52) into (45) yields an identity $0 = 0$, that means no additional solution. The only solution of (53) and (54) is given by (see Appendix B)

$$\Phi = 0\,, \tag{56}$$

while Eq. (55) has obviously no solution. In view of the solutions (49) - (52) and (56) we obtain the extremal points $P_{\rm e1} : (z = 0, \Phi = 0)$ and $P_{\rm e2} : (z = 1, \Phi = 0)$. These extremal points are only two points on one of the boundaries. The boundaries are given by

$$f_5\Big|_{z=0} = 15\,\Phi \le 15\,\pi\,, \tag{57}$$

$$f_5\Big|_{z=\infty} = \left|\,-15\,\pi + 15\,\Phi\,\right| \le 15\,\pi\,, \tag{58}$$

$$f_5\Big|_{\Phi=0} = 15\,\pi\left(1 - \arccos\frac{1}{\sqrt{1+z^2}}\right) \le 15\,\pi\,, \tag{59}$$

$$f_5\Big|_{\Phi=\pi} = \frac{15\,\pi}{1+z} \le 15\,\pi\,. \tag{60}$$

From Eqs. (57) - (60) we conclude the validity of inequality (43).

VII. ESTIMATE OF FUNCTION f_6

In this section we want to proof the inequality (C2) from Appendix C of [2]:

$$f_6 = \left(\frac{1 - z\cos\Phi}{\sqrt{1+z^2-2z\cos\Phi}} + 1\right)\frac{z\,(1-\cos\Phi)}{1+z^2-2z\cos\Phi} \le \frac{4z}{(1+z)^2} \le 1\,. \tag{61}$$

The second inequality has been shown in Eq. (3), thus we focus on the first inequality only. By inserting (1) into first inequality of (61), we recognize that we have only to show the considerably simpler inequality

$$\left|\frac{1 - z\cos\Phi}{\sqrt{1+z^2-2z\cos\Phi}}\right| \le 1\,, \tag{62}$$

or

$$(1 - z\cos\Phi)^2 \le 1 + z^2 - 2\,z\cos\Phi\,. \tag{63}$$

The inequality (63) is, however, obviously valid. Thus we have shown the validity of first inequality of (61).

VIII. ESTIMATE OF FUNCTION f_7

In this section we want to proof the inequality (C4) from Appendix C of [2]:

$$f_7 = \Big| 16 \sin\Psi + \sin\Psi \cos\Psi - 2\sin^3\Psi \cos\Psi - 15\pi + 15\Psi \Big| \leq 15\pi\,. \tag{64}$$

The extremal condition $f_{7,\Psi} = 0$ yields

$$(1+\cos\Psi)^2 \left(\cos^2\Psi - 2\cos\Psi + 2\right) = 0\,. \tag{65}$$

The only solution of Eq. (65) is $\Psi = \pi$ that means the extremal point is $P_{\mathrm{e}} : (\Psi = \pi)$, which is basically the boundary given below in Eq. (67). The boundaries are given by

$$f_7 \Big|_{\Psi=0} = 15\pi\,, \tag{66}$$

$$f_7 \Big|_{\Psi=\pi} = 0\,. \tag{67}$$

From Eqs. (66) and (67) we conclude the validity of (64).

IX. ESTIMATE OF FUNCTION f_8

In this section we want to proof the inequality (D2) from Appendix D of [2]:

$$f_8 = \frac{z\,(1-\cos\Phi)}{\sqrt{1+z^2-2z\cos\Phi}} \leq \frac{2z}{1+z}\,, \tag{68}$$

which is equivalent to the inequality

$$h_4 = w + 2\,w\,z + z^2\,w - 3 + 2\,z - 3\,z^2 \leq 0\,. \tag{69}$$

The extremal conditions $h_{4,z} = 0$ and $h_{4,w} = 0$ yield

$$w + w\,z + 1 - 3\,z = 0\,, \tag{70}$$

$$(1+z)^2 = 0\,. \tag{71}$$

Eq. (71) has obviously no real solution for variable $z \geq 0$, and, therefore, the function h_4 has no extremal points. The boundaries of h_4 are given by

$$h_4 \Big|_{z=0} = w - 3 \leq 0\,, \tag{72}$$

$$h_4 \Big|_{z=\infty} = (w-3) \lim_{z\to\infty} z^2 \leq 0\,, \tag{73}$$

$$h_4 \Big|_{w=-1} = -4\,(1+z)^2 \leq 0\,, \tag{74}$$

$$h_4 \Big|_{w=1} = -2\,(1-z)^2 \leq 0\,. \tag{75}$$

From Eqs. (72) - (75) we conclude the validity of (69) and (68).

X. ESTIMATE OF FUNCTION f_9

In this section we want to proof the inequality (D4) from Appendix D of [2]:

$$f_9 = \frac{z^2\,(1+z)\,(1-\cos\Phi)^2}{(1+z^2-2z\,\cos\Phi)^2} \leq \frac{4z^2}{(1+z)^3}\,, \tag{76}$$

which is equivalent to the inequality

$$h_5 = z^2\,w - 3\,z^2 + 6\,w\,z - 2\,z + w - 3 \leq 0\,, \tag{77}$$

where $w = \cos\Phi$. The extremal conditions $h_{5,z} = 0$ and $h_{5,w} = 0$ yield

$$w\,z - 3\,z + 3\,w - 1 = 0\,, \tag{78}$$

$$z^2 + 6\,z + 1 = 0\,. \tag{79}$$

Eq. (79) has obviously no real solution for variable $z \geq 0$, and, therefore, the function h_5 has no extremal points. The boundaries of h_5 are given by

$$h_5\Big|_{z=0} = w - 3 \leq 0\,, \tag{80}$$

$$h_5\Big|_{z=\infty} = (w-3)\lim_{z\to\infty} z^2 \leq 0\,, \tag{81}$$

$$h_5\Big|_{w=-1} = -4\,(1+z)^2 \leq 0\,, \tag{82}$$

$$h_5\Big|_{w=1} = -2\,(1-z)^2 \leq 0\,. \tag{83}$$

From Eqs. (80) - (83) we conclude the validity of (77) and (76).

XI. ESTIMATE OF FUNCTION f_{10}

In this section we want to proof the inequality (D6) from Appendix D of [2]:

$$f_{10} = \left|\frac{z\,(16z - z\,\cos\Phi - 15)\,\sin\Phi}{1+z^2-2z\,\cos\Phi} + \frac{z(1-3z^2+2z^3\cos\Phi)\,\sin^3\Phi}{(1+z^2-2z\,\cos\Phi)^2} + \frac{15z\,(\cos\Phi - z)\,\Phi}{1+z^2-2z\,\cos\Phi}\right| \leq 15\,\pi\,. \tag{84}$$

With $|a+b| \leq |a| + |b|$, and the inequality (see Appendix C)

$$\left|\frac{z(1-3z^2+2z^3\cos\Phi)\,\sin^3\Phi}{(1+z^2-2z\,\cos\Phi)^2}\right| \leq 8\,\sin\Phi\,, \tag{85}$$

we get

$$f_{10} \leq \left|\frac{z\,(16z - z\,\cos\Phi - 15)\,\sin\Phi + 15z\,(\cos\Phi - z)\,\Phi}{1+z^2-2z\,\cos\Phi}\right| + 8\,\sin\Phi\,. \tag{86}$$

Due to the inequality (see Appendix D)

$$z\,(16\,z - z\cos\Phi - 15)\sin\Phi + 15\,z\,\left(\cos\Phi - z\right)\Phi \le 0\,, \tag{87}$$

and since $\sin\Phi \ge 0$, we can write

$$f_{10} \le \left|\sin\Phi\;\frac{16z^2 - 15z - z^2\cos\Phi}{1+z^2-2z\cos\Phi} - 8\,\sin\Phi + 15\;\frac{z\;\Phi\;\left(\cos\Phi - z\right)}{1+z^2-2z\cos\Phi}\right|\,. \tag{88}$$

Since the expression in the parentheses of Eq. (88) is negative, we can replace $\cos\Phi$ by 1 in the nominator of the first term (note, that $1+z^2-2\,z\,\cos\Phi \ge 0$) and obtain

$$f_{10} \le \left|15\;\sin\Phi\;\frac{z\;(z-1)}{1+z^2-2z\cos\Phi} - 8\;\sin\Phi + 15\;\frac{z\;\Phi\;\left(\cos\Phi - z\right)}{1+z^2-2z\cos\Phi}\right|\,. \tag{89}$$

This expression can further be simplified by means of the inequality (see Appendix E)

$$\left|\;\frac{z\;\Phi\;\left(\cos\Phi - z\right)}{1+z^2-2z\cos\Phi}\;\right| \le \frac{z\;\pi}{1+z}\,. \tag{90}$$

Thus we obtain

$$f_{10} \le \left|15\;\sin\Phi\;\frac{z\;(z-1)}{1+z^2-2z\cos\Phi} - 15\;\sin\Phi - 15\;\frac{z\;\pi}{1+z}\;\right|\,, \tag{91}$$

where we also made the replacement $-8\;\sin\Phi$ by $-15\;\sin\Phi$ for getting an expression more convenient for subsequent considerations. This expression can simplified with the aid of the inequality (see Appendix G)

$$\left|\sin\Phi\;\frac{z\;(z-1)}{1+z^2-2z\cos\Phi} - \sin\Phi\right| \le \frac{2}{1+z}\;, \tag{92}$$

by means of which we obtain

$$f_{10} \le 15\;\left|\;\frac{2}{1+z} + \frac{z\;\pi}{1+z}\;\right| \le 15\;\left|\;\frac{\pi}{1+z} + \frac{z\;\pi}{1+z}\;\right| = 15\,\pi\,. \tag{93}$$

Thus, we have shown the validity of inequality (84).

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[1] Sergei A. Klioner, Sven Zschocke, report GAIA-CA-TN-LO-SK-002-2.

[2] Sven Zschocke, Sergei A. Klioner, report GAIA-CA-TN-LO-SZ-002-2.

APPENDIX A: PROOF THE NON-EXISTENCE OF OTHER SOLUTIONS OF EQ. (44)

If we set the parentheses of Eq. (44) to zero, we obtain the following equation:

$$0 = -3\,z^2\,\sin\Phi\,\cos\Phi + 15\,z^2\,\Phi + 2\,z\,\sin\,\Phi\cos^2\Phi \\ +4\,z\,\sin\Phi - 30\,z\,\Phi\,\cos\Phi - 3\,\sin\Phi\,\cos\Phi + 15\,\Phi\,. \tag{A1}$$

We want to show that Eq. (A1) has the only solution $P_\mathrm{e} : (z = 1, \Phi = 0)$. The both solutions of Eq. (A1) for variable z are given by

$$z_{1,2} = \frac{1}{3}\,\frac{\sin^3\Phi - 3\,\sin\Phi + 15\,\Phi\,\cos\Phi \pm \sqrt{T_1}}{5\,\Phi - \sin\Phi\,\cos\Phi}\,, \tag{A2}$$

where the discriminant is defined by

$$T_1 = -\sin^2\Phi\left(-\cos^4\Phi + 5\,\cos^2\Phi - 30\,\Phi\,\sin\Phi\,\cos\Phi + 225\,\Phi^2 - 4\right) \leq 0\,. \tag{A3}$$

The inequality (A3) can also be expressed by

$$h_6 = -w^4 + 5\,w^2 - 30\,\sqrt{1-w^2}\,w\,\arccos w + 225\,\arccos^2 w - 4 \geq 0\,, \tag{A4}$$

where $w = \cos\Phi$. The extremal condition $h_{6,w} = 0$ leads to

$$0 = 120\,\arccos w - 10\,w\,\sqrt{1-w^2} - 15\,w^2\,\arccos w + w^3\,\sqrt{1-w^2}\,, \tag{A5}$$

with the only solution $w = 1$, that means $\Phi = 0$ (it is straightforward to show, that the first derivative of expression (A5) is always negative; thus the expression (A5) represents a monotonically decreasing function and since $w = 1$ is obviously a solution of equation (A5) it is, therefore, the only one). Inserting $\Phi = 0$ into Eq. (A2) yields $z_1 = z_2 = 1$. Thus, the extremal point is given by

$$P_\mathrm{e} : (z = 1, w = 1)\,. \tag{A6}$$

The boundaries of function h_6 are given by

$$h_6\Big|_{w=-1} = 225\,\pi^2 > 0\,, \tag{A7}$$

$$h_6,\Big|_{w=1} = 0\,. \tag{A8}$$

From Eqs. (A7) and (A8) we conclude the validity of inequality (A4) and (A3). That means, the only real solution of Eq. (A1) is given by Eq. (A6), that means $P_\mathrm{e} : (z = 1, \Phi = 0)$.

APPENDIX B: PROOF THAT EQ. (56) IS THE ONLY SOLUTION OF EQ. (53) AND EQ. (54)

Eq. (53) is given by

$$0 = -\sin^3\Phi + 15\,\sin\Phi - 15\,\Phi\,\cos\Phi\,, \tag{B1}$$

and Eq. (54) is given by

$$0 = (1 - \cos\Phi)^2 \;\; (15\,\Phi + \sin\Phi\,(1 + \cos\Phi) + 15\,\sin\Phi)\;. \tag{B2}$$

The only solution of (B2) is (it is straightforward to show that the first derivative of $15\,\Phi + \sin\Phi\,(1 + \cos\Phi) + 15\,\sin\Phi$ is always positive, that means this expression represents a monotonically increasing function; thus (B3) is, therefore, the only solution of (B2))

$$\Phi = 0\;. \tag{B3}$$

Inserting the solution (B3) into Eq. (B1) yields an identity $0 = 0$. Thus, the only solution of Eqs. (B1) and (B2) is given by (B3).

APPENDIX C: PROOF OF INEQUALITY (85)

The inequality in Eq. (85) is given by

$$\frac{\sin^3\Phi}{(1 + z^2 - 2\,z\,\cos\Phi)^2}\left|z - 3\,z^3 + 2\,z^4\cos\Phi\right| \leq 8\,\sin\Phi\;, \tag{C1}$$

which can be written as

$$\frac{1 - w^2}{(1 + z^2 - 2\,w\,z\,)^2}\left|\,z - 3\,z^3 + 2\,w\,z^4\,\right| \leq 8\;, \tag{C2}$$

where $w = \cos\Phi$. In order to show the validity of Eq. (C2) it is convenient to simplify this expression with the aid of the following inequality:

$$\frac{1 - w^2}{(1 + z^2 - 2\,w\,z\,)^2}\left|\,z - 3\,z^3 + 2\,w\,z^4\,\right| \leq 2\,\frac{1 - w}{(1 + z^2 - 2\,w\,z\,)^2}\left|\,z - 3\,w\,z^3 + 2\,z^4\,\right|. \tag{C3}$$

The proof of inequality (C3) can be shown as follows. Due to $1 - w^2 \leq 2\,(1 - w)$ for $-1 \leq w \leq 1$, we have only to show $|\,z - 3\,z^3 + 2\,w\,z^4\,| \leq |\,z - 3\,w\,z^3 + 2\,z^4\,|$. Squaring both of these sides and subtracting from each other leads to the inequality

$$z^4\,(w - 1)\;\;(3 + 2\,z)\left(2\,w\,z^3 - 3\,w\,z^2 + 2 + 2\,z^3 - 3\,z^2\right) \leq 0\,. \tag{C4}$$

That means, in order to proof (C3), one has actually to show the inequality

$$g = 2\,w\,z^3 - 3\,w\,z^2 + 2 + 2\,z^3 - 3\,z^2 \geq 0\,. \tag{C5}$$

The extremal point is given by $P_{\rm e}\,(z = 3/2, w = -1)$ where $g\,|_{P_{\rm e}} = 2$, and the boundaries are $g\,|_{z=0} = 2 \geq 0$, $g\,|_{z=\infty} = 2\,(1 + w)\,z^3 \geq 0$, $g\,|_{w=-1} = 2$, $g\,|_{w=1} = 4\,z^3 - 6\,z^2 + 2 \geq 0$. Thus, we have shown the validity of (C5) and (C3), respectively. Let us turn back to the original inequality (C1),

$$h_7 = \frac{1 - w}{(1 + z^2 - 2\,w\,z\,)^2}\left|\,z - 3\,w\,z^3 + 2\,z^4\,\right| \leq 4\,, \tag{C6}$$

which follows from the combination of (C2) and (C3). In order to show the validity of (C6), we consider the extremal conditions $h_{7,z}=0$ and $h_{7,w}=0$ which yield

$$(1-w)\,\left(1+2\,w\,z-3\,z^2-9\,w\,z^2+8\,z^3+6\,w^2\,z^3-5\,w\,z^4\right)=0\,, \tag{C7}$$

$$z\,\;(z-1)^3\,\left(-2\,z^2-z+2\,w\,z+1\right)=0\,. \tag{C8}$$

The solutions of (C8) are given by

$$z_1=0\,, \tag{C9}$$

$$z_2=1\,, \tag{C10}$$

$$z_{3,4}=\frac{2\,w-1}{4}\pm\frac{\sqrt{9-4\,w+4\,w^2}}{4}\,, \tag{C11}$$

$$w=\frac{2\,z^2+z-1}{2\,z}\,. \tag{C12}$$

Inserting (C9) - (C12) into extremal condition (C7) yields the relations

$$1-w=0\,, \tag{C13}$$

$$(1-w)^3=0\,, \tag{C14}$$

$$(7\,w+20)\;(4\,w+5)\;(w-1)^4=0\,, \tag{C15}$$

$$(z-1)^3\;(z+2)\;(2\,z+1)\;(2\,z+7)=0\,. \tag{C16}$$

The relevant solutions of (C13) - (C16), respectively, are given by

$$w=1\,, \tag{C17}$$

$$z=1\,. \tag{C18}$$

Thus, the extremal point is given by $P_{\rm e}:(z=1,w=1)$, which is just one point of one of the boundaries. The boundaries of h_7 are given by

$$h_7\Big|_{z=0}=0\,, \tag{C19}$$

$$h_7\Big|_{z=\infty}=2\;(1-w)\leq 4\,, \tag{C20}$$

$$h_7\Big|_{w=-1}=2\;z\;\frac{1+3\,z^2+2\,z^3}{(1+z)^4}\leq 4\,, \tag{C21}$$

$$h_7\Big|_{w=1}=0\,. \tag{C22}$$

From Eqs. (C19) - (C22) we conclude the validity of inequality (C6), that means the validity of (C1) and (C2), respectively.

APPENDIX D: PROOF OF INEQUALITY (87)

Since $z\geq 0$, Eq. (87) can also be written by

$$h_8=\sin\Phi\,(16\,z-15-z\cos\Phi)+15\;(\cos\Phi-z)\;\Phi\leq 0\,. \tag{D1}$$

The extremal conditions $h_{8,z} = 0$ and $h_{8,\Phi} = 0$ yield

$$0 = 16 \sin\Phi - \sin\Phi \cos\Phi - 15\,\Phi\,, \tag{D2}$$
$$0 = 16\, z \cos\Phi - z \cos^2\Phi + z \sin^2\Phi - 15\, z - 15\,\Phi \sin\Phi\,. \tag{D3}$$

The only solution of Eq. (D2) is given by (note, that it is straightforward to show that the first derivative of (D1) is negative, i.e. (D1) represents a monotonically decreasing function; thus the given solution (D4) is indeed the only possible solution)

$$\Phi = 0\,. \tag{D4}$$

Inserting the solution (D4) into (D3) yields an identity $0 = 0$. That means the function h_8 does not have an extremal point, while h_8 takes extremal values at the boundary $\Phi = 0$. Especially, the boundaries are given by

$$h_8\Big|_{z=0} = -15 \sin\Phi + 15\,\Phi \cos\Phi \le 0\,, \tag{D5}$$
$$h_8\Big|_{z=\infty} = \lim_{z\to\infty} \left(-15\,\Phi - \sin\Phi\cos\Phi + 16\cos\Phi\right) z \le 0\,, \tag{D6}$$
$$h_8\Big|_{\Phi=0} = 0\,, \tag{D7}$$
$$h_8\Big|_{\Phi=\pi} = -15\,\pi\,(1+z) \le 0\,. \tag{D8}$$

From Eqs. (D5) - (D8) we conclude the validity of inequality (D1).

APPENDIX E: PROOF OF INEQUALITY (90)

Eq. (90) can be written by

$$\left|\frac{\Phi\,(\cos\Phi - z)}{1 + z^2 - 2\,z\cos\Phi}\right| \le \frac{\pi}{1+z}\,, \tag{E1}$$

that means we have to show the validity of

$$\frac{\Phi^2\,(\cos\Phi - z)^2}{(1 + z^2 - 2\,z\cos\Phi)^2} - \frac{\pi^2}{(1+z)^2} \le 0\,. \tag{E2}$$

Obviously, the following inequality is valid:

$$\frac{(\cos\Phi - z)^2}{(1 + z^2 - 2\,z\cos\Phi)} \le 1\,. \tag{E3}$$

Thus, that means by inserting (E3) into (E2), we have to show the inequality

$$h_9 = \frac{\Phi^2}{1 + z^2 - 2\,z\cos\Phi} - \frac{\pi^2}{(1+z)^2} \le 0\,. \tag{E4}$$

The extremal conditions $h_{9,z} = 0$ and $h_{9,\Phi} = 0$ lead to

$$0 = -z\,\Phi - 2\,z^2\,\Phi - z^3\,\Phi + \Phi\,\cos\Phi + 2\,z\,\Phi\,\cos\Phi + z^2\,\Phi\,\cos\Phi + \pi\left(1 + z^2 - 2\,z\,\cos\Phi\right)^{3/2}, \tag{E5}$$

$$0 = 1 + z^2 - 2\,z\,\cos\Phi - z\,\Phi\,\sin\Phi\,. \tag{E6}$$

The relation (E6) represents an quadratic equation in variable z and has the both solutions

$$z_{1,2} = \cos\Phi + \frac{1}{2}\,\sin\Phi \pm \frac{1}{2}\,\sqrt{T_2}\,, \tag{E7}$$

where the discriminant T_2 is given by

$$T_2 = \sin\Phi\left(-4\,\sin\Phi + 4\,\Phi\,\cos\Phi + \Phi^2\,\sin\Phi\right) \le 0\,, \tag{E8}$$

where the inequality (E8) is shown in Appendix F. The discriminant $T_2 = 0$ at $\Phi = 0$. Thus, in view of Eq. (E7) and (E8), the only real solution of Eq. (E6) is given by $P\,(z = 1, \Phi = 0)$. Inserting this solution into Eq. (E5) yields

$$0 = \pi\,\left(z - 1\right)^3. \tag{E9}$$

Thus, the extremal point is given by

$$P_{\rm e} : (z = 1, \Phi = 0)\,, \tag{E10}$$

which is just one point on one of the boundaries. The boundaries of function h_9 are given by

$$h_9\Big|_{z=0} = \Phi^2 - \pi^2 \le 0\,, \tag{E11}$$

$$h_9\Big|_{z=\infty} = 0\,, \tag{E12}$$

$$h_9\Big|_{\Phi=0} = -\frac{\pi^2}{\left(1+z\right)^2} \le 0\,, \tag{E13}$$

$$h_9\Big|_{\Phi=\pi} = 0\,. \tag{E14}$$

From Eqs. (E11) - (E14) we conclude the validity of inequality (E4) and (E1), respectively.

APPENDIX F: PROOF OF INEQUALITY (E8)

Inequality (E8) is given by

$$T_2 = \sin\Phi\,\left(-4\,\sin\Phi + 4\,\Phi\,\cos\Phi + \Phi^2\,\sin\Phi\right) \le 0\,, \tag{F1}$$

that means, due to $\sin\Phi \ge 0$, we have to show the inequality

$$h_{10} = -4\,\sin\Phi + 4\,\Phi\,\cos\Phi + \Phi^2\,\sin\Phi \le 0\,. \tag{F2}$$

The extremal condition $h_{10,\Phi} = 0$ leads to

$$0 = \Phi \left(\Phi \, \cos\Phi - 2 \, \sin\Phi \right) , \tag{F3}$$

with the only solution (the first derivative of (F3) is always negative, thus the expression on the r.h.s. of Eq. (F3) represents a monotonically decreasing function and, therefore, the given solution (F4) is in fact the only possible solution)

$$P_{\rm e} : (\Phi = 0) \, . \tag{F4}$$

The extremal point (F4) is just one point on one of the both boundaries. The boundaries are given by

$$h_{10} \bigg|_{\Phi = 0} = 0 \, , \tag{F5}$$

$$h_{10} \bigg|_{\Phi = \pi} = -4 \, \pi \, . \tag{F6}$$

From Eqs. (F5) and (F6) we conclude the validity of inequality (F1).

APPENDIX G: PROOF OF INEQUALITY (92)

Eq. (92) is given by

$$\left| \sin\Phi \, \frac{z \, (z-1)}{1 + z^2 - 2z \cos\Phi} - \sin\Phi \right| \leq \frac{2}{1+z} \, , \tag{G1}$$

which is equivalent to the inequality

$$h_{11} = \frac{\left(1 - w^2\right) \, \left(2 \, w \, z - z - 1\right)^2 \, \left(1 + z\right)^2}{\left(1 + z^2 - 2 \, w \, z\right)^2} - 4 \leq 0 \, . \tag{G2}$$

The extremal conditions $h_{11,z} = 0$ and $h_{11,w} = 0$ lead to the both relations

$$0 = (1+z) \left(1 - w^2\right) (2 \, w \, z - z - 1) \left(2 \, z^2 \, w^2 - 2 \, w \, z - z^2 + 1\right) , \tag{G3}$$

$$\begin{aligned} 0 = {} & (1+z)^2 \, \left(2 \, w \, z - z - 1\right) \\ & \times \left(4 \, w^3 \, z^2 - 4 \, w^2 \, z^3 - 4 \, w^2 \, z + w \, z^3 + w \, z^2 + w \, z + w + 2 \, z^3 - 2 \, z^2\right) . \end{aligned} \tag{G4}$$

The solutions of (G3) are given by

$$w_1 = -1 \, , \tag{G5}$$

$$w_2 = 1 \, , \tag{G6}$$

$$w_3 = \frac{1 + z}{2 \, z} \, , \tag{G7}$$

$$w_{4,5} = \frac{1 \pm \sqrt{2 \, z^2 - 1}}{2 \, z} \, , \tag{G8}$$

$$z_1 = \frac{1}{2 \, w - 1} \, , \tag{G9}$$

$$z_{2,3} = \frac{w \pm \sqrt{1 - w^2}}{2 \, w^2 - 1} \, . \tag{G10}$$

Inserting (G5), (G6), (G8) and (G10) into (G4) yields

$$0 = \left(7\,z^2 + 3\,z^3 + 5\,z + 1\right), \tag{G11}$$

$$0 = (z-1)^4\,, \tag{G12}$$

$$0 = 2\,z^5 + z^4 - 2\,z^3 - 2\,z^2 + 1 \pm \sqrt{2\,z^2 - 1}\left(-z^4 - 2\,z^3 + 2\,z^2 + 2\,z - 1\right), \tag{G13}$$

$$\begin{aligned} 0 = w\,\left(1-w^2\right)\,\left(1+w^2\right)\Big[&\left(8\,w^7 - 16\,w^5 + 12\,w^4 + 2\,w^3 - 12\,w^2 + 6\,w - 1\right)\sqrt{1-w^2} \\ &\pm\left(8\,w^7 - 12\,w^6 - 16\,w^5 + 24\,w^4 + 2\,w^3 - 11\,w^2 + 6\,w - 1\right)\Big]\,, \end{aligned} \tag{G14}$$

while inserting (G7) or (G9) into (G4) yields an identity $0 = 0$. Obviously, Eq. (G11) has no real solution for variable $z \geq 0$. The only solution of Eq. (G12) is given by

$$z_1 = 1\,. \tag{G15}$$

In order to find the solutions of (G13), we first have to bring all those terms proportional to $\sqrt{2\,z^2 - 1}$ on the left side, then squaring both sides and obtain

$$0 = (1+z)^4\,\,(z-1)^6\,\,, \tag{G16}$$

according to which it follows that the only solution of (G13) is already given by (G15). In order to find the solutions of (G14), we first have to bring all those terms proportional to $\sqrt{1-w^2}$ on the left side, then squaring both sides and obtain

$$0 = w^2\,\left(1-w^2\right)\,\left(2\,w^2 - 1\right)^6\,. \tag{G17}$$

The solutions of Eq. (G17) are given by (G5), (G6) and

$$w_{6,7} = \pm\frac{1}{\sqrt{2}}\,. \tag{G18}$$

Collecting the results (G5) - (G8), (G15) and (G18) together, we obtain the extremal points

$$P_{\rm e1} : (z = 1, w = 0)\,\,, \tag{G19}$$

$$P_{\rm e2} : (z = 1, w = 1)\,\,. \tag{G20}$$

While the extremal point $P_{\rm e2}$ is just one point on one of the boundaries, the numerical value of function h_{11} at the extremal point $P_{\rm e1}$ is given by

$$h_{11}\Big|_{P_{\rm e1}} = 0\,. \tag{G21}$$

The boundaries are given by

$$h_{11}\Big|_{z=0} = -3 - w^2 \leq 0\,, \tag{G22}$$

$$h_{11}\Big|_{z=\infty} = \left(1-w^2\right)\,(2\,w-1)^2 - 4 \leq 0\,, \tag{G23}$$

$$h_{11}\Big|_{w=-1} = -4\,, \tag{G24}$$

$$h_{11}\Big|_{w=1} = -4\,. \tag{G25}$$

From Eqs. (G21) - (G25) we conclude the validity of inequality (G2) and (G1), respectively.