Light-propagation in the gravitational field of moving quadrupoles

Sven Zschocke

*Lohrmann Observatory,
Dresden Technical University,
Mommsen Str. 13,
D-01062 Dresden, Germany*

A simplified formula for light-deflection in the quadrupole field of moving massive bodies has been obtained in [1, 2, 3], which will be applied for Gaia data reduction. So far, in Gaia data reduction it has been assumed that the positions of the giant planets should be computed at the retarded instant of time. The problem of light-deflection due to quadrupole field of moving planets has been re-considered in [4]. According to their solution, the position and velocity of the massive body have to be taken at retarded time. We show that the solution given in [4] coincides with our simplified quadrupole formula obtained in [1, 2, 3]. This coincidence implies that the positions of giant planets have, in fact, to be taken at retarded time.

Contents

I. INTRODUCTION

An essential experiment of Gaia mission to test relativity concerns the new effect of light-deflection at giant planets due to their quadrupole field. Analytical solutions of light deflection by a quadrupole field of a massive body are well-known and have been investigated by many authors [5, 6, 7, 8, 9, 10, 11, 12]. Because these formulas are rather complicated, they imply massive computations of quadrupole light deflection and are too time-consuming for Gaia data reduction. Therefore, we have derived a simplified quadrupole formula in [1, 2, 3], which is suitable for a time-efficient computation of quadrupole light-deflection on microarcsecond level of accuracy.

In reality, while the light-signal is being emitted at a position $\boldsymbol{x}_0$ at time moment t_0 and received at position $\boldsymbol{x}_1$ at a time moment t_1, the massive body chances the position from $\boldsymbol{x}_A(t_0)$ to $\boldsymbol{x}_A(t_1)$. Therefore, it is not obvious at which coordinate time $t_0 \le t \le t_1$ the position of the massive body $\boldsymbol{x}_A(t)$ has to be chosen in the formula of quadrupole light-deflection. Up to now, in Gaia data reduction it is tacitly and implicitly assumed that the positions of multipoles (Jupiter and Saturn) should be computed at the retarded moments of time, given by the implicit light-cone equation:

$$
s_1^A = t_1 - \frac{\left| \boldsymbol{x}_1 - \boldsymbol{x}_A \left(s_1^A \right) \right|}{c}, \tag{1}
$$

where $\boldsymbol{x}_1$ is the position of observer at observation time t_1 and $\boldsymbol{x}_A \left(s_1^A \right)$ is the coordinate of massive body at retarded time moment s_1^A. However, no theoretical proof of this assumption has been given. This problem has recently been solved by [4]. In our report we show that the solutions in [1, 2, 3] and in [4] agree with each other, implying that in the quadrupole formula the position of massive body has indeed to be taken at the retarded instant of time.

II. SIMPLIFIED QUADRUPOLE FORMULA

In [1, 2, 3] a simplified quadrupole formula has been derived which takes into account only those terms relevant for microarcsecond level of accuracy. In this Section we will give the main steps and results. Consider a gravitational field of N massive bodies A, and the positions of these individual massive bodies are $\boldsymbol{x}_A$. A light-ray is being emitted at a position $\boldsymbol{x}_0$ at time moment t_0 and received at position $\boldsymbol{x}_1$ at a time moment t_1. The unit coordinate direction of the light propagation at the moment of observation reads $\boldsymbol{n} = \dfrac{\dot{\boldsymbol{x}}(t_1)}{|\dot{\boldsymbol{x}}(t_1)|}$ and the unit tangent vector of light path at infinitely past is $\boldsymbol{\sigma} = \lim\limits_{t \to -\infty} \dfrac{\dot{\boldsymbol{x}}(t)}{c}$. Then, in post-Newtonian order, the transformation $\boldsymbol{\sigma}$ to $\boldsymbol{n}$ is given by [9, 10]

$$
\boldsymbol{n} = \boldsymbol{\sigma} + \sum_i \delta\boldsymbol{\sigma}_i + \mathcal{O}\left(c^{-4} \right), \tag{2}
$$

where the sum runs over individual terms of various physical origin, that means monopole gravitational field, quadrupole field and higher multipole fields. The spherical symmetric part (monopole field) due to one massive body A is given, for instance, by Eq. (102) in [13].

Here, we are only interested at the quadrupole light deflection term. For one massive body A, the quadrupole light-deflection is given as follows (see Eq. (40) in [2] or Eq. (9) in

[3]):

$$\delta\boldsymbol{\sigma}_{\mathrm{Q}} = -\frac{G\,M_A}{c^2}\,J_2^A\,\frac{P_A^2}{d_A^3}\,\left(2 + 3\,\cos\psi - \cos^3\psi\right)$$

$$\times \left[\left(1 - (\boldsymbol{\sigma}\cdot\boldsymbol{e}_3)^2 - 4\,(\boldsymbol{n}_A\cdot\boldsymbol{e}_3)^2\right)\boldsymbol{n}_A + 2\,(\boldsymbol{n}_A\cdot\boldsymbol{e}_3)\,\boldsymbol{e}_3 - 2\,(\boldsymbol{\sigma}\cdot\boldsymbol{e}_3)\,(\boldsymbol{n}_A\cdot\boldsymbol{e}_3)\,\boldsymbol{\sigma}\right]\,. \quad (3)$$

Here, M_A is the mass of body A, c is the speed of light, G is the gravitational constant, and the impact vector

$$\boldsymbol{d}_A = \boldsymbol{\sigma}\times\left(\boldsymbol{r}_1^A\times\boldsymbol{\sigma}\right), \quad (4)$$

where $\boldsymbol{r}_1^A = \boldsymbol{x}_1 - \boldsymbol{x}_A$, and the absolute value $d_A = |\boldsymbol{d}_A|$. Furthermore, the angle $\psi = \delta\left(\boldsymbol{\sigma},\boldsymbol{r}_1^A\right)$, the unit vector along the axis of symmetry (rotational axis of massive body) is denoted by $\boldsymbol{e}_3$, P_A denotes the equatorial radius, and J_2^A is the coefficient of second zonal harmonic of the gravitational field of massive body A. The unit vector $\boldsymbol{n}_A$ is defined by

$$\boldsymbol{n}_A = \frac{\boldsymbol{d}_A}{d_A}\,. \quad (5)$$

Since the massive bodies move, the question arises at which instant of time the positions of the massive bodies have to be chosen. This problem has been solved in [4], and the main results of this work will be subject of the next Section.

III.   QUADRUPOLE FORMULA BY KOPEIKIN & MAKAROV

A.   Description of the approach

The light-deflection at moving monopoles and quadrupoles has been re-investigated by in [4]. In this Subsection we describe the basic steps of this approach. The gravitational field is described by $g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$ where $h_{\alpha\beta}$ is the metric perturbation in post-Minkowski apprximation. Using harmonic gauge, the linearized Einstein equations for the pertubation $h_{\alpha\beta}$ are homogeneus wave equations,

$$\left(-\frac{\partial^2}{\partial t^2} + \boldsymbol{\nabla}^2\right)h_{\alpha\beta} = 0\,. \quad (6)$$

A general solution of (6) is given in terms of multipole expansion [14, 15], where the terms in the perturbation $h_{\alpha\beta}$ are taken at the retarded instant of time s_1^A. The retardation follows directly from the retarded (causal) solution of the homogeneus wave equations (6).

In the approach [4] the monopole, dipole and quadrupole terms of the general multipole expansion of $h_{\alpha\beta}$, given in Eqs. (12) - (14) in [4], are taken into account. The center of coordinate system is shifted from the mass center of the massive body by a spatial distance. Then, by means of parallel axis theorem (Huygens-Steiner theorem), they apply general expressions for the quadrupole moment and their time derivative, see Eq. (17) and Eq. (20) in [4].

Then, in [4] the geodesic equation is rewritten into a considerably simpler form, given by Eq. (19) in [16]. The integration leads to the expressions (28) and (29) in [4]. Using their developed integration method, presented in part by Eqs. (30) and (31) in [4], they succeed to integrate analytically the differential equation (29) in [4]. Furthermore, in [4] all those terms are neglected which they proof to contribute less than 1 microarcsecond. The result of their approach is finally given by Eqs. (39) - (41) in [4].

B.   The analytical solution for quadrupole light-deflection

In Eq. (44) in [4], the following form for the light-deflecetion of sources at infinite distances due to one moving quadrupole A has been given:

$$\delta\boldsymbol{\sigma}_{\mathrm{Q}}^A = 4\left(1 - \frac{\boldsymbol{\sigma}\cdot\boldsymbol{v}_A}{c}\right)\frac{G\,M_A}{c^2}\,J_2^A\,\frac{P_A^2}{d_A^3}$$

$$\times\left[\left[(\boldsymbol{e}_3\cdot\boldsymbol{n}_A)^2 - (\boldsymbol{e}_3\cdot\boldsymbol{m}_A)^2\right]\,\boldsymbol{n}_A - 2\,(\boldsymbol{e}_3\cdot\boldsymbol{n}_A)(\boldsymbol{e}_3\cdot\boldsymbol{m}_A)\,\boldsymbol{m}_A\right]. \tag{7}$$

It is essential to noticed, that all time-dependent quantities in Eq. (7), that is $\boldsymbol{x}_A$, $\boldsymbol{v}_A$ and $\boldsymbol{e}_3$, and therefore also $\boldsymbol{n}_A$ and $\boldsymbol{m}_A$, are computed at retarded instant of time s_1^A given by Eq. (1). It should be noticed that in Eq. (7) we have omitted a term proportional to the displacement of the planetary center from the origin of the coordinate system which is an artificial term of the approach and not relevant for Gaia data reduction. Furthermore, the unit vector $\boldsymbol{m}_A$ is defined by

$$\boldsymbol{m}_A = \boldsymbol{\sigma}\times\boldsymbol{n}_A. \tag{8}$$

In Eq. (7) we have used our notational conventions in [2, 3]. As mentioned in the introductionary Section, we have replaced the vector $\boldsymbol{k}$ by $\boldsymbol{\sigma}$, since $\boldsymbol{\sigma} = \boldsymbol{k} + \mathcal{O}\,(m)$, that means such a replacement would cause effects of higher order beyond the post-Newtonian approximation.

C.   Comparison of the formula of quadrupole light-deflection

The quadrupole light-deflection (7) can be estimated by

$$\left|\delta\boldsymbol{\sigma}_{\mathrm{Q}}^A\right| \le 4\left(1 - \frac{\boldsymbol{\sigma}\cdot\boldsymbol{v}_A}{c}\right)\frac{G\,M_A}{c^2}\,\frac{\left|J_2^A\right|}{P_A}. \tag{9}$$

Accordingly, for Jupiter and Saturn we have

$$\left|\boldsymbol{\sigma}_{\mathrm{Q}}^{\mathrm{Jupiter}}\right| \le \left(1 - \frac{\boldsymbol{\sigma}\cdot\boldsymbol{v}_{\mathrm{Jupiter}}}{c}\right)240\,\mu\mathrm{as}, \tag{10}$$

$$\left|\boldsymbol{\sigma}_{\mathrm{Q}}^{\mathrm{Saturn}}\right| \le \left(1 - \frac{\boldsymbol{\sigma}\cdot\boldsymbol{v}_{\mathrm{Saturn}}}{c}\right)95\,\mu\mathrm{as}, \tag{11}$$

while for other massive bodies of the solar system we find considerably smaller values. Furthermore, the orbital speed of these planets in respect to the barycenter is of the order $v_A \sim 10^{-4}\,c$. Hence, the contribution of the velocity-term in the quadrupole light-deflection is by far less than about $0.1\,\mu$as and can be neglected for Gaia astrometric accuracy. Thus, the quadrupole light-deflection formula (7) can be simplified as follows:

$$\delta\boldsymbol{\sigma}_{\mathrm{Q}}^A = 4\,\frac{G\,M_A}{c^2}\,J_2^A\,\frac{P_A^2}{d_A^3}$$

$$\times\left[\left[(\boldsymbol{e}_3\cdot\boldsymbol{n}_A)^2 - (\boldsymbol{e}_3\cdot\boldsymbol{m}_A)^2\right]\,\boldsymbol{n}_A - 2\,(\boldsymbol{e}_3\cdot\boldsymbol{n}_A)(\boldsymbol{e}_3\cdot\boldsymbol{m}_A)\,\boldsymbol{m}_A\right]. \tag{12}$$

Eq. (12) determines the quadrupole light-deflection for moving massive bodies, while the rotational axis $\boldsymbol{e}_3$ and the unit vectors $\boldsymbol{n}_A$ and $\boldsymbol{m}_A$ have to be computed at retarded instant of time s_1^A given by Eq. (1).

The expression given by Eq. (12) in [4] coincides with our expression given in Eq. (3), see [1, 2, 3]. This can be shown by means of the relation $\cos^2 x = 1 - \sin^2 x$, that is $(\boldsymbol{e}_3 \cdot \boldsymbol{m}_A)^2 = 1 - (\boldsymbol{e}_3 \times \boldsymbol{m}_A)^2$. Then, by taking into account $\boldsymbol{e}_3 \times \boldsymbol{m}_A = \boldsymbol{e}_3 \times (\boldsymbol{\sigma} \times \boldsymbol{n}_A)$ we obtain the relation

$$(\boldsymbol{e}_3 \cdot \boldsymbol{m}_A)^2 = 1 - (\boldsymbol{e}_3 \cdot \boldsymbol{n}_A)^2 - (\boldsymbol{e}_3 \cdot \boldsymbol{\sigma})^2 \ . \tag{13}$$

Inserting (13) into (12) we obtain

$$\delta\boldsymbol{\sigma}_{\mathrm{Q}} = -4\, \frac{G\, M_A}{c^2}\, J_2^A\, \frac{P_A^2}{d_A^3}$$

$$\times \left[\left[1 - 2\, (\boldsymbol{e}_3 \cdot \boldsymbol{n}_A)^2 - (\boldsymbol{e}_3 \cdot \boldsymbol{\sigma})^2 \right]\, \boldsymbol{n}_A + 2\, (\boldsymbol{e}_3 \cdot \boldsymbol{n}_A)\, (\boldsymbol{e}_3 \cdot \boldsymbol{m}_A)\, \boldsymbol{m}_A \right] \ . \tag{14}$$

Furthermore, for the vector $\boldsymbol{e}_3$ we have to use the linear combination in terms of the unit vectors $\boldsymbol{d}_A$, $\boldsymbol{\sigma}$ and $\boldsymbol{m}_A$, that means $\boldsymbol{e}_3 = (\boldsymbol{e}_3 \cdot \boldsymbol{\sigma})\, \boldsymbol{\sigma} + (\boldsymbol{e}_3 \cdot \boldsymbol{n}_A)\, \boldsymbol{n}_A + (\boldsymbol{e}_3 \cdot \boldsymbol{m}_A)\, \boldsymbol{m}_A$, from which we conclude

$$(\boldsymbol{e}_3 \cdot \boldsymbol{m}_A)\, \boldsymbol{m}_A = \boldsymbol{e}_3 - (\boldsymbol{e}_3 \cdot \boldsymbol{\sigma})\, \boldsymbol{\sigma} - (\boldsymbol{e}_3 \cdot \boldsymbol{n}_A)\, \boldsymbol{n}_A \ . \tag{15}$$

Inserting (15) into (14), we obtain

$$\delta\boldsymbol{\sigma}_{\mathrm{Q}}^A = -4\, \frac{G\, M_A}{c^2}\, J_2^A\, \frac{P_A^2}{d_A^3}$$

$$\times \left[\left(1 - (\boldsymbol{\sigma} \cdot \boldsymbol{e}_3)^2 - 4\, (\boldsymbol{n}_A \cdot \boldsymbol{e}_3)^2 \right) \boldsymbol{n}_A + 2\, (\boldsymbol{n}_A \cdot \boldsymbol{e}_3)\, \boldsymbol{e}_3 - 2\, (\boldsymbol{\sigma} \cdot \boldsymbol{e}_3)\, (\boldsymbol{n}_A \cdot \boldsymbol{e}_3)\, \boldsymbol{\sigma} \right] \ . \tag{16}$$

This expression coincides with our quadrupole light-deflection for a massive body, given by Eq. (3) if we approximate $2 + 3 \cos \psi - \cos^3 \psi$ by a factor 4. This approximation follows from $\sin \psi = \dfrac{d_A}{r_1^A}$, that means $2 + 3 \cos \psi - \cos^3 \psi = 4 + \mathcal{O}\left(\dfrac{d_A^4}{r_1^{A4}} \right)$. Thus, we have shown the coincidence of our quadrupole formula given by Eq. (3) (that means Eq. (40) in [2] or Eq. (9) in [3]) with the quadrupole formula in [4] given by Eq. (12). This coincidence implies that for Gaia data reduction the position of giant planets has, in fact, to be taken at the retarded instant of time, given by the implicit light-cone equation (1).

IV. SUMMARY

A simplified quadrupole formula for massive bodies has been derived in [1, 2, 3], which will be used for a time-efficient computation of quadrupole light-deflection in Gaia data reduction. However, since the massive bodies move during the light-signal travelling, it is not obvious at which coordinate time the position of the massive body $\boldsymbol{x}_A(t)$ has to be taken in this formula of quadrupole light-deflection. So far, in Gaia data reduction it has been tacitly assumed that the positions of the giant planets should be computed at the retarded instant of time $\boldsymbol{x}_A(s_1^A)$. However, no theoretical proof of this assumption has been performed.

Therefore, in this report, we have re-considered the results of [4] where the light-deflection at moving quadrupoles has been determined. We have shown that the quadrupole formula obtained in [4] given by Eq. (12) coincides with our simplified quadrupole formula obtained in [1, 2, 3] and given by Eq. (3). This coincidence implies that the position of massive body has, in fact, to be taken at the retarded instant of time.

Acknowledgements

---

[1] S. Zschocke, S.A. Klioner, (2006) "*Efficient computation of the quadrupole light deflection*" GAIA-CA-TN-LO-SZ-001-1; available from Gaia document archive http://www.rssd.esa.int/llink/livelink.

[2] S. Zschocke, S.A. Klioner, (2010) "*Efficient computation of the quadrupole light deflection*" GAIA-CA-TN-LO-SZ-001-2; available from Gaia document archive http://www.rssd.esa.int/llink/livelink; preprint available at arXiv:0907.4318v3.

[3] S. Zschocke, S.A. Klioner, Class. Quantum Grav. 28 (2011) 015009.

[4] S.M. Kopeikin, V.V. Makarov, Phys. Rev. D **75** 062002 (2007).

[5] O.S. Ivanitskaya (1979) *Lorentz Basis and Gravitational Effects in Einstein's Theory of Gravitation* (Minsk, Nauka i Tekhnika, in Russian).

[6] R. Epstein, I.I. Shapiro, Phys. Rev. D **22** (1980) 2947.

[7] G.W. Richter, A. Matzner Phys. Rev. D **26** (1982) 1219; **26** (1982) 2549; **28** (1982) 3007.

[8] S.A. Cowling, Mon. Not. R. astr. Soc. **209** (1984) 415.

[9] S.A. Klioner, Sov. Astron. **35** (1991) 523.

[10] S.A. Klioner, Astron. J. **125** (2003) 1580.

[11] S.A. Klioner, S.M. Kopejkin, Astron J. **104** (1992) 897.

[12] S.A. Klioner, R. Blankenburg (2003) Gaia report, available from Gaia document archive http://www.rssd.esa.int/llink/livelink.

[13] S.A. Klioner, S. Zschocke, Class. Quantum Grav. 27 (2010) 075015.

[14] K.S. Thorne, Rev. Mod. Phys. **52** (1980) 299.

[15] L. Blanchet, T. Damour, Phil. Trans. R. Soc. London A **320** (1986) 379.

[16] S.M. Kopeikin, G. Schäfer, Phys. Rev. D **60** (1999) 124002.