A generalized lens equation for small light-deflection angles

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A generalized lens equation for weak gravitational fields of Schwarzschild metric and valid for finite distances of source and observer from the light deflecting body is suggested. The magnitude of neglected terms in the generalized lens equation is estimated to be smaller than or equal to $\dfrac{15\,\pi}{4}\,\dfrac{m^2}{d^{\,\prime 2}}$, where m is the Schwarzschild radius of masive body and $d^{\,\prime}$ is Chandrasekhar's impact parameter. The main applications of this generalized lens equation are extreme astrometrical configurations, where *Standard post-Newtonian approach* as well as *Classical lens equation* cannot be applied. It is shown that in the appropriate limits the proposed lens equation yields the known post-Newtonian terms, 'enhanced' post-post-Newtonian terms and the Classical lens equation, thus provides a link between these both essential approaches for determining the light-deflection.

Contents

I. INTRODUCTION

The accuracy of GAIA mission necessitates theoretical predictions of light-deflection by massive bodies on microarcsecond (μas) level. In principle, such an astrometric precision on microarcsecond level can be achieved by numerical integration of geodesic equation of light-propagation. However, the GAIA mission determines the positions and proper motions of approximately one billion objects, each of which is observed about one hundred times. The data reduction of such huge amount of observations implies the need of analytical solutions, because the numerical investigation of geodesic equation is by far too time-consuming.

The metric of a massive body can be expanded in terms of multipoles, i.e. monopole term, quadrupole term and higher multipoles. Usually, the largest contributions of light-deflection originates from the spherically symmetric part (Schwarzschild) of the massive body under consideration. The exact analytical solution of light-propagation in Schwarzschild metric [1] inherits elliptic integrals, but their evaluation becomes comparable with the time effort needed for a numerical integration of geodesic equation. Thus, approximative analytical solutions valid on microarcsecond level of accuracy are indispensible for a highly time-efficient data reduction.

In the same way, exact lens equations of light-deflection have been obtained in [9, 10, 11]. Such exact relations are also given in terms of elliptic integrals and imply numerical efforts comparable with a numerical solution of geodesic equation. Therefore, also approximations of these exact solutions which are valid up to a given astrometric accuracy are very welcome. An excellent overview of such approximative lens equations has recently been presented in [12].

Basically, two essential approximative approaches for determining the light-deflection in weak gravitational fields are known:

The first one is the standard parameterized post-Newtonian approach (PPN) [7, 8] which is of the order $\mathcal{O}(m)$. During the last decades, it has been the common understanding that the higher order terms $\mathcal{O}(m^2)$ are negligible even on microarcsecond level, except for observations in the vicinity of the Sun. Recent investigations [2, 3, 4, 5] have revealed that the post-post-Newtonian approximation [6, 7], which is of the order $\mathcal{O}(m^2)$, is needed for such high accuracy. Both approximations are applicable for $d \gg m$, where d being the impact parameter of the unperturbed light ray.

The second one is the standard weak-field approximative lens equation, which usually is called classical lens equation, see Eq. (67) in [11] or Eq. (24) in [12]. One decisive advantage of classical lens equation is it's validity for arbitrarily small values of impact parameter d. The classical lens equation is valid for astrometrical configurations where source and observer are far ernough from the lens, especially in case of $A \gg d$ and $B \gg d$, where $A = \boldsymbol{k} \cdot \boldsymbol{x}_1$ and $B = -\boldsymbol{k} \cdot \boldsymbol{x}_0$, where $\boldsymbol{x}_0$ and $\boldsymbol{x}_1$ are the three-vectors from the center of the massive body to the source and observer, respectively, and $\boldsymbol{k}$ is the unit vector from the source to the observer. However, the classical lens equation is not applicable for determine the light-deflection of moons of giant planets in the solar system, because astrometrical configurations with $B = 0$ are possible.

Moreover, there are astrometric configurations where neither the standard post-Newtonian approach nor the classical lens equation are applicable. In order to investigate the light-deflection in such systems a link between these both approaches is needed. Such a link can be provided by a generalized lens equation which, in the appropriate limits, coincides with standard post-Newtonian approach and classical lens equation. Accordingly, the aim of our investigation is an analytical expression for the generalized lens equation having a form very similar to the classical lens equation. We formulate the following conditions under which our generalized lens equation should be applicable:

1. valid for $d = 0\,,\; A = x_1 \gg m\,,\; B = x_0 \gg m$,
2. valid for $A = 0\,,\; d \gg m\,,\; B \neq 0$,
3. valid for $B = 0\,,\; d \gg m\,,\; A \neq 0$.

These conditions imply that the light-path is always far enough from the lens, thus inherit weak gravitational fields, i.e. small light deflection angles. In order to control the numerical accuracy, the generalized lens equation is compared with the numerical solution of exact geodesic equation in the Schwarzschild metric (throughout the paper, we work in harmonic gauge):

$$g_{00} = -\frac{1-a}{1+a}\,, \qquad g_{i0} = 0\,,$$
$$g_{ij} = (1+a)^2\,\delta_{ij} + \frac{a^2}{x^2}\,\frac{1+a}{1-a}\,x^i\,x^j\,. \tag{1}$$

Here, $a = \dfrac{m}{x}$ and $m = \dfrac{G\,M}{c^2}$ is the Schwarzschild radius and M is the mass of massive body, G is Newtonian constant of gravitation and c is the speed of light. Latin indices take values $1, 2, 3$, and the Euclidean metric $\delta_{ij} = 1(0)$ for $i = j$ $(i \neq j)$. The absolute value of a three-vector is denoted by $x = |\boldsymbol{x}| = \sqrt{x_1^2 + x_2^2 + x_3^2}$. The exact geodesic equation in Schwarzschild metric reads, cf. [4]

$$\ddot{\boldsymbol{x}} = \frac{a}{x^2}\left[-c^2\frac{1-a}{(1+a)^3} - \dot{\boldsymbol{x}}\cdot\dot{\boldsymbol{x}} + a\frac{2-a}{1-a^2}\left(\frac{\boldsymbol{x}\cdot\dot{\boldsymbol{x}}}{x}\right)^2\right]\boldsymbol{x} + 2\frac{a}{x^2}\,\frac{2-a}{1-a^2}\left(\boldsymbol{x}\cdot\dot{\boldsymbol{x}}\right)\dot{\boldsymbol{x}}\,, \tag{2}$$

where a dot denotes time derivative in respect to the coordinate time t, and $\boldsymbol{x}$ is the three-vector pointing from the center of mass of the massive body to the photon trajectory at time moment t. The scalar product of two three-vectors with respect to Euclidean metric δ_{ij} is $\boldsymbol{a}\cdot\boldsymbol{b} = \sum\limits_{i,j=1}^{3}\delta_{ij}\,a^i b^j$. The numerical solution of this equation will be used in order to determine the accuracy of approximative solutions. We abbreviate the angle between two three-vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ by $\delta(\boldsymbol{a},\boldsymbol{b})$, which can be computed by means of $\delta(\boldsymbol{a},\boldsymbol{b}) = \arccos\dfrac{\boldsymbol{a}\cdot\boldsymbol{b}}{a\,b}$.

The report is based on our recent article [20] and is organized as follows: In Section II the standard post-Newtonian approach is presented. In Section III the steps of post-post-Newtonian approach are shown which are relevant for this investigation, and some main results of our work [4] are summarized. The generalized lens equation is obtained in Section IV and discussed in Section V. A summary is given in Section VI.

II. POST-NEWTONIAN APPROXIMATION

Let us consider the trajectory of a light-signal in post-Newtonian Schwarzschild metric:

$$g_{00} = -1 + 2\,a + \mathcal{O}\left(c^{-4}\right)\,, \quad g_{i0} = 0\,,$$
$$g_{ij} = \delta_{ij} + 2\,\gamma\,a\,\delta_{ij} + \mathcal{O}\left(c^{-4}\right). \tag{3}$$

Here, γ is the parameter of the Parametrized Post-Newtonian (PPN) formalism, which characterizes possible deviation of the physical reality from general relativity theory where $\gamma = 1$. The light-ray is being emitted at a position $\boldsymbol{x}_0$ at time moment t_0 and received at position $\boldsymbol{x}_1$ at a time moment t_1, see FIG. 1.

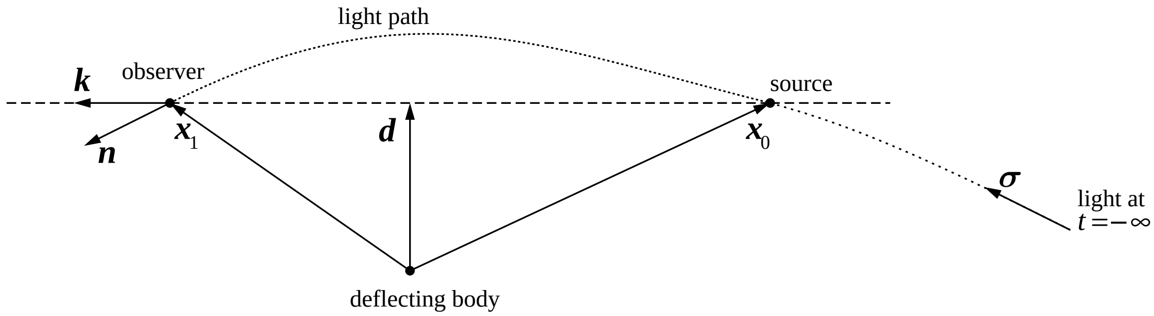

FIG. 1: A geometrical representation of the boundary problem under consideration for a light-propagation from the source to the observer.

Light propagation is governed by geodesic equation, in post-Newtonian order given by

$$\ddot{\boldsymbol{x}} = -(1+\gamma)\, c^2\, \frac{a\, \boldsymbol{x}}{x^2} + 2\,(1+\gamma)\, \frac{a\, \dot{\boldsymbol{x}}\, (\dot{\boldsymbol{x}} \cdot \boldsymbol{x})}{x^2} + \mathcal{O}(c^{-2})\,. \tag{4}$$

The unit tangent vector at the point of observation is $\boldsymbol{n} = \dfrac{\dot{\boldsymbol{x}}(t_1)}{|\dot{\boldsymbol{x}}(t_1)|}$, and the unit tangent vector $\boldsymbol{k} = \dfrac{\boldsymbol{R}}{R}$, where $\boldsymbol{R} = \boldsymbol{x}_1 - \boldsymbol{x}_0$ and the absolute value is $R = |\boldsymbol{R}|$. Furthermore, the unit tangent vector at remote past $\boldsymbol{\sigma} = \lim\limits_{t \to -\infty} \dfrac{\dot{\boldsymbol{x}}(t)}{c}$ is introduced.

Up to post-Newtonian order, the differential equation (4) can be solved analytically. The solution for the transformation between $\boldsymbol{n}$ and $\boldsymbol{k}$ reads

$$\boldsymbol{n} = \boldsymbol{k} - (1+\gamma)\, \frac{m}{d'}\, \frac{\boldsymbol{d}'}{d'}\, \frac{x_0\, x_1 - \boldsymbol{x}_0 \cdot \boldsymbol{x}_1}{R\, x_1} + \mathcal{O}\left(m^2\right), \tag{5}$$

in terms of the coordinate-independent impact vector $\boldsymbol{d}'$, cf. Eq. (57) of [4]:

$$\boldsymbol{d}' = \lim_{t \to -\infty} \boldsymbol{\sigma} \times (\boldsymbol{x}(t) \times \boldsymbol{\sigma})\,. \tag{6}$$

This impact parameter is identical to Chandrasekhar's impact parameter [4, 14] which in vectorial form is given by $\boldsymbol{d}' = \dfrac{\boldsymbol{L}}{E}$, where $\boldsymbol{L}$ is the orbital three-momentum and E is the energy of the photon on the light-trajectory; cf. Eq. (215) in chapter 20 of [1].

By means of $\sin\varphi = |\boldsymbol{n} \times \boldsymbol{k}|$, we find the light-deflection angle $\varphi = \delta(\boldsymbol{n}, \boldsymbol{k})$ in post-Newtonian approximation:

$$\varphi = (1+\gamma)\, \frac{m}{d'}\, \frac{x_0\, x_1 - \boldsymbol{x}_0 \cdot \boldsymbol{x}_1}{R\, x_1} + \mathcal{O}\left(m^2\right). \tag{7}$$

Note that $\dfrac{x_0\,x_1 - \boldsymbol{x}_0\cdot\boldsymbol{x}_1}{R\,x_1} \leq 2$, and therefore $\varphi \leq \dfrac{4\,m}{d'}$. One problem of post-Newtonian solution (5) or (7) is, that one can only state that the higher order terms are of order $\mathcal{O}\left(m^2\right)$, but their magnitude remains unclear. In order to make a statement about the upper estimate of the higher order terms one needs to consider the geodesic equation in post-post-Newtonian approximation.

III. POST-POST NEWTONIAN APPROXIMATION

Now let us consider the trajectory of a light-signal in post-post-Newtonian Schwarzschild metric:

$$\begin{aligned} g_{00} &= -1 + 2\,a - 2\,\beta\,a^2 + \mathcal{O}\left(c^{-6}\right)\;, \quad g_{i0} = 0\,, \\ g_{ij} &= \delta_{ij} + 2\,\gamma\,a\,\delta_{ij} + \epsilon\left(\delta_{ij} + \frac{x^i\,x^j}{x^2}\right)a^2 + \mathcal{O}\left(c^{-6}\right). \end{aligned} \tag{8}$$

The geodesic equation of light-propagation in post-post-Newtonian approximation is given by [4]

$$\begin{aligned} \ddot{\boldsymbol{x}} &= -(1+\gamma)\,c^2\,\frac{a\,\boldsymbol{x}}{x^2} + 2\,(1+\gamma)\,\frac{a\,\dot{\boldsymbol{x}}\,(\dot{\boldsymbol{x}}\cdot\boldsymbol{x})}{x^2} \\ &+ 2\,c^2\,\left(\beta - \epsilon + 2\,\gamma\,(1+\gamma)\right)\,\frac{a^2\,\boldsymbol{x}}{x^2} + 2\,\epsilon\,\frac{a^2\,\boldsymbol{x}\,(\dot{\boldsymbol{x}}\cdot\boldsymbol{x})^2}{x^4} \\ &+ 2\,(2(1-\beta) + \epsilon - 2\,\gamma^2)\,\frac{a^2\,\dot{\boldsymbol{x}}\,(\dot{\boldsymbol{x}}\cdot\boldsymbol{x})}{x^2} + \mathcal{O}\left(c^{-4}\right). \end{aligned} \tag{9}$$

In general relativity the parameters β, γ and ϵ characterize possible deviation of physical reality from general relativity theory where $\beta = \gamma = \epsilon = 1$. The solution of (9) and the transformation between the unit vectors $\boldsymbol{n}$ and $\boldsymbol{k}$ in post-post-Newtonian order has been given in [4], cf. Eqs. (108) and (109) ibid., and reads

$$\boldsymbol{n} = \boldsymbol{k} - (1+\gamma)\frac{m}{d'}\frac{\boldsymbol{d}'}{d'}\frac{x_0\,x_1 - \boldsymbol{x}_0\cdot\boldsymbol{x}_1}{R\,x_1} + \mathcal{O}\left(\frac{m^2}{d'^2}\right). \tag{10}$$

The terms of the order $\mathcal{O}\left(\dfrac{m^2}{d'^2}\right)$ can be estimated to be smaller or equal than $\dfrac{15\,\pi}{4}\,\dfrac{m^2}{d'^2}$. From Eq. (10) the expression

$$\varphi = (1+\gamma)\,\frac{m}{d'}\,\frac{x_0\,x_1 - \boldsymbol{x}_0\cdot\boldsymbol{x}_1}{R\,x_1} + \mathcal{O}\left(\frac{m^2}{d'^2}\right) \tag{11}$$

is obtained. The solutions (10) and (11) are identical to the post-Newtonian solution (5) and (7), respecticvely. This fact means that the post-post-Newtonian terms in the metric (8) and also the post-post-Newtonian terms in the geodesic equation (9) contribute only terms which can be estimated to be smaller or equal than $\dfrac{15\,\pi}{4}\,\dfrac{m^2}{d'^2}$. Therefore, the only difference between (11) and (7) here is, that the post-post-Newtonian approximation allows to estimate the magnitude of the regular post-post-Newtonian terms.

IV. GENERALIZED LENS EQUATION

Usually, in practical astrometry the position of observer $\boldsymbol{x}_1$ and the position of light-deflecting body is known (here, the position of massive body coincides with the center of coordinate system), but the impact parameter $d\,'$ is not accessible. Therefore, the solutions (10) or (11) are not applicable in the form presented. Instead, one has to rewrite these solutions in terms of the impact vector of unperturbed light-trajectory:

$$\boldsymbol{d} = \boldsymbol{k} \times (\boldsymbol{x}_1 \times \boldsymbol{k}) \,. \tag{12}$$

For that one needs a relation between impact vector $\boldsymbol{d}\,'$ defined in Eq. (6) and impact vector $\boldsymbol{d}$ defined in Eq. (12). Such a relation has been given in [4], cf. (62) ibid., and reads (note, $d\,' = d + \mathcal{O}\,(m)$):

$$d\,' = d + (1+\gamma)\,\frac{m}{d'}\,\frac{x_0 + x_1}{R}\,\frac{x_0\,x_1 - \boldsymbol{x}_0 \cdot \boldsymbol{x}_1}{R} + \mathcal{O}\left(m^2\right). \tag{13}$$

Eq. (13) is actually an quadratic equation for $d\,'$, and their both solutions correspond to the two possible light-trajectories. A comparison of (13) with (11) yields the relation

$$d\,' = d + x_1\,\varphi + \frac{x_0 + x_1 - R}{R}\,x_1\,\varphi + \mathcal{O}\left(m^2\right)\,, \tag{14}$$

where φ is given by Eq. (11) and a term $\dfrac{x_0 + x_1 - R}{R}\,x_1\,\varphi$ has been separated, which can be shown to contribute to the light-deflection φ only to order $\mathcal{O}\left(\dfrac{m^2}{d\,'^2}\right)$, see Appendix A. By inserting (14) into (11) an quadratic equation is obtained which has the solution

$$\varphi_{1,2} = \frac{1}{2}\left(\sqrt{\frac{d^2}{x_1^2} + 4\,(1+\gamma)\,\frac{m}{x_1}\,\frac{x_0\,x_1 - \boldsymbol{x}_0 \cdot \boldsymbol{x}_1}{R\,x_1}} \mp \frac{d}{x_1}\right) + \mathcal{O}\left(\frac{m^2}{d\,'^2}\right). \tag{15}$$

The solution with the upper (lower) sign is denoted by φ_1 (φ_2). For astrometry the solution φ_1 can be considered to be the more relevant solution, because φ_2 represents the second image of one and the same source. Eq. (15) represents the generalized lens equation, which is valid in all those extreme astrometrical configurations defined in 1. - 3. in the introductionary Section. One can show, by means of the inequalities in [16], that the terms $\mathcal{O}\left(\dfrac{m^2}{d\,'^2}\right)$ are smaller or equal than $\dfrac{15\,\pi}{4}\,\dfrac{m^2}{d\,'^2}$. The generalized lens equation (15) is applicable for extreme astrometric configurations. Especially, it allows an analytical investigation of light-deflection in binary systems [21]. In the following Section it will be shown that the formula (15) represents a link between standard post-Newtonian approach and classical lens equation.

V. DISCUSSION OF GENERALIZED LENS EQUATION

A. Comparison with standard post-Newtonian and post-post-Newtonian approach

In this Section we compare the generalized lens equation (15) with the standard post-Newtonian and post-post-Newtonian approach of light-deflection. A series expansion of the solution φ_1 in Eq. (15) for $d \gg m$ yields

$$\varphi_1 = \varphi_{\rm pN} + \varphi_{\rm ppN} + \mathcal{O}\left(m^3\right) + \mathcal{O}\left(\frac{m^2}{d^{\,\prime 2}}\right), \tag{16}$$

with

$$\varphi_{\rm pN} = (1+\gamma)\,\frac{m}{d}\,\frac{x_0\,x_1 - \boldsymbol{x}_0\cdot\boldsymbol{x}_1}{R\,x_1} \le 4\,\frac{m}{d}\,, \tag{17}$$

$$\varphi_{\rm ppN} = -(1+\gamma)^2\,\frac{m^2}{d^2}\,\frac{\left(x_0\,x_1 - \boldsymbol{x}_0\cdot\boldsymbol{x}_1\right)^2}{R^2\,d\,x_1} \le 16\,\frac{m^2}{d^2}\,\frac{x_1}{d}\,. \tag{18}$$

Expression (17) is called *Standard post-Newtonian solution*, cf. Eq. (24) in [4]. The expression (18) is just the 'enhanced' post-post-Newtonian term, cf. Eqs. (3) and (4) in [17]. It should be noticed that the difference between Eqs. (3) and (4) in [17] and Eqs. (92) and (93) in [4] is just of order $\mathcal{O}\left(\frac{m^2}{d^{\,\prime 2}}\right)$, as it has been pointed out in [17].

The 'enhanced' term (18) can be arbitrarily large for small d and large x_1. That is the reason why the standard post-Newtonian and post-post-Newtonian solution is not applicable for extreme configurations like binary stars. It is essential to realize that the terms $\mathcal{O}\left(m^3\right)$ can be larger than the neglected terms $\mathcal{O}\left(\frac{m^2}{d^{\,\prime 2}}\right)$, as we will see below.

B. Comparison of generalized lens equation and classical lens equation

The standard weak-field lens equation is usually called *classical lens equation* and given, for instance, in Eq. (67) in [11] or Eq. (24) in [12]. Let us briefly reconsider the classical lens equation.

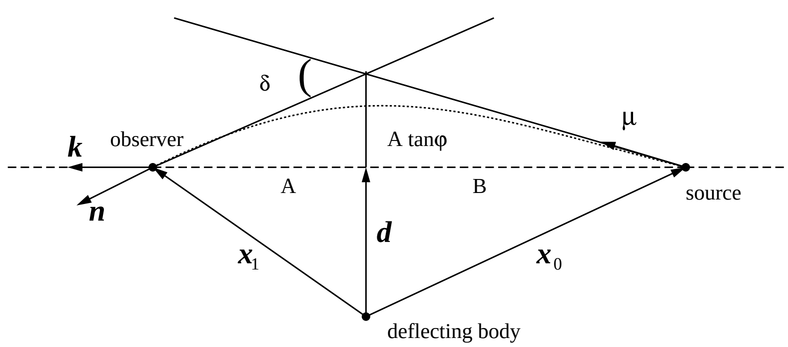

FIG. 2: A geometrical representation for the classical lens equation (23).

According to the scheme in FIG. 2, we obtain the following geometrical relations

$$\varphi + \psi = \delta\,, \tag{19}$$

$$A\,\tan\varphi = B\,\tan\psi\,, \tag{20}$$

where $\psi = \delta(\boldsymbol{\mu}, \boldsymbol{k})$ and $\boldsymbol{\mu} = \dfrac{\dot{\boldsymbol{x}}(t_0)}{|\dot{\boldsymbol{x}}(t_0)|}$. If source and observer are infinitely far from the massive body, then the total light-deflection angle $\delta = \delta\,(\boldsymbol{n}, \boldsymbol{\mu})$ in Schwarzschild metric reads [18]

$$\delta = 2\,(1+\gamma)\,\frac{m}{d^{\,\prime}} + \mathcal{O}\left(\frac{m^2}{d^{\,\prime 2}}\right), \tag{21}$$

which is a coordinate independent result. The terms of order $\mathcal{O}\left(\dfrac{m^2}{d^{\,\prime 2}}\right)$ can be estimated to be smaller or equal than $\dfrac{15\,\pi}{4}\,\dfrac{m^2}{d^{\,\prime 2}}$, see [18]. In the classical lens approach, the approximation $d^{\,\prime} \simeq d + A\,\tan\varphi$ is used, see FIG. 2. Inserting this relation into (21), by means of geometrical relations (19) and (20), and using $\tan\varphi = \varphi + \mathcal{O}(\varphi^3)$ and $\tan\psi = \psi + \mathcal{O}(\varphi^3)$, we obtain the quadratic equation

$$\varphi^2 + \frac{d}{A}\,\varphi - 2\,(1+\gamma)\,\frac{m}{A}\,\frac{B}{A+B} = 0\,. \tag{22}$$

The solution of Eq. (22) is the classical lens equation:

$$\varphi_{1,2}^{\rm class} = \frac{1}{2}\left(\sqrt{\frac{d^2}{A^2} + 8(1+\gamma)\,\frac{m}{A}\,\frac{B}{A+B}} \mp \frac{d}{A}\right), \tag{23}$$

which is valid in case of $A, B \gg d$; the solution with the upper (lower) sign is denoted by $\varphi_1^{\rm class}$ ($\varphi_2^{\rm class}$). In the limit of large distances of source and observer Eq. (23) coincides with Eq. (67) of Ref. [11].

It is important to notice, that in (23) not only the light-deflection angle φ is assumed to be small, but also that source and observer are far from the massive body, i.e. $\delta(\boldsymbol{x}_0, \boldsymbol{x}_1) \simeq \pi$. Therefore, the classical lens equation is not applicable for extreme configurations like binary systems or light-deflection of moons of giant planets in the solar system.

It can easily be shown that the classical lens equation (23) follows straightforward from the generalized lens equation (15). That means, if we rewrite (15) in terms of $A = \boldsymbol{k} \cdot \boldsymbol{x}_1$ and $B = -\boldsymbol{k} \cdot \boldsymbol{x}_0$ and perform a corresponding series expansion of generalized lens equation (15), then we just obtain the classical lens equation (23) as the leading term in this series.

Furthermore, in the limit $d \rightarrow 0$, known as Einstein ring solution, the generalized lens equation (15) and the classical lens equation (23) yield the same result:

$$\lim_{d \rightarrow 0} \varphi_{1,2} = \sqrt{2\,(1+\gamma)\,\frac{m}{x_1}\,\frac{x_0}{x_0+x_1}} = \lim_{d \rightarrow 0}\,\varphi_{1,2}^{\rm class}\,. \tag{24}$$

Finally, we note that in the extreme configuration $B = 0$ (in this limit φ_2 does not exist) we obtain from (15) the result

$$\lim_{B \rightarrow 0} \varphi_1 = \frac{1}{2}\left(\sqrt{\frac{d^2}{x_1^2} + 4\,(1+\gamma)\,\frac{m}{x_1}\,\frac{d\,A}{(x_1+d)\;x_1}} - \frac{d}{x_1}\right) \le \sqrt{(1+\gamma)\,\frac{m}{x_1}}\,, \tag{25}$$

while the classical lens equation yields simply $\varphi_1^{\rm class} = 0$. Obviously, in the limit $A \rightarrow 0$ the second expression in (25) yields zero as it has to be because in this limit the distance between source and observer vanishes, that means no light deflection.

C. Comparison with exact solution

The accuracy of (15) and the stated estimate that the neglected terms are smaller or equal than $\dfrac{15\,\pi}{4}\,\dfrac{m^2}{d\,'^2}$ has also been confirmed by a comparison with the exact numerical solution of (2).

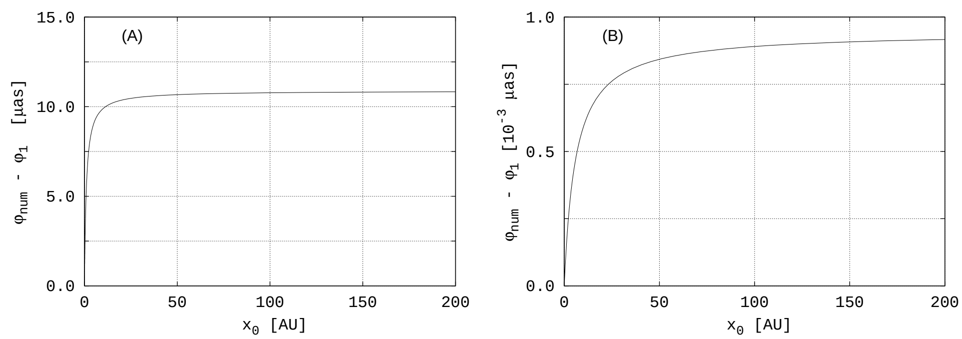

FIG. 3: Comparison of solution φ_1 of generalized lens equation (15) with exact numerical solution $\varphi_{\rm num}$ for the case of a grazing ray at Sun (A) $(\boldsymbol{x}_1 = (-1\,{\rm a.u.}, 0, 0), m_{\odot} = 1476.6\,{\rm m},\ d\,' = 696.0 \times 10^6\,{\rm m})$ and Jupiter (B) $(\boldsymbol{x}_1 = (-6.0\,{\rm a.u.}, 0, 0), m_{♃} = 1.40987\,{\rm m},\ d\,' = 71.492 \times 10^6\,{\rm m})$, where a.u. $= 1.496 \times 10^{11}$ m denotes astronomical unit.

For that, we have solved the geodesic equation (2) in Schwarzschild metric by numerical integrator ODEX [19] for several extreme astrometrical configurations. Using forth and back integration a numerical accuracy of at least 10^{-24} in the components of position and velocity of the photon is guaranteed. Thus, the numerical integration can be considered as an exact solution of geodesic equation, which we denote by $\varphi_{\rm num}$. This numerical approach has been described in some detail in [4]. In all considered extreme configurations the validity of (15) and the given estimate of neglected terms have been confirmed. As example, in FIG. 3 we present the results for light-deflection of a grazing ray at Sun and Jupiter. Especially, the accuracy of generalized lens equation (15) for a grazing ray at Jupiter in FIG. 3 (B) is much beyond what is needed for GAIA accuracy. Moreover, the accuracy shown in FIG. 3 (B) is considerably better than the post-post-Newtonian solution investigated in detail in [4, 15], cf. FIG. 3 (B) with FIG. 2 in [15]. In order to understand the numerical difference between the here shown FIG. 3 (B) and FIG. 2 in [15], we perform a further series expansion of Eq. (15) up to terms of order m^4, that means

$$\varphi_1 = \varphi_{\rm pN} + \varphi_{\rm ppN} + \varphi_{\rm pppN} + \mathcal{O}\left(m^4\right) + \mathcal{O}\left(\frac{m^2}{d\,'^2}\right), \tag{26}$$

where the 'enhanced' terms beyond post-post-Newtonian terms are:

$$\varphi_{\rm pppN} = 2\,\left(1+\gamma\right)^3\,\frac{m^3}{d^3}\,\frac{\left(x_0\,x_1 - \boldsymbol{x}_0\cdot\boldsymbol{x}_1\right)^3}{R^3\,d^2\,x_1} \le 128\,\frac{m^3}{d^3}\,\frac{x_1^2}{d^2}\,. \tag{27}$$

The given estimation in (27) shows, that for large x_1 this term can be considerably larger than the neglected terms of order $\mathcal{O}\left(\dfrac{m^2}{d\,'^2}\right)$. Moreover, it can be shown that the numerical difference between the here shown FIG. 3 (B) and FIG. 2 in [15] is just given by this term, while the terms of higher order $\mathcal{O}\left(m^4\right)$ are negligible in this case.

VI. SUMMARY

The astrometric mission GAIA needs highly precise approximative solutions for light-deflection on microarcsecond level of accuracy. In our investigation we have suggested a generalized lens equation (15) for Schwarzschild metric which is valid for weak gravitational fields, i.e. for areas where $\frac{m}{d\,'} \ll 1$, and which is valid for finite distances of source and observer from the light deflecting body. The results of this report were recently published in our article [20].

The derivation is based on the solution of geodesic equation (11) in post-Newtonian metric and Chandrasekhar's coordinate independent impact parameter $d\,'$ (6) and it's relation to the light-deflection angle φ given in (14). The neglected terms in (15) can be estimated to be smaller or equal than $\dfrac{15\,\pi}{4}\,\dfrac{m^2}{d\,'^2}$. The accuracy of generalized lens equation (15) is considerably better than the standard post-Newtonian and post-post-Newtonian approach, which has been investigated in some detail in [4, 15] and the reason for this fact has been pointed out. Furthermore, the distance of source and observer from light-deflecting body is finite and can be chosen arbitrarily.

The generalized lens equation satisfies three conditions formulated in the introductionary Section. Moreover, we have shown that in the appropriate limits we obtain the post-Newtonian terms, 'enhanced' post-post-Newtonian terms and the classical lens equation. Thus, the generalized lens equation (15) provides also a link between these essential approaches to determine the light-deflection. Numerical investigations have confirmed the analytical results obtained.

The generalized lens equation (15) allows an analytical understanding and investigation of light-deflection in extreme astrometric configurations. Especially, the determination of light-deflection in binary systems using of generalized lens equation (15) have been investigated in a further report [21].

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APPENDIX A: ESTIMATE OF THE THIRD TERM IN EQ. (14)

Let us consider the third term in Eq. (14), given by

$$\frac{x_0 + x_1 - R}{R}\, x_1\, \varphi \leq (1+\gamma)\; m\, f_1\,, \tag{A1}$$

where the expression for the light-deflection angle φ in Eq. (11) and the inequality $d\,' \geq d$ has been used, and the function is given by

$$f_1 = \frac{\left(1 + z - \sqrt{1 + z^2 - 2\,z\,\cos\delta(\boldsymbol{x}_0\,,\,\boldsymbol{x}_1)}\right)\left(1 - \cos\delta(\boldsymbol{x}_0\,,\,\boldsymbol{x}_1)\right)}{\sin\delta(\boldsymbol{x}_0\,,\,\boldsymbol{x}_1)} \leq 1\,. \tag{A2}$$

Using the inequalities (A1) and (A2), we conclude the inequality $\dfrac{x_0 + x_1 - R}{R}\, x_1\, \varphi \leq 2\,m$, and obtain

$$d\,' = d + x_1\,\varphi + \mathcal{O}\,(m)\,, \tag{A3}$$

where we have omitted the term of order $\mathcal{O}\,(m^2)$. By inserting this expression into (11) we obtain an quadratic equation for the light-deflection angle φ:

$$\varphi^2 + \frac{d}{x_1}\,\varphi - (1+\gamma)\,\frac{m}{x_1}\,\frac{x_0\,x_1 - \boldsymbol{x}_0\cdot\boldsymbol{x}_1}{R\,x_1} = \epsilon\,, \tag{A4}$$

where $\epsilon_1 = \mathcal{O}\left(\dfrac{m}{x_1}\,\varphi\right)$. The both solutions of (A4) are

$$\varphi_{1,2} = \frac{1}{2}\left(\sqrt{\frac{d^2}{x_1^2} + 4\,(1+\gamma)\,\frac{m}{x_1}\,\frac{x_0\,x_1 - \boldsymbol{x}_0\cdot\boldsymbol{x}_1}{R\,x_1} + \epsilon_1} \mp \frac{d}{x_1}\right). \tag{A5}$$

Since ϵ is much smaller in comparison with the other terms in the square root, we can perform a series expansion of (A5) in terms of ϵ_1 and obtain

$$\varphi_{1,2} = \frac{1}{2}\left(\sqrt{\frac{d^2}{x_1^2} + 4\,(1+\gamma)\,\frac{m}{x_1}\,\frac{x_0\,x_1 - \boldsymbol{x}_0\cdot\boldsymbol{x}_1}{R\,x_1}} \mp \frac{d}{x_1}\right) + \epsilon_2\,. \tag{A6}$$

The term ϵ_2 can be estimated to be of the order

$$\epsilon_2 = \mathcal{O}\left(\frac{m^2}{d\,'^2}\right), \tag{A7}$$

where we have used expression (11), and an expression for $d\,'$ which follows from the quadratic equation (13).