Testing Local Lorentz Invariance with High-accuracy Astrometrical Observations

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This is a draft aimed at collecting all relevant formulas for the real calculations.

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I. INTRODUCTION

II. DIRECT AND INVERSE TRANSFORMATIONS BETWEEN A PREFERRED AND NON-PREFERRED FRAME

The preferred frame where the light velocity is isotropic and equal to c has coordinates (T, X^a). An arbitrary inertial non-preferred frame (where the light velocity is not necessarily equal to c and not necessarily isotropic) is denoted as (t, x^i).

It is assumed here that the transformation between preferred coordinates (T, X^a) and non-preferred ones (t, x^i) read

$$c\,t = \Lambda^0_0\, c\,T + \Lambda^0_a\, X^a, \tag{1}$$
$$x^i = \Lambda^i_0\, c\,T + \Lambda^i_a\, X^a, \tag{2}$$

where

$$\Lambda^0_0 = A, \tag{3}$$
$$\Lambda^0_a = E^a, \tag{4}$$
$$\Lambda^i_0 = F\,K^i, \tag{5}$$
$$\Lambda^i_a = D\,\delta^{ai} + B\,\frac{K^a\,K^i}{K^2}. \tag{6}$$

Here $K^a = V^a/c$ where V^a are the spatial components of the velocity of the origin of the non-preferred reference system as seen from the preferred one. Therefore, events with coordinates $(t, x^a = 0)$ should have preferred coordinates as $(T, X^a = V^a\,T)$, t and T being related by the transformations given above. From this condition one infers

$$F = -(B + D). \tag{7}$$

On the other hand considering the events with preferred coordinates $(T, X^a = 0)$ one finds the velocity of the origin of the preferred coordinates as seen in the non-preferred one

$$\widetilde{V}^i = \frac{F}{A}\,V^i = -\frac{B+D}{A}\,V^i. \tag{8}$$

Here c is the light velocity (in the preferred frame), and A, B, D, E, and F are arbitrary functions of $V = |\boldsymbol{V}|$. It is straightforward to demonstrate that the inverse transformations read

$$c\,T = \widetilde{\Lambda}^0_0\, c\,t + \widetilde{\Lambda}^0_i\, x^i, \tag{9}$$
$$X^a = \widetilde{\Lambda}^a_0\, c\,t + \widetilde{\Lambda}^a_i\, x^i, \tag{10}$$

where

$$\widetilde{\Lambda}^0_0 = \frac{1}{A + \boldsymbol{E}\cdot\boldsymbol{K}}\,, \tag{11}$$
$$\widetilde{\Lambda}^0_i = \frac{1}{A + \boldsymbol{E}\cdot\boldsymbol{K}}\,\frac{1}{D}\left(\frac{B}{B+D}\,\frac{\boldsymbol{E}\cdot\widetilde{\boldsymbol{K}}}{\widetilde{K}^2}\,\widetilde{K}^i - E^i\right), \tag{12}$$
$$\widetilde{\Lambda}^a_0 = -\frac{1}{A + \boldsymbol{E}\cdot\boldsymbol{K}}\,\frac{A}{B+D}\,\widetilde{K}^a\,, \tag{13}$$
$$\widetilde{\Lambda}^a_i = \frac{1}{D}\,\delta^{ai} - \frac{1}{D}\,\frac{B}{B+D}\left(1 - \frac{1}{A + \boldsymbol{E}\cdot\boldsymbol{K}}\,\boldsymbol{E}\cdot\boldsymbol{K}\right)\frac{\widetilde{K}^a\,\widetilde{K}^i}{\widetilde{K}^2} + \frac{1}{A + \boldsymbol{E}\cdot\boldsymbol{K}}\,\frac{A}{D}\,\frac{1}{B+D}\,E^i\,\widetilde{K}^a\,. \tag{14}$$

Here, $\widetilde{K^i} = \widetilde{V}^i/c$. Note that both V^a and $\widetilde{V}^i$ appear in these formulas. In order to complete the inversion one should re-parametrize A, B, D, and E by $\widetilde{V} = |\widetilde{\boldsymbol{V}}|$. Using (8) one can easily transform $\boldsymbol{V}$ into $\widetilde{\boldsymbol{V}}$. Below, following usual practice (see, e.g., Mansouri & Sexl (1977)) we will use only $\boldsymbol{V}$ in both direct and inverse transformations. Special relativistic Lorentz transformations can be restored here with the following values

$$A = \gamma\,, \tag{15}$$
$$B = \gamma - 1\,, \tag{16}$$
$$D = 1\,, \tag{17}$$
$$E^i = -K^i\gamma\,, \tag{18}$$

where

$$\gamma = \frac{1}{\sqrt{1-K^2}}. \tag{19}$$

One can easily see that with these values, both the direct transformation (1)–(7) and the inverse one (9)–(14) coincide with the usual Lorentz transformation.

III. COMPARISON WITH THE MANSOURI-SEXL FORMULATION

It will be useful to compare the coordinate transformation matrices (3)–(6) with the transformation matrices used by Mansouri & Sexl (1977) (note the misprint in the second term of the right-hand side of the second of the equations (6.14) in Mansouri & Sexl (1977): $\boldsymbol{X}$ should appear instead of $\boldsymbol{x}$) and given by

$$\Lambda^0_0 = a - b\,(\boldsymbol{\epsilon}\cdot\boldsymbol{K})\,, \tag{20}$$
$$\Lambda^0_a = d\,\epsilon^a + (b-d)\frac{\boldsymbol{\epsilon}\cdot\boldsymbol{K}}{K^2}\,K^a\,, \tag{21}$$
$$\Lambda^i_0 = -b\,K^i\,, \tag{22}$$
$$\Lambda^i_a = d\delta^{ia} + (b-d)\frac{K^a\,K^i}{K^2}\,. \tag{23}$$

A direct comparision of (20)–(23) with the transformation (3)–(6) yields the following relation between the functions used in this report and those functions used by Mansouri and Sexl:

$$A = a - b\,(\boldsymbol{\epsilon}\cdot\boldsymbol{K})\,, \tag{24}$$
$$B = b - d\,, \tag{25}$$
$$D = d\,, \tag{26}$$
$$\boldsymbol{E} = d\,\boldsymbol{\epsilon} + (b-d)\frac{(\boldsymbol{\epsilon}\cdot\boldsymbol{K})\,\boldsymbol{K}}{K^2}\,. \tag{27}$$

We also note the inverse transformation matrices of Mansouri and Sexl,

$$\widetilde{\Lambda}^0_0 = \frac{1}{a}\,, \tag{28}$$
$$\widetilde{\Lambda}^0_i = -\frac{1}{a}\,\epsilon^i\,, \tag{29}$$
$$\widetilde{\Lambda}^a_0 = \frac{1}{a}\,K^a\,, \tag{30}$$

$$\widetilde{\Lambda}^a_i = \frac{1}{d}\,\delta^{ai} - \left(\frac{1}{d} - \frac{1}{b}\right)\frac{K^a\,K^i}{K^2} - \frac{1}{a}\,\epsilon^i\,K^a\,. \tag{31}$$

The matrices of inverse transformation (28)–(31) can also be obtained just by inserting the relations (24)–(27) into the inverse transformation (11)–(14). We note by passing, that inverse transformation (28)–(31) coincides with Eqs. (56)–(57) of Lämmerzahl (1992).

The inverse relation of Eqs. (24)–(27), that is, the relation between the functions of Mansouri-Sexl and the functions used in this report read

$$a = A + (\boldsymbol{E}\cdot\boldsymbol{K})\,, \tag{32}$$
$$b = B + D\,, \tag{33}$$
$$d = D\,, \tag{34}$$
$$\boldsymbol{\epsilon} = \frac{\boldsymbol{E}}{D} - \frac{B}{D}\,\frac{1}{B+D}\,\frac{(\boldsymbol{E}\cdot\boldsymbol{K})\,\boldsymbol{K}}{K^2}\,. \tag{35}$$

We also note the case of Lorentz transformation,

$$a = \gamma^{-1}\,, \tag{36}$$
$$b = \gamma\,, \tag{37}$$
$$d = 1\,, \tag{38}$$
$$\boldsymbol{\epsilon} = -\boldsymbol{K}\,. \tag{39}$$

IV. ADDITION OF VELOCITIES

In order to obtain the relation of addition of velocities we consider three systems Σ, S and S' with corresponding relative velocities as shown in Fig. 1.

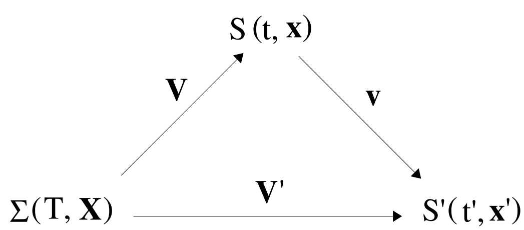

FIG. 1: Three systems Σ, S and S' with corresponding velocities $\boldsymbol{V}, \boldsymbol{V}'$ and $\boldsymbol{v}$.

The coordinate transformation from S to Σ is given by (9) - (10) with coefficents $\widetilde{\Lambda}^0_0$, $\widetilde{\Lambda}^0_i$, $\widetilde{\Lambda}^a_0$ and $\widetilde{\Lambda}^a_i$ given by Eqs. (28) – (31). The functions a, b, d, ϵ are functions of V, being the absolute value of velocity of the non-preferred system S as seen from the prefereed system Σ. Similarly, for the coordinate transformation S' to Σ we have

$$c\,T = \widetilde{\Lambda}^{0\,\prime}_0\,c\,t' + \widetilde{\Lambda}^{0\,\prime}_i\,x^{i\,\prime}\,, \tag{40}$$
$$X^a = \widetilde{\Lambda}^{a\,\prime}_0\,c\,t' + \widetilde{\Lambda}^{a\,\prime}_i\,x^{i\,\prime}\,. \tag{41}$$

The transformation matrices are again given by Eqs. (28) – (31), where the velocity is denoted by $\boldsymbol{V}'$. The corresponding free functions are denoted as a', b', d', ϵ' and are now now functions of V', the absolute value of velocity of non-preferred system S' as seen from the preferred system Σ.

Let us denote the velocity of S' as measured from S as $\boldsymbol{v}$. To relate $\boldsymbol{V}$, $\boldsymbol{V}'$ and $\boldsymbol{v}$ we consider the worldline of the origin of S' with coordinates $,x^{i\prime} \equiv 0$ in S' and $x^i = v^i t$ in S. Equating the right-hand sides of the transformations (9)–(10) and (40)–(41) for an arbitrary event on the worldline of the origin of S' one gets

$$\widetilde{\Lambda}^{0\prime}_0 \, c\,t' = \widetilde{\Lambda}^{0}_0 \, c\ t + \widetilde{\Lambda}^{0}_i \, v^i \, t \ , \tag{42}$$

$$\widetilde{\Lambda}^{a\prime}_0 \, c\,t' = \widetilde{\Lambda}^{a}_0 \, c\,t + \widetilde{\Lambda}^{a}_i \ v^i \, t \ , \tag{43}$$

from which we deduce

$$\frac{\widetilde{\Lambda}^{a\prime}_0}{\widetilde{\Lambda}^{0\prime}_0} = \frac{\widetilde{\Lambda}^{a}_0 + \widetilde{\Lambda}^{a}_i \, k^i}{\widetilde{\Lambda}^{0}_0 + \widetilde{\Lambda}^{0}_i \, k^i} \, , \tag{44}$$

where $k^i = v^i/c$. Eqs. (30) and (28) show that $\widetilde{\Lambda}^{a\prime}_0/\widetilde{\Lambda}^{0\prime}_0 = K^{a\prime} = V^{a\prime}/c$. Therefore, Eq. (44) above gives the following relation between the three velocities

$$\boldsymbol{K}' = \boldsymbol{K} + \frac{a}{d} \, (1 - \boldsymbol{\epsilon} \cdot \boldsymbol{k})^{-1} \, \left(\boldsymbol{k} - (1-f) \, \frac{\boldsymbol{k} \cdot \boldsymbol{K}}{K^2} \, \boldsymbol{K} \right) \tag{45}$$

where $f = d/b$. It is straightforward to check that in case of Lorentz transformation when the Mansouri-Sexl functions take the values (36)–(39), the known special-relativistic addition of velocities (Jackson 1975) is reproduced

$$\boldsymbol{V}' = \frac{\gamma \, \boldsymbol{V} + \boldsymbol{v} + (\gamma - 1) \, \dfrac{\boldsymbol{v} \cdot \boldsymbol{V}}{V^2} \, \boldsymbol{V}}{\gamma \, \left(1 + \dfrac{\boldsymbol{v} \cdot \boldsymbol{V}}{c^2} \right)} \, . \tag{46}$$

V. ABERRATIONAL FORMULA BETWEEN PREFERRED AND NON-PREFERRED REFERENCE FRAMES

Using this transformation one can get the relation between the unit directions of light propagation in the preferred frame S^a and that in the non-preferred one s^i. Here we consider the same light ray as seen by an observer at rest relative to (T, X^i) and another observer (co-located with the first one) at rest relative to (t, x^i). Taking the differentials along the light ray we have

$$S^a = \frac{1}{c} \, \frac{dX^a}{dT}, \tag{47}$$

for the preferred frame ($\boldsymbol{S} \cdot \boldsymbol{S} = 1$) and

$$p^i = \frac{1}{c} \, \frac{dx^i}{dt} \, , \tag{48}$$

$$s^i = p^i/|\boldsymbol{p}| \tag{49}$$

for the non-preferred frame. The last normalization is needed since the light velocity is not equal to c in the non-preferred frames and therefore vector p^i is not unit. Substituting here the transformations (1)–(7) one gets

$$p^i = \frac{\Lambda^i_0 + \Lambda^i_a\, S^a}{\Lambda^0_0 + \Lambda^0_a\, S^a} = \frac{d\, S^i + (b-d)\, \dfrac{K^i\, (\boldsymbol{K}\cdot\boldsymbol{S})}{K^2} - b\, K^i}{\Lambda^0_0 + \Lambda^0_a\, S^a}\,. \tag{50}$$

where $K^a = V^a/c$. The absolute value of the light velocity in the non-preferred frame can computed as $c\,|\boldsymbol{p}|$

$$p = |\boldsymbol{p}| = \frac{1}{\Lambda^0_0 + \Lambda^0_a\, S^a}\, \left(d^2 + b^2\, (K^2 - 2\, \boldsymbol{K}\cdot\boldsymbol{S}) + \frac{(b^2-d^2)\, (\boldsymbol{K}\cdot\boldsymbol{S})^2}{K^2}\right)^{1/2} \tag{51}$$

Note that because of the normalization the value of $\Lambda^0_0 + \Lambda^0_a\, S^a$ plays no role here. Finally, one gets

$$\boldsymbol{s} = \frac{f\, \boldsymbol{S} + (1-f)\, \dfrac{\boldsymbol{K}\, (\boldsymbol{K}\cdot\boldsymbol{S})}{K^2} - \boldsymbol{K}}{\left(f^2 + K^2 - 2\, \boldsymbol{K}\cdot\boldsymbol{S} + (1-f^2)\, \dfrac{(\boldsymbol{K}\cdot\boldsymbol{S})^2}{K^2}\right)^{1/2}}, \tag{52}$$

where $f = d/b$. We see that the transformation is a function of only one parameter f. The transformation is independent of a and $\boldsymbol{\epsilon}$. Substituting (36)–(39) into (52) one gets the usual special relativistic aberrational formula following from the Lorentz transformation. Equation (51) with (36)–(39) demonstrates that in special relativity $p = 1$.

VI. INVERSION OF THE ABERRATIONAL FORMULA

Performing the same calculations taking the inverse transformation matrices (28)–(31) one get the inverse relation

$$\begin{aligned} S^a &= \frac{\widetilde{\Lambda}^a_0 + \widetilde{\Lambda}^a_i\, p^i}{\widetilde{\Lambda}^0_0 + \widetilde{\Lambda}^0_i\, p^i} \\ &= \frac{K^a + \dfrac{a}{d}\, p^a - \dfrac{a(b-d)}{b\, d}\, \dfrac{K^a\, (\boldsymbol{K}\cdot\boldsymbol{p})}{K^2} - K^a\, (\boldsymbol{\epsilon}\cdot\boldsymbol{p})}{1 - \boldsymbol{\epsilon}\cdot\boldsymbol{p}} \\ &= K^a + h\, \left(s^a - (1-f)\, \frac{K^a\, (\boldsymbol{K}\cdot\boldsymbol{s})}{K^2}\right), \end{aligned} \tag{53}$$

where

$$h = \frac{a}{d\, (p^{-1} - \boldsymbol{\epsilon}\cdot\boldsymbol{s})}\,. \tag{54}$$

The value of h can be derived from the condition that $\boldsymbol{S}$ is a unit vector. Taking the last formula in (53) one gets

$$1=\boldsymbol{S}\cdot\boldsymbol{S}=K^2+2\,f\,(\boldsymbol{K}\cdot\boldsymbol{s})\,h+\left(1-\left(1-f^2\right)\left(\frac{\boldsymbol{K}\cdot\boldsymbol{s}}{K}\right)^2\right)\,h^2\,. \qquad (55)$$

From this formula we see immediately that h does not depend on $\boldsymbol{\epsilon}$ and, therefore, as expected the aberrational formula does not depend on the synchronization convention. Moreover, h does not depend also on a. Equation (55) is quadratic and has two solutions. From the definition (54) of h it is clear that for $\boldsymbol{V}=0$ one should have $h=1$. Therefore,

$$\begin{aligned} h=&\frac{K}{K^2-(1-f^2)\,(\boldsymbol{K}\cdot\boldsymbol{s})^2}\\ &\times\left[\left(f^2\,(\boldsymbol{K}\cdot\boldsymbol{s})^2+(1-K^2)\,\left(K^2-(\boldsymbol{K}\cdot\boldsymbol{s})^2\right)\right)^{1/2}-f\,K\,(\boldsymbol{K}\cdot\boldsymbol{s})\right]\,. \end{aligned} \qquad (56)$$

This equation should considered as an equation for p appearing in (54). One can check by direct computations that (54) and (56) give the same expression for p as (51) provided that $\boldsymbol{S}$ and $\boldsymbol{s}$ are related by (52). Inserting Eq. (56) into (53) one finally obtains the inversion of aberrational formula,

$$\begin{aligned} \boldsymbol{S}=\boldsymbol{K}+&\left(\left(f^2\,(\boldsymbol{K}\cdot\boldsymbol{s})^2+(1-K^2)\,\left(K^2-(\boldsymbol{K}\cdot\boldsymbol{s})^2\right)\right)^{1/2}-f\,K\,(\boldsymbol{K}\cdot\boldsymbol{s})\right)\\ &\times\frac{K}{K^2-(1-f^2)\,(\boldsymbol{K}\cdot\boldsymbol{s})^2}\,\left(\boldsymbol{s}-(1-f)\,\frac{\boldsymbol{K}\,(\boldsymbol{K}\cdot\boldsymbol{s})}{K^2}\right)\,. \end{aligned} \qquad (57)$$

This transformation, like in aberrational formula Eq. (52), is also a function of f only. No other Mansouri-Sexl parameters appear here. A direct substitution of (52) und (57) into each other demostrates that these two transformations are indeed inversions of each other.

VII. ABERRATIONAL FORMULA BETWEEN TWO NON-PREFERED REFERENCE FRAMES

Now let us consider a satellite moving in the space of solar system and performing positional observations of stars. The standard general-relativistic data reduction of this satellite assumes that the stars have known positions in the Barycentric Celestial Reference System, BCRS (Soffel et al. 2003). The observed direction towards a star is defined in the locally inertial reference system momentarily co-moving with the satellite at the moment of observation (Klioner 2003, 2004). In BCRS one introduces a fictitious observer momentarily co-located with the satellite. The direction of the light propagation from the star as seen by that fictitious observer and the direction as seen by the observer co-moving with the satellite are related by the aberrational formula. That aberrational formula is given, e.g., by Eq. (10)-(11) of Klioner (2003) and follows directly from the Lorentz transformation relating the two reference frames in question.

From the point of view of the kinematical test theories for Local Lorentz Invariance the BCRS can be interpreted as the non-preferred coordinate system $(t,\boldsymbol{x})$ as introduced above. That reference frame has velocity $\boldsymbol{V}$ relative to the preferred frame $(T,\boldsymbol{X})$. The unit direction of the propagation of a light ray coming from the star is $\boldsymbol{s}$. The reference frame moving together with the satellite can be associated with $(t', \boldsymbol{x}')$ in Section IV. That latter frame has velocity $\boldsymbol{V}'$ relative to the preferred frame, and velocity $\boldsymbol{v}$ relative to $(t, \boldsymbol{x})$. The relation between $\boldsymbol{s}$ and $\boldsymbol{s}'$ as function of velocities $\boldsymbol{V}$ and $\boldsymbol{v}$ and the parameters of Mansouri-Sexl transformations can be derived by combining (57), (52) and (45). Namely, Eq. (57) gives a formula for $\boldsymbol{S}$ as function of $\boldsymbol{s}$ and $\boldsymbol{k} = \boldsymbol{v}/c$. Changing $\boldsymbol{s}$ and $\boldsymbol{K}$ in (52) to $\boldsymbol{s}'$ and $\boldsymbol{K}' = \boldsymbol{V}'/c$ one gets a formula for $\boldsymbol{s}'$ as function of $\boldsymbol{S}$ and $\boldsymbol{K}'$. Note that performing the above-mentioned substitution one should remember that f is also a function of $\boldsymbol{K}$ and should be changed to f'. Substituting that latter formula and (45) into (57) results into the final formula for $\boldsymbol{s}'$ as functions of $\boldsymbol{s}$, $\boldsymbol{v}$, $\boldsymbol{V}$ and Mansouri-Sexl parameters.

Although it is straightforward to complete these calculation in closed form, it leads to a complicated and rather useless formula. We prefer to expand the functions a, b, d and $\boldsymbol{\epsilon}$ in powers of K and b', d' in powers of K'. Note that functions a' and $\boldsymbol{\epsilon}'$ do not appear in our calculations. The expansions read

$$a = 1 + \sum_{k=1}^{\infty} \left(\alpha_i - \frac{(2i-3)!!}{(2i)!!} \right) K^{2i} = 1 + \left(\alpha_1 - \frac{1}{2} \right) K^2 + \left(\alpha_2 - \frac{1}{8} \right) K^4 + \mathcal{O}(K^6)\,, \tag{58}$$

$$b = 1 + \sum_{k=1}^{\infty} \left(\beta_i + \frac{(2i-1)!!}{(2i)!!} \right) K^{2i} = 1 + \left(\beta_1 + \frac{1}{2} \right) K^2 + \left(\beta_2 + \frac{3}{8} \right) K^4 + \mathcal{O}(K^6)\,, \tag{59}$$

$$d = 1 + \sum_{k=1}^{\infty} \delta_i\, K^{2i}\,, \tag{60}$$

$$\boldsymbol{\epsilon} = (\epsilon - 1)\, \boldsymbol{K} \left(1 + \sum_{k=1}^{\infty} \epsilon_i\, K^{2i} \right)\,. \tag{61}$$

Here we assumed that $(-1)!! = 1$. The parameters α_i, β_i, δ_i, ϵ, ϵ_i are assumed to be numerical. The Standard Mansouri-Sexl parameters read $\alpha = \alpha_1 - \frac{1}{2}$, $\beta = \beta_1 + \frac{1}{2}$ and $\delta = \delta_1$. Parameters α_i, β_i, δ_i, ϵ and ϵ_i are all zero in case of special relativity with Einstein synchronization. The expansions for b' and d' coincide with (59) and (60), respectively, with K' instead of K, and with the same coefficients β_i and δ_i. Using these expansions we get

$$\boldsymbol{s}' = \mathbf{P}\, \boldsymbol{s}''\,, \tag{62}$$

where $\mathbf{P}$ is the matrix of Thomas-like precession

$$\begin{aligned} P^{ij} = {} & \delta^{ij} + K^{[i}\, k^{j]} - \frac{1}{8}\, K^2\, k^i\, k^j + \frac{1}{4}\, (\boldsymbol{k} \cdot \boldsymbol{K})\, K^{(i}\, k^{j)} - \frac{1}{8}\, k^2\, K^i\, K^j \\ & - \frac{1}{4}\, K^{[i}\, k^{j]} \left(k^2\, (6\eta - 1) + (1 - 4\epsilon + 4\eta\, (1 - \eta))\, \boldsymbol{k} \cdot \boldsymbol{K} - (1 + 4\theta + 4\eta^2)\, K^2 \right) \\ & + \mathcal{O}\left(c^{-5} \right), \end{aligned} \tag{63}$$

where $\eta = \frac{1}{2} - \beta + \delta_1 = -\beta_1 + \delta_1$ is the parameter that is measured in Michelson-Morley-type experiments, and $\theta = \alpha_1 - \delta_1$. This formula for the Thomas-like precession coincides with the special relativistic Thomas precession in the limit of special relativity. Direction $\boldsymbol{s}''$ is related to $\boldsymbol{s}$ by the following formula that does not involve any rotational matrix (we define $\zeta = \beta_2 - \delta_2 - 1/2\, \delta_1$):

$$
\begin{aligned}
\boldsymbol{s}'' = \boldsymbol{s} & \\
& +(\boldsymbol{s}\cdot\boldsymbol{k})\,\boldsymbol{s} - \boldsymbol{k} \\
& -\frac{1}{2}\,(\boldsymbol{s}\cdot\boldsymbol{k})\,\boldsymbol{k} - \frac{1}{2}\,k^2\,\boldsymbol{s} + (\boldsymbol{s}\cdot\boldsymbol{k})^2\;\boldsymbol{s} \\
& -\eta\,(\boldsymbol{s}\cdot\boldsymbol{K})\,\boldsymbol{k} - \eta\,(\boldsymbol{s}\cdot\boldsymbol{k})\,(\boldsymbol{k}+\boldsymbol{K}) + \eta\;(\boldsymbol{s}\cdot\boldsymbol{k})^2\;\boldsymbol{s} + 2\eta\,(\boldsymbol{s}\cdot\boldsymbol{k})\,(\boldsymbol{s}\cdot\boldsymbol{K})\,\boldsymbol{s} \\
& -\frac{1}{2}\;(\boldsymbol{s}\cdot\boldsymbol{k})^2\;\boldsymbol{k} - \frac{1}{2}\,k^2\,(\boldsymbol{s}\cdot\boldsymbol{k})\,\boldsymbol{s} + (\boldsymbol{s}\cdot\boldsymbol{k})^3\;\boldsymbol{s} \\
& +\eta\,k^2\,(\boldsymbol{k}+\boldsymbol{K}) - 3\,\eta\,(\boldsymbol{s}\cdot\boldsymbol{k})(\boldsymbol{s}\cdot\boldsymbol{K})\,\boldsymbol{k} - \eta\;(\boldsymbol{s}\cdot\boldsymbol{K})^2\;\boldsymbol{k} + (\eta-\epsilon)\,(\boldsymbol{k}\cdot\boldsymbol{K})\,\boldsymbol{k} \\
& -\eta\;(\boldsymbol{s}\cdot\boldsymbol{k})^2\;(2\boldsymbol{k}+\boldsymbol{K}) + (\epsilon-2\,\eta)\,(\boldsymbol{s}\cdot\boldsymbol{k})(\boldsymbol{k}\cdot\boldsymbol{K})\,\boldsymbol{s} - 2\eta\,k^2\,(\boldsymbol{s}\cdot\boldsymbol{K})\,\boldsymbol{s} \\
& +6\eta\;(\boldsymbol{s}\cdot\boldsymbol{k})^2\;(\boldsymbol{s}\cdot\boldsymbol{K})\,\boldsymbol{s} + \eta\,(\boldsymbol{s}\cdot\boldsymbol{k})\;(\boldsymbol{s}\cdot\boldsymbol{K})^2\;\boldsymbol{s} + 3\eta\;(\boldsymbol{s}\cdot\boldsymbol{k})^3\;\boldsymbol{s} - \theta\,K^2\,\boldsymbol{k} \\
& +\left(\theta\,K^2 - 2\,\eta\,k^2\right)\;(\boldsymbol{s}\cdot\boldsymbol{k})\,\boldsymbol{s} \\
& +\frac{1}{8}\,k^2\,(\boldsymbol{s}\cdot\boldsymbol{k})\,\boldsymbol{k} - \frac{1}{8}\,k^4\,\boldsymbol{s} - \frac{1}{2}\,(\boldsymbol{s}\cdot\boldsymbol{k})^3\;\boldsymbol{k} - \frac{1}{2}\,k^2\,(\boldsymbol{s}\cdot\boldsymbol{k})^2\,\boldsymbol{s} + (\boldsymbol{s}\cdot\boldsymbol{k})^4\,\boldsymbol{s} \\
& +\left[\zeta - \theta + \eta\left(\frac{1}{2} - 2\,\theta + \delta_1\right)\right]\;K^2\,(\boldsymbol{s}\cdot\boldsymbol{k})\,\boldsymbol{k} \\
& +\left[\zeta + \eta\left(\frac{7}{2} + \delta_1\right)\right]\;k^2\,(\boldsymbol{s}\cdot\boldsymbol{k})\,\boldsymbol{k} - \frac{1}{2}\,\eta(7+2\eta)\,(\boldsymbol{s}\cdot\boldsymbol{k})^3\,\boldsymbol{k} \\
& -3\,\eta\,(2+\eta)\,(\boldsymbol{s}\cdot\boldsymbol{k})^2\,(\boldsymbol{s}\cdot\boldsymbol{K})\,\boldsymbol{k} - 2\,\eta\,(1+\eta)\,(\boldsymbol{s}\cdot\boldsymbol{k})\,(\boldsymbol{s}\cdot\boldsymbol{K})^2\,\boldsymbol{k} \\
& +\left[2\zeta - \eta\;\left(\frac{3}{2}\eta - 3 + \epsilon - 2\delta_1\right)\right]\;(\boldsymbol{K}\cdot\boldsymbol{k})\,(\boldsymbol{s}\cdot\boldsymbol{K})\,\boldsymbol{k} \\
& +\left[2\zeta - \epsilon(1+2\,\eta) + \frac{1}{2}\,\eta\,(9+4\,\delta_1)\right]\;(\boldsymbol{K}\cdot\boldsymbol{k})\,(\boldsymbol{s}\cdot\boldsymbol{k})\,\boldsymbol{k} \\
& +\left[\zeta + \eta\;\left(\frac{11}{4} + \delta_1\right)\right]\;(\boldsymbol{s}\cdot\boldsymbol{K})\,k^2\,\boldsymbol{k} \\
& +\left[\zeta - \eta\;\left(\frac{1}{2}\,\eta + \theta - \frac{1}{2} - \delta_1\right)\right]\;(\boldsymbol{s}\cdot\boldsymbol{K})\,K^2\,\boldsymbol{k} \\
& -\eta\,(1+\eta)\,(\boldsymbol{s}\cdot\boldsymbol{k})^3\,\boldsymbol{K} - 2\,\eta^2\,(\boldsymbol{s}\cdot\boldsymbol{k})^2\,(\boldsymbol{s}\cdot\boldsymbol{K})\,\boldsymbol{K} \\
& +\left[\zeta + \eta\left(\frac{3}{2} + \delta_1\right)\right]\;k^2(\boldsymbol{s}\cdot\boldsymbol{K})\,\boldsymbol{K} \\
& +\left[\zeta + \eta\left(\frac{11}{4} + \delta_1\right)\right]\;k^2(\boldsymbol{s}\cdot\boldsymbol{k})\,\boldsymbol{K} \\
& +\left[\zeta - \frac{1}{2}\eta\,(\eta + 2\theta - 1 - 2\delta_1)\right]\;K^2(\boldsymbol{s}\cdot\boldsymbol{k})\,\boldsymbol{K} \\
& +[2\zeta - \eta\,(-3 + 3\eta - 2\delta_1)]\;(\boldsymbol{s}\cdot\boldsymbol{K})\,(\boldsymbol{K}\cdot\boldsymbol{k})\,\boldsymbol{K} \\
& +\left[2\zeta - \eta\left(-3 + \frac{3}{2}\eta + \epsilon - 2\delta_1\right)\right]\;(\boldsymbol{s}\cdot\boldsymbol{k})\,(\boldsymbol{K}\cdot\boldsymbol{k})\,\boldsymbol{K} \\
& -\theta\,K^2\,k^2\boldsymbol{s} + \eta k^4\,\boldsymbol{s} + k^2(2\eta - \epsilon)\,(\boldsymbol{K}\cdot\boldsymbol{k})\,\boldsymbol{s} \\
& +\frac{3}{2}\,\eta\,(4+\eta)(\boldsymbol{s}\cdot\boldsymbol{k})^4\,\boldsymbol{s} + 6\,\eta\,(2+\eta)\,(\boldsymbol{s}\cdot\boldsymbol{k})^3\,(\boldsymbol{s}\cdot\boldsymbol{K})\,\boldsymbol{s} + 3\,\eta\,(1+2\eta)\,(\boldsymbol{s}\cdot\boldsymbol{k})^2\,(\boldsymbol{s}\cdot\boldsymbol{K})^2\,\boldsymbol{s}
\end{aligned}
$$

$$
\begin{aligned}
&- \left[\zeta - 2\theta + \frac{1}{2}\,\eta\,(1+\eta-4\,\theta+2\,\delta_1)\right]\,K^2\,(\boldsymbol{s}\cdot\boldsymbol{k})^2\,\boldsymbol{s}\\
&- \left[\zeta + \frac{1}{2}\eta\,(14+\eta+2\delta_1)\right]\,k^2\,(\boldsymbol{s}\cdot\boldsymbol{k})^2\,\boldsymbol{s}\\
&- \left[\zeta + \frac{1}{2}\eta\,(5+\eta+2\delta_1)\right]\,k^2\,(\boldsymbol{s}\cdot\boldsymbol{K})^2\,\boldsymbol{s}\\
&- [4\zeta - 2\eta\,(-3+\epsilon+\eta-2\delta_1)]\,(\boldsymbol{s}\cdot\boldsymbol{K})\,(\boldsymbol{s}\cdot\boldsymbol{k})\,(\boldsymbol{K}\cdot\boldsymbol{k})\,\boldsymbol{s}\\
&- [2\zeta - \eta\,(-3+3\eta-2\delta_1)]\,(\boldsymbol{s}\cdot\boldsymbol{K})^2\,(\boldsymbol{K}\cdot\boldsymbol{k})\,\boldsymbol{s}\\
&- [2\zeta - 2\epsilon(1+\eta)+\eta\,(7+\eta+2\delta_1)]\,(\boldsymbol{s}\cdot\boldsymbol{k})^2\,(\boldsymbol{K}\cdot\boldsymbol{k})\,\boldsymbol{s}\\
&- [2\zeta - \eta\,(\eta+2\theta-1-2\,\delta_1)]\,K^2\,(\boldsymbol{s}\cdot\boldsymbol{K})\,(\boldsymbol{s}\cdot\boldsymbol{k})\,\boldsymbol{s}\\
&- [2\zeta + \eta\,(\eta+10+2\delta_1)]\,k^2\,(\boldsymbol{s}\cdot\boldsymbol{K})\,(\boldsymbol{s}\cdot\boldsymbol{k})\,\boldsymbol{s}\\
&+\mathcal{O}\left(c^{-5}\right). \qquad (64)
\end{aligned}
$$

The terms in this formula are grouped by the orders of $1/c$. The groups are separated by additional empty line. Within each group the first line represents terms that survive in the limit of special relativity and the rest (if any) represents the additional Mansouri-Sexl terms.

In the limit of special relativity Eq. (64) coincides with the expansion of the special-relativistic aberrational formula. The dependence of this formula on ϵ and α_1 comes solely from the relation (45) between the velocities.

Several points are unclear here:

(1) How does the Thomas-like precession given by P^{ij} above relate to the Thomas precession derived from the coordinate transformations (see below)?

(2) These calculations were performed with arbitrary ϵ. Should we put ϵ to some particular value to make the whole calculation "physically meaningful"? This question may be related to the first one and to the questions (in boldface) given in the following sections.

(3) Another more "practical questions". Velocity $\boldsymbol{v}$ is taken from processing of orbital data (Doppler, ranging and positional observations of the satellite). That orbital data are processed using general-relativistic models assuming the LLI to be correct. The resulting orbit is the only orbit, i.e. the only $\boldsymbol{v}$ available for our analysis. Should we put all Mansouri-Sexl parameters to their special-relativistic values in (45) because of this circumstances?

VIII. THOMAS PRECESSION

In special relativity, two successive Lorentz transformations are equivalent to one Lorentz transformation plus a three-dimensional rotation, the so-called Thomas precession. Two successive coordinate transformations from preferred system $\Sigma(T,\boldsymbol{X})$ to the non-preferred system $S(t,\boldsymbol{x})$ and afterwards to the non-preferred system $S'(t',\boldsymbol{x}')$ can be represented by

$$ct' = A^0_0(\boldsymbol{K}',\boldsymbol{K})\;ct + A^0_i(\boldsymbol{K}',\boldsymbol{K})\;x^i\,, \qquad (65)$$

$$x^{i\,\prime} = A^i_0(\boldsymbol{K}',\boldsymbol{K})\;ct + A^i_j(\boldsymbol{K}',\boldsymbol{K})\;x^j\,. \qquad (66)$$

The vectors $\boldsymbol{K}$ and $\boldsymbol{K}'$ are related by the addition of velocities in Eq. (45). According to

Eqs. (1), (2) and Eqs. (9), (10), the spatial components A^{ij} of the matrix $A^{\mu\nu}$ follow as

$$A^i_j = \Lambda^i_0(\boldsymbol{K}')\ \tilde{\Lambda}^0_j(\boldsymbol{K})\ +\ \Lambda^i_a(\boldsymbol{K}')\ \tilde{\Lambda}^a_j(\boldsymbol{K})\,. \tag{67}$$

The explicit form of Λ^i_0 and Λ^i_a are given in Eqs. (22) and (23), respectively, and the explicit form of $\tilde{\Lambda}^0_j$ and $\tilde{\Lambda}^a_j$ are given in Eqs. (29) and (31), respectively. Inserting the expansions of Mansouri-Sexl-functions given in Eqs. (58) – (61) and the relation between three velocities in Eq. (45), we obtain (for notation see Appendix)

$$a_1^{(ij)} = (2\,\delta_1\,\boldsymbol{K}\cdot\boldsymbol{k} + \delta_1\,k^2)\,\delta^{ij} + \left(\frac{1}{2} + \beta_1 - \delta_1\right)\,k_i k_j + (2\beta_1 - 2\delta_1 + \epsilon)\,K^{(i}\,k^{j)}\,, \tag{68}$$

$$a_1^{[ij]} = (1-\epsilon)\ K^{[i}\,k^{j]} \tag{69}$$

$$\begin{aligned} a_2^{[ij]} = &\left[(1-\epsilon)\epsilon_1 + \left(\frac{1}{2} + \beta_1 - \delta_1\right)^2\right]\ K^2\ K^{[i}k^{j]} + (1-\epsilon)\ \left(\frac{1}{2} + \beta_1\right)\ k^2\ K^{[i}k^{j]} \\ &+ \left[(1-\epsilon)\left(\epsilon - \frac{1}{2} + \beta_1 + \delta_1\right) + \left(\frac{1}{2} + \beta_1 - \delta_1\right)^2\right]\boldsymbol{k}\cdot\boldsymbol{K}\ K^{[i}k^{j]}\,. \end{aligned} \tag{70}$$

The matrix of Thomas precession follows as (for details see Appendix),

$$\begin{aligned} P^{ij} = &\,\delta^{ij} + (1-\epsilon)K^{[i}\,k^{j]} \\ &+(1-\epsilon)^2\ \left[-\frac{1}{8}\,K^2\,k^i\,k^j + \frac{1}{4}\,(\boldsymbol{k}\cdot\boldsymbol{K})\,K^{(i}\,k^{j)} - \frac{1}{8}\,k^2\,K^i\,K^j\right] \\ &-K^{[i}\,k^{j]}\ \left(\left[(\epsilon-1)\epsilon_1 - \left(\frac{1}{2} - \eta\right)^2\right]\ K^2 + \left[\frac{1}{2}\,(\epsilon-1)^2 - \left(\frac{1}{2} - \eta\right)^2\right]\boldsymbol{k}\cdot\boldsymbol{K} \right. \\ &\left. +\frac{1}{2}(\epsilon-1)\left(\frac{1}{2} - \eta\right)\ k^2\right) + \mathcal{O}(c^{-6})\,. \end{aligned} \tag{71}$$

In Einstein synchronization (78) we have

$$\begin{aligned} \epsilon &= 0\,, \\ \epsilon_1 &= \alpha_1 - \beta_1\,, \end{aligned} \tag{72}$$

and obtain for Eq. (71) the Thomas precession

$$\begin{aligned} P^{ij} = &\,\delta^{ij} + K^{[i}\,k^{j]} \\ &-\frac{1}{8}\,K^2\,k^i\,k^j + \frac{1}{4}\,(\boldsymbol{k}\cdot\boldsymbol{K})\,K^{(i}\,k^{j)} - \frac{1}{8}\,k^2\,K^i\,K^j \\ &-\frac{1}{4}\,K^{[i}\,k^{j]}\ \left(k^2\,(2\eta - 1) + (1 + 4\eta\,(1-\eta))\,\boldsymbol{k}\cdot\boldsymbol{K} - (1 + 4\theta + 4\eta^2)\,K^2\right) \\ &+\mathcal{O}(c^{-5})\,. \end{aligned} \tag{73}$$

Up to $\mathcal{O}(c^{-4})$ the antisymmetric part $A^{[ij]}$ should represent the "Thomas precession". However, $A^{[ij]}$ does not coincide with Eq. (63) for $\epsilon \neq 0$.

IX. METRIC TENSOR

The covariant components of metric tensor in the non-preferred system S can be determined by the tensor relation

$$g_{\mu\nu} = \left(\frac{\partial X^\alpha}{\partial x^\mu}\right)\ \left(\frac{\partial X^\beta}{\partial x^\nu}\right)\ G_{\alpha\beta}\,, \tag{74}$$

where $G_{\alpha\beta} = \eta_{\alpha\beta} = \mathrm{diag}(-1, +1, +1, +1)$ is the Minkowski metric tensor in the preferred system Σ. Implementation of coordinate transformations (9) – (10) in Mansouri-Sexl parametrization (28) – (31) yields

$$g_{00} = -\frac{1}{a^2}\,(1-K^2)\,, \tag{75}$$

$$g_{0i} = \frac{1}{a^2}\,(1-K^2)\,\epsilon^i + \frac{1}{a\,b}\,K^i\,, \tag{76}$$

$$g_{ij} = -\frac{1}{a^2}\,(1-K^2)\,\epsilon^i\,\epsilon^j + \frac{1}{d^2}\,\delta^{ij} + \left(\frac{1}{b^2} - \frac{1}{d^2}\right)\frac{K^i\,K^j}{K^2} - \frac{1}{a\,b}\left(\epsilon^i\,K^j + \epsilon^j\,K^i\right)\,. \tag{77}$$

The nondiagonal components g_{0i} vanish with the following synchronization condition:

$$\begin{aligned}\epsilon^i &= -\frac{a}{b}\,\frac{K^i}{1-K^2}\,,\\ &= -K^i + (\beta_1-\alpha_1)\,K^2\,K^i + \left(\alpha_1\,\beta_1 + \beta_2 - \alpha_2 - \frac{\alpha_1}{2} - \frac{\beta_1}{2} - \beta_1^2\right)\,K^4\,K^i + \mathcal{O}(K^7)\,,\end{aligned} \tag{78}$$

which is just the condition for the Einstein synchronization as given by Eq. (6.12) in Mansouri & Sexl (1977) and Eq. (74) of Lämmerzahl (1992). In case of Einstein synchronization, the metric components read:

$$g_{00} = -\frac{1}{a^2}\,(1-K^2)\,, \tag{79}$$

$$g_{0i} = 0\,, \tag{80}$$

$$g_{ij} = \frac{1}{d^2}\,\delta^{ij} + \left(\frac{1}{b^2}\,\frac{1}{1-K^2} - \frac{1}{d^2}\right)\frac{K^i\,K^j}{K^2}\,. \tag{81}$$

This metric goes to Minkowski one for (and only for) the values of a, b and d given by (36)–(38) and (19).

Are (t, x^i) observables? For any ϵ or for particular one? Probably only for the Einstein synchronization (78)?

X. APPLICATION FOR GAIA: NUMERICAL SIMULATIONS AND COVARIANCE ANALYSIS

The expected accuracy for η is 10^{-8} (it could be better). Although it is 100 times lower than what can be done now in a laboratory it is a different type of experiment and is still interesting.

XI. CONCLUSION

APPENDIX A: REPRESENTING ANY MATRIX AS PRODUCT OF OTHOGONAL AND SYMMETRIC ONES

Let us consider an arbitrary matrix of the form

$$A^i_j = \delta^{ij} + \epsilon\,a_1^{ij} + \epsilon^2\,a_2^{ij} + \mathcal{O}(\epsilon^3)\,, \tag{A1}$$

where ϵ is some small parameter. Our goal is to represent this matrix as a product of an orthogonal matrix P^{ij} and a symmetric matrix S^{ij}:

$$A^i_j = P^{ik}\, S^k_j\,. \tag{A2}$$

Now let us assume that both matrices are represented as expansions:

$$S^i_j = \delta^{ij} + \epsilon\, s_1^{ij} + \epsilon^2\, s_2^{ij} + \mathcal{O}(\epsilon^3)\,, \tag{A3}$$

where s_1^{ij} and s_2^{ij} are symmetric matrices,

$$s_1^{[ij]} = 0\,, \tag{A4}$$

$$s_2^{[ij]} = 0\,, \tag{A5}$$

and

$$P^{ij} = \delta^{ij} + \epsilon\, p_1^{ij} + \epsilon^2\, p_2^{ij} + \mathcal{O}(\epsilon^3)\,, \tag{A6}$$

where the condition that P^{ij} is orthogonal ($P^{ik}P^{jk} = \delta^{ij}$) imposes the following properties on p_1^{ij} and p_2^{ij}:

$$p_1^{(ij)} = 0\,, \tag{A7}$$

$$p_2^{(ij)} = -\frac{1}{2} p_1^{ik} p_1^{jk}\,. \tag{A8}$$

Computing $P^{ik}S^{kj}$ from these expansions and considering the properties of s_1^{ij}, s_2^{ij}, p_1^{ij} and p_2^{ij} one gets:

$$p_1^{ij} = a_1^{[ij]}\,, \tag{A9}$$

$$s_1^{ij} = a_1^{(ij)}\,, \tag{A10}$$

$$p_2^{(ij)} = -\frac{1}{2} p_1^{ik} p_1^{jk}\,, \tag{A11}$$

$$p_2^{[ij]} = b^{[ij]}\,, \tag{A12}$$

$$s_2^{ij} = b^{(ij)}\,, \tag{A13}$$

$$\begin{aligned} b^{ij} &= a_2^{ij} + \frac{1}{2} p_1^{ia} p_1^{ja} - p_1^{ia} s_1^{ja} \\ &= a_2^{ij} + \frac{1}{2} a_1^{[ia]} a_1^{[ja]} - a_1^{[ia]} a_1^{(ja)}\,. \end{aligned} \tag{A14}$$

Clearly,

$$p_2^{ij} = p_2^{(ij)} + p_2^{[ij]} \;=\; b^{[ij]} - \frac{1}{2} p_1^{ik} p_1^{jk}\,. \tag{A15}$$

These formulas allow us to compute explicitly the orthogonal and symmetric matrices P^{ij} and S^{ij}.

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