Using a coordinate-independent impact parameter in the post-post-Newtonian light deflection formulas

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In [3] we have seen that among the post-post-Newtonian terms in the light deflection formulas there are "enhanced" ones that may become much larger than the other post-post-Newtonian terms. It is demonstrated here that these "enhanced" terms result from an inadequate choice of the impact parameter. Introducing another impact parameter, that can be considered as coordinate-independent, we demonstrate that all "enhanced" terms disappear from the formulas.

Contents

I. Introduction 2

II. Impact parameters 2

III. Transformation between k and σ 3

IV. Transformation between σ and n 4

V. Transformation k to n 5

VI. Transformation σ to n for stars and quasars 5

VII. Summary 6

Acknowledgments 6

References 6

A. Proof of (16) 6

I. INTRODUCTION

In [2] an analytical post-post-Newtonian solution of light propagation in the Schwarzschild metric generalized using PPN and post-linear parameters has been presented. In [3] various terms in the transformations between the units vectors $\boldsymbol{\sigma}$, $\boldsymbol{n}$, and $\boldsymbol{k}$ characterizing light propagation have been analytically estimated. These estimates reveal that in each transformation "peculiar" or "enhanced" post-post-Newtonian terms exist that can become much larger that the other "regular" post-post-Newtonian terms. In each case the sum of the "regular" post-post-Newtonian terms can be estimated as $\frac{15}{4}\,\pi\,\frac{m^2}{d^2}$, m being the Schwarzschild radius of the central body and d is the impact parameter. The "enhanced" terms can be much larger (being, however, of order m^2). Below we demonstrate that the "enhanced" terms result from an inadequate choice of parametrization of the light ray. Indeed, one can demonstrate that the "enhanced" terms disappear if the light deflection formulas are expressed through a coordinate-independent impact parameter.

II. IMPACT PARAMETERS

The analytical solutions of the four basic transformations $\boldsymbol{k}$ to $\boldsymbol{\sigma}$, $\boldsymbol{\sigma}$ to $\boldsymbol{n}$, $\boldsymbol{k}$ to $\boldsymbol{n}$, and $\boldsymbol{\sigma}$ to $\boldsymbol{n}$ for stars and quasars, given in [2, 3], are expressed through one of the two following impact parameters:

$$\boldsymbol{d}_\sigma = \boldsymbol{\sigma} \times (\boldsymbol{x} \times \boldsymbol{\sigma})\,, \tag{1}$$

$$\boldsymbol{d} \;= \boldsymbol{k} \times (\boldsymbol{x}_0 \times \boldsymbol{k}) = \boldsymbol{k} \times (\boldsymbol{x} \times \boldsymbol{k})\,, \tag{2}$$

where $\boldsymbol{x}_0$ is the position of the source and $\boldsymbol{x}$ is the position of the observer. The definitions of $\boldsymbol{\sigma}$ and $\boldsymbol{k}$ are given in [2]. The absolute values of these impact vectors are denoted by $d = |\boldsymbol{d}|$ and $d_\sigma = |\boldsymbol{d}_\sigma|$.

In Section III.B of [2] the absolute value $d' = |\boldsymbol{d}'|$ of yet another impact parameter

$$\boldsymbol{d}' = \lim_{t\to-\infty} \boldsymbol{\sigma} \times (\boldsymbol{x}(t) \times \boldsymbol{\sigma}) \tag{3}$$

has been introduced in order to compare the expression of total light deflection derived there with the results found by other authors. Note that here $\boldsymbol{x}(t)$ is the position of the photon at some arbitrary moment of time t. For $\boldsymbol{d}'$ one also has

$$\boldsymbol{d}' = \lim_{t\to-\infty} \frac{1}{c}\,\dot{\boldsymbol{x}}(t) \times \left(\boldsymbol{x}(t) \times \frac{1}{c}\,\dot{\boldsymbol{x}}(t)\right). \tag{4}$$

Here one should take into account that $\boldsymbol{\sigma} = \lim\limits_{t\to-\infty} \frac{1}{c}\,\dot{\boldsymbol{x}}(t)$, $\dot{\boldsymbol{x}}(t)$ being the velocity of the photon at time t. For a similar impact parameter defined at $t \to +\infty$

$$\boldsymbol{d}'' = \lim_{t\to+\infty} \frac{1}{c}\,\dot{\boldsymbol{x}}(t) \times \left(\boldsymbol{x}(t) \times \frac{1}{c}\,\dot{\boldsymbol{x}}(t)\right) = \lim_{t\to+\infty} \boldsymbol{\nu} \times (\boldsymbol{x}(t) \times \boldsymbol{\nu})\,, \tag{5}$$

where $\boldsymbol{\nu} = \lim\limits_{t\to+\infty} \frac{1}{c}\,\dot{\boldsymbol{x}}(t)$, one has $|\boldsymbol{d}'| = |\boldsymbol{d}''|$. It is also clear that the angle between $\boldsymbol{d}'$ and $\boldsymbol{d}''$ is equal to the full light deflection. Since both $\boldsymbol{d}'$ and $\boldsymbol{d}''$ resided at time-like infinity

(since they are defined for $|t| \to \infty$) and since the gravitational system under study is asymptotically flat, these parameters can be called coordinate-independent.

One can show that $d' = d''$ coincides with the impact parameter D introduced, e.g., by Eq. (215) of Section 20 of [1] in terms of full energy and angular momentum of the photon. Indeed, in polar coordinates (x, φ) the Chandrasekhar's impact parameter $D = f(x)\, x^2\, \dot{\varphi}$, where $\lim\limits_{x\to\infty} f(x) = 1$. Clearly, $x^2\, \dot{\varphi} = |\dot{\boldsymbol{x}}(t) \times \boldsymbol{x}(t)|$ and it is obvious that $d' = d'' = D$. Interestingly, this discussion allows one to find an exact integral of the equations of motion for a photon in Schwarzschild field. In the exact Schwarzschild solution used in [3] in harmonic coordinates, Eq. (11) of [3] has an integral

$$\boldsymbol{D} = \frac{(1+a)^3}{1-a}\, \frac{1}{c}\, \dot{\boldsymbol{x}}(t) \times \boldsymbol{x}(t) = \text{const}, \tag{6}$$

while for the parametrized post-post-Newtonian equations of motion given by Eq. (24) of [2] one has

$$\begin{aligned} \boldsymbol{D} &= \exp\left(2(1+\gamma)\, a + \alpha\, \left(2\,(1-\beta) + \epsilon - 2\gamma^2\right)\, a^2\right) \frac{1}{c}\, \dot{\boldsymbol{x}}(t) \times \boldsymbol{x}(t) \\ &= \left(1 + 2(1+\gamma)\, a + \left(2(1+\gamma)^2 + \alpha\, \left(2\,(1-\beta) + \epsilon - 2\gamma^2\right)\right)\, a^2\right)\, \frac{1}{c}\, \dot{\boldsymbol{x}}(t) \times \boldsymbol{x}(t) + \mathcal{O}(c^{-6}) \\ &= \text{const}, \end{aligned} \tag{7}$$

where all the notations are as in the corresponding equations in [2, 3]. First line of (7) represents an exact integral of the [approximate] equation of motion (24) of [3]. In both cases the Chandrasekhar's D is the absolute value of $\boldsymbol{D}$ as given above.

The aim of this investigation is to express all the transformations between vectors $\boldsymbol{\sigma}$, $\boldsymbol{n}$, and $\boldsymbol{k}$ using the coordinate-independent impact vector $\boldsymbol{d}'$. Therefore, we need to have a relation between impact parameters (2) and (1), respectively, and (3). Relation between $\boldsymbol{d}'$ and $\boldsymbol{d}_\sigma$ can be derived using the post-Newtonian solution for light propagation given in Section III.A of [2] and has been partially given by Eq. (43) in Section III.B of [2]. One gets

$$\boldsymbol{d}' = \boldsymbol{d}_\sigma \left(1 + (1+\gamma)\, \frac{m}{d_\sigma^2}\, \left(x + \boldsymbol{\sigma} \cdot \boldsymbol{x}\right)\right) + \mathcal{O}\left(m^2\right). \tag{8}$$

Relation of $\boldsymbol{d}'$ and $\boldsymbol{d}$ can be derived by considering Eq. (31) in Section III.C of [2] in post-Newtonian approximation. Substituting this relation into the definition of $\boldsymbol{d}_\sigma$ and considering (8) one gets

$$\boldsymbol{d}' = \boldsymbol{d} \left(1 + (1+\gamma)\, \frac{m}{d^2}\, \frac{x + x_0}{R}\, \frac{R^2 - (x - x_0)^2}{2\, R}\right) - (1+\gamma)\, m\, \boldsymbol{k}\, \frac{x - x_0 + R}{R} + \mathcal{O}\left(m^2\right). \tag{9}$$

Now let us consider the transformations between $\boldsymbol{\sigma}$, $\boldsymbol{n}$, and $\boldsymbol{k}$.

III. TRANSFORMATION BETWEEN $\boldsymbol{k}$ AND $\boldsymbol{\sigma}$

The transformation between $\boldsymbol{k}$ and $\boldsymbol{\sigma}$ is given by Eq. (29) of [3] or, retaining only "enhanced" post-post-Newtonian terms, by Eqs. (35)–(36) of [3]. These latter equations

can be re-written as follows

$$\boldsymbol{d}\,S\left(1-S\,\frac{1}{2}\,(x+x_0)\left(1+\frac{x_0-x}{R}\right)\right)=\boldsymbol{d}\,'\,S\,' \\ +(1+\gamma)^2\,m^2\,\frac{(x-x_0+R)^2}{|\boldsymbol{x}\times\boldsymbol{x}_0|^2}\,\boldsymbol{k}+\mathcal{O}\left(m^3\right), \quad (10)$$

where S is defined by Eq. (36) of [3], and $S\,'$ has the same functional form, but with d replaced by $d\,'$:

$$S\,'=(1+\gamma)\,\frac{m}{d\,'^2}\left(1-\frac{x_0-x}{R}\right). \quad (11)$$

The term in (10) proportional to $\boldsymbol{k}$ is of type "scaling" as explained in Section V.B of [3] and plays no role for the light deflection (its absolute value can be estimated as $8\,m^2/d^2$). Therefore, the transformation between $\boldsymbol{k}$ and $\boldsymbol{\sigma}$ can be finally written as

$$\boldsymbol{\sigma}=\boldsymbol{k}\,+\boldsymbol{d}\,'\,S\,'+\mathcal{O}\left(\frac{m^2}{d^2}\right)+\mathcal{O}\left(m^3\right). \quad (12)$$

Note that the terms of order m^2/d^2 in (12) have the same upper estimate given by Eq. (33) of [3].

IV. TRANSFORMATION BETWEEN $\boldsymbol{\sigma}$ AND $\boldsymbol{n}$

The transformation between $\boldsymbol{\sigma}$ and $\boldsymbol{n}$ is given by Eq. (37) of [3] or, retaining only "enhanced" post-post-Newtonian terms, by Eqs. (43)–(44) of [3]. The latter equations can be rewritten in terms of $\boldsymbol{d}\,'$ as

$$\boldsymbol{d}\,T\left(1+T\,x\,\frac{R+x_0-x}{R+x_0+x}\right)=\boldsymbol{d}\,'\,T'-(1+\gamma)^2\,\frac{m^2}{d^2}\left(1+\frac{\boldsymbol{k}\cdot\boldsymbol{x}}{x}\right)\frac{x-x_0+R}{R}\,\boldsymbol{k} \\ +\boldsymbol{\varphi}_{\rm corr}+\mathcal{O}\left(m^3\right) \quad (13)$$

where T is defined by Eq. (44) of [3] and

$$T'=-(1+\gamma)\,\frac{m}{d\,'^2}\left(1+\frac{\boldsymbol{k}\cdot\boldsymbol{x}}{x}\right), \quad (14)$$

$$\boldsymbol{\varphi}_{\rm corr}=-(1+\gamma)^2\,\frac{m^2}{d^2}\frac{x-x_0+R}{R}\,\frac{\boldsymbol{d}}{x}. \quad (15)$$

In [3] the sum of all "regular" post-post-Newtonian terms in the transformation from $\boldsymbol{\sigma}$ to $\boldsymbol{n}$ was denoted by $\boldsymbol{\varphi}_{\rm ppN}$ and estimated to have absolute value less than $\frac{15}{4}\,\pi\,\frac{m^2}{d^2}$ (see Section V.C of [3]). One can also demonstrate (see Appendix A) that

$$\left|\,\boldsymbol{\varphi}_{\rm ppN}+\boldsymbol{\varphi}_{\rm corr}\,\right|\le\frac{15}{4}\,\pi\,\frac{m^2}{d^2}. \quad (16)$$

The term in (13) proportional to vector $\boldsymbol{k}$ is again a "scaling" term and can be omitted since it does not influence the direction of $\boldsymbol{n}$. Again its absolute value is of order m^2/d^2. Finally, the transformation between $\boldsymbol{\sigma}$ and $\boldsymbol{n}$ can be written as

$$\boldsymbol{n} = \boldsymbol{\sigma} + \boldsymbol{d}\,' T\,' + \mathcal{O}\left(\frac{m^2}{d^2}\right) + \mathcal{O}\left(m^3\right) . \tag{17}$$

Again the terms of order m^2/d^2 in (17) have the same upper estimate given by Eq. (41) of [3].

V. TRANSFORMATION k TO n

The transformation between $\boldsymbol{n}$ and $\boldsymbol{k}$ is given by Eq. (45) of [3] or, retaining only "enhanced" post-post-Newtonian terms, by Eqs. (52)–(53) of [3]. The latter equations can be rewritten in terms of $\boldsymbol{d}\,'$ as

$$\boldsymbol{d}\,P\left(1 + P\,x\,\frac{x_0 + x}{R}\right) = \boldsymbol{d}\,'\,P\,' - (1+\gamma)^2\,\frac{m^2}{d^2}\,\frac{x - x_0 + R}{R}\left(\frac{x_0 - x}{R} + \frac{\boldsymbol{k}\cdot\boldsymbol{x}}{x}\right)\boldsymbol{k} + \mathcal{O}\left(m^3\right) , \tag{18}$$

where P is defined by Eq. (53) of [3], and P' has the same functional form as P, but with d replaced by $d\,'$:

$$P' = -(1+\gamma)\,\frac{m}{d'^2}\left(\frac{x_0 - x}{R} + \frac{\boldsymbol{k}\cdot\boldsymbol{x}}{x}\right) . \tag{19}$$

The term in (18) proportional to vector $\boldsymbol{k}$ is again "scaling" and plays no role for the light deflection. Finally, the transformation between $\boldsymbol{n}$ and $\boldsymbol{k}$ can be written as

$$\boldsymbol{n} = \boldsymbol{k} + \boldsymbol{d}\,'\,P\,' + \mathcal{O}\left(\frac{m^2}{d^2}\right) + \mathcal{O}\left(m^3\right) . \tag{20}$$

Also in this equation terms of order m^2/d^2 have the same upper estimate given by Eq. (50) of [3].

VI. TRANSFORMATION σ TO n FOR STARS AND QUASARS

Finally, let us consider transformation from $\boldsymbol{\sigma}$ to $\boldsymbol{n}$ for stars and quasars. This transformation is given by Eq. (58) of [3] or, retaining only "enhanced" post-post-Newtonian terms by Eqs. (62)–(63), of [3]. The latter equations can be rewritten in terms of $\boldsymbol{d}\,'$ as

$$\boldsymbol{d}_\sigma\,Q\,(1 + Q\,x) = \boldsymbol{d}\,'\,Q' + \mathcal{O}\left(m^3\right) , \tag{21}$$

where Q is defined by Eq. (63) of [3], and Q' has the same functional form as Q with d_σ replaced by $d\,'$:

$$Q' = -(1+\gamma)\,\frac{m}{d'^2}\left(1 + \frac{\boldsymbol{\sigma}\cdot\boldsymbol{x}}{x}\right) . \tag{22}$$

Therefore, transformation from $\boldsymbol{\sigma}$ to $\boldsymbol{n}$ can be written as

$$\boldsymbol{n} = \boldsymbol{\sigma} + \boldsymbol{d}\,'\,Q\,' + \mathcal{O}\left(\frac{m^2}{d_\sigma^2}\right) + \mathcal{O}\left(m^3\right) . \tag{23}$$

VII. SUMMARY

Thus, we have demonstrated that the source of the "enhanced" post-post-Newtonian terms discussed in [3] is an inadequate choice of the impact parameter. Using the coordinate-independent coordinate parameter $\boldsymbol{d}'$ discussed above one can eliminate the "enhanced" post-post-Newtonian terms from the analytical formulas. Above we have demonstrated that the four relevant transformations between vectors $\boldsymbol{\sigma}$, $\boldsymbol{n}$, and $\boldsymbol{k}$ can be expressed by Eqs. (12), (17), (20), and (23). In all these formulas the omitted post-post-Newtonian terms can be estimated to be less than $\frac{15}{4}\,\pi\,\frac{m^2}{d^2}$ and, therefore, the formulas guarantee numerical accuracy of 1 μas in each case as long as observations of sources within 5 angular radii from the Sun are not considered.

Although this investigation elucidates the origin of the "enhanced" post-post-Newtonian terms, the results given by Eqs. (12), (17), (20), and (23) are strictly equivalent to those derived in [3] and are not necessarily more convenient from the computational point of view than the formulas in terms of $\boldsymbol{d}$ and $\boldsymbol{d}_\sigma$.

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APPENDIX A: PROOF OF (16)

Eq. (C3) of [2] demonstrates that

$$\left|\,\boldsymbol{\varphi}_{\rm ppN}\right| = \frac{m^2}{d^2}\,\frac{1}{4}\,f_7, \tag{A1}$$

$$f_7 = \left|16\,\sin\Psi + \sin\Psi\,\cos\Psi - 2\,\sin^3\Psi\,\cos\Psi - 15\,(\pi-\Psi)\right|\,, \tag{A2}$$

where Ψ is the angle between vectors $\boldsymbol{k}$ and $\boldsymbol{n}$ ($0 \le \Psi \le \pi$). For $\boldsymbol{\varphi}_{\rm corr}$ defined by (15) one can write (as usual we assume $\gamma = 1$ here)

$$|\boldsymbol{\varphi}_{\rm corr}| = 4\,\frac{m^2}{d^2}\,\frac{d}{x}\,\frac{x - x_0 + R}{R} = 4\,\frac{m^2}{d^2}\,\frac{d}{x}\,f_3, \tag{A3}$$

$$f_3 = \frac{1-z}{\sqrt{1+z^2-2z\cos\Phi}} + 1, \tag{A4}$$

where $\Phi = \delta(\boldsymbol{x}, \boldsymbol{x}_0)$ is the angle between $\boldsymbol{x}$ and $\boldsymbol{x}_0$, and $z = x_0/x$. Function f_3 has been already considered in [3] (see Eq. (B2) of that report) and found to be less than 2 (it is

obvious that $f_3 \geq 0$). Therefore, one has

$$|\boldsymbol{\varphi}_{\rm corr}| \leq 8\,\frac{m^2}{d^2}\,\sin\Psi. \tag{A5}$$

Finally, combining (A1) and (A5) one gets

$$|\boldsymbol{\varphi}_{\rm ppN} + \boldsymbol{\varphi}_{\rm corr}| \leq |\boldsymbol{\varphi}_{\rm ppN}| + |\boldsymbol{\varphi}_{\rm corr}| \leq \frac{m^2}{d^2}\,\left(\frac{1}{4}\,f_7 + 8\,\sin\Psi\right). \tag{A6}$$

It is not difficult to see that

$$f_7^{\rm corr} = \frac{1}{4}\,f_7 + 8\,\sin\Psi \leq \frac{15\,\pi}{4}, \tag{A7}$$

and this immediately gives (16).