On the Generating Functional of the special case of S-Stopped Branching Processes

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Outline

1. Set up of the process
2. Main properties of the transition matrix
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- $(X, \mathcal{A})$ – measured state space with $\mathcal{A}$ – σ-algebra of Borel sets on X
- Ω defines the set of all nonnegative measures α on $\mathcal{X}$, that are concentrated on the finite subsets from $\mathcal{A}$
- For $\alpha \in \Omega$, $\alpha = (x_1, n_1; \ldots; x_k, n_k)$ where $\{x_1, \ldots, x_k\}$ is that finite subset $\mathcal{A}$ on which α is concentrated.
- n_i is a nonnegative integer, and corresponds to the number of particles of a specific type.
- Let $\mathcal{Y}$ be a Kolmogorov σ-algebra on Ω, which is the smallest σ-algebra, that contains all cylindric sets $\{\alpha \in \Omega : \alpha(\{x\}) = n\}$.

- Let $P(t_1, x, t_2, A)$ be the transition probability, where $t_1, t_2 \in \mathbb{R}$ is time, $x \in X,\ A \in \mathcal{Y}$.
- lifetime τ of a particle is random.
- in the end of life every particle promptly gives rise to a random number of offsprings.
- random measures $\mu_{xt_0t}(A)$ and $\mu_{t_0t}(A)$, for $A \in \mathcal{Y}$.
- $X = \mathbb{R}^+$.
- in the beginning there is a finite number of particles.

Based on measure $\mu_{xt_0t}(A)$ we introduce a multivariate measure $\boldsymbol{\mu}_{\mathbf{x}t_0t}(A)$

$$\boldsymbol{\mu}_{\alpha(\mathbf{x})t_0t}(A) = \int_X \mu_{xt_0t}(A)\,dx,$$

where $\alpha \in \Omega$, $\mathbf{x} \subset X$ is the set of types of particles, which is the argument for the function α. For the short hand writing let $\boldsymbol{\mu}_{\cdot}(t_0, t) = \boldsymbol{\mu}_{\cdot t_0 t}(X)$.

Having $P(t_1, x, t_2, A)$ let us introduce $\widehat{P}(t_1, \alpha_1, t_2, \alpha_2)$, $\alpha_1, \alpha_2 \in \Omega$.

It is obvious, that

$$\widehat{P}(t_1, \alpha_1, t_2, \alpha_2) = P\{\boldsymbol{\mu}_{\alpha_1}(t_1, t_2) = \alpha_2\}.$$

Let us fix the the finite $S \in \Omega$, $0 \notin S$, which as the generalization can be of Lebesgue measure zero.

Stopped, or *S-stopped* multitype branching process is the process $\boldsymbol{\xi}_{\alpha t}(X)$, defined for $t \in \mathbb{R}^+$ and $\alpha \in \Omega$ by equations

$$\boldsymbol{\xi}_{\mathbf{x}}(t_0, t) = \begin{cases} \boldsymbol{\mu}_{\mathbf{x}}(t_0, t), & \text{if } \forall v,\ 0 \le v < t,\ \boldsymbol{\mu}_{\mathbf{x}}(t_0, v) \notin S; \\ \boldsymbol{\mu}_{\mathbf{x}}(t_0, u), & \text{if } \forall v,\ 0 \le v < u,\ \boldsymbol{\mu}_{\mathbf{x}}(t_0, v) \notin S, \\ & \quad \boldsymbol{\mu}_{\mathbf{x}}(t_0, u) \in S,\ u < t. \end{cases}$$

Let $\hat{P}_S(t_1, \alpha, t_2, A)$ be trans. prob. for S-stopped process and $\hat{P}(t_1, \alpha, t_2, A)$ for ordinary process defined for all $t_1 \le t_2$, $\alpha \in \Omega$ and $A \subset \mathcal{Y}$.

Both $\hat{P}_S(t_1, \alpha, t_2, A)$ and $\hat{P}(t_1, \alpha, t_2, A)$ define branch processes with cont time if they fulfill following properties:

- $\hat{P}(t_1, \alpha, t_2, \Omega) = \hat{P}_S(t_1, \alpha, t_2, \Omega) = 1$
- $\hat{P}(t_1, \alpha, t_1, A) = \hat{P}_S(t_1, \alpha, t_1, A) = \mathbf{I}\{\alpha \in A\}$
- Kolmogorov-Chapman equation:

$$\hat{P}(t_1, \alpha, t_3, A) = \int_{\Omega} \hat{P}(t_2, \alpha', t_3, A) \hat{P}(t_1, \alpha, t_2, d\alpha'), \ \forall t_1 \le t_2 \le t_3,$$

and a small modification for the S-stopped

$$\hat{P}_S(t_1, \alpha, t_3, A) = \int_{\Omega \backslash (S \backslash A)} \hat{P}(t_2, \alpha', t_3, A) \hat{P}(t_1, \alpha, t_2, d\alpha'), \ \forall t_1 \le t_2 \le t_3.$$

As all particles in the process evolute independently from each other, we may write following relationships

$$\begin{aligned}\hat{P}(t_1, \alpha_1 + \alpha_2, t_2, A) &= \int_\Omega \int_\Omega \mathrm{I}\{\alpha_1' + \alpha_2' \in A\} \hat{P}(t_1, \alpha_1, t_2, d\alpha_1') \\ &\times \hat{P}(t_1, \alpha_2, t_2, d\alpha_2'), \\ \hat{P}_S(t_1, \alpha_1 + \alpha_2, t_2, A) &= \int_{\Omega\setminus(S\setminus A)} \int_{\Omega\setminus(S\setminus A)} \mathrm{I}\{\alpha_1' + \alpha_2' \in A\} \hat{P}(t_1, \alpha_1, t_2, d\alpha_1') \\ &\times \hat{P}(t_1, \alpha_2, t_2, d\alpha_2').\end{aligned}$$

Under the given conditions, for the process with the continuous time, it is natural to assume, that for $\triangle \to 0$ both processes are equal $\hat{P}(t, \alpha, t + \triangle, A) = \hat{P}_S(t, \alpha, t + \triangle, A)$ and

$$\hat{P}(t, \alpha, t + \triangle, A) = \begin{cases} 1 + p(t, \alpha, A)t + o(t), & \text{for } p(t, \alpha, A) < 0, \ \alpha \in A; \\ p(t, \alpha, A)t + o(t), & \text{for } p(t, \alpha, A) \geq 0, \ \alpha \notin A. \end{cases}$$

Thus upper defined assumption can be reformulated as follows

$$\begin{aligned}\hat{P}(t_1,\alpha,t_2,A) &= p(t_1,\alpha,A)(t_2-t_1)+o(t_2-t_1),\\ &\qquad t_2\to t_1^-,\ \alpha\notin A,\\ \lim_{t_2\to t_1^+}\frac{\hat{P}(t_1,\alpha,t_2,A)-\hat{P}(t_1,\alpha,t_1,A)}{t_2-t_1} &= p(t_1,\alpha,A).\end{aligned}$$

Similarly one can show this for the right limit

$$\lim_{t_2\to t_1^-}\frac{\hat{P}(t_2,\alpha,t_1,A)-\hat{P}(t_1,\alpha,t_1,A)}{t_2-t_1}=p(t_1,\alpha,A).$$

This is equivalent to the fact, that $\frac{\partial}{\partial t}\hat{P}(t,\alpha,t,A)=p(t,\alpha,A)$. Function $p(t,\alpha,A)$ will be called a *transition density* of the branching process.

Let us introduce ordinal and logarithmic *Laplace functional* for both processes, based on the main transition probabilities
$([f, \alpha] = \int_X f(x)\alpha(dx))$

$$
\begin{aligned}
F(t_1, \alpha, t_2, s) &= \int_\Omega \exp\left\{\int_X s(x)\alpha'(dx)\right\} \hat{P}(t_1, \alpha, t_2, d\alpha') \\
\Psi(t_1, \alpha, t_2, s) &= \log F(t_1, \alpha, t_2, s) = \log E_{\hat{P}} \exp[s, \alpha], \\
F_S(t_1, \alpha, t_2, s) &= \int_\Omega \exp\left\{\int_X s(x)\alpha'(dx)\right\} \hat{P}_S(t_1, \alpha, t_2, d\alpha') \\
&= E_{\hat{P}_S} \exp[s, \alpha] \\
&= \int_{\Omega\setminus(S\setminus A)} \exp\left\{\int_X s(x)\alpha'(dx)\right\} \hat{P}_S(t_1, \alpha, t_2, d\alpha'), \\
\Psi_S(t_1, \alpha, t_2, s) &= \log F_S(t_1, \alpha, t_2, s) = \log E_{\hat{P}_S} \exp[s, \alpha].
\end{aligned}
$$

Know fact from Jirina M. (1967) "**General branching processes with continuous time parameter**", *Proc. Fifth Berkeley Symp. on Math. Statist. and Prob., Vol. 2, Pt. 1 (Univ. of Calif. Press, 1967)*, 389-399.:

Theorem

Functional equation for the ordinary branching processed is given through

$$\frac{\partial}{\partial s}F(s,w,t,f) = -\int_{\Omega} F(s,w',t,f)p(s,w,dw').$$

Theorem

Functional equation for S-stopped branching processed became

$$\frac{\partial}{\partial s}F(s,w,t,f) = -\int_{\Omega} F(s,w',t,f)p_{\bar{S}}(s,w,dw') + \mathcal{B}$$

where $\mathcal{B} = \partial B/\partial s$ and $p_{\bar{S}}(t,\alpha,A) = p(t,\alpha,A \setminus S)$.

Part B does not contain any recursions and infinite sums, and is also continuous and differentiable with respect to s.

- Immigration influence on S-stopped branching processes
- Critical extension
- Other properties

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