Light-deflection in binary systems on microarcsecond level

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GAIA will detect about 10^8 binaries. We investigate the light-deflection among the components of a binary system on microarcsecond level, an effect which has not been observed so far. In order to determine the total number of such binaries, an inclination formula has been derived by means of generalized lens equation. It turns out that there exist about 10^3 binaries having orbital parameters that the light-deflection amounts to be on microarcsecond level $\varphi \geq 1\,\mu$as, but only a very few systems with $\varphi \geq 25\,\mu$as being the best positional accuracy of GAIA in the ideal case of a bright star. The orbital parameters of such possibly relevant systems are determined, which however, take rather extreme values. Thus, only in a very few and rather extreme binary systems the light-deflection effect might be detectable by GAIA. This conclusion is supported by additional numerical investigations.

Contents

I. INTRODUCTION

The light-deflection effect in binary systems has not been observed so far. GAIA will observe 10^9 stars brighter than 20^{th} apparent magnitude and with an astrometric accuracy on microarcsecond level; for the end-of-mission parallax-standard-error see [1]. Detailed numerical simulations predict the detection of about 10^8 (resolved, astrometric, eclipsing, spectroscopic) binary systems by GAIA mission [2, 3], which is a considerable increase compared to the 10^5 binary systems known so far; 109087 systems in the " *Washington Double Star Catalog*" in 2011 [4]. Thus the question arises about the existence of binaries having such parameters that the light-deflection among the components is on microarcsecond level and, therefore, might be detectable by GAIA mission, hence providing a new observational science of the effects in general theory of relativity.

In order to investigate the light-deflection effect in binaries one needs a simple analytical formula which allows to determine the light-deflection on microarcsecond level. As we will see, a post-Newtonian or post-post-Newtonian solution of light-propagation in Schwarzschild metric, as given for instance in [5–8], does not allow to study light-deflection in binary systems. Moreover, the classical lens equation is also not appropriate to study the light-deflection in binary systems. Recently, in [9, 10] we have derived a generalized lens equation which allows to determine the light-deflection of binary systems on microarcsecond level. By means of this generalized lens equation we derive an expression for the inclination of a binary system having a given light-deflection angle φ. This inclination formula allows to determine the number of relevant binaries having a given light-deflection angle. Furthermore, in order to determine the number of relevant binaries, we have taken into account the distribution of stellar masses and semi-major axes in binary systems. With the aid of these three basic tools (1. generalized lens equation, 2. inclination formula, and 3. probability distribution of stellar masses and semi-major axes) we are in the position to determine the number of relevant binaries observable by GAIA mission.

The report is organized as follows: In Section II some basics about orbital elements of binary systems are given. The generalized lens equation and the inclination formula are presented in Section III. In Section IV we present two stringent conditions on the orbital elements of any binary system (astrometric, spectroscopic, eclipsing and resolved binaries) in order to have a light-deflection which is observable by GAIA mission. In Section V we have considered the special case of resolved binareis and present an additional condition for such kind of systems. The total number of binaries which have a given light-deflection is estimated in Section VI. A Summary is given in Section VII.

Without loss of generality, throughout our investigation we consider the light-deflection of component B at component A. Hence, component A is considered to be the massive body, while component B is the light-source. We will use fairly standard notations:

- G is the Newtonian constant of gravitation.
- c is the velocity of light.
- γ is the parameter of the Parametrized Post-Newtonian (PPN) formalism which characterize possible deviation of the physical reality from general relativity theory ($\gamma = 1$ in general relativity).
- The 3-dimensional coordinate quantities ("3-vectors") referred to the spatial axes of the corresponding reference system are set in boldface: $\boldsymbol{a}$.

- The absolute value (Euclidean norm) of a "3-vector" $\boldsymbol{a}$ is denoted as $|\boldsymbol{a}|$ or, simply, a and can be computed as $a = |\boldsymbol{a}| = (a^1\,a^1 + a^2\,a^2 + a^3\,a^3)^{1/2}$.

- The scalar product of any two "3-vectors" $\boldsymbol{a}$ and $\boldsymbol{b}$ with respect to the Euclidean metric δ_{ij} is denoted by $\boldsymbol{a}\,\cdot\,\boldsymbol{b}$ and can be computed as $\boldsymbol{a}\,\cdot\,\boldsymbol{b} = \delta_{ij}\,a^i\,b^j = a^i\,b^i$, where latin indices $i = 1, 2, 3$.

- For any two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, the angle between them is designated as $\delta(\boldsymbol{a}, \boldsymbol{b})$. Clearly, for an angle between two vectors one has $0 \leq \delta(\boldsymbol{a}, \boldsymbol{b}) \leq \pi$. The angle $\delta(\boldsymbol{a}, \boldsymbol{b})$ can be computed by $\delta(\boldsymbol{a}, \boldsymbol{b}) = \arccos \dfrac{\boldsymbol{a} \cdot \boldsymbol{b}}{a\,b}$.

- a.u. $= 1.496 \times 10^{11}$ m stands for astronomical unit.

- pc $= 3.086 \times 10^{16}$ m stands for parallax of one arcsecond.

- $m_{\odot} = 1.476 \times 10^3\,m$ is the gravitational radius of the Sun.

- $1\,\mu\text{as} = \frac{\pi}{180\times 60\times 60} \times 10^{-6}\,\text{rad}$ is one microarcsecond.

II. ORBITAL ELEMENTS OF A BINARY SYSTEM

We consider a binary system, component A with mass M_A at coordinate $\boldsymbol{r}_A$ and component B with mass M_B at coordinate $\boldsymbol{r}_B$. In order to express the light-deflection effect in terms of orbital elements, we introduce spherical coordinates, illustrated in FIG. 1.

The center of coordinate system is situated at the center of mass (CMS), i.e.

$$\boldsymbol{r}_{\text{CMS}} \;=\; \frac{1}{M_A + M_B}\,(M_A\,\boldsymbol{r}_A + M_B\,\boldsymbol{r}_B)\;. \tag{1}$$

Thus, the vector $\boldsymbol{r}$, which points from CMS to the observer, is given by

$$\boldsymbol{r} = \begin{pmatrix} r\,\cos\omega\;\sin i \\ \\ r\,\sin\omega\;\sin i \\ \\ r\,\cos i \end{pmatrix}, \tag{2}$$

where $r = |\,\boldsymbol{r}\,|$, the argument of periapsis is denoted by ω, and i is the inclination, see FIG. 1. The solution of equation of motion yields for vectors $\boldsymbol{r}_A$ and $\boldsymbol{r}_B$ the expression given by Eqs. (A22) - (A26). The vector $\boldsymbol{x}_1$ points from the mass center of massive body to the observer, and vector $\boldsymbol{x}_0$ points from the mass center of massive body to the source, see also FIG. 1. The coordinates of these vectors can be expressed by the orbital elements of the binary star as follows:

$$\boldsymbol{x}_1 = \boldsymbol{r} - \boldsymbol{r}_{\text{A}} = \begin{pmatrix} r\,\cos\omega\;\sin i - \dfrac{A\,(\cos E - e)}{1 + \frac{M_A}{M_B}} \\ \\ r\,\sin\omega\;\sin i - \dfrac{A\,\sqrt{1-e^2}\,\sin E}{1 + \frac{M_A}{M_B}} \\ \\ r\,\cos i \end{pmatrix}, \tag{3}$$

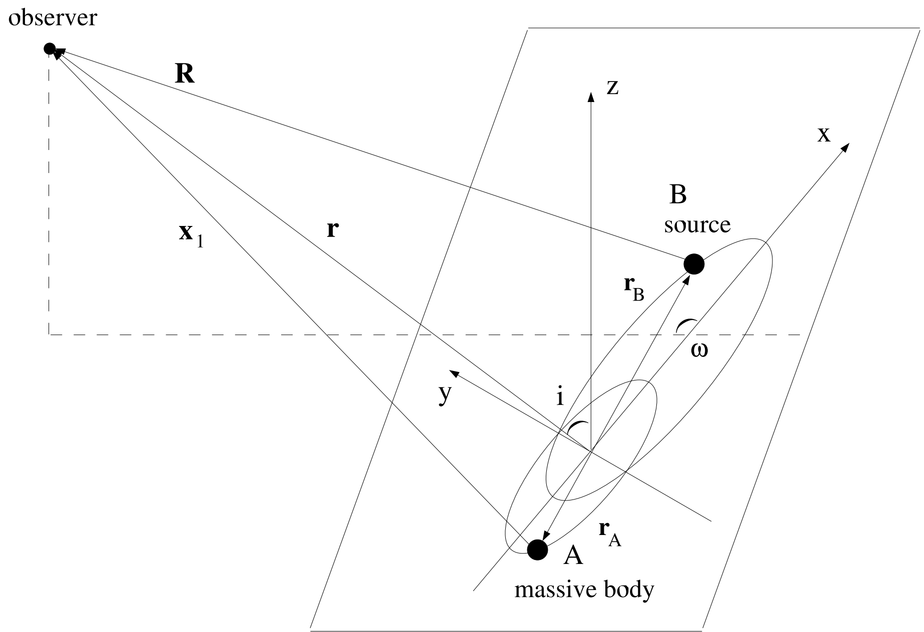

FIG. 1: The seven orbital elements which define the orbit of a binary system: distance vector $\boldsymbol{r}$, semi-major axis A, inclination i, eccentricity e, eccentric anomaly E, periapsis ω, and mass ration M_A/M_B. The orbit of the binary system spans the (x, y)-plane and the z-axis is perpendicular to the orbital plane. The x-axis is oriented along the semi-major axis of the orbit of the binary system, while the y-axis is perpendicular to the x-axis. The vector $\boldsymbol{r}$ is directed from the center-of-mass (CMS) of the binary system, see Eq. (1), to the observer. The center of spherical coordinate system is situated at the CMS of binary system, that means $\boldsymbol{r}_{\rm CMS} = \mathbf{0}$. The inclination $0 \leq i \leq \pi$ is the angle between $\boldsymbol{r}$ and z-axis; $i = \pi/2$ is called edge-on and $i > \pi/2$ corresponds to retrograd orbit. The dotted line indicates the projection of $\boldsymbol{r}$ onto orbital (x, y)-plane, i.e. z-component of $\boldsymbol{r}$ equals zero. The angle between this projection and x-axis is called argument of periapsis $0 \leq \omega \leq \pi$. The orbital elements semi-major axis A, eccentricity $0 \leq e \leq 1$ and mass ratio M_A/M_B govern uniquely the geometric shape of both ellipses. The eccentric anomaly $0 \leq E \leq 2\pi$ (not plotted here), is defined in Eq. (A16) of Appendix A and determines the actual position of the bodies A and B on their orbit.

$$\boldsymbol{x}_0 = \boldsymbol{r}_{\rm B} - \boldsymbol{r}_{\rm A} = - \begin{pmatrix} A\; (\cos E - {\rm e}) \\ \\ A\,\sqrt{1 - {\rm e}^2}\,\sin E \\ \\ 0 \end{pmatrix} . \tag{4}$$

Here, A is the semi-major axis, e is the eccentricity, and E is the eccentric anomaly, see Appendix A. The vectors (3) and (4) will be used to express the light-deflection in terms of orbital elements of the binary system.

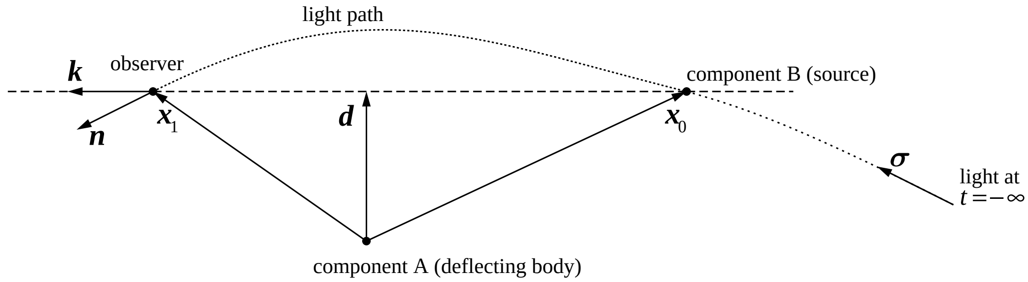

FIG. 2: Binary star composed of component A being the massive body, and component B considered to be the light-source.

III. INCLINATION FORMULA FROM GENERALIZED LENS EQUATION

The light-deflection angle $\varphi = \delta\,(\boldsymbol{k}, \boldsymbol{n})$, where $\boldsymbol{k}$ is the unit vector pointing from source to observer, and $\boldsymbol{n}$ is the unit tangent vector of light-trajectory at position of observer.

Recently, we have derived a generalized lens equation in [9, 10], which allows to determine the light-deflection is extreme astrometric configurations like binary systems. This generalized lens equation reads

$$\varphi_{1,2} \;=\; \frac{1}{2}\left(\sqrt{\frac{d^2}{x_1^2} + 4\,(1+\gamma)\,\frac{m}{x_1}\,\frac{x_0\,x_1 - \boldsymbol{x}_0\cdot\boldsymbol{x}_1}{R\,x_1}} \mp \frac{d}{x_1}\right) + \left(\frac{m^2}{d\,'^2}\right), \tag{5}$$

where the impact vector $\boldsymbol{d} = \boldsymbol{k}\times(\boldsymbol{x}_1\times\boldsymbol{k})$ and its absolute value $d = |\boldsymbol{d}|$, see also FIG. 2. The vector $\boldsymbol{x}_1$ points from the mass center of massive body to the observer, and vector $\boldsymbol{x}_0$ points from the mass center of massive body to the source; $\boldsymbol{R} = \boldsymbol{x}_0 - \boldsymbol{x}_1$ and its absolute value $R = |\boldsymbol{R}|$, and $m = \dfrac{G\,M}{c^2}$ is the Schwarzschild radius of massive body, i.e. of component A of the binary system. In Eq. (5) there are two solutions φ_1 (φ_2) belonging to the minus (plus) sign according to the two light-trajectories, however in what follows we will consider only φ_1 which is the relevant solution, because φ_2 is just the second image of one and the same source.

Here, $d\,' = \dfrac{L}{E}$ is Chandrasekhar's impact parameter [12], where L being the orbital momentum and E is the energy of the photon in the gravitational field of massive body. Basically, the light-ray of component B cannot be observed if $d\,'$ is smaller than the radius of massive body A. For stars, the radius is much larger than Schwarzschild radius m, hence $\dfrac{m^2}{d\,'^2} \ll 1$. Furthermore, the generalized lens equation (5) is finite for $d\rightarrow 0$ and $b \rightarrow 0$, both of which are possible astrometric configurations in binary systems.

In the following, we will apply (5) in order to determine the light-deflection in binary systems. For that, we will use the coordinates $\boldsymbol{x}_0$ and $\boldsymbol{x}_1$ in the form as given by Eqs. (3) and (4), respectively. A typical light-curve of a binary system, calculated by means of generalized lens equation (5) is shown in FIG. 3.

Now we derive an inclination formula from the generalized lens equation (5). The impact of eccentricity is neglected, i.e. we consider circular orbite $e = 0$ which implies $\omega = 0$. Thus, we obtain the coordinates (B1) and (B2) given in Appendix B. Furthermore, we are interested in the maximal value of light-deflection of a binary system, i.e. we consider here the astrometric configuration $E = 0$. Then, by inserting these coordinates in the generalized

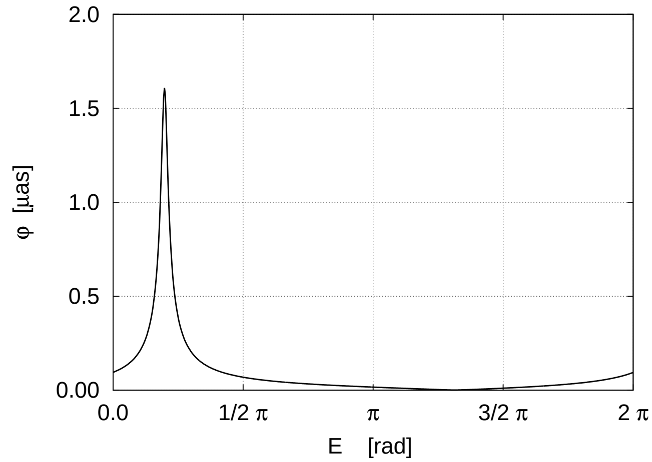

FIG. 3: A typical lightcurve of a binary system, determined using generalized lens equation (5) or (B11), respectively. The parameters chosen are: distance $r = 1\,\mathrm{pc}$, semi-major axis $A = 100\,a.u.$ inclination $i = \frac{31}{64}\,\pi$, mass $M_A = 2\,M_{\odot}$, mass ratio $\frac{M_A}{M_B} = 2.0$, eccentricity $e = 0.25$, argument of periapsis $\omega = \frac{\pi}{4}$.

lens equation (5) we obtain (see Eq. (B14) in Appendix B):

$$\varphi = \frac{1}{2}\left(\sqrt{\frac{A^2}{r^2}\,\cos^2 i + 8\,\frac{m}{r}\,\frac{A}{r}\,\left(1+\sin i\right)} - \frac{A}{r}\,\left|\cos i\right|\right) + \mathcal{O}\left(\frac{A}{r}\,\sqrt{\frac{m}{r}\,\frac{A}{r}}\right), \tag{6}$$

where for simplicity we take $\gamma = 1$. The minimal value and maximal value of light-deflection for the astrometric position $E = 0$ follow from Eq. (6):

$$\varphi_{\rm min} = \varphi\left(i=0\right) = \frac{1}{2}\left(\sqrt{\frac{A^2}{r^2} + 8\,\frac{m}{r}\,\frac{A}{r}} - \frac{A}{r}\right) = 2\,\frac{m}{r} + \mathcal{O}\left(\frac{m^2}{r\,A}\right), \tag{7}$$

$$\varphi_{\rm max} = \varphi\left(i = \frac{\pi}{2}\right) = 2\,\frac{\sqrt{m\,A}}{r}\,. \tag{8}$$

The expression (6) can be reconverted in terms of inclination (see Appendix C):

$$\left|\frac{\pi}{2} - i\right| = \arccos\left(-\frac{p}{2} + \sqrt{\frac{p^2}{4} - q}\right), \tag{9}$$

where

$$p = \frac{8\,m^2\,A - 4\,m\,r^2\,\varphi^2}{A\,(r^2\,\varphi^2 + 4\,m^2)}\,, \tag{10}$$

$$q = -\frac{A^2\,r^2\,\varphi^2 + 4\,m\,A\,r^2\,\varphi^2 - 4\,m^2\,A^2 - r^4\,\varphi^4}{A^2\,(r^2\,\varphi^2 + 4\,m^2)}\,. \tag{11}$$

For a given value of light-deflection φ, the inclination formula (9) yields the corresponding value of maximal inclination i of a binary system. Note, that the values of φ cannot be chosen arbitrarily, but they are restricted by $\varphi_{\rm min}$ and $\varphi_{\rm max}$ given by Eqs. (7) and (8), respectively.

The inclination formula (9) can considerably be simplified. From (10) and (11) we obtain by series expansion

$$p \;=\; -4\,\frac{m}{A} + 8\,\frac{m^2}{r^2\,\varphi^2} + \mathcal{O}\left(m^3\right), \tag{12}$$

$$q \;=\; -1 - 4\,\frac{m}{A} + 8\,\frac{m^2}{r^2\,\varphi^2} - 4\,\frac{m^2}{A^2} + \frac{r^2\varphi^2}{A^2} + \mathcal{O}\left(m^3\right). \tag{13}$$

By means of (8), the last term in (13) can be estimated to be smaller than $4\,\dfrac{m}{A} \ll 1$. Here, we underline that $\frac{m}{A} \ll \frac{m}{r\,\varphi}$ even at large distances $r \simeq 10^3\,{\rm pc}$ and small values for semi-major axis $A \simeq 1\,a.u.$. Thus, we neglect all terms of order $\mathcal{O}\left(\dfrac{m}{A}\right)$ and obtain

$$\left|\frac{\pi}{2} - i\right| \;=\; \arccos\left(1 - 8\,\frac{m^2}{r^2\,\varphi^2}\right) + \mathcal{O}\left(\frac{m}{A}\right) = 2\,\arctan\left(2\,\frac{m}{r\,\varphi}\right) + \mathcal{O}\left(\frac{m^3}{r^3\,\varphi^3}\right) + \mathcal{O}\left(\frac{m}{A}\right), \tag{14}$$

where we have used $\arccos\left(1 - 8\,x^2\right) = 2\,\arctan 2\,x + \mathcal{O}\left(x^3\right)$ for $x \ll 1$. We underline, that the applicability of (14) is restricted by the condition (D5). Here, it should be noticed that $x = 2\,\frac{m}{r\,\varphi} \ll 1$, even in such an extreme case like $r = 1\,{\rm pc}$, $m = m_{\odot}$ and $\varphi \simeq 1.0\,\mu{\rm as}$ we obtain $x = 0.019$; for an analytical proof use the exact expression for $\varphi_{\rm min}$. Due to $\frac{m}{A} \ll \frac{m}{r\,\varphi}$, the impact of semi-major axis is of lower order and can be neglected in the inclination formula. We notice that expression (14) agrees with an inclination formula derived in [13]; because the manuscript [13] has not been published, the arguments of this work are represented in Appendix D.

IV. STRINGENT CONDITIONS ON ORBITAL PARAMETERS FOR BINARY SYSTEMS

In this Section we present two stringent conditions on the orbital elements of binary systems to have a light-deflection which could be observed by GAIA mission. This strict conditions are valid for any binary system, that means for astrometric, spectroscopic, eclipsing and resolved binaries.

The first stringent condition on the orbital elements folows from the inclination formula in the simplified form as given by Eq. (14). For a better illustution we will present this condition in terms of angular degrees instead of radians. Using $\arctan x = x + \mathcal{O}(x^3)$ we obtain

$$\left|\,90^{\circ} - i\,\right| \;\le\; 2.25^{\circ}\;\frac{m}{m_{\odot}}\;\frac{pc}{r}\;\frac{\mu{\rm as}}{\varphi}\;. \tag{15}$$

According to this strict condition, the inclination i of a binary system with mass m and at distance r must not exceed the given value in order to have a light-deflection φ.

Two remarks are in order. First, the GAIA accuracy for one individual positional measurement in ideal case amounts to be $25\,\mu{\rm as}$, which implies $\varphi \ge 25\,\mu{\rm as}$. Second, inside a sphere of $10\,pc$ almost every star or binary system is known by RECONS data [14].

Since inside that sphere there is no such a binary system having a light-deflection beyond microarcsecond level, we have to assume $r \geq 10\,pc$. According to condition (15), these both remarks imply in total a factor of $\dfrac{1}{250}$. Therefore, even in the best case we conclude $|\,90^\circ - i\,| \leq 0.01^\circ \,\dfrac{m}{m_\odot}$, that means the binaries must be in fact almost edge-on in order to have a light-deflection observable by the GAIA facility.

The second stringent condition follows from the maximal light-deflection angle (8), given by

$$\varphi \;\leq\; 200\,\mu\text{as}\,\sqrt{\frac{m}{m_\odot}\;\frac{A}{a.u.}}\;\frac{pc}{r}\;. \tag{16}$$

Also for this restrictive condition two remarks are in order. First, even rather extreme values of $m = 10\,m_\odot$ and $A = 10^3\,a.u.$ yield an maximal upper distance of $r \leq 825\,pc$ of the binaries in order to to have a light-deflection observable by the GAIA instrumentation. And second, like for condition (15) we have to assume $r \geq 10\,pc$ because of the RECONS data. This value implies actually in the best case $\dfrac{m}{m_\odot}\,\dfrac{A}{a.u.} \geq 2$.

The observability of light-deflection effect in binaries implies the realization of both these stringent conditions (15) and (16) simultaneously for a given binary system. But even in case a given binary system satisfies the both condition, the observability of light-deflection effect is no guaranteed, because the astrometric position $E = 0$ has to be reached during mission time and we have assumed the highest accuracy of $25\,\mu$as for one positional measurement, which is only valid for bright stars. Nonetheless, as soon as the orbital elements r, A, m and i of the binaries are known, these both stringent conditions (15) and (16) allow to scan the GAIA data in order to find a possible candidate for being a relevant binary system where the light-deflection effect might be observable by GAIA mission. However, as we will see in Section VI, the existence of such systems is highly unprobable.

V. SPECIAL CASE: CONDITIONS ON ORBITAL PARAMETERS FOR RESOLVED BINARIES

In this Section we consider the special case of a resolved binary system.

A. Resolving power of GAIA

The core of GAIA optical instrumentation consists of two identical mirror telescopes, ASTRO-1 and ASTRO-2, with a rectangular pupil whose dimensions are A = 0.50 m, B = 1.45 m, and $f = 35$ m is the effective focal length. The intensity is given by [26, 27]:

$$I(z_{\rm A}, z_{\rm B}) \;=\; I_0 \left(\frac{\sin^2(z_{\rm A})}{z_{\rm A}^2}\;\frac{\sin^2(z_{\rm B})}{z_{\rm B}^2}\right)\,, \tag{17}$$

where $z_{\rm A} = \pi\ {\rm A}/\lambda\sin\Theta_{\rm A}$, $z_{\rm B} = \pi\ {\rm B}/\lambda\sin\Theta_{\rm B}$, A and B the width and lenght of recangular mirror, λ is the wavelength of incident light-ray, and $\Theta_{\rm A}, \Theta_{\rm B}$ are the angle of observation, i.e. the angle between the axis of the rectangular aperture and the line between aperture center and observation point. The intensity of incident light-ray at $\Theta_{\rm A} = 0, \Theta_{\rm B} = 0$ is denoted by I_0. The function $I(z_{\rm A}, z_{\rm B})/I_0$ in Eq. (17) is the (by $\Theta_{\rm A} = \Theta_{\rm B} = 0$ normalized) Point

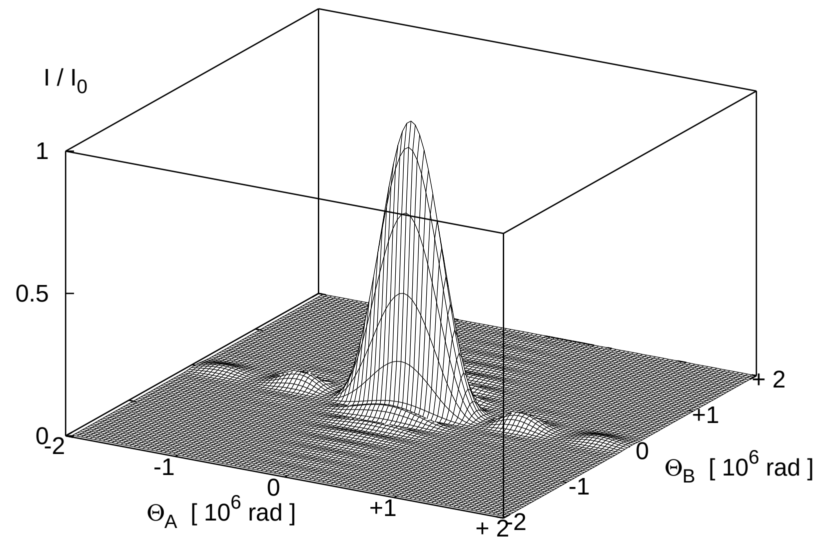

FIG. 4: Point Spread Function (PSF) for a rectangular telescope according to Eq. (17). The incident monochromatic light-ray has a wavelenght of $\lambda = 350\,\mathrm{nm}$. The parameters of the rectangular telescope are: $\mathrm{A} = 0.5\,\mathrm{m}$, $\mathrm{B} = 1.45\,\mathrm{m}$.

Spread Function (PSF) for monochromatic incident light with wavelenght λ for a rectangular aperture. The PSF of ASTRO-1 and ASTRO-2 has been discussed in [26, 27]. The optical spectrum of stars is $\lambda = (350 - 750)$ nm. In FIG. 4 the PSF for an incident monochromatic light-ray with $\lambda = 350\,\mathrm{nm}$ is represented for GAIA telescopes.

Most of the light is concentrated in the central bright rectangular shaped pattern. The lenght l_{A} and width l_{B} of this rectangle is determined by the first zero-roots of (17) at $z_{\mathrm{A}} \simeq \pi$ and $z_{\mathrm{B}} \simeq \pi$, respectively. From this follows, that $\sin\Theta_{\mathrm{A}} = \pi\ \lambda/(\pi\ \mathrm{A}) = \lambda/\mathrm{A}$, and $\sin\Theta_{\mathrm{B}} = \pi\ \lambda/(\pi\ \mathrm{B}) = \lambda/\mathrm{B}$. Furthermore, if the diffraction pattern is shown on a screen at a distance f, then the lenght and width is given by [28]

$$L_{\mathrm{A}} \; = \; 2\,\frac{f\,\lambda}{\mathrm{A}}\,, \tag{18}$$

$$L_{\mathrm{B}} \; = \; 2\,\frac{f\,\lambda}{\mathrm{B}}\,, \tag{19}$$

where f is the focal length of the optic, i.e. of the rectangular GAIA mirror. Accordingly, by means of the given numerical values $\mathrm{A} = 0.50\,\mathrm{m}$, $\mathrm{B} = 1.45\,\mathrm{m}$, and $\lambda = 350$ nm results in $L_{\mathrm{A}} = 49.0\ \mu\mathrm{m}$ and $L_{\mathrm{B}} = 16.9\ \mu\mathrm{m}$ for the lenght and width of "*Airy rectangle*" of GAIA optics. Note, that the "*Airy rectangle*" has the same order of magnitude than the pixel size ($10\mu\mathrm{m} \times 30\mu\mathrm{m}$) of the 110 CCDs of astrometric field part of the focal plane. In order to separate two pointlike sources, the distance between their centers of the rectangle has to be larger than either L_{A} or L_{B}. Since $L_{\mathrm{B}} < L_{\mathrm{A}}$, in our study we will take the better resolution value L_{B}, which corresponds to a resolution angle of

$$\delta \; = \; \frac{L_{\mathrm{B}}}{2\,f} = \frac{\lambda}{\mathrm{B}}\,. \tag{20}$$

The resolving power is the minimal angular distance between two objects to get separable by GAIA instrumentation. With the parameters given above we obtain the resolving power δ of GAIA optics:

$$\delta \;=\; 0.24 \times 10^{-6}\,\mathrm{rad} = 49.7\,\mathrm{mas}\,. \tag{21}$$

In what follows this parameter is of fundamental importance in order to determine the ability of GAIA to determine the light-deflection in binary systems.

B. Orbital parameters of resolved binaries observable by GAIA

In this Section the question is addressed, which and how many binary systems can be separated by GAIA instrumentation among all those relevant binaries found in the previous Section; see FIG. 6. In average, GAIA will observe each object 80 times, but will not constantly observe these objects during mission time. However, for simplicity we approximate the scanning law of GAIA by assuming a permanent observation of all objects during the whole mission time.

Furthermore, we consider visual binaries, i.e. binaries which are separable by GAIA telescopes. The both largest telescopes of GAIA have a resolution angle δ discussed in the previous Section, see Eqs. (20) and (21). For binary systems, this resolution angle δ corresponds to a minimal distance between the components A and B to get separable within GAIA optics. Using (B6) and (20) we obtain the condition

$$d \;=\; A\,\left|\cos i\right| \;\ge\; \delta\, r \;=\; \frac{\lambda}{B}\, r\,, \tag{22}$$

where r is the distance between the remote objects and GAIA observer. Here we note, that this condition is by far much more important than taking into account the effect of finite radius of the stars, which would imply $A\,\left|\cos i\right| \;\ge\; R_A$, where R_A being the radius of component A. If we insert the extreme case $A\,\left|\cos i\right| = \delta\, r$ into (6), we obtain

$$\varphi \;=\; \frac{1}{2}\left(\sqrt{\delta^2 + 8\left(1+\sin i\right)\,\frac{m}{r}\,\frac{A}{r}} - \delta\right) \simeq \frac{1}{2}\left(\sqrt{\delta^2 + 16\,\frac{m}{r}\,\frac{A}{r}} - \delta\right) \simeq 4\,\frac{m}{r}\,\frac{A}{r}\,\frac{1}{\delta}\,. \tag{23}$$

Here, in the second term we have used $i \simeq \dfrac{\pi}{2}$, that means $\sin i = 1 + \mathcal{O}\left(\left(i-\dfrac{\pi}{2}\right)^2\right)$, and in the last term we have neglected higher order $\mathcal{O}\left(m^2\right)$ in the series expansion. Relation (23) is an expression for the maximal light-deflection angle of a binary system when taking into account the resolving power of GAIA. Eq. (23) is a much stricter restriction than the generalized lens equation (6), because (23) determines the light-deflection angle only of those binary systems having a resolution angle δ of GAIA optical instrumentation, while (6) determines the light-deflection angle of any possible binary system. We notice, that from Eq. (23) follows the maximal possible distance of visual binaries:

$$r \;\le\; \sqrt{4\,\frac{m\,A}{\varphi\,\delta}} \;=\; 0.18\,\mathrm{pc}\,\sqrt{\frac{m}{m_{\odot}}\,\frac{A}{a.u.}}\,, \tag{24}$$

where in the second expression we have used the optimal values $\delta = 0.24 \times 10^{-6}\,\mathrm{rad}$ and $\varphi = 25\,\mu\mathrm{as}$. We note, that condition (24) can also be written by

$$A \;\ge\; 30\,a.u.\,\frac{m_{\odot}}{m}\,\frac{r^2}{pc^2}\,. \tag{25}$$

These both conditions (24) and (25) imply rather extreme orbital parameters on visual binaries. For instance, condition (24) implies a maximal distance of $r \le 18\,pc$ for solar-mass type binaries even with a huge semi-major axis of $A = 10^4\,a.u.$, while condition (25) implies a large semi-major axis for solar-mass type binaries aoutside a sphere of $r \ge 10\,pc$. It is almost for sure, that such extreme parameters will not be realized in reality.

VI. TOTAL NUMBER OF BINARIES WITH A GIVEN LIGHT-DEFLECTION

By means of numerical simulations it has been estimated that GAIA will detect $\sim 10^8$ (resolved, astrometric, eclipsing, spectroscopic) binaries [2, 3]. This is an enormous increase of known binaries compared to the so far discovered $\sim 10^5$ systems; 109087 systems in the "*Washington Double Star Catalog*" in 2011 [4]. However, it is obvious that by far not all binaries are relevant objects in respect to an observable light-deflection effect of one component at the other. In the previous Sections we have determined the orbital conditions for a binary system in oder to have a light-deflection which might be observed during GAIA mission. In this Section we will consider the possible total number of such relevant binaries. In order to estimate the total number of binaries having a given light-deflection φ we will apply the following formula:

$$N(\varphi) = \int\limits_{R_{\rm min}}^{R_{\rm max}} d^3r\,\rho(r) \int\limits_{A_{\rm min}}^{A_{\rm max}} dA\, f(A) \int\limits_{\mu_{\rm min}}^{\mu_{\rm max}} d\mu\, f(\mu)\, P(i)\,. \qquad (26)$$

Here, $\rho(r)$ is the density of binaries, $f(A)$ is the semi-major axis distribution of binary systems, $f(\mu)$ is the mass distribution of stars where $\mu = \dfrac{M}{M_\odot}$ is the mass-ratio of massive body (component A) and solar mass. The probability distribution $P(i)$ to find a binary system with given inclination $0 \le i \le \pi$, is a function of μ, A and the given light-deflection angle φ. According to (9), the probability distribution $P(i)$ is given by (the inclination of binary systems is of course a random distribution)

$$P(i) = \frac{2}{\pi}\,\arccos\left(-\frac{p}{2} + \sqrt{\frac{p^2}{4} - q}\right), \qquad (27)$$

where p and q are given by Eqs. (10) and (11).

For the minimal distance of a binary system from the Sun we may safely assume $R_{\rm min} = 1\,{\rm pc}$. From (8) follows that beyond a sphere with radius $R_{\rm max} = 2000\,{\rm pc}$ only a very few exceptional binary systems might have a light-deflection of more than $1\,\mu$as. The Sun is located in the *Orion arm* which has about 1000 pc across and approximately 3000 pc in length. For the estimate according to (26), we will assume that the stars are homogeneously distributed inside the *Orion arm.* For the uniform star density we take $\rho_{\rm stars} = 0.1\,{\rm star/pc^3}$, a value which is in line with the data of Research Consortium on Nearby Stars (RECONS) [14]. Furthermore, we take the common presumption, that about 50 percent of all stars are components of a binary or multiple system [15, 16]. Then we obtain for the density of binaries

$$\rho(r) \simeq 0.025\ {\rm binaries/pc^3}\,. \qquad (28)$$

Let us now consider the distribution of semi-major axis A in binary systems. Statistical investigations show that the distribution of binary semi-major axis is flat in a logarithmic scale over the range of six orders of magnitude, that means assumed to be valid in the large range $\mathrm{A}_{\rm min} = 10\, R_{\odot} \le a \le 10^4\, a.u. = \mathrm{A}_{\rm max}$, [17]. The lower limit $\mathrm{A}_{\rm min}$ is determined by the semi-major axis at which Roche lobe overflow occurs, while the upper limit $\mathrm{A}_{\rm max}$ depends on how large the averaged star density is. The logarithmic distribution is known as "*Öpik's law*" (1924) after its discoverer and given by $f(\mathrm{A}) \sim \dfrac{1}{\mathrm{A}}$, a law which has also been confirmed by recent investigations, e.g. [18]. This distribution is a consequence of the process of star formation as well as of the dynamical history of the binary system. Here we will take this law as a given fact in our numerical study. Accordingly, we have (see Appendix E)

$$f(\mathrm{A}) \;=\; \frac{1}{\mathrm{A}}\,\left(\ln \frac{\mathrm{A}_{\rm max}}{\mathrm{A}_{\rm min}}\right)^{-1}\,. \tag{29}$$

Furthermore, for the mass distribution $f\,(\mu)$ we recall the initial mass function (IMF) which is the probablity that a star is newly formed with a stellar mass M and is frequently assumed to be a power law $f(M) \sim M^{-\alpha}$. Originally, the IMF has been introduced by Salpeter in 1955 [19] for solar neighborhood region who gave the value $\alpha = 2.35$ and a validity region for stars with masses between $0.4\,M_{\odot}$ and $10\,M_{\odot}$. During the past decades the IMF has been refined by several investigations. Especially, the numerical values of slope parameter α and regions of validity have been proposed in subsequent investigations, e.g. [20–24], for a review see [25]. Moreover, the IMF does not necessarily coincide with the real mass distribution of stars, because IMF describes mass distribution of a star formation, while the solar neighboorhood mainly consists of evolved stars of main sequence. Here, for simplicity we will use this distribution as a given fact with $\alpha = 2.35$ and take the proposed region of validity $\mu_{\rm min} = 0.4$, and $\mu_{\rm max} = 10$. According to IMF, we find for $\alpha \neq 1$ (see Appendix E)

$$f(\mu) \;=\; \frac{(1-\alpha)\;\mu^{-\alpha}}{\mu_{\rm max}^{(1-\alpha)} - \mu_{\rm min}^{(1-\alpha)}}\,. \tag{30}$$

In order to motivate that distribution further, we have also compared (30) with the RECONS data [14] and have found a fair agreement. Using (27) - (30), the results of the estimate (26) are shown in FIG. 5. According to FIG. 5, in total there are about $N \sim 10^3$ binaries having a light-deflection of at least $\varphi = 1\,\mu$as.

In Eq. (26) we have determined the number of binaries with a given maximal possible light-deflection φ, just by taking for eccentric anomaly the value $E = 0$, that means the ideal configuration where the light-deflection take its maximal value (note, the eccentricity $e = 0$). It is, however, obvious that during the most part of the orbital motion we will have $E \neq 0$ and the light-deflection will be much smaller than the maximal possible light-deflection angle φ. On the other side, the mission time of GAIA is about $T_{\rm mission} \simeq 5\,$years while the orbital period T of relevant binaries, given by Eq. (A21), might easily exceed the mission time of GAIA. Therefore, it will be not very probable, that the component B will be just at the relevant position near the value $E = 0$, where the light-deflection becomes observable on microarcsecond level. In order to determine that number of observable relevant binaries, we have to extend Eq. (26) as follows,

$$N\,(\varphi) \;=\; \int\limits_{R_{\rm min}}^{R_{\rm max}} d^3r\;\rho(r) \int\limits_{A_{\rm min}}^{A_{\rm max}} dA\;f(A) \int\limits_{\mu_{\rm min}}^{\mu_{\rm max}} d\mu\;f(\mu)\;P(i)\;P(E)\,. \tag{31}$$

Here, $P(E)$ is the probability for the binary system to be in the region E, where the light deflection is larger than a given value for φ.

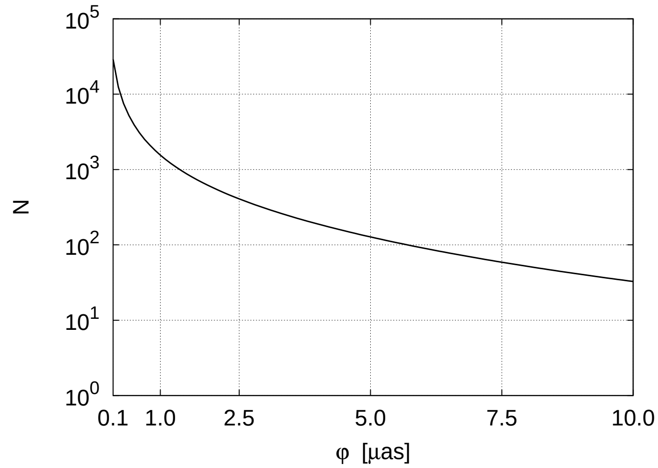

FIG. 5: Total number of binaries according to Eq. (26), having parameters such that the light-deflection of component B at component A is larger than a given value for φ.

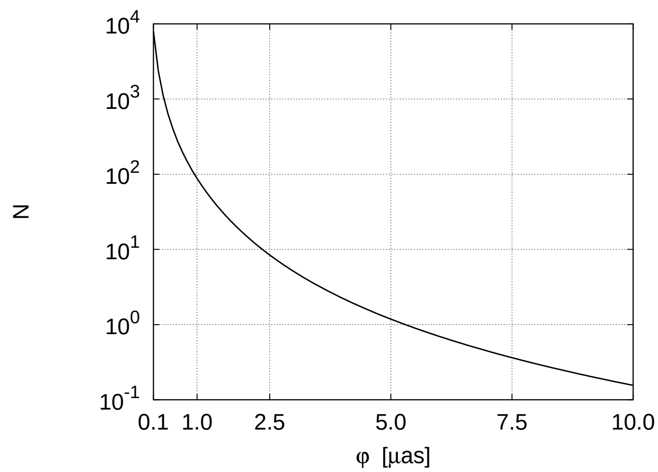

FIG. 6: Total number of binaries according to Eq. (31) where the binary system reaches the configuration $E = 0$ and having orbital such that the light-deflection of component B at component A is larger than a given value for φ.

In the very same way, as we have derived the inclination formula (9), we reconvert (B15) in terms of eccentric anomaly E and find the eccentric anomaly formula:

$$E \;=\; \pm\;\arccos\left(-\frac{p}{2}+\sqrt{\frac{p^2}{4}-q}\right)\,, \tag{32}$$

where p and q are given by Eqs. (10) and (11). For a given value of light-deflection φ, the formula (32) yields the value of eccentric anomaly E of a binary system characterized by semi-major axis A and mass m at a distance r. However, the values of φ cannot be chosen arbitrarily, instead they are restricted by $\varphi_{\rm min}$ and $\varphi_{\rm max}$ given by (of course, we take into account only astrometric positions with $0 \le E \le \dfrac{\pi}{2}$, because for the area $\dfrac{\pi}{2} \le E \le \pi$ the light-deflection is negligible):

$$\varphi_{\min} \;=\; \varphi\left(E=\pm\frac{\pi}{2}\right)=\frac{1}{2}\left(\sqrt{\frac{A^2}{r^2}+8\,\frac{m}{r}\,\frac{A}{r}}-\frac{A}{r}\right)=2\,\frac{m}{r}+\mathcal{O}\left(\frac{m^2}{r\,A}\right), \tag{33}$$

$$\varphi_{\max} \;=\; \varphi\left(E=0\right)=2\,\frac{\sqrt{m\,A}}{r}\,. \tag{34}$$

These expressions resemble the corresponding expressions in Eqs. (7) and (8). According to Eq. (32), the region where the binary system has a light-deflection larger or equal φ is given by $2\,E$. We have also to take into account that during GAIA mission time $T_{\rm mission}$ the component B moves along the orbit and could reach into the region $2\,E$. Therefore, the probability $P(E)$ that the binary system is during GAIA mission time at least ones inside the relevant astrometric position with the value E in (32), is given by

$$P\left(E\right) \;=\; \mathcal{P}_1\left(\frac{1}{\pi}\;\arccos\left(-\frac{p}{2}+\sqrt{\frac{p^2}{4}-q}\right)+\frac{T_{\rm mission}}{T}\right)\,, \tag{35}$$

where the operator is given by $\mathcal{P}_1\left(x\right)=x+\Theta\left(x-1\right)\left(1-x\right)$ with Θ-function defined by $\Theta(y)=0$ for $y<0$ and $\Theta(y)=1$ for $y\geq 0$, that means

$$\mathcal{P}_1\left(x\right)=\begin{array}{ccc} x & \text{if} & x<1\,, \\ 1 & \text{if} & x\geq 1\,. \end{array} \tag{36}$$

The probability distribution (35) has to be implemented in Eq. (31) in order to determine the number of binary systems having a given light-deflection φ and to be observable during GAIA mission time $T_{\rm mission}$. The results of Eq. (31) are shown in FIG. 6. According to FIG. 6, there are only a very few binaries $\sim 10^2$ having a light-deflection of at least $\varphi=1\,\mu$as.

VII. SUMMARY

In this study, the light-deflection in binary systems has been considered. So far, this effect has not been observed and, since GAIA will detect about 10^8 binary systems, might be considered as a challenge for the astrometric GAIA mission. To investigate this effect of light-deflection, an inclination formula (9) has been derived by means of generalized lens equation (5), and these both equations are the theoretical basis for investigating the light-deflection effect in binary systems.

In Section IV two stringent conditions are given by Eqs. (15) and (16) and valid for any kind of binary system in order to have a light-deflection which could be detected by GAIA msission. These conditions allow to scan the GAIA data to search for relevant binary systems as far as the orbital elements r, A, m and i are known. The special case of resolved binaries has been considered in Section V Two conditions were presented in Eqs. (24) and (25) for such special kind of systems. It has been shown that these systems take rather extreme orbital parameters for binary systems and their existence is highly unprobable.

Furthermore, the inclination formula allows to estimate the number of relevant binaries with the aid of Eq. (26). We have estimated the total number of relevant binaries using a semi-major axis distribution according to "*Öpik's law*" and a mass distribution according to "*Salpeter's mass distribution*". The Monte-Carlo simulation shows that there are only a very few binaries having a light-deflection which reaches the the technical limit of GAIA mission, see FIG. 5. Even more, by taking into account the probability to find the system in the ideal astrometric position $E = 0$ where the light-deflection becomes maximal, it has been found that there is not any relevant binary system during GAIA mission time, see FIG. 6.

In summary, the main results are presented by the conditions (15) and (16) and by the diagram FIG. 6. Accordingly, we come to the conclusion that the detectability of light-deflection in binary systems reaches the technical limit of GAIA mission and might be detected only in case of a very few and highly exotic binary systems. It is, however, very unlikely that such extreme binaries might exist.

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APPENDIX A: TWO-BODY PROBLEM

The calculations in this Appendix follow mainly Ref. [29]. Consider two massive bodies, one component having a mass M_A and spatial coordinate $\boldsymbol{r}_A$, and second component with a mass M_B and spatial coordinate $\boldsymbol{r}_B$, respectively. They orbit around their common center of mass $\boldsymbol{r}_{\rm CMS}$,

$$\boldsymbol{r}_{\rm CMS} = \frac{1}{M_A + M_B} \left(M_A\, \boldsymbol{r}_A + M_B\, \boldsymbol{r}_B\right) . \tag{A1}$$

The Lagrangian $\mathcal{L}$ of the two-body problem is given by

$$\mathcal{L} = \frac{M_A}{2}\, \dot{\boldsymbol{r}}_A^2 + \frac{M_B}{2}\, \dot{\boldsymbol{r}}_B^2 - U(|\boldsymbol{r}_A - \boldsymbol{r}_B|)\,, \tag{A2}$$

where U is the potential. With the aid of relative coordinate $\boldsymbol{r}_{AB} = \boldsymbol{r}_A - \boldsymbol{r}_B$ and reduced mass $\overline{M} = M_A\, M_B/(M_A + M_B)$, the two-body problem can be transformed into into one-body problem,

$$\mathcal{L} = \frac{\overline{M}}{2}\, \dot{\boldsymbol{r}}_{AB}^2 - U(r_{AB})\,. \tag{A3}$$

Polar coordinates (r_{AB}, ϕ) yield (polar angle ϕ should not be confused with light-deflection angle φ)

$$\mathcal{L} = \frac{1}{2} \left(\overline{M}\, \dot{r}_{AB}^2 + r_{AB}^2\, \dot{\phi}^2\right) - U\left(r_{AB}\right). \tag{A4}$$

The orbital angular momentum L is conserved

$$L = \overline{M}\, r_{AB}^2\, \dot{\phi} = {\rm const}\,, \tag{A5}$$

by means of which we obtain for the total energy of the two-body system the expression

$$E = \frac{\overline{M}}{2}\, \dot{r}_{AB}^2 + \frac{L^2}{2\, \overline{M}\, r_{AB}^2} + U\left(r_{AB}\right) . \tag{A6}$$

From Eq. (A6) we deduce

$$\dot{r}_{AB} = \left(\frac{2}{\overline{M}} \left[E - U\left(r_{AB}\right)\right] - \frac{L^2}{\overline{M}^2\, r_{AB}^2}\right) , \tag{A7}$$

and from Eq. (A7) we obtain

$$t = \int dr_{AB} \left(\frac{2}{\overline{M}} \left[E - U\left(r_{AB}\right)\right] - \frac{L^2}{\overline{M}^2\, r_{AB}^2}\right)^{-1/2} + {\rm const}\ , \tag{A8}$$

$$\phi = \int dr_{AB}\, \frac{\overline{M}}{r_{AB}^2} \left(2\, \overline{M} \left[E - U\left(r_{AB}\right)\right] - \frac{L^2}{r_{AB}^2}\right)^{-1/2} + {\rm const}\ , \tag{A9}$$

where in the second relation we have used (A5); note that (A9) is the relation between r_{AB} and ϕ and is called orbital equation. The Eqs. (A8) and (A9) are the general solutions. In order to integrate Eqs. (A8) and (A9) we have to specify the potential U. In case of Kepler problem we have

$$U(r) = -\frac{\alpha}{r_{AB}} \quad \text{with} \quad \alpha = G\, M_A\, M_B\,, \tag{A10}$$

where G being Newtonian gravitational constant. The Eq. (A9) can be integrated and yields

$$\phi = \arccos\left(\frac{L}{r_{AB}} - \frac{\gamma\,\overline{M}\,M_A\,M_B}{L}\right)\left(2\overline{M}\,E + \frac{\gamma^2\,\overline{M}^2\,M_A^2\,M_B^2}{L^2}\right)^{-1/2}, \tag{A11}$$

where the axis are chosen such that the integration constant vanishes. Furthermore, by introducing the eccentricity e (we are interested in closed orbits, i.e. possible values of eccentricity are between $0 \le \mathrm{e} < 1$; $\mathrm{e} = 0$ corresponds to a circular orbit),

$$\mathrm{e} = \left(1 + \frac{2\;E\;L^2\;(M_A + M_B)}{\gamma^2\;M_A^3\;M_B^3}\right)^{1/2}, \tag{A12}$$

the solution (A11) can be written as

$$\frac{1}{r_{AB}}\,\frac{L^2}{\gamma\,\overline{M}\,M_A\,M_B} = 1 + \mathrm{e}\;\cos\phi\,. \tag{A13}$$

We note the expressions of semi-major axis A and semi-minor axis B,

$$\mathrm{A} = \frac{L^2}{(1-\mathrm{e}^2)\;\gamma\,\overline{M}\,M_A\,M_B}\,, \tag{A14}$$

$$\mathrm{B} = \frac{L^2}{\sqrt{1-\mathrm{e}^2}\;\gamma\,\overline{M}\,M_A\,M_B}\,. \tag{A15}$$

To solve the integral (A8), we substitute

$$r_{AB} - \mathrm{A} = -\mathrm{A}\;\mathrm{e}\;\cos E\,, \tag{A16}$$

where E is called eccentric anomaly. Then, we obtain for the integral in Eq. (A8) the expression

$$t = \left(\frac{A^3}{\gamma\;(M_A + M_B)}\right)^{1/2} \int dE\;(1 - \mathrm{e}\;\cos E)\,, \tag{A17}$$

and the solution is given by

$$t = \left(\frac{\mathrm{A}^3}{\gamma\;(M_A + M_B)}\right)^{1/2}\;(E - \mathrm{e}\;\sin E)\;, \tag{A18}$$

where the integration constant vanishes, i.e. the particle at $t = 0$ is in periastron. The Eqs. (A13) and (A18) are the general solutions of two-body problem. They can be rewritten as

$$r_{AB} = A\,(1 - \mathrm{e}\;\cos E)\;, \tag{A19}$$

$$t = \left(\frac{\mathrm{A}^3}{\gamma\,(M_A + M_B)}\right)^{1/2}\;(E - \mathrm{e}\;\sin E)\;. \tag{A20}$$

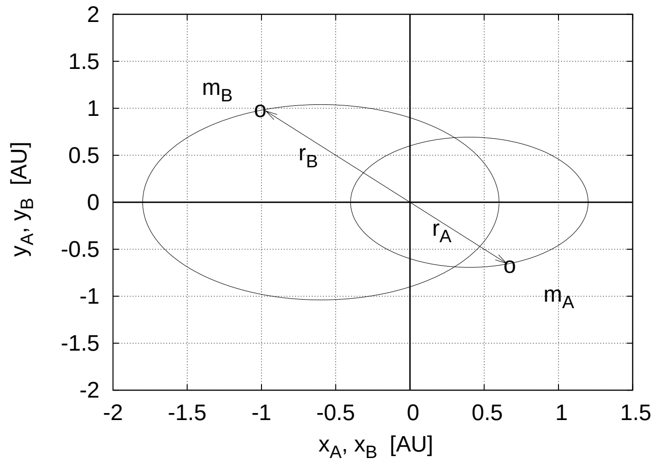

FIG. 7: Geometrical representation of the coordinates of a binary star. In the example considered, the masses are $M_A = 1.5\ M_\odot$ and $M_B = 1.0\ M_\odot$, respectively. The semi-major axis of the binary system is chosen $\mathrm{A} = 2\ a.u.$ and eccentricity is taken $\mathrm{e} = 0.5$. The coordinates of mass center are $\boldsymbol{r}_{\mathrm{CMS}} = \boldsymbol{0}$. The massive bodies A and B are always in opposition to each other.

In case of ellipse, $E = 0$ in periastron, $E = \pi$ in apastron, and for a complete orbit E runs from $E = 0$ to $E = 2\pi$. Thus, we obtain for the orbital period the expression

$$T \;=\; 2\pi \left(\frac{\mathrm{A}^3}{\gamma\,(M_A + M_B)}\right)^{1/2} . \tag{A21}$$

We also note the solution $\boldsymbol{r}$ in cartesian coordinates, $x = r_{AB}\ \cos\phi$ and $y = r_{AB}\ \sin\phi$:

$$\boldsymbol{r}_{AB} \;=\; \begin{pmatrix} x \\ y \end{pmatrix} , \tag{A22}$$

$$x \;=\; \mathrm{A}\,(\cos E - \mathrm{e})\ , \tag{A23}$$

$$y \;=\; \mathrm{A}\left(1 - \mathrm{e}^2\right)^{1/2}\ \sin E\,. \tag{A24}$$

The coordinates of the bodies A and B, i.e. their orbits, are given by

$$\boldsymbol{r}_A \;=\; \boldsymbol{r}_{\mathrm{CMS}} + \frac{\boldsymbol{r}_{AB}}{1 + \frac{M_A}{M_B}} , \tag{A25}$$

$$\boldsymbol{r}_B \;=\; \boldsymbol{r}_{\mathrm{CMS}} - \frac{\boldsymbol{r}_{AB}}{1 + \frac{M_B}{M_A}} . \tag{A26}$$

Accordingly, the form of orbit is determined by two orbital parameters: semi-major axis a and eccentricity e. In order to konw the position of one celestial body, either component A or component B, two additional orbital parameters are needed, namely orbital period T and true anomaly ν. A geometrical representation of the coordinates of the components of a binary star is given in FIG. 7 for the case of $M_A = 1.5\,M_\odot$, $M_B = 1.0\,M_\odot$, $\mathrm{e} = 0.5$, $\mathrm{A} = 2\,a.u.$.

APPENDIX B: DERIVATION OF EQ. (6)

In the inclination formula the impact of eccentricity on light-deflection is neglected, thus $e = 0$, implying that $\omega = 0$ is taken. Then, for the vectors from massive body to observer $\boldsymbol{x}_1$ and from massive body to source $\boldsymbol{x}_0$, we have

$$\boldsymbol{x}_1 = r \begin{pmatrix} \sin i - \epsilon_1 \cos E \\ \\ -\epsilon_1 \sin E \\ \\ \cos i \end{pmatrix} , \tag{B1}$$

$$\boldsymbol{x}_0 = -A \begin{pmatrix} \cos E \\ \\ \sin E \\ \\ 0 \end{pmatrix} , \tag{B2}$$

where we have introduced the small parameter

$$\epsilon_1 = \frac{A}{r} \frac{m_B}{m_A + m_B} \ll 1 . \tag{B3}$$

From Eqs. (B1) and (B2) we obtain for vector $\boldsymbol{k} = \boldsymbol{R}/R$, where $\boldsymbol{R} = \boldsymbol{x}_1 - \boldsymbol{x}_0$, the expression

$$\boldsymbol{k} = \frac{1}{\sqrt{1 + 2\,\epsilon_2 \sin i \cos E + \epsilon_2^2}} \begin{pmatrix} \sin i + \epsilon_2 \cos E \\ \\ \epsilon_2 \sin E \\ \\ \cos i \end{pmatrix} , \tag{B4}$$

where we have introduced the small parameter

$$\epsilon_2 = \frac{A}{r} \frac{m_A}{m_A + m_B} \ll 1 . \tag{B5}$$

Here, we notice that (B2) and (B4) yields

$$d \; = \; |\boldsymbol{k} \times \boldsymbol{x}_0| = A \; |\cos i| \left(1 + \mathcal{O}\left(\epsilon_2\right)\right) . \tag{B6}$$

Using (B1) - (B5), the generalized lens equation (5) reads

$$\varphi_{1,2} = \frac{1}{2} \frac{1}{T_1\,T_2} \left(\sqrt{\frac{A^2}{r^2}\,\left(1 - T_0^2\right) + 4\left(1+\gamma\right) \frac{m}{r} \frac{A}{r} \,\left(T_0 + T_1 - \epsilon_1\right)\, T_2} \mp \frac{A}{r} \sqrt{1 - T_0^2}\right) , \tag{B7}$$

where

$$T_0 \;=\; \sin i \cos E\,, \tag{B8}$$

$$T_1 \;=\; \sqrt{1 - 2\,\epsilon_1 \sin i \cos E + \epsilon_1^2}\,, \tag{B9}$$

$$T_2 \;=\; \sqrt{1 + 2\,\epsilon_2 \sin i \cos E + \epsilon_2^2}\,. \tag{B10}$$

In this investigation the light-deflection in binbary systems is considered, but not the lens effect. Therefore, in what follows the solution $\varphi \equiv \varphi_1$ is taken. By series expansion we obtain

$$\varphi = \frac{1}{2}\left(\sqrt{\frac{A^2}{r^2}\,(1-w^2)+4\,(1+\gamma)\,\frac{m}{r}\frac{A}{r}\,(1+w)}-\frac{A}{r}\sqrt{1-w^2}\right)+\mathcal{O}\left(\frac{A}{r}\sqrt{\frac{m}{r}\frac{A}{r}}\right), \tag{B11}$$

where we have introduced the abbreviation $w = \sin i\,\cos E$. The minmal and maximal light-deflection angle are

$$\varphi_{\rm min} = \varphi\left(i=\frac{\pi}{2}\,,E=\pi\right) = 0\,, \tag{B12}$$

$$\varphi_{\rm max} = \varphi\left(i=\frac{\pi}{2}\,,E=0\right) = 2\,\frac{\sqrt{m\,A}}{r}\,. \tag{B13}$$

In our stydy we are interested in the maximal possible light-deflection effext. Accordingly, two situtations are relevant: namely

$$\varphi\,(E=0) = \frac{1}{2}\left(\sqrt{\frac{A^2}{r^2}\,\cos^2 i+4\,(1+\gamma)\,\frac{m}{r}\,\frac{A}{r}\,(1+\sin i)}-\frac{A}{r}\,|\cos i|\right)+\mathcal{O}\left(\frac{A}{r}\sqrt{\frac{m}{r}\,\frac{A}{r}}\right), \tag{B14}$$

which is just Eq. (6), and

$$\varphi\left(i=\frac{\pi}{2}\right) = \frac{1}{2}\left(\sqrt{\frac{A^2}{r^2}\sin^2 E+4\,(1+\gamma)\,\frac{m}{r}\frac{A}{r}\,(1+\cos E)}-\frac{A}{r}\,|\sin E|\right)+\mathcal{O}\left(\frac{A}{r}\sqrt{\frac{m}{r}\frac{A}{r}}\right), \tag{B15}$$

Furthermore, it is useful to take into account only astrometric positions with $0 \le E \le \frac{\pi}{2}$, because otherwise the light-deflection is for sure negligible.

APPENDIX C: DERIVATION OF EQ. (9)

In this Section we will use $\gamma = 1$. From (B14) we obtain

$$\left(2\,\varphi+\frac{A}{r}\,|\cos i|\right)^2 = \frac{A^2}{r^2}\,\cos^2 i+8\,\frac{m}{r}\,\frac{A}{r}\,(1+\sin i)\,. \tag{C1}$$

From (C1) we obtain

$$\left(\varphi^2+4\,\frac{m^2}{r^2}\right)\frac{A^2}{r^2}\sin^2 i+4\frac{m}{r}\,\frac{A}{r}\left(2\,\frac{m}{r}\frac{A}{r}-\varphi^2\right)\sin i = \left(\frac{A^2}{r^2}+4\,\frac{m}{r}\frac{A}{r}\right)\varphi^2-4\frac{m^2}{r^2}\,\frac{A^2}{r^2}-\varphi^4. \tag{C2}$$

Eq. (C2) represents an quadratic equation for the expression $|\sin i\,|$, which has the following both solutions for the inclination i:

$$\sin i = \left(-\frac{p}{2}\pm\sqrt{\frac{p^2}{4}-q}\right), \tag{C3}$$

where

$$p = \frac{8\,m^2\,A - 4\,m\,r^2\,\varphi^2}{A\,(r^2\,\varphi^2 + 4\,m^2)}\,, \tag{C4}$$

$$q = -\frac{A^2\,r^2\,\varphi^2 + 4\,m\,A\,r^2\,\varphi^2 - 4\,m^2\,A^2 - r^4\,\varphi^4}{A^2\,(r^2\,\varphi^2 + 4\,m^2)}\,. \tag{C5}$$

Eq. (C3) represents two solutions, however only the one with the plus-sign is valid. This can be shown as follows. For the value $i = \frac{\pi}{2}$ the light-deflection has to be $\varphi = \varphi_{\rm max} = 2\sqrt{m\,A}/r$, according to (8). Inserting $\varphi_{\rm max}$ in Eqs. (C4) and (C5) we obtain $p = -2\,m/(A+m)$ and $q = -(A-m)/(A+m)$. If we insert $i = \frac{\pi}{2}$ for p and q into Eq. (C3) we obtain the relation

$$1 = \frac{m}{A+m} \pm \sqrt{\frac{m^2}{(A+m)^2} + \frac{A-m}{A+m}} = \frac{m}{A+m} \pm \frac{A}{A+m}\,. \tag{C6}$$

Obviously, relation (C6) is only correct for the upper sign. We note, that a very similar proof can also be done using $\varphi_{\rm min}$ which also yields that the upper sign is the correct and only solution. Therefore, the inclination formula is given by (note, that in the region under consideration $\sin i = \sin{(\pi - i)}$)

$$i = \begin{cases} \arcsin\left(-\dfrac{p}{2} + \sqrt{\dfrac{p^2}{4} - q}\right) & \text{for} \quad 0 \le i \le \dfrac{\pi}{2}\,, \\[2ex] \pi - \arcsin\left(-\dfrac{p}{2} + \sqrt{\dfrac{p^2}{4} - q}\right) & \text{for} \quad \dfrac{\pi}{2} < i \le \pi\,. \end{cases} \tag{C7}$$

For the complete region $0 \le i \le \pi$ we obtain for the inclination formula the following expression:

$$\left|\frac{\pi}{2} - i\right| = \arccos\left(-\frac{p}{2} + \sqrt{\frac{p^2}{4} - q}\right), \tag{C8}$$

where p and q are given by Eqs. (C4) and (C5).

APPENDIX D: DERIVATION OF EQ. (14)

In this Appendix we briefly summarize the relevant steps which lead us to the inclination formula given in Ref. [13], because that investigation has not been published. According to [7], the transformation of $\boldsymbol{k}$ to the unit tangent vector $\boldsymbol{n}$ of light-trajectory at observer is in post-Newtonian order given by

$$\boldsymbol{n} = \boldsymbol{k} - (1+\gamma)\,m\,\frac{\boldsymbol{k} \times (\boldsymbol{x}_0 \times \boldsymbol{x}_1)}{x\,(x_0\,x_1 + \boldsymbol{x}_0 \cdot \boldsymbol{x}_1)} + \mathcal{O}\left(m^2\right). \tag{D1}$$

This expression is valid as long as $d \gg m$, but diverges for $d \rightarrow 0$. By means of (D1) we obtain for the light-deflection angle φ, i.e. for the angle between $\boldsymbol{n}$ and $\boldsymbol{k}$, the expression

$$\varphi = (1+\gamma)\,\frac{m}{r}\,\tan\frac{\psi}{2}\,, \tag{D2}$$

where we have used $\dfrac{\sin\alpha}{1+\cos\alpha} = \tan\frac{\alpha}{2}$, $x = r + \mathcal{O}\left(A\right)$, amd ψ is the angle $\delta\left(\boldsymbol{x}_0, \boldsymbol{x}_1\right)$. The expression (D2) diverges for $\psi \rightarrow \pi$, which coresponds with the mentioned divergence of (D1) for $d \rightarrow 0$. Obviously, $\psi \le i + \frac{\pi}{2}$ (from Eq. (32) it is obvious that eccentric anomaly E of binary system should be very close to zero for the light-deflection effect to be observable at the level of microarcsecond, i.e. we actually could even assume $\psi \simeq i + \frac{\pi}{2}$) and we obtain

$$\varphi \;\le\; \left(1+\gamma\right)\frac{m}{r}\,\tan\left(\frac{i}{2}+\frac{\pi}{4}\right) = \left(1+\gamma\right)\frac{m}{r}\,\cot\left(\frac{\pi}{4}-\frac{i}{2}\right), \tag{D3}$$

where we have used that $\tan\left(\alpha+\frac{\pi}{2}\right) = -\cot\alpha$, $\cot\alpha = \tan^{-1}\alpha$, and the antisymmetry of function $\cot\alpha$. From (D3) we obtain

$$\left|\frac{\pi}{2} - i\right|_{\rm KMS} \;\le\; 2\,\arctan\left(\left(1+\gamma\right)\frac{m}{r\,\varphi}\right), \tag{D4}$$

which, for $\gamma = 1$, is just the inclination formula (14). However, we have to underline that due to the divergence for $d \rightarrow 0$, which corresponds to $\psi \rightarrow \pi$, the applicability of (D4) is restricted by the condition $d \gg m$. Using $d = A\,\left|\cos i\right|$ we obtain the validity condition for the applicability of (D4):

$$\left|\frac{\pi}{2} - i\right|_{\rm KMS} \;\gg\; \arcsin\frac{m}{A}\,. \tag{D5}$$

APPENDIX E: PROBABILITY DISTRIBUTION

Let us assume we have a probability distribution of any quantity x, given by $f(x)$. The probability P, to find a system in the intervall $x_i \le x \le x_i + \Delta x$ is given by

$$P(x_i \le x \le x_i + \Delta x) \;=\; \frac{\displaystyle\int\limits_{x_i}^{x_i+\Delta x} dz\; f(z)}{\displaystyle\int\limits_{x_{\rm min}}^{x_{\rm max}} dz\; f(z)}\,, \tag{E1}$$

where the region of validity of probability distribution $f(x)$ is given by $x_{\rm min}$ and $x_{\rm max}$. In the infinitesimal limit $\Delta x \rightarrow dx$, we obtain by series expansion the following explicit form for the here used probability distributions: for a power law $f(x) \sim x^{-\alpha}$ with $\alpha \neq 1$ we find

$$f(x) \;=\; \frac{\left(1-\alpha\right)\;x^{-\alpha}}{x_{\rm max}^{(1-\alpha)} - x_{\rm min}^{(1-\alpha)}}\,, \tag{E2}$$

and for a logarithmic law $f(x) \sim x^{-1}$ we have

$$f(x) \;=\; \frac{1}{x}\,\left(\ln\frac{x_{\rm max}}{x_{\rm min}}\right)^{-1}\,. \tag{E3}$$

The normalization is $\int\limits_{x_{\rm min}}^{x_{\rm max}} f(x) = 1$ and we also note the averaged value $\overline{x} = \int\limits_{x_{\rm min}}^{x_{\rm max}} f(x)\,x\,dx$.