Further simplification of the light deflection formula for solar system objects

Sven Zschocke, Sergei A. Klioner

Lohrmann Observatory, Dresden Technical University,
Mommsenstr. 13, 01062 Dresden, Germany

GAIA-CA-TN-LO-SZ-005-1

July 29, 2010

The transformation $n$ to $k$ in post-post-Newtonian order is simplified. All post-post-Newtonian terms of the order $O\left(\frac{m^2}{d^2}\right)$ are neglected and we show that the total sum of these terms is smaller than $\frac{15}{4} \pi \frac{m^2}{d^2}$. This simpler transformation will improve the efficiency of Gaia data reduction.
The approximative analytical solution of the problem of light deflection has been presented in [1–3]. One of the main results of these investigations is the transformation \( n \) to \( k \) for solar system objects in post-post-Newtonian approximation. A detailed analysis [3, 4] has shown that most of the terms in this transformation can be neglected at the microarcsecond level of accuracy, leading to a simplified formula \( n \) to \( k \) for the data reduction. This simplified formula \( n \) to \( k \) has been given in Eqs. (92) and (93) in [1] and in Eqs. (52) and (53) in [3]. In this report we will show that this transformation can be further simplified. The report is organized as follows. In Section II we will present the transformation \( n \) to \( k \) in post-post-Newtonian order. The estimate of post-post-Newtonian terms and the new simplified transformation \( n \) to \( k \) is given in Section III. A new estimation will be given in Section IV. A summary is given in Section V. Detailed proofs of the estimates used are given in the appendices.

II. TRANSFORMATION \( n \) TO \( k \) IN POST-POST-NEWTONIAN ORDER

The transformation \( n \) to \( k \) in post-post-Newtonian order has been given in Eq. (87) in [1], Eq. (57) in [2], and in Eq. (45) in [3]. We will present this transformation in the following equivalent form:

\[
\begin{align*}
N & \quad n = k \\
\text{pN} & \quad -(1 + \gamma) m \frac{k \times (x_0 \times x_1)}{x_1 (x_1 x_0 + x_1 \cdot x_0)} \\
\Delta \text{pN} & \quad + (1 + \gamma)^2 m^2 \frac{k \times (x_0 \times x_1)}{(x_1 x_0 + x_1 \cdot x_0)^2} \frac{R}{x_1} \\
s\text{scaling} & \quad - \frac{1}{8} (1 + \gamma)^2 m^2 \frac{k ((x_1 - x_0)^2 - R^2)^2}{|x_1 \times x_0|^2} \\
\text{ppN} & \quad + m^2 k \times (x_0 \times x_1) \left[ \frac{1}{2} (1 + \gamma)^2 \frac{R^2 - (x_1 - x_0)^2}{x_1^2 |x_1 \times x_0|^2} \right] \\
\text{ppN} & \quad + \frac{1}{4} \alpha \epsilon \left( \frac{1}{R x_0^3} - \frac{1}{R x_1^3} - 2 \frac{k \cdot x_1}{x_1^3} \right) \\
\text{ppN} & \quad - \frac{1}{4} \left( 8(1 + \gamma - \alpha \gamma)(1 + \gamma) - 4\alpha \beta + 3 \alpha \epsilon \right) \frac{R}{x_1^3} \frac{k \cdot x_1}{|x_1 \times x_0|^2} \\
\text{ppN} & \quad + \frac{1}{8} \left( 8(1 + \gamma - \alpha \gamma)(1 + \gamma) - 4\alpha \beta + 3 \alpha \epsilon \right) \frac{x_1^2 - x_0^2 - R^2}{|x_1 \times x_0|^3} \delta(x_1, x_0) \\
\text{ppN} & \quad + (1 + \gamma)^2 m^2 \frac{k \times (x_0 \times x_1)}{(x_1 x_0 + x_1 \cdot x_0)^2} \frac{x_1 + x_0 - R}{x_1} \\
& \quad + O(e^{-6}).
\end{align*}
\]

Here we have classified the nature of the individual terms by labels \( N \) (Newtonian), \( \text{pN} \) (post-Newtonian), \( \text{ppN} \) (post-post-Newtonian) and \( \Delta \text{pN} \) (terms that are formally of post-
post-Newtonian order, but may numerically become significantly larger than other post-post-Newtonian terms, see estimates in (6)).

III. SIMPLIFIED TRANSFORMATION $n$ TO $k$

The effect of all the “ppN” terms in (1) can be estimated as (cf. Eq. (91) in [1] or Eq. (50) in [3])

$$\left| \omega_{ppN}' \right| \leq \frac{15}{4} \pi \frac{m^2}{d^2}. $$

(2)

The proof of (2) is given in Appendix A. These terms can attain $1 \mu\text{as}$ only for observations within about 3.3 angular radii from the Sun and can be neglected. Accordingly, we obtain a simplified formula for the transformation from $k$ to $n$ keeping only the post-Newtonian and “enhanced” post-post-Newtonian terms labelled as “pN” and “$\Delta pN$” in (1):

$$n = k + d \, P \left( 1 + P \, x_1 \right) + \mathcal{O} \left( \frac{m^2}{d^2} \right) + \mathcal{O} \left( m^3 \right),$$

(3)

$$P = - \left( 1 + \gamma \right) \frac{m}{d^2} \left( \frac{x_0 - x_1}{R} + \frac{k \cdot x_1}{x_1} \right).$$

(4)

The simplified transformation $n$ to $k$ given in Eq. (3) has now simpler structure than the former expression given in Eq. (92) in [1] or in Eq. (52) in [3]. Therefore, (3) is more efficient for the data reduction. Furthermore, the transformation in Eq. (3) has now similar structure as the simplified transformation $n$ to $\sigma$ given in Eq. (102) in [1] or in Eq. (62) in [3].

IV. A NEW ESTIMATION

The enhanced post-post-Newtonian term $\left| \omega_{\Delta pN}' \right|$ in Eq. (1) is, for $\gamma = 1$, given by (cf. Eq. (89) in [1] or Eq. (48) in [3])

$$\left| \omega_{\Delta pN}' \right| = 4 \, m^2 \frac{\left| k \times (x_0 \times x_1) \right|}{(x_1 \, x_0 + \, x_1 \cdot x_0)^2} \, \frac{R}{x_1}.$$

(5)

This term differs from the corresponding term $\left| \omega_{\Delta pN} \right|$ defined in Eq. (89) in [1] or Eq. (48) in [3] only by a factor $\frac{R}{x_0 + x_1} \leq 1$. Therefore, we conclude that the estimates given in Eqs. (89) and (90) of [1] or in Eqs. (48) and (49) of [3] are also valid for $\left| \omega_{\Delta pN}' \right|$, that means:

$$\left| \omega_{\Delta pN}' \right| \leq 16 \, \frac{m^2}{d^3} \frac{R^2 \, x_1 \, x_0^2}{(x_1 + x_0)^4} \leq 16 \, \frac{m^2}{d^3} \frac{R \, x_1 \, x_0^2}{(x_1 + x_0)^3} \leq 16 \, \frac{m^2}{d^3} \frac{x_1 \, x_0^2}{(x_1 + x_0)^2} \leq 16 \, \frac{m^2}{d^2} \frac{x_1}{d},$$

(6)

where the first expression given in (6) represents a new estimation. Another estimation can be given, namely (cf. Eq. (90) in [1] or Eq. (49) in [3])
\[ |\omega'_{\Delta pN}| \leq \frac{64}{27} \frac{m^2 R}{d^2} \frac{1}{d}, \]

which cannot be related to the estimations in (6) and reflect different properties of \( |\omega'_{\Delta pN}| \) as function of multiple variables.

V. SUMMARY

In Eq. (57) in [2] the complete transformation \( \mathbf{n} \) to \( \mathbf{k} \) in post-post-Newtonian order has been given. In [3] we have shown that most of the terms can be neglected because they are of the order \( \mathcal{O} \left( \frac{m^2}{d^2} \right) \) and can attain 1 \( \mu \)as only for observations within about 3.3 angular radii from the Sun. These investigations have yielded a simplified transformation, given in Eqs. (92) and (93) in [1] or in Eqs. (52) and (53) in [3], and applicable for an efficient data reduction. In this report we have shown that Eq. (92) in [1] or Eq. (52) in [3] can further be simplified. The main result of this report is Eq. (3), where we give a new simplified transformation \( \mathbf{n} \) to \( \mathbf{k} \) which will improve the efficiency of Gaia data reduction. We have shown that the total sum of the neglected ppN-terms is smaller than \( \frac{15}{4} \pi \frac{m^2}{d^2} \). Furthermore, estimations of the enhanced post-post-Newtonian term has been given in Eqs. (6) and (7).

APPENDIX A: PROOF OF INEQUALITY (2)

The sum of all ppN-terms in Eq. (1) can be written as follows (here $\alpha = \beta = \gamma = \epsilon = 1$):

$$\left| \omega_{\text{ppN}}' \right| = \frac{1}{4} \frac{m^2}{d^2} f_{10}' ,$$  \hspace{1cm} (A1)

where the function is defined by (cf. with $f_{10}$ defined in Eq. (84) in [4])

$$f_{10}' = \left| \frac{z (16z - z \cos \Phi - 15) \sin \Phi}{1 + z^2 - 2z \cos \Phi} + \frac{z(1 - 3z^2 + 2z^3 \cos \Phi) \sin^3 \Phi}{(1 + z^2 - 2z \cos \Phi)^2} + \frac{15z (\cos \Phi - z) \Phi}{1 + z^2 - 2z \cos \Phi} + 16 \frac{z (1 - \cos \Phi)^2 \left(1 + z - \sqrt{1 + z^2 - 2z \cos \Phi}\right)}{(1 + z^2 - 2z \cos \Phi) \sin \Phi} \right| .$$  \hspace{1cm} (A2)

Here we have used the notation $\Phi = \delta (x_0, x_1)$ and $z = \frac{x_0}{x_1}$. By means of the inequalities (note that (A4) improves the inequality given in Eq. (C1) in [4])

$$f_2 = 16 \frac{z (1 - \cos \Phi)^2 \left(1 + z - \sqrt{1 + z^2 - 2z \cos \Phi}\right)}{(1 + z^2 - 2z \cos \Phi) \sin \Phi} \leq 8 \sin \Phi ,$$  \hspace{1cm} (A3)

$$f_3 = \left| \frac{z (1 - 3z^2 + 2z^3 \cos \Phi) \sin^3 \Phi}{(1 + z^2 - 2z \cos \Phi)^2} \right| \leq 3 \sin \Phi ,$$  \hspace{1cm} (A4)

(proof of (A3) and (A4) are shown in Appendices B and C, respectively) we obtain

$$f_{10}' \leq \left| \frac{z (16z - z \cos \Phi - 15) \sin \Phi}{1 + z^2 - 2z \cos \Phi} + \frac{15z (\cos \Phi - z) \Phi}{1 + z^2 - 2z \cos \Phi} \right| + 11 \sin \Phi .$$  \hspace{1cm} (A5)

In [4] we have shown $z (16z - z \cos \Phi - 15) \sin \Phi + 15z (\cos \Phi - z) \Phi \leq 0$. Accordingly, due to $\sin \Phi \geq 0$, we obtain

$$f_{10}' \leq \left| \frac{z (16z - z \cos \Phi - 15) \sin \Phi}{1 + z^2 - 2z \cos \Phi} + \frac{15z (\cos \Phi - z) \Phi}{1 + z^2 - 2z \cos \Phi} - 15 \sin \Phi \right| ,$$  \hspace{1cm} (A6)

where, for convenience, we have replaced the term $11 \sin \Phi$ by the larger term $15 \sin \Phi$. Furthermore, in [4] we have shown that

$$\left| \frac{z (16z - z \cos \Phi - 15) \sin \Phi}{1 + z^2 - 2z \cos \Phi} + \frac{15z (\cos \Phi - z) \Phi}{1 + z^2 - 2z \cos \Phi} - 15 \sin \Phi \right| \leq 15 \pi .$$  \hspace{1cm} (A7)

Thus, we obtain

$$f_{10}' \leq 15 \pi .$$  \hspace{1cm} (A8)

The inequality (A8) in combination with (A1) shows the validity of inequality (2).
APPENDIX B: PROOF OF INEQUALITIES (A3)

In order to show (A3), we rewrite this inequality as follows:

\[
\frac{z (1 - \cos \Phi)}{1 + z^2 - 2z \cos \Phi} \leq \frac{1 + z - \sqrt{1 + z^2 - 2z \cos \Phi}}{1 + \cos \Phi} \leq \frac{1}{2}.
\]  

(B1)

The inequality (B1) can be split into two factors satisfying the following inequalities:

\[
\frac{z (1 - \cos \Phi)}{1 + z^2 - 2z \cos \Phi} \leq \frac{1}{2}, \quad \text{(B2)}
\]

\[
\frac{1 + z - \sqrt{1 + z^2 - 2z \cos \Phi}}{1 + \cos \Phi} \leq 1. \quad \text{(B3)}
\]

The inequality (B2) is obviously valid, because by multiplying (B2) with the denominator we obtain \(-(1 - z)^2 \leq 0\). The inequality (B3) is also straightforward, because it can be rewritten as \(z - \cos \Phi \leq \sqrt{1 + z^2 - 2z \cos \Phi}\), which is obviously valid due to \(z - \cos \Phi \leq |z - \cos \Phi|\). Thus we have shown the validity of inequality (B1) and (A3), respectively.
APPENDIX C: PROOF OF INEQUALITY (A4)

Using the notation \( w = \cos \Phi \), the inequality (A4) can be written as follows:

\[
f_3 = \frac{z \left| 1 - 3 z^2 + 2 z^3 w \right| (1 - w^2)}{(1 + z^2 - 2 w z)^2} \leq 3. \tag{C1}
\]

Using the inequality (proof see below)

\[
\left| 1 - 3 z^2 + 2 z^3 \right| \leq 1 - 3 w z^2 + 2 z^3, \tag{C2}
\]

we obtain

\[
f_3 \leq \frac{z \left( 1 - 3 w z^2 + 2 z^3 \right) (1 - w^2)}{(1 + z^2 - 2 w z)^2} = h_1 + h_2 \leq 3. \tag{C3}
\]

In (C3) the relation \( 1 - 3 w z^2 + 2 z^3 = (1 + z^2 - 2 w z) + (3 w z^2 + 2 z^3 - z^2 + 2 w z) \) has been used. The functions are defined by

\[
h_1 = \frac{z \left( 1 - w^2 \right)}{1 + z^2 - 2 w z} \leq \frac{2 z \left( 1 - w \right)}{1 + z^2 - 2 w z} \leq 1, \tag{C4}
\]

\[
h_2 = \frac{z^2 \left| -3 w z + 2 z^2 - z + 2 w \right| (1 - w^2)}{(1 + z^2 - 2 w z)^2} \leq 2. \tag{C5}
\]

The inequality (C4) has been shown in [4]. In order to show (C5), we factorize the function \( h_2 \) as follows:

\[
h_2 = h_2^A h_2^B, \tag{C6}
\]

\[
h_2^A = \frac{z^2 \left( 1 - w^2 \right)}{1 + z^2 - 2 w z} \leq 1, \tag{C7}
\]

\[
h_2^B = \frac{\left| -3 w z + 2 z^2 - z + 2 w \right| (1 - w^2)}{1 + z^2 - 2 w z} \leq 2. \tag{C8}
\]

Thus, by means of the inequalities (C2) and (C4) - (C8) we have shown the validity of inequality (C1) and (A4), respectively. We still have to proof of inequalities (C2), (C7) and (C8).

Let us consider (C2). First, we remark that \( 1 - 3 w z^2 + 2 z^3 \geq 0 \) because of \( 1 - 3 z^2 + 2 z^3 \geq 0 \). Then, squaring both sides of (C2) and subtracting from each other leads to

\[
h_3 = 2 z^3 + 2 w z^3 - 3 z^2 - 3 w z^2 + 2 \geq 0. \tag{C9}
\]

The boundaries of \( h_3 \) are
\[ \lim_{w \to -1} h_3 = 2 \geq 0 \], \quad (C10) \\
\[ \lim_{w \to +1} h_3 = 2 (2z + 1) (z - 1)^2 \geq 0 \], \quad (C11) \\
\[ \lim_{z \to 0} h_3 = 2 \geq 0 \], \quad (C12) \\
\[ \lim_{z \to \infty} h_3 = 2 (1 + w) \lim_{z \to \infty} z^3 \geq 0 \]. \quad (C13)

The extremal conditions \( h_3, w = 0 \) and \( h_3, z = 0 \) lead to

\[ z^2 (2z - 3) = 0 \], \quad (C14) \\
\[ z (1 + w) (z - 1) = 0 \]. \quad (C15)

The common solutions of (C14) and (C15) are given by

\[ P_1 \left( w = -1, z = 0 \right) \], \quad (C16) \\
\[ P_2 \left( w = -1, z = \frac{3}{2} \right) \]. \quad (C17)

The numerical values of \( h_3 \) at these turning points are

\[ h_3 \left( P_1 \right) = 2 \geq 0 \], \quad (C18) \\
\[ h_3 \left( P_2 \right) = 2 \geq 0 \]. \quad (C19)

Thus, we have shown (C9) and, therefore, the validity of inequality (C2).

Now we consider the inequality (C7). Multiplying both sides of this relation with the denominator leads to the inequality

\[ h_4 = -z^2 w^2 - 1 + 2 w z \leq 0 \]. \quad (C20)

The boundaries of \( h_4 \) are

\[ \lim_{w \to -1} h_4 = - (1 + z)^2 \leq 0 \], \quad (C21) \\
\[ \lim_{w \to +1} h_4 = - (1 - z)^2 \leq 0 \], \quad (C22) \\
\[ \lim_{z \to 0} h_4 = -1 \leq 0 \], \quad (C23) \\
\[ \lim_{z \to \infty} h_4 = -w^2 \lim_{z \to \infty} z^2 \leq 0 \]. \quad (C24)
The extremal conditions \( h_{4,w} = 0 \) and \( h_{4,z} = 0 \) lead to

\[
z (1 - wz) = 0, \quad (C25)
\]

\[
w (1 - wz) = 0. \quad (C26)
\]

The common solution of (C25) and (C26) is given by

\[
P_3 (w = 0, z = 0), \quad (C27)
\]

and the numerical value of \( h_4 \) at this turning point is

\[
h_4 (P_3) = -1 \leq 0. \quad (C28)
\]

Thus, we have shown (C20) and, therefore, the inequality (C7).

Now we consider the inequality (C8). Squaring both sides of (C8) and subtracting from each other leads to the inequality

\[
h_5 = 4 z^2 - z - 7 wz + 2 + 2w \geq 0. \quad (C29)
\]

The boundaries of \( h_5 \) are

\[
\lim_{w \to -1} h_5 = 2z (3 + 2z) \geq 0, \quad (C30)
\]

\[
\lim_{w \to +1} h_5 = 4 (z - 1)^2 \geq 0, \quad (C31)
\]

\[
\lim_{z \to 0} h_5 = 2 (1 + w) \geq 0, \quad (C32)
\]

\[
\lim_{z \to \infty} h_5 = 4 \lim_{z \to \infty} z^2 \geq 0. \quad (C33)
\]

The extremal conditions \( h_{5,w} = 0 \) and \( h_{5,z} = 0 \) lead to

\[
-7z + 2 = 0, \quad (C34)
\]

\[
8z - 1 - 7w = 0. \quad (C35)
\]

The common solution of (C34) and (C35) is given by

\[
P_4 \left( w = \frac{9}{49}, z = \frac{2}{7} \right), \quad (C36)
\]

and the numerical value of \( h_5 \) at this turning point is

\[
h_5 (P_4) = \frac{100}{49} \geq 0. \quad (C37)
\]

Thus, we have shown (C29) and, therefore, the inequality (C8).