Asymptotic behavior of S-stopped branching processes

Kyrychynska
Okhrin
Yeleyko

Institute for Statistics and Econometrics
Humboldt-Universität zu Berlin
http://ise.wiwi.hu-berlin.de

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Definition I

- $(X, \mathcal{A})$ – measured state space with $\mathcal{A}$ – σ-algebra on X
- $P(t, x, A)$ – transition probability, with t – time, $x \in X$ and $A \in \mathcal{A}$
- time is **discrete**
- $T_1, \ldots, T_n, \ldots$ – types of particles
- only **one** particle at the point $x \in X$ at time point $t = 0$
- $\mu_{xt}(A)$ – a random measure
- $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$
- $\mathbb{N}_0^\infty = \mathbb{N}_0 \times \mathbb{N}_0 \times \ldots$
- $\mathcal{E}(i)$ – the particle of type i

Definition II

Measure which describe the number of particles of each type at time point t.
Multivariate measure $\boldsymbol{\mu}$ is based on μ

$$\boldsymbol{\mu}_{\mathbf{x}t}(A) = \left\{ \begin{array}{ll} \sum_{i=0}^{\infty} \sum_{j=1}^{n_i} \mu_{x_{ij}t}(x_m), & \text{if } x_m \in A \\ 0, & \text{else} \end{array} \right\}_{m=0}^{\infty},$$

where $\mathbf{x} = \{x_{11}, \ldots, x_{1n_1}, x_{21}, \ldots, x_{2n_2}, \ldots\}$ and $x_{ij} \in X$ is the j-th element of the i-th type.

Definition III

Definition

Let us fix countable subset $S \subset \mathbb{N}_0^\infty$, $0 \notin S$. *Stopped*, or *S-stopped* branching process is the process $\boldsymbol{\xi}_{\mathbf{x}t}(X)$, defined for $t = 1, 2, \ldots$ and $\mathbf{x} \in \mathbb{N}_0^\infty$ by equalities

$$\boldsymbol{\xi}_{\mathbf{x}t}(X) = \begin{cases} \boldsymbol{\mu}_{\mathbf{x}t}(X), & \text{if } \forall v,\ 0 \le v < t,\ \boldsymbol{\mu}_{\mathbf{x}v}(X) \notin S \\ \boldsymbol{\mu}_{\mathbf{x}u}(X), & \text{if } \forall v,\ 0 \le v < u,\ \boldsymbol{\mu}_{\mathbf{x}v}(X) \notin S, \\ & \boldsymbol{\mu}_{\mathbf{x}u}(X) \in S,\ u < t \end{cases}$$

Definition IV

Definition

Let

$$q_{\mathbf{r}}^{\mathbf{n}}(t) = P\{\xi_{\mathbf{n}t}(X) = \mathbf{r}\}$$

be the probability of *extinction* of the S-stopped branching process $\xi_{\mathbf{x}t}(X)$ into state $\mathbf{r} \in S$ till time t, which starts from state $\mathbf{n} \in \mathbb{N}_0^\infty$ and $q_{\mathbf{r}}^{\mathbf{n}} = \lim_{t\to\infty} q_{\mathbf{r}}^{\mathbf{n}}(t)$.

Definition V

$$\widehat{P}(t,\mathbf{x},\mathbf{y}) = P\{\boldsymbol{\mu}_{\mathbf{x}t}(X)=\mathbf{y}\},\ \mathbf{x},\mathbf{y}\in\mathbb{N}_0^\infty$$

$$\widetilde{P}(t,\boldsymbol{\alpha},\mathbf{r}) = \begin{cases} \widehat{P}(1,\boldsymbol{\alpha},\mathbf{r}), & t=1; \\ \sum_{\boldsymbol{\beta}\notin S}\widehat{P}(1,\boldsymbol{\alpha},\boldsymbol{\beta})\widetilde{P}(t-1,\boldsymbol{\beta},\mathbf{r}), & t\geq 2. \end{cases}$$

where $\boldsymbol{\alpha}\notin S,\ \boldsymbol{\alpha}\neq 0,\ r\in S,\ \widetilde{P}$ is the conditional probability of the event

$$\{\boldsymbol{\mu}_{\boldsymbol{\alpha}t}(X)=\mathbf{r}\}\bigcap\bigcap_{l'=1}^{t-1}\{\boldsymbol{\mu}_{\boldsymbol{\alpha}l'}(X)\notin S\}.$$

generated functional

$$h(t,s(\cdot)) = \mathrm{E}\exp\left\{\int_X \ln s(z)\boldsymbol{\mu}_t(dz)\right\},\ t\in\mathbb{N}$$

where $s(\cdot)$ is an measurable bounded function

$$h(t+\tau,s) = h(t,h(\tau,s))$$

$$h(s) = h(1,s)$$

First Main Theorem

Theorem

$\forall \mathbf{n} \notin S,\ \mathbf{n} \neq 0,\ \mathbf{r} \in S,\ t \geq 1$ *holds*

$$q_{\mathbf{r}}^{\mathbf{n}}(t) = \sum_{\boldsymbol{\alpha} \in S} \sum_{l=1}^{t} c_{\boldsymbol{\alpha}\mathbf{r}}(t,l) \widehat{P}(l, \mathbf{n}, \boldsymbol{\alpha}),$$

$$\begin{aligned} \textit{where } c_{\boldsymbol{\alpha}\mathbf{r}}(t+1, l+1) &= c_{\boldsymbol{\alpha}\mathbf{r}}(t,l), \\ c_{\boldsymbol{\alpha}\mathbf{r}}(t+1, 1) &= \delta_{\boldsymbol{\alpha}\mathbf{r}} - \sum_{l=1}^{t-1} \widetilde{P}(l, \boldsymbol{\alpha}, \mathbf{r}) \\ c_{\boldsymbol{\alpha}\mathbf{r}}(1,1) &= \delta_{\boldsymbol{\alpha}\mathbf{r}}. \end{aligned}$$

sketch of the proof.

based on the moment of the first attendance in S, the recursive use of the definitions of both transitions probabilities and that $q_{\mathbf{r}}^{\mathbf{n}}(t) = \sum_{l=1}^{t} \widetilde{P}(l, \mathbf{n}, \mathbf{r})$ □

Assumptions

Assumption (I)

The process is indecomposable, noncyclic and subcritical.

Assumption (II)

$\forall i, j = 1, 2, \ldots \ \mathrm{E}\{\boldsymbol{\mu}_{\mathcal{E}(j)1}(x_i) \log \boldsymbol{\mu}_{\mathcal{E}(j)1}(x_i)\}$ is finite, for $x_i \in X$.

Assumption (III)

If $h_{ij}(s) = \frac{\partial h_i(s)}{\partial s_j}$, then for all j, $1 \leq j < \infty$ there exists such i, $1 \leq i < \infty$, that $h_{ij}(0)$ are positive.

Main Result I ($t \to \infty$)

Lemma

Under Assumptions [I],[II] and [III]
$\lim_{t\to\infty} P\{\mu_{\mathbf{n}t}(X) = \mathcal{E}(i)|\mathbf{n} \neq 0\} = p^*_{\mathcal{E}(i)} > 0$, *for all* $i = 1, 2, \ldots$

sketch of the proof.

based on the existence of the limit $\lim_{t\to\infty} P\{\boldsymbol{\mu}_{\mathbf{n}t}(X) = \mathbf{k}|\mathbf{n} \neq 0\} = p^*_{\mathbf{k}}$
and that generating function $h^*(s) = \sum_{\mathbf{k}\in\mathcal{N}_{\mathbf{0}}^{\infty}} p^*_{\mathbf{k}} s^{\mathbf{k}}$ fulfills
$1 - h^*(h(\cdot)) = \delta(1 - h^*(s))$ and fulfills the main differential equation. □

Theorem

If $c_{\boldsymbol{\alpha}\mathbf{r}} = \lim\limits_{t\to\infty} c_{\boldsymbol{\alpha}\mathbf{r}}(t, l) = \delta_{\boldsymbol{\alpha}\mathbf{r}} - \sum\limits_{u=1}^{\infty} \widetilde{P}(u, \boldsymbol{\alpha}\mathbf{r})$ *then under the Assumption*
[I] $q^{\mathbf{n}}_{\mathbf{r}} = \sum_{l=1}^{\infty} \sum_{\boldsymbol{\alpha}\in S} c_{\boldsymbol{\alpha}\mathbf{r}} \widehat{P}(l, n, \boldsymbol{\alpha})$, $\forall \mathbf{n} \notin S$, $\mathbf{r} \in S$

Main Result II ($t \to \infty$ and $\overline{\mathbf{n}} \to \infty$)

Theorem

Let Assumptions [I],[II] are fulfilled and $\lim_{\overline{\mathbf{n}}\to\infty}(n_i/\overline{\mathbf{n}}) = a_i$, *where* $a = (a_1, a_2, \ldots)$. *In this case for* $\mathbf{r} \in S,\ \mathbf{n} \notin S,\ \mathbf{n} \neq 0$

$$q_{\mathbf{r}}^{\mathbf{n}} - H(\log_\delta \overline{\mathbf{n}}) \to 0,\ \textit{for}\ \overline{\mathbf{n}} \to \infty.$$

$H(x)$ *is a periodic function with period* 1

Lemma

Under Assumptions (1), (2), there exists such constant $\Theta > 0$, *that for some number* n_0 *holds* $q_{\mathbf{r}}^{\mathbf{n}} > \Theta$, *for* $\forall \mathbf{n}$ *with* $\overline{\mathbf{n}} \geq n_0$ *and* $\forall \mathbf{r} \in S$.

Further Research

- Uncountable number of types of particles
- Immigration influence on S-stopped branching processes
- Critical extension
- Other properties

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