Further simplification of the light deflection formula for solar system objects

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The transformation $\boldsymbol{n}$ to $\boldsymbol{k}$ in post-post-Newtonian order is simplified. All post-post-Newtonian terms of the order $\mathcal{O}\left(\frac{m^2}{d^2}\right)$ are neglected and we show that the total sum of these terms is smaller than $\frac{15}{4}\,\pi\,\frac{m^2}{d^2}$. This simpler transformation will improve the efficiency of Gaia data reduction.

I. INTRODUCTION

The approximative analytical solution of the problem of light deflection has been presented in [1–3]. One of the main result of these investigations is the transformation $\boldsymbol{n}$ to $\boldsymbol{k}$ for solar system objects in post-post-Newtonian approximation. A detailed analysis [3, 4] has shown that most of the terms in this transformation can be neglected at the microarcsecond level of accuracy, leading a simplified formula $\boldsymbol{n}$ to $\boldsymbol{k}$ for the data reduction. This simplified formula $\boldsymbol{n}$ to $\boldsymbol{k}$ has been given in Eqs. (92) and (93) in [1] and in Eqs. (52) and (53) in [3]. In this report we will show that this transformation can be further simplified. The report is organized as follows. In Section II we will present the transformation $\boldsymbol{n}$ to $\boldsymbol{k}$ in post-post-Newtonian order. The estimate of post-post-Newtonian terms and the new simplified transformation $\boldsymbol{n}$ to $\boldsymbol{k}$ in given in Section III. A new estimation will be given in Section IV. A summary is given in Section V. Detailed proofs of the estimates used are given in the appendices.

II. TRANSFORMATION $\boldsymbol{n}$ TO $\boldsymbol{k}$ IN POST-POST-NEWTONIAN ORDER

The transformation $\boldsymbol{n}$ to $\boldsymbol{k}$ in post-post-Newtonian order has been given in Eq. (87) in [1], Eq. (57) in [2], and in Eq. (45) in [3]. We will present this transformation in the following equivalent form:

$$
\begin{aligned}
\text{N} \;\Big| \quad & \boldsymbol{n} = \boldsymbol{k} \\
\text{pN} \;\Big| \quad & \quad - (1+\gamma)\, m \, \frac{\boldsymbol{k} \times (\boldsymbol{x}_0 \times \boldsymbol{x}_1)}{x_1 \, (x_1 \, x_0 + \boldsymbol{x}_1 \cdot \boldsymbol{x}_0)} \\
\Delta\text{pN} \;\Big| \quad & \quad + (1+\gamma)^2 \, m^2 \, \frac{\boldsymbol{k} \times (\boldsymbol{x}_0 \times \boldsymbol{x}_1)}{(x_1 \, x_0 + \boldsymbol{x}_1 \cdot \boldsymbol{x}_0)^2} \, \frac{R}{x_1} \\
\text{scaling} \;\Big| \quad & \quad - \frac{1}{8} \, (1+\gamma)^2 \, \frac{m^2}{x_1^2} \, \boldsymbol{k} \, \frac{\left((x_1 - x_0)^2 - R^2\right)^2}{|\boldsymbol{x}_1 \times \boldsymbol{x}_0|^2} \\
\text{ppN} \;\Big| \quad & \quad + m^2 \, \boldsymbol{k} \times (\boldsymbol{x}_0 \times \boldsymbol{x}_1) \left[\frac{1}{2} \, (1+\gamma)^2 \, \frac{R^2 - (x_1 - x_0)^2}{x_1^2 \, |\boldsymbol{x}_1 \times \boldsymbol{x}_0|^2} \right. \\
\text{ppN} \;\Big| \quad & \quad + \frac{1}{4} \, \alpha \, \epsilon \, \frac{1}{R} \left(\frac{1}{R \, x_0^2} - \frac{1}{R \, x_1^2} - 2 \, \frac{\boldsymbol{k} \cdot \boldsymbol{x}_1}{x_1^4} \right) \\
\text{ppN} \;\Big| \quad & \quad - \frac{1}{4} \, \left(8(1+\gamma-\alpha\gamma)(1+\gamma) - 4\alpha\beta + 3\,\alpha\,\epsilon \right) \, R \, \frac{\boldsymbol{k} \cdot \boldsymbol{x}_1}{x_1^2 \, |\, \boldsymbol{x}_1 \times \boldsymbol{x}_0 \,|^2} \\
\text{ppN} \;\Big| \quad & \quad \left. + \frac{1}{8} \, (8(1+\gamma-\alpha\,\gamma)(1+\gamma) - 4\,\alpha\,\beta + 3\alpha\,\epsilon) \, \frac{x_1^2 - x_0^2 - R^2}{|\boldsymbol{x}_1 \times \boldsymbol{x}_0|^3} \, \delta(\boldsymbol{x}_1, \boldsymbol{x}_0) \right] \\
\text{ppN} \;\Big| \quad & \quad + (1+\gamma)^2 \, m^2 \, \frac{\boldsymbol{k} \times (\boldsymbol{x}_0 \times \boldsymbol{x}_1)}{(x_1 \, x_0 + \boldsymbol{x}_1 \cdot \boldsymbol{x}_0)^2} \, \frac{x_1 + x_0 - R}{x_1} \\
& \quad + \mathcal{O}(c^{-6}) \, .
\end{aligned} \tag{1}
$$

Here we have classified the nature of the individual terms by labels N (Newtonian), pN (post-Newtonian), ppN (post-post-Newtonian) and ΔpN (terms that are formally of post-

post-Newtonian order, but may numerically become significantly larger than other post-post-Newtonian terms, see estimates in (6)).

III. SIMPLIFIED TRANSFORMATION $\boldsymbol{n}$ TO $\boldsymbol{k}$

The effect of all the "ppN" terms in (1) can be estimated as (cf. Eq. (91) in [1] or Eq. (50) in [3])

$$\left|\boldsymbol{\omega}^{\prime}_{\rm ppN}\right| \le \frac{15}{4}\,\pi\,\frac{m^2}{d^2}\,. \tag{2}$$

The proof of (2) is given in Appendix A. These terms can attain 1 μas only for observations within about 3.3 angular radii from the Sun and can be neglected. Accordingly, we obtain a simplified formula for the transformation from $\boldsymbol{k}$ to $\boldsymbol{n}$ keeping only the post-Newtonian and "enhanced" post-post-Newtonian terms labelled as "pN" and "ΔpN" in (1):

$$\boldsymbol{n} = \boldsymbol{k} + \boldsymbol{d}\,P\,\left(1 + P\,x_1\right) + \mathcal{O}\left(\frac{m^2}{d^2}\right) + \mathcal{O}(m^3)\,, \tag{3}$$

$$P = -(1+\gamma)\,\frac{m}{d^2}\,\left(\frac{x_0 - x_1}{R} + \frac{\boldsymbol{k}\cdot\boldsymbol{x}_1}{x_1}\right)\,. \tag{4}$$

The simplified transformation $\boldsymbol{n}$ to $\boldsymbol{k}$ given in Eq. (3) has now simpler structure than the former expression given in Eq. (92) in [1] or in Eq. (52) in [3]. Therefore, (3) is more efficient for the data reduction. Furthermore, the transformation in Eq. (3) has now similar structure as the simplified transformation $\boldsymbol{n}$ to $\boldsymbol{\sigma}$ given in Eq. (102) in [1] or in Eq. (62) in [3].

IV. A NEW ESTIMATION

The enhanced post-post-Newtonian term $\left|\boldsymbol{\omega}^{\prime}_{\Delta pN}\right|$ in Eq. (1) is, for $\gamma = 1$, given by (cf. Eq. (89) in [1] or Eq. (48) in [3])

$$\left|\boldsymbol{\omega}^{\prime}_{\Delta pN}\right| = 4\,m^2\,\frac{\left|\,\boldsymbol{k}\times(\boldsymbol{x}_0\times\boldsymbol{x}_1)\,\right|}{\left(x_1\,x_0 + \boldsymbol{x}_1\cdot\boldsymbol{x}_0\right)^2}\,\frac{R}{x_1}\,. \tag{5}$$

This term differs from the corresponding term $|\,\boldsymbol{\omega}_{\Delta pN}\,|$ defined in Eq. (89) in [1] or Eq. (48) in [3] only by a factor $\dfrac{R}{x_0 + x_1} \le 1$. Therefore, we conclude that the estimates given in Eqs. (89) and (90) of [1] or in Eqs. (48) and (49) of [3] are also valid for $\left|\boldsymbol{\omega}^{\prime}_{\Delta pN}\right|$, that means:

$$\left|\boldsymbol{\omega}^{\prime}_{\Delta pN}\right| \le 16\,\frac{m^2}{d^3}\,\frac{R^2\,x_1\,x_0^2}{\left(x_1 + x_0\right)^4} \le 16\,\frac{m^2}{d^3}\,\frac{R\,x_1\,x_0^2}{\left(x_1 + x_0\right)^3} \le 16\,\frac{m^2}{d^3}\,\frac{x_1\,x_0^2}{\left(x_1 + x_0\right)^2} \le 16\,\frac{m^2}{d^2}\,\frac{x_1}{d}\,, \tag{6}$$

where the first expression given in (6) represents a new estimation. Another estimation can be given, namely (cf. Eq. (90) in [1] or Eq. (49) in [3])

$$
|\,\boldsymbol{\omega}'_{\Delta\mathrm{pN}}\,| \leq \frac{64}{27}\,\frac{m^2}{d^2}\,\frac{R}{d}\,, \tag{7}
$$

which cannot be related to the estimations in (6) and reflect different properties of $\left|\boldsymbol{\omega}'_{\Delta\mathrm{pN}}\right|$ as function of multiple variables.

V. SUMMARY

In Eq. (57) in [2] the complete transformation $\boldsymbol{n}$ to $\boldsymbol{k}$ in post-post-Newtonian order has been given. In [3] we have shown that most of the terms can be neglected because they are of the order $\mathcal{O}\left(\frac{m^2}{d^2}\right)$ and can attain $1\,\mu$as only for observations within about 3.3 angular radii from the Sun. These investigations have yielded a simplified transformation, given in Eqs. (92) and (93) in [1] or in Eqs. (52) and (53) in [3], and applicable for an efficient data reduction. In this report we have shown that Eq. (92) in [1] or Eq. (52) in [3] can further be simplified. The main result of this report is Eq. (3), where we give a new simplified transformation $\boldsymbol{n}$ to $\boldsymbol{k}$ which will improve the efficiency of Gaia data reduction. We have shown that the total sum of the neglected ppN-terms is smaller than $\frac{15}{4}\,\pi\,\frac{m^2}{d^2}$. Furthermore, estimations of the enhanced post-post-Newtonian term has been given in Eqs. (6) and (7).

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[4] Sven Zschocke, Sergei A. Klioner, GAIA-CA-TN-LO-SZ-003-1, *Formal proof of some inequalities used in the analysis of the post-post-Newtonian light propagation theory*, available on arXiv:astro-ph/0907.4281.

APPENDIX A: PROOF OF INEQUALITY (2)

The sum of all ppN-terms in Eq. (1) can be written as follows (here $\alpha = \beta = \gamma = \epsilon = 1$):

$$\left|\boldsymbol{\omega}'_{\rm ppN}\right| = \frac{1}{4}\,\frac{m^2}{d^2}\,f'_{10}\,, \tag{A1}$$

where the function is defined by (cf. with f_{10} defined in Eq. (84) in [4])

$$f'_{10} = \Bigg|\,\frac{z\,(16z - z\,\cos\Phi - 15)\,\sin\Phi}{1 + z^2 - 2z\,\cos\Phi} + \frac{z(1 - 3z^2 + 2z^3\cos\Phi)\,\sin^3\Phi}{\left(1 + z^2 - 2z\,\cos\Phi\right)^2}$$
$$+ \frac{15z\,(\cos\Phi - z)\,\Phi}{1 + z^2 - 2z\,\cos\Phi} + 16\,\frac{z\,\left(1 - \cos\Phi\right)^2\,\left(1 + z - \sqrt{1 + z^2 - 2\,z\,\cos\Phi}\right)}{\left(1 + z^2 - 2\,z\,\cos\Phi\right)\,\sin\Phi}\,\Bigg|\,. \tag{A2}$$

Here we have used the notation $\Phi = \delta\left(\boldsymbol{x}_0, \boldsymbol{x}_1\right)$ and $z = \frac{x_0}{x_1}$. By means of the inequalities (note that (A4) improves the inequality given in Eq. (C1) in [4])

$$f_2 = 16\,\frac{z\,\left(1 - \cos\Phi\right)^2\,\left(1 + z - \sqrt{1 + z^2 - 2\,z\,\cos\Phi}\right)}{\left(1 + z^2 - 2\,z\,\cos\Phi\right)\,\sin\Phi} \le 8\,\sin\Phi\,, \tag{A3}$$

$$f_3 = \frac{|\,z\,\left(1 - 3z^2 + 2z^3\cos\Phi\right)|\,\sin^3\Phi}{\left(1 + z^2 - 2z\,\cos\Phi\right)^2} \le 3\,\sin\Phi\,, \tag{A4}$$

(proof of (A3) and (A4) are shown in Appendices B and C, respectively) we obtain

$$f'_{10} \le \left|\,\frac{z\,(16z - z\,\cos\Phi - 15)\,\sin\Phi}{1 + z^2 - 2z\,\cos\Phi} + \frac{15z\,(\cos\Phi - z)\,\Phi}{1 + z^2 - 2z\,\cos\Phi}\,\right| + 11\,\sin\Phi\,. \tag{A5}$$

In [4] we have shown $z\,(16z - z\,\cos\Phi - 15)\,\sin\Phi + 15z\,(\cos\Phi - z)\,\Phi \le 0$. Accordingly, due to $\sin\Phi \ge 0$, we obtain

$$f'_{10} \le \left|\,\frac{z\,(16z - z\,\cos\Phi - 15)\,\sin\Phi}{1 + z^2 - 2z\,\cos\Phi} + \frac{15z\,(\cos\Phi - z)\,\Phi}{1 + z^2 - 2z\,\cos\Phi} - 15\,\sin\Phi\,\right|\,, \tag{A6}$$

where, for convenience, we have replaced the term $11\,\sin\Phi$ by the larger term $15\,\sin\Phi$. Furthermore, in [4] we have shown that

$$\left|\,\frac{z\,(16z - z\,\cos\Phi - 15)\,\sin\Phi}{1 + z^2 - 2z\,\cos\Phi} + \frac{15z\,(\cos\Phi - z)\,\Phi}{1 + z^2 - 2z\,\cos\Phi} - 15\,\sin\Phi\,\right| \le 15\,\pi\,. \tag{A7}$$

Thus, we obtain

$$f'_{10} \le 15\,\pi\,. \tag{A8}$$

The inequality (A8) in combination with (A1) shows the validity of inequality (2).

APPENDIX B: PROOF OF INEQUALITIES (A3)

In order to show (A3), we rewrite this inequality as follows:

$$\frac{z\,(1-\cos\Phi)}{1+z^2-2\,z\,\cos\Phi}\;\frac{1+z-\sqrt{1+z^2-2\,z\,\cos\Phi}}{1+\cos\Phi}\le\frac{1}{2}\,. \tag{B1}$$

The inequality (B1) can be splitted into two factors satisfying the following inequalities:

$$\frac{z\,(1-\cos\Phi)}{1+z^2-2\,z\,\cos\Phi}\le\frac{1}{2}\,, \tag{B2}$$

$$\frac{1+z-\sqrt{1+z^2-2\,z\,\cos\Phi}}{1+\cos\Phi}\le 1\,. \tag{B3}$$

The inequality (B2) is obviously valid, because by multiplying (B2) with the denominator we obtain $-\left(1-z\right)^2\le 0$. The inequality (B3) is also straightforward, because it can be rewritten as $z-\cos\Phi\le\sqrt{1+z^2-2\,z\,\cos\Phi}$, which is obviously valid due to $z-\cos\Phi\le|\,z-\cos\Phi\,|$. Thus we have shown the validity of inequality (B1) and (A3), respectively.

APPENDIX C: PROOF OF INEQUALITY (A4)

Using the notation $w = \cos\Phi$, the inequality (A4) can be written as follows:

$$f_3 = \frac{z\,|\,1 - 3\,z^2 + 2\,z^3\,w\,|\,(1 - w^2)}{(1 + z^2 - 2\,w\,z)^2} \leq 3\,. \tag{C1}$$

Using the inequality (proof see below)

$$\Big|\,1 - 3\,z^2 + 2\,z^3\,w\,\Big| \leq 1 - 3\,w\,z^2 + 2\,z^3\,, \tag{C2}$$

we obtain

$$f_3 \leq \frac{z\,(1 - 3\,w\,z^2 + 2\,z^3)\,(1 - w^2)}{(1 + z^2 - 2\,w\,z)^2} = h_1 + h_2 \leq 3\,. \tag{C3}$$

In (C3) the relation $1 - 3\,w\,z^2 + 2\,z^3 = (1 + z^2 - 2\,w\,z) + (-3\,w\,z^2 + 2\,z^3 - z^2 + 2\,w\,z)$ has been used. The functions are defined by

$$h_1 = \frac{z\,(1 - w^2)}{1 + z^2 - 2\,w\,z} \leq \frac{2\,z\,(1 - w)}{1 + z^2 - 2\,w\,z} \leq 1\,, \tag{C4}$$

$$h_2 = \frac{z^2\,|-3\,w\,z + 2\,z^2 - z + 2\,w\,|\,(1 - w^2)}{(1 + z^2 - 2\,w\,z)^2} \leq 2\,. \tag{C5}$$

The inequality (C4) has been shown in [4]. In order to show (C5), we factorize the function h_2 as follows:

$$h_2 = h_2^A\, h_2^B\,, \tag{C6}$$

$$h_2^A = \frac{z^2\,(1 - w^2)}{1 + z^2 - 2\,w\,z} \leq 1\,, \tag{C7}$$

$$h_2^B = \frac{|-3\,w\,z + 2\,z^2 - z + 2\,w\,|}{1 + z^2 - 2\,w\,z} \leq 2\,. \tag{C8}$$

Thus, by means of the inequalities (C2) and (C4) - (C8) we have shown the validity of inequality (C1) and (A4), respectively. We still have to proof of inequalities (C2), (C7) and (C8).

Let us consider (C2). First, we remark that $1 - 3\,w\,z^2 + 2\,z^3 \geq 0$ because of $1 - 3\,z^2 + 2\,z^3 \geq 0$. Then, squaring both sides of (C2) and subtracting from each other leads to

$$h_3 = 2\,z^3 + 2\,w\,z^3 - 3\,z^2 - 3\,w\,z^2 + 2 \geq 0\,. \tag{C9}$$

The boundaries of h_3 are

$$\lim_{w \to -1} h_3 = 2 \geq 0\,, \tag{C10}$$

$$\lim_{w \to +1} h_3 = 2\ (2\,z + 1)\ (z - 1)^2 \geq 0\,, \tag{C11}$$

$$\lim_{z \to 0} h_3 = 2 \geq 0\,, \tag{C12}$$

$$\lim_{z \to \infty} h_3 = 2\ (1 + w)\ \lim_{z \to \infty} z^3 \geq 0\,. \tag{C13}$$

The extremal conditions $h_{3\,,\,w} = 0$ and $h_{3\,,\,z} = 0$ lead to

$$z^2\ (2\,z - 3) = 0\,, \tag{C14}$$

$$z\ (1 + w)\ (z - 1) = 0\,. \tag{C15}$$

The common solutions of (C14) and (C15) are given by

$$P_1\,(w = -1\,,\ z = 0)\ , \tag{C16}$$

$$P_2\left(w = -1\,,\ z = \frac{3}{2}\right)\ . \tag{C17}$$

The numerical values of h_3 at these turning points are

$$h_3\,(P_1) = 2 \geq 0\,, \tag{C18}$$

$$h_3\,(P_2) = 2 \geq 0\,. \tag{C19}$$

Thus, we have shown (C9) and, therefore, the validity of inequality (C2).

Now we consider the inequality (C7). Multiplying both sides of this relation with the denominator leads to the inequality

$$h_4 = -z^2\,w^2 - 1 + 2\,w\,z \leq 0\,. \tag{C20}$$

The boundaries of h_4 are

$$\lim_{w \to -1} h_4 = -\,(1 + z)^2 \leq 0\,, \tag{C21}$$

$$\lim_{w \to +1} h_4 = -\,(1 - z)^2 \leq 0\,, \tag{C22}$$

$$\lim_{z \to 0} h_4 = -1 \leq 0\,, \tag{C23}$$

$$\lim_{z \to \infty} h_4 = -w^2\ \lim_{z \to \infty} z^2 \leq 0\,. \tag{C24}$$

The extremal conditions $h_{4\,,\,w} = 0$ and $h_{4\,,\,z} = 0$ lead to

$$z\,(1 - w\,z) = 0\,, \tag{C25}$$

$$w\,(1 - w\,z) = 0\,. \tag{C26}$$

The common solution of (C25) and (C26) is given by

$$P_3\,(w = 0\,,\,z = 0)\,, \tag{C27}$$

and the numerical value of h_4 at this turning point is

$$h_4\,(P_3) = -1 \leq 0\,. \tag{C28}$$

Thus, we have shown (C20) and, therefore, the inequality (C7).

Now we consider the inequality (C8). Squaring both sides of (C8) and subtracting from each other leads to the inequality

$$h_5 = 4\,z^2 - z - 7\,w\,z + 2 + 2\,w \geq 0\,. \tag{C29}$$

The boundaries of h_5 are

$$\lim_{w\to -1} h_5 = 2\,z\,(3 + 2\,z) \geq 0\,, \tag{C30}$$

$$\lim_{w\to +1} h_5 = 4\,(z - 1)^2 \geq 0\,, \tag{C31}$$

$$\lim_{z\to 0} h_5 = 2\,(1 + w) \geq 0\,, \tag{C32}$$

$$\lim_{z\to \infty} h_5 = 4\,\lim_{z\to \infty} z^2 \geq 0\,. \tag{C33}$$

The extremal conditions $h_{5\;,w} = 0$ and $h_{5\;z} = 0$ lead to

$$-7\,z + 2 = 0\,, \tag{C34}$$

$$8\,z - 1 - 7\,w = 0\,. \tag{C35}$$

The common solution of (C34) and (C35) is given by

$$P_4\left(w = \frac{9}{49}\,,\,z = \frac{2}{7}\right)\,, \tag{C36}$$

and the numerical value of h_5 at this turning point is

$$h_5\,(P_4) = \frac{100}{49} \geq 0\,. \tag{C37}$$

Thus, we have shown (C29) and, therefore, the inequality (C8).