Localising temperature risk

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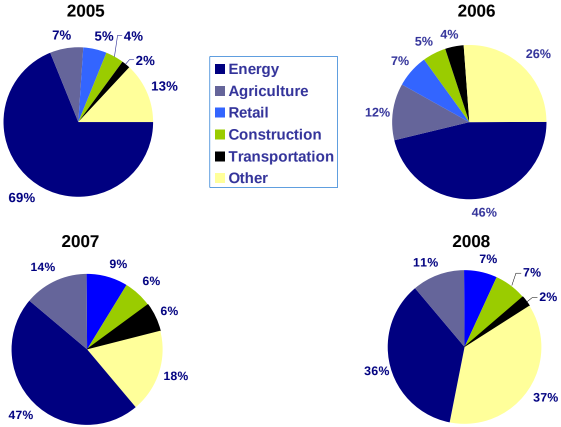
Weather

- Influences our daily lives and choices
- Impact on corporate revenues and earnings
- Meteorological institutions: business activity is weather dependent
  - British Met Office: daily beer consumption gain 10% if temperature increases by 3° C
  - If temperature in Chicago is less than 0° C consumption of orange juice declines 10% on average

Weather

Top 5 sectors in need of financial instruments to hedge weather risk, PwC survey for WRMA:



What are Weather Derivatives (WD)?

Hedge weather related risk exposures

- Payments based on weather related measurements
- Underlying: temperature, rainfall, wind, snow, frost

Chicago Mercantile Exchange (CME)

- Monthly/seasonal/weekly temperature Futures/Options
- 24 US, 6 Canadian, 9 European, 3 Australian, 3 Asian cities
- From 2.2 billion USD in 2004 to 15 billion USD through March 2009

Weather Derivatives

Temperature CME products

- $\text{HDD}(\tau_1, \tau_2) = \int_{\tau_1}^{\tau_2} \max(18°\text{C} - T_t, 0)dt$
- $\text{CDD}(\tau_1, \tau_2) = \int_{\tau_1}^{\tau_2} \max(T_t - 18°\text{C}, 0)dt$
- $\text{CAT}(\tau_1, \tau_2) = \int_{\tau_1}^{\tau_2} T_t dt$, where $T_t = \frac{T_{t,max} + T_{t,min}}{2}$
- $\text{AAT}(\tau_1, \tau_2) = \int_{\tau_1}^{\tau_2} \widetilde{T}_t dt$, where $\widetilde{T}_t = \frac{1}{24} \int_1^{24} T_{t_i} dt_i$ and T_{t_i} denotes the temperature of hour t_i,

Algorithm

Econometrics

$$T_t$$
$$\downarrow$$
$$X_t = T_t - \Lambda_t$$
$$\downarrow$$
$$X_{t+p} = a^\top X_t + \sigma_t \varepsilon_t$$
$$\downarrow$$
$$\hat{\varepsilon}_t = \frac{\hat{X}_t}{\hat{\sigma}_t} \sim \mathrm{N}(0,1)$$

Fin. Mathematics.

$$CAR(p)$$
$$\downarrow$$
$$F_{CAT(t,\tau_1,\tau_2)} = \mathrm{E}^{Q_\lambda}\left[CAT(\tau_1,\tau_2)\right]$$

- How to smooth the seasonal mean & variance curve?
- How close are the residuals to $\mathbf{N}(0, 1)$?
- How to infer the market price of weather risk?
- How to price no CME listed cities?

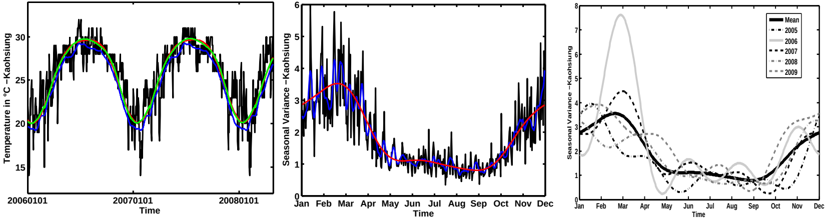


Figure 1: Kaohsiung daily average temperature, seasonal mean (left) & seasonal variation function (middle) with a Fourier truncated, the corrected Fourier and local linear estimation, seasonal variation over years (right).

Outline

1. Motivation ✓
2. Weather Dynamics
3. Stochastic Pricing
4. Localising temperature risk
5. Conclusion

CAT and AAT Indices

Can we make money?

WD type	Trading date	Measurement Period			Realised T_t
	t	τ_1	τ_2	CME[1]	$I^2_{(\tau_1,\tau_2)}$
Berlin-CAT	20070316	20070501	20070531	457.00	494.20
		20070601	20070630	529.00	574.30
		20070701	20070731	616.00	583.00
Tokyo-AAT	20081027	20090401	20090430	592.00	479.00
		20090501	20090531	682.00	623.00
		20090601	20090630	818.00	679.00

Table 1: Berlin and Tokyo contracts listed at CME. Source: Bloomberg. CME[1] WD Futures listed on CME, $I^2_{(\tau_1,\tau_2)}$ index values computed from the realized temperature data.

Weather Dynamics

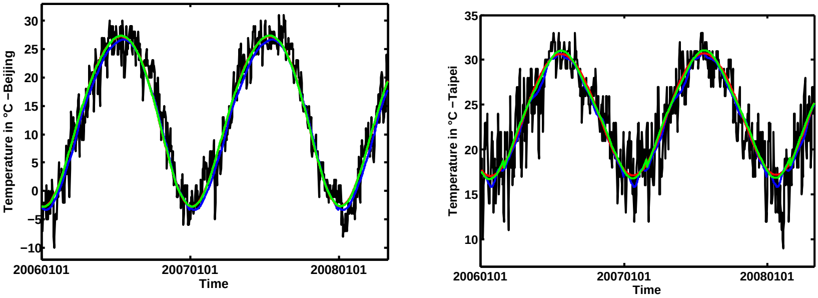


Figure 2: The Fourier truncated, the corrected Fourier and the the local linear seasonal component for daily average temperatures.

Go to details

AR(p): $X_t = \sum_{l=1}^{L} \beta_l X_{t-l} + \varepsilon_t, \; \varepsilon_t = \sigma_t e_t$

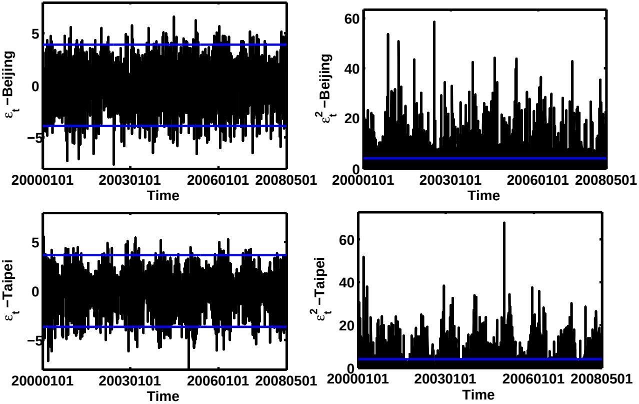


Figure 3: (square) Residuals $\hat{\varepsilon}_t$ (left), $\hat{\varepsilon}_t^2$ (right). No rejection of H_0 that residuals are uncorrelated at 0% significance level, (Li-McLeod Portmanteau test)

Go to details

ACF of (Squared) Residuals after Correcting Seasonal Volatility

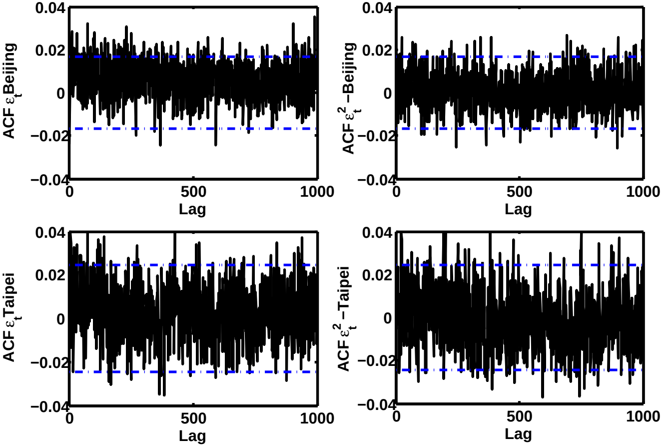


Figure 4: (Left) Right: ACF for temperature (squared) residuals $\frac{\hat{\varepsilon}_t}{\hat{\sigma}_{t,LLR}}$

Residuals $\left(\frac{\hat{\varepsilon}_t}{\hat{\sigma}_t}\right)$ become normal

City		JB	Kurt	Skew	KS	AD
Berlin	$\frac{\hat{\varepsilon}_t}{\hat{\sigma}_{t,FTSG}}$	304.77	3.54	-0.08	0.01	7.65
	$\frac{\hat{\varepsilon}_t}{\hat{\sigma}_{t,LLR}}$	279.06	3.52	-0.08	0.01	7.29
Kaohsiung	$\frac{\hat{\varepsilon}_t}{\hat{\sigma}_{t,FTSG}}$	2753.00	4.68	-0.71	0.06	79.93
	$\frac{\hat{\varepsilon}_t}{\hat{\sigma}_{t,LLR}}$	2252.50	4.52	-0.64	0.06	79.18
Tokyo	$\frac{\hat{\varepsilon}_t}{\hat{\sigma}_{t,FTSG}}$	133.26	3.44	-0.10	0.02	8.06
	$\frac{\hat{\varepsilon}_t}{\hat{\sigma}_{t,LLR}}$	148.08	3.44	-0.13	0.02	10.31

Table 2: Skewness, kurtosis, Jarque Bera (JB), Kolmogorov Smirnov (KS) and Anderson Darling (AD) test statistics (365 days). Critical values JB: 5%(5.99), 1%(9.21), KS: 5%(0.07), 1%(0.08), AD: 5%(2.49),1% (3.85)

Temperature Dynamics

Temperature time series:

$$T_t = \Lambda_t + X_t$$

with seasonal function Λ_t. X_t can be seen as a discretization of a continuous-time process AR(p) (CAR(p)).

This stochastic model allows CAR(p) futures/options pricing.

CAT Futures

For $0 \leq t \leq \tau_1 < \tau_2$, the future Cumulative Average Temperature:

$$
\begin{aligned}
F_{CAT(t,\tau_1,\tau_2)} &= \mathsf{E}^{Q_\lambda}\left[\int_{\tau_1}^{\tau_2} T_s ds | \mathcal{F}_t\right] \\
&= \int_{\tau_1}^{\tau_2} \Lambda_u du + \mathbf{a}_{t,\tau_1,\tau_2}\mathbf{X}_t + \int_t^{\tau_1} \lambda_u \sigma_u \mathbf{a}_{t,\tau_1,\tau_2}\mathbf{e}_L du \\
&+ \int_{\tau_1}^{\tau_2} \lambda_u \sigma_u \mathbf{e}_1^\top \mathbf{A}^{-1}\left[\exp\left\{\mathbf{A}(\tau_2 - u)\right\} - I_L\right]\mathbf{e}_L du
\end{aligned}
$$

with $\mathbf{a}_{t,\tau_1,\tau_2} = \mathbf{e}_1^\top \mathbf{A}^{-1}\left[\exp\left\{\mathbf{A}(\tau_2 - t)\right\} - \exp\left\{\mathbf{A}(\tau_1 - t)\right\}\right]$, $I_L : L \times L$ identity matrix, λ_u MPR inferred from data, Benth et al. (2007). Λ_u, σ_u to be localised.

Go to details

Local Temperature Risk

Normality of ε_t requires estimating the function $\theta(t) = \{\Lambda_t, \sigma_t^2\}$ with $t = 1, \ldots, 365$ days, $j = 0, \ldots, J$ years. Recall:

$$
\begin{aligned}
X_{365j+t} &= T_{t,j} - \Lambda_t, \\
X_{365j+t} &= \sum_{l=1}^{L} \beta_{lj} X_{365j+t-l} + \varepsilon_{t,j}, \\
\varepsilon_{t,j} &= \sigma_t e_{t,j}, \\
e_{t,j} &\sim \mathrm{N}(0,1), i.i.d.
\end{aligned}
$$

Adaptation Scale (for variance)

Fix $s \in 1, 2, \ldots, 365$, sequence of ordered weights:
$W^k(s) = \{w(s, 1, h_k), w(s, 2, h_k), \ldots, w(s, 365, h_k)\}^\top$.
Define $w(s, t, h_k) = K_{h_k}(s - t), (h_1 < h_2 < \ldots < h_K)$.

$$
\begin{aligned}
\hat{\varepsilon}_{365j+t} &= X_{365j+t} - \sum_{l=1}^{L} \hat{\beta}_l X_{365j+t-l} \\
\tilde{\theta}_k(s) &\stackrel{\text{def}}{=} \arg\max_{\theta \in \Theta} L\{W^k(s), \theta\} \\
&= \arg\min_{\theta \in \Theta} \sum_{t=1}^{365} \sum_{j=0}^{J} \{\log(2\pi\theta)/2 + \hat{\varepsilon}_{t,j}^2 / 2\theta\} w(s, t, h_k) \\
&= \sum_{t,j} \hat{\varepsilon}_{t,j}^2 w(s, t, h_k) / \sum_{t,j} w(s, t, h_k)
\end{aligned}
$$

Parametric Exponential Bounds

$$
\begin{aligned}
L(W^k, \tilde{\theta}_k, \theta^*) &\stackrel{\text{def}}{=} N_k \mathcal{K}(\tilde{\theta}_k, \theta^*) \\
&= -\{\log(\tilde{\theta}_k/\theta^*) + 1 - \theta^*/\tilde{\theta}_k\}/2,
\end{aligned}
$$

where $\mathcal{K}\{\tilde{\theta}_k, \theta^*\}$ is the Kullback-Leibler divergence between $\tilde{\theta}_k$ and θ^* and $N_k = J \cdot \Sigma_{t=1}^{365} w(s,t,h_k)$. For any $\mathfrak{z} > 0$,

$$
\begin{aligned}
\mathrm{P}_{\theta^*}\{L(W^k, \tilde{\theta}_k, \theta^*) > \mathfrak{z}\} &\leq 2\exp(-\mathfrak{z}) \\
\mathsf{E}_{\theta^*}\left|L(W^k, \tilde{\theta}_k, \theta^*)\right|^r &\leq \mathfrak{r}_r
\end{aligned}
$$

where $\mathfrak{r}_r = 2r \int_{\mathfrak{z} \geq 0} \mathfrak{z}^{r-1} \exp(-\mathfrak{z}) d\mathfrak{z}$.

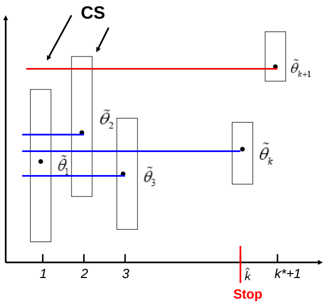
LMS Procedure

Construct an estimate $\hat{\theta} = \hat{\theta}(s)$, on the base of $\tilde{\theta}_1(s), \tilde{\theta}_2(s), \ldots, \tilde{\theta}_K(s)$.

- Start with $\hat{\theta}_1 = \tilde{\theta}_1$.
- For $k \geq 2$, $\tilde{\theta}_k$ is accepted and $\hat{\theta}_k = \tilde{\theta}_k$ if $\tilde{\theta}_{k-1}$ was accepted and

$$L(W^k, \tilde{\theta}_\ell, \tilde{\theta}_k) \leq \mathfrak{z}_\ell, \ell = 1, \ldots, k-1$$

$\hat{\theta}_k$ is the the latest accepted estimate after the first k steps.



Propagation Condition

A bound for the risk associated with first kind error:

$$E_{\theta^*} |L(W^k, \tilde{\theta}_k, \hat{\theta}_k)|^r \leq \alpha \mathfrak{r}_r \tag{1}$$

where $k = 1, \ldots, K$ and $\mathfrak{r}_r$ is the parametric risk bound.

Sequential Choice of Critical Values

- Consider first $\mathfrak{z}_1$ letting $\mathfrak{z}_2 = \ldots = \mathfrak{z}_{K-1} = \infty$. Leads to the estimates $\hat{\theta}_k(\mathfrak{z}_1)$ for $k = 2, \ldots, K$.
- The value $\mathfrak{z}_1$ is selected as the minimal one for which

$$\sup_{\theta^*} \mathsf{E}_{\theta^*} \left| L\{W^k, \tilde{\theta}_k, \hat{\theta}_k(\mathfrak{z}_1)\} \right|^r \leq \frac{\alpha}{K-1} \mathfrak{r}_r, k = 2, \ldots, K.$$

- Set $\mathfrak{z}_{k+1} = \ldots = \mathfrak{z}_{K-1} = \infty$ and fix $\mathfrak{z}_k$ lead the set of parameters $\mathfrak{z}_1, \ldots, \mathfrak{z}_k, \infty, \ldots, \infty$ and the estimates $\hat{\theta}_m(\mathfrak{z}_1, \ldots, \mathfrak{z}_k)$ for $m = k+1, \ldots, K$. Select $\mathfrak{z}_k$ s.t.

$$\sup_{\theta^*} \mathsf{E}_{\theta^*} \left| L\{W^k, \tilde{\theta}_m, \hat{\theta}_m(\mathfrak{z}_1, \mathfrak{z}_2, \ldots, \mathfrak{z}_k)\} \right|^r \leq \frac{k\alpha}{K-1} \mathfrak{r}_r,$$

$m = k+1, \ldots, K$.

Critical Values

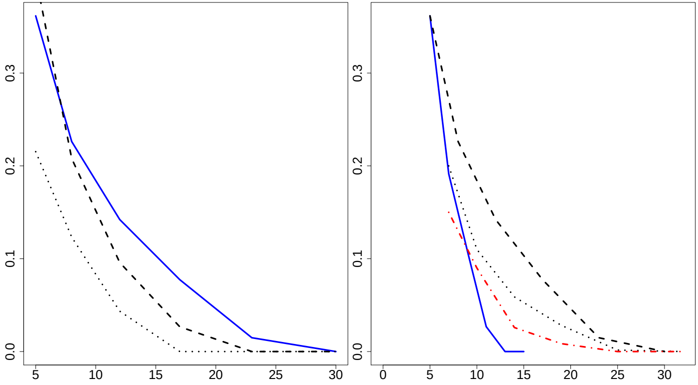


Figure 5: Simulated CV with $\theta^* = 1$, $r = 0.5$, $MC = 5000$ with $\alpha = 0.3$, 0.5, 0.7 (left), with different bandwidth sequences (right).

Small Modeling Bias (SMB) Condition and Oracle Property

$$\Delta(W^k, \theta) = \sum_{t=1}^{365} \mathcal{K}\{\theta(t), \theta\} \mathbf{1}\{w(s, t, h_k) > 0\} \le \Delta, \forall k < k^*$$

k^* is the maximum k satisfying the SMB condition.

Propagation Property:
For any estimate $\tilde{\theta}_k$ and θ satisfying SMB, it holds:

$$\mathrm{E}_{\theta(.)} \log\{1 + |L(W^k, \tilde{\theta}_k, \theta)|^r / \mathfrak{r}_r\} \le \Delta + \alpha$$

Stability Property

The attained quality of estimation during "propagation" can not get lost at further steps.

$$L(W^{k^*}, \tilde{\theta}_{k^*}, \hat{\theta}_{\hat{k}}) \mathbf{1}\{\hat{k} > k^*\} \leq \mathfrak{z}_{k^*}$$

$\hat{\theta}_{\hat{k}}$ delivers at least the same accuracy of estimation as the "oracle" $\tilde{\theta}_{k^*}$

Oracle Property

Theorem

Let $\Delta(W^k, \theta) \leq \Delta$ *for some* $\theta \in \Theta$ *and* $k \leq k^*$. *Then*

$$E_{\theta(.)} \log \left\{1 + |L(W^{k^*}, \tilde{\theta}_{k^*}, \theta)|^r / \mathfrak{r}_r \right\} \leq \Delta + 1$$

$$E_{\theta(.)} \log \left\{1 + |L(W^{k^*}, \tilde{\theta}_{k^*}, \hat{\theta}_{\hat{k}})|^r / \mathfrak{r}_r \right\} \leq \Delta + \alpha + \log\{1 + \mathfrak{z}_{k^*} / \mathfrak{r}_r\}$$

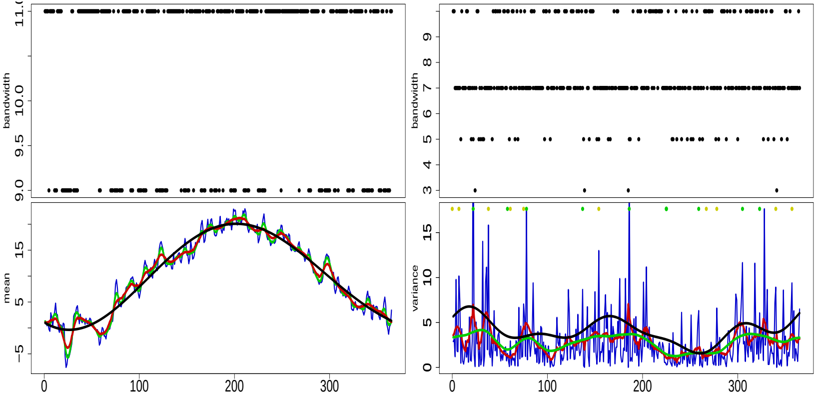


Figure 6: Estimation of mean 2007 (left) and variance 20050101-20071231 (right) for *Berlin*. Bandwidths sequences (upper panel), nonparametric function estimation, with fixed bandwidth, adaptive bandwidth and truncated Fourier (bottom panel), $\alpha = 0.7$, $r = 0.5$.

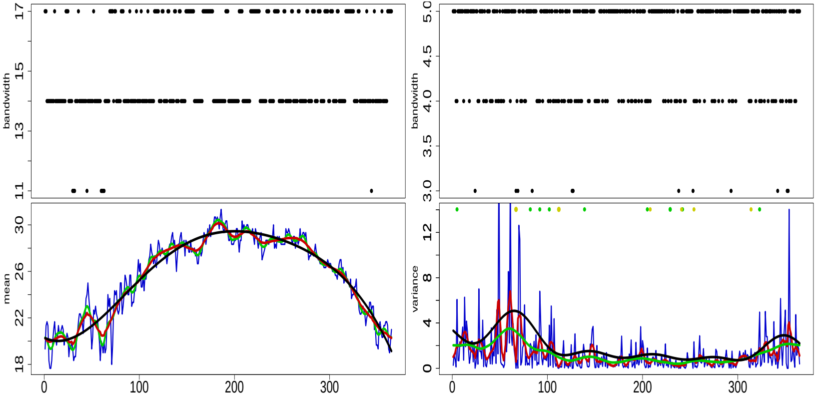


Figure 7: Estimation of mean 2008 (left) and variance 20060101-20081231 (right) for *Kaohsiung*. Bandwidths sequences (upper panel), nonparametric function estimation, with fixed bandwidth, adaptive bandwidth and truncated Fourier (bottom panel), $\alpha = 0.7$, $r = 0.5$.

Iterative approach, $\theta(t) = \{\Lambda_t, \sigma_t^2\}$

Step 1. Estimate $\hat{\beta}$ in an initial Λ_t^0 using a truncated Fourier series or any other deterministic function;

Step 2. For fixed $\hat{\Lambda}_{s,\nu} = \{\hat{\Lambda}'_{s,\nu}, \hat{\Lambda}''_{s,\nu}\}^\top, s = \{1, \ldots, 365\}$ from last step ν, and fixed $\hat{\beta}$, get $\hat{\sigma}^2_{s,\nu+1}$ by

$$\begin{aligned}\hat{\sigma}^2_{s,\nu+1} &= \arg\min_{\sigma^2} \sum_{t=1}^{365} \sum_{j=0}^{J} [\{T_{365j+t} - \hat{\Lambda}'_{s,\nu} - \hat{\Lambda}''_{s,\nu}(t-s) \\ &- \sum_{l=1}^{L} \hat{\beta}_l X_{365j+t-l}\}^2 / 2\sigma^2 + \log(2\pi\sigma^2)/2] w(s, t, h'_k);\end{aligned}$$

Iterative approach

Step 3. For fixed $\hat{\sigma}^2_{s,\nu+1}$ and $\hat{\beta}$, we estimate $\hat{\Lambda}_{s,\nu+1}, s = \{1, \ldots, 365\}$ via another a local adaptive procedure:

$$\hat{\Lambda}_{s,\nu+1} = \arg\min_{\{\Lambda',\Lambda''\}^\top} \sum_{t=1}^{365} \sum_{j=0}^{J} \Big\{ T_{365j+t} - \Lambda' - \Lambda''(t-s) - \sum_{l=1}^{L} \hat{\beta}_l X_{365j+t-l} \Big\}^2 w(s,t,h'_k)/2\hat{\sigma}^2_{s,\nu+1},$$

where $\{h'_1, h'_2, h'_3, \ldots, h'_{K'}\}$ is a sequence of bandwidths;

Step 4. Repeat steps 2 and 3 till both $|\hat{\Lambda}_{t,\nu+1} - \hat{\Lambda}_{t,\nu}| < \pi_1$ and $|\hat{\sigma}^2_{t,\nu+1} - \hat{\sigma}^2_{t,\nu}| < \pi_2$ for some constants π_1 and π_2.

Aggregated approach

Let $\hat{\theta}^j(t)$ the localised observation at time t of year j, the aggregated local function is given by:

$$\hat{\theta}_\omega(t) = \sum_{j=1}^{J} \omega_j \hat{\theta}^j(t) \tag{2}$$

$\arg\min_{\omega} \sum_{j=1}^{J} \sum_{t=1}^{365} \{\hat{\theta}_\omega(t) - \hat{\theta}_j^o(t)\}^2$ s.t. $\sum_{j=1}^{J} \omega_j = 1; \omega_j > 0$,
$\hat{\theta}_j^o$ defined as:

1. (Locave) $\hat{\theta}_j^o(t) = J^{-1} \sum_{j=1}^{J} \hat{\sigma}_j^2(t)$
2. (Locsep) $\hat{\theta}_j^o(t) = \hat{\sigma}_j^2(t)$
3. (Locmax) maximising p-values of AD-test over a year.

Normalized Residuals

		2 years			3 years		
		KS	JB	AD	KS	JB	AD
Berlin	Adaptive BW	5.06e-06	1.91e-01	0.55	0.01	2.41e-01	0.56
	Fixed BW	3.49e-03	1.81e-10	0.06	0.09	6.55e-08	0.13
	Locmax	9.79e-01	3.30e-01	0.94	0.87	1.60e-02	0.47
	Fourier	3.14e-01	0.00	0.01	0.60	2.22e-16	0.01
	Campbell&Diebold	4.94e-07	0.00	0.00	0.00	0.00	0.01
Kaohsiung	Adaptive BW	1.55e-07	9.90e-03	1.78e-02	2.38e-05	1.04e-11	1.57e-07
	Fixed BW	1.83e-05	0.00	2.76e-09	2.25e-03	0.00	1.13e-14
	Locmax	5.92e-02	1.11e-04	4.44e-04	9.05e-03	1.57e-05	4.46e-06
	Fourier	6.29e-03	0.00	3.03e-10	3.89e-04	0.00	2.01e-14
	Campbell&Diebold	1.49e-05	0.00	1.95e-10	0.00	0.00	6.72e-20

Table 3: *p*-values for different models and GoF tests for Berlin and Kaohsiung.

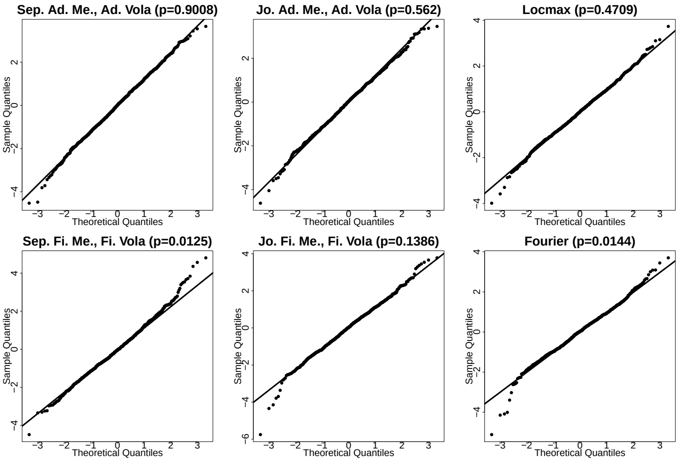


Figure 8: QQ-plot for standardized residuals from Berlin using different methods for the data from 2005-2007 (3 years)

Can we make money?

Trading date	MP		Future Prices			Real. T_t	
t	τ_1	τ_2	CME	$\lambda_t = 0$	$\lambda_t = \lambda$	$I_{(\tau_1,\tau_2)}$	Strategy
Berlin-CAT							
20070316	20070501	20070531	457.00	450.67	442.58	494.20	6.32(C)
20070316	20070601	20070630	529.00	538.46	542.92	574.30	-9.46(P)
20070316	20070701	20070731	616.00	628.36	618.89	583.00	-12.36(P)
Tokyo-AAT							
20081027	20090401	20090430	592.00	442.12	458.47	479.00	149.87(C)
20081027	20090501	20090531	682.00	577.98	602.75	623.00	104.01(C)
20081027	20090601	20090630	818.00	688.28	692.54	679.00	129.71(C)

Table 4: Weather contracts listed at CME. (Source: Bloomberg). Future prices $\hat{F}_{t,\tau_1,\tau_2,\lambda,\theta}$ estimated prices with MPR (λ_t) under different localisation schemes ($\hat{\theta}$ under Locmax for Berlin (20020101-20061231), Tokyo (20030101-20081231)), Strategy (CME- $\hat{F}_{t,\tau_1,\tau_2,\lambda=0}$), P(Put), C(Call), MP(Measurement Period)

References

W. K. Härdle and B. López Cabrera (2011)
Implied market price of weather risk
Applied Mathematical Finance, Issue 5, 1-37.

F.E Benth and J.S. Benth and S. Koekebakker (2007)
Putting a price on temperature
Scandinavian Journal of Statistics 34: 746-767

P.J. Brockwell
Continuous time ARMA Process
Handbook of Statistics 19: 248-276, 2001

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Appendix A

Li-McLeod Portmanteau Test– modified Portmanteau test statistic Q_L to check the uncorrelatedness of the residuals:

$$Q_L = n \sum_{k=1}^{L} r_k^2(\hat{\varepsilon}) + \frac{L(L+1)}{2n},$$

where r_k, $k = 1, \ldots, L$ are values of residuals ACF up to the first L lags and n is the sample size. Then,

$$Q_L \sim \chi^2_{(L-p-q)}$$

Q_L is χ^2distributed on $(L - p - q)$ degrees of freedom where p,q denote AR and MA order respectively and L is a given value of considered lags.

Appendix

Consider 2 prob. measures P & Q. Assume that $\frac{dQ}{dP}|_{\mathcal{F}_t} = Z_t > 0$ is a positive Martingale. By $Ito's$ Lemma, then:

$$\begin{aligned} Z_t &= \exp\{\log(Z_t)\} \\ &= \exp\left\{\int_0^t (Z_s)^{-1} dZ_s - \frac{1}{2}\int_0^t (Z_s)^{-2} d < Z, Z >_s\right\} \end{aligned} \quad (3)$$

Let $dZ_s = Z_s \cdot \theta_s \cdot dB_s$, then:

$$Z_t = \exp\left(\int_0^t \theta_s dB_s - \frac{1}{2}\int_0^t \theta_s^2 ds\right) \quad (4)$$

Appendix B

Let B_t, Z_t be Martingales under P, then by Girsanov theorem:

$$\begin{aligned} B_t^\theta &= B_t - \int_0^t (Z_s)^{-1} d < Z, B >_s \\ &= B_t - \int_0^t (Z_s)^{-1} d < \int_0^s \theta_u Z_u dB_u, B_s > \\ &= B_t - \int_0^t (Z_s)^{-1} \theta_s Z_s d < B_s, B_s > \\ &= B_t - \int_0^t \theta_s ds \end{aligned} \tag{5}$$

is a Martingale unter Q.

Black-Scholes Model

Asset price follows:

$$dS_t = \mu S_t dt + \sigma_t S_t dB_t$$

Note that S_t is not a Martingale unter P, but it is under Q!
Explicit dynamics:

$$\begin{aligned} S_t &= S_0 + \int_0^t \mu S_s ds + \int_0^t \sigma_s S_s dB_s \\ &= S_0 + \int_0^t \mu S_s ds + \int_0^t \sigma_s S_s dB_s^\theta + \int_0^t \theta_s \sigma_s S_s ds \\ &= S_0 + \int_0^t S_s(\mu + \theta_s \sigma_s) ds + \int_0^t \sigma_s S_s dB_s^\theta \end{aligned} \tag{6}$$

Market price of Risk and Risk Premium

By the no arbitrage condition, the risk free interest rate r should be equal to the drift $\mu + \theta_s \sigma_s$, so that:

$$\theta_s = \frac{r - \mu}{\sigma_s} \qquad (7)$$

In practice:
$B_t^\theta = B_t - \int_0^t \left(\frac{\mu - r}{\sigma_s} \right) ds$ is a Martingale under Q and then $e^{-rt} S_t$ is also a Martingale.

Under risk taking, the risk premium is defined as:

$$r + \Delta$$

Stochastic Pricing

The process $X_t = T_t - \Lambda_t$ can be seen as a discretization of a continuous-time process AR(L) (CAR(L)): Ornstein-Uhlenbeck process $\mathbf{X}_t \in \mathbb{R}^L$:

$$d\mathbf{X}_t = \mathbf{A}\mathbf{X}_t dt + \mathbf{e}_L \sigma_t dB_t$$

$\mathbf{e}_l$: lth unit vector in $\mathbb{R}^L$ for $l = 1, \ldots, L$, $\sigma_t > 0$, $\mathbf{A}$: $(L \times L)$-matrix

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & \ldots & \ldots & 0 & 1 \\ -\alpha_L & -\alpha_{L-1} & \ldots & & -\alpha_1 \end{pmatrix}$$

$\mathbf{X}_t$ can be written as a Continuous-time AR(p) (CAR(p)):

For $p = 1$,

$$dX_{1t} = -\alpha_1 X_{1t} dt + \sigma_t dB_t$$

For $p = 2$,

$$\begin{aligned} X_{1(t+2)} &\approx (2-\alpha_1) X_{1(t+1)} \\ &+ (\alpha_1 - \alpha_2 - 1) X_{1t} + \sigma_t (B_{t-1} - B_t) \end{aligned}$$

For $p = 3$,

$$\begin{aligned} X_{1(t+3)} &\approx (3-\alpha_1) X_{1(t+2)} + (2\alpha_1 - \alpha_2 - 3) X_{1(t+1)} \\ &+ (-\alpha_1 + \alpha_2 - \alpha_3 + 1) X_{1t} + \sigma_t (B_{t-1} - B_t) \end{aligned}$$

Proof $CAR(3) \approx AR(3)$

Let

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\alpha_3 & -\alpha_2 & -\alpha_1 \end{pmatrix}$$

- use $B_{t+1} - B_t = \varepsilon_t$
- assume a time step of length one $dt = 1$
- substitute iteratively into X_1 dynamics

Proof $CAR(3) \approx AR(3)$:

$$
\begin{aligned}
X_{1(t+1)} - X_{1(t)} &= X_{2(t)}dt \\
X_{2(t+1)} - X_{2(t)} &= X_{3(t)}dt \\
X_{3(t+1)} - X_{3(t)} &= -\alpha_1 X_{1(t)}dt - \alpha_2 X_{2(t)}dt - \alpha_3 X_{3(t)}dt + \sigma_t \varepsilon_t \\
X_{1(t+2)} - X_{1(t+1)} &= X_{2(t+1)}dt \\
X_{2(t+2)} - X_{2(t+1)} &= X_{3(t+1)}dt \\
X_{3(t+2)} - X_{3(t+1)} &= -\alpha_1 X_{1(t+1)}dt - \alpha_2 X_{2(t+1)}dt \\
&\quad -\alpha_3 X_{3(t+1)}dt + \sigma_{t+1}\varepsilon_{t+1} \\
X_{1(t+3)} - X_{1(t+2)} &= X_{2(t+2)}dt \\
X_{2(t+3)} - X_{2(t+2)} &= X_{3(t+2)}dt \\
X_{3(t+3)} - X_{3(t+2)} &= -\alpha_1 X_{1(t+2)}dt - \alpha_2 X_{2(t+2)}dt \\
&\quad -\alpha_3 X_{3(t+2)}dt + \sigma_{t+2}\varepsilon_{t+2}
\end{aligned}
$$

return

Temperature: $T_t = X_t + \Lambda_t$ Seasonal function with trend:

$$\hat{\Lambda}_t = a + bt + \sum_{l=1}^{L} \hat{c}_l \cdot \cos\left\{\frac{2\pi l(t - \hat{d}_l)}{l \cdot 365}\right\} + \mathcal{I}(t \in \omega) \cdot \sum_{i=l}^{p} \hat{c}_i \cdot \cos\left\{\frac{2\pi(i-4)(t - \hat{d}_i)}{i \cdot 365}\right\} \quad (8)$$

$\hat{a}$: average temperature, $\hat{b}$: global Warming. $\mathcal{I}(t \in \omega)$ an indicator for Dec., Jan. and Feb

City	Period	$\hat{a}$	$\hat{b}$	$\hat{c}_1$	$\hat{d}_1$
Tokyo	19730101-20081231	15.76	7.82e-05	10.35	-149.53
Osaka	19730101-20081231	15.54	1.28e-04	11.50	-150.54
Beijing	19730101-20081231	11.97	1.18e-04	14.91	-165.51
Taipei	19920101-20090806	23.21	1.68e-03	6.78	-154.02

Table 5: Seasonality estimates of daily average temperatures in Asia. All coefficients are nonzero at 1% significance level. Data source: Bloomberg

City(Period)	$\hat{a}$	$\hat{b}$	$\hat{c}_1$	$\hat{d}_1$	$\hat{c}_2$	$\hat{d}_2$	$\hat{c}_3$	$\hat{d}_3$
Tokyo								
(730101-081231)	15.7415	0.0001	8.9171	-162.3055	-2.5521	-7.8982	-0.7155	-15.0956
(730101-821231)	15.8109	0.0001	9.2855	-162.6268	-1.9157	-16.4305	-0.5907	-13.4789
(830101-921231)	15.4391	0.0004	9.4022	-162.5191	-2.0254	-4.8526	-0.8139	-19.4540
(930101-021231)	16.4284	0.0001	8.8176	-162.2136	-2.1893	-17.7745	-0.7846	-22.2583
(030101-081231)	16.4567	0.0001	8.5504	-162.0298	-2.3157	-18.3324	-0.6843	-16.5381
Taipei								
(920101-081231)	23.2176	0.0002	1.9631	-164.3980	-4.8706	-58.6301	-0.2720	39.1141
(920101-011231)	23.1664	0.0002	3.8249	-150.6678	-2.8830	-68.2588	0.2956	-41.7035
(010101-081231)	24.1295	-0.0001	1.8507	-149.1935	-5.1123	-67.5773	-0.3150	22.2777
Osaka								
(730101-081231)	15.2335	0.0002	10.0908	-162.3713	-2.5653	-7.5691	-0.6510	-19.4638
(730101-821231)	15.9515	-0.0001	9.7442	-162.5119	-2.1081	-17.9337	-0.5307	-18.9390
(830101-921231)	15.7093	0.0003	10.1021	-162.4248	-2.1532	-10.7612	-0.7994	-24.9429
(930101-021231)	16.1309	0.0003	10.3051	-162.4181	-2.0813	-21.9060	-0.7437	-27.1593
(030101-081231)	16.9726	0.0002	10.5863	-162.4215	-2.1401	-14.3879	-0.8138	-17.0385
Kaohsiung								
(730101-081231)	24.2289	0.0001	0.9157	-145.6337	-4.0603	-78.1426	-1.0505	10.6041
(730101-821231)	24.4413	0.0001	2.1112	-129.1218	-3.3887	-91.1782	-0.8733	20.0342
(830101-921231)	25.0616	0.0003	2.0181	-135.0527	-2.8400	-89.3952	-1.0128	20.4010
(930101-021231)	25.3227	0.0003	3.9154	-165.7407	-0.7405	-51.4230	-1.1056	19.7340
Beijing								
(730101-081231)	11.8904	0.0001	14.9504	-165.2552	0.0787	-12.8697	-1.2707	4.2333
(730101-821231)	11.5074	0.0003	14.8772	-165.7679	0.6253	15.8090	-1.2349	1.8530
(830101-921231)	12.4606	0.0002	14.9616	-165.7041	0.5327	14.3488	-1.2630	4.8809
(930101-021231)	13.6641	-0.0003	14.8970	-166.1435	0.9412	16.9291	-1.1874	-4.5596
(030101-081231)	12.8731	0.0003	14.9057	-165.9098	0.7266	16.5906	-1.5323	1.8984

Table 6: Seasonality estimates $\hat{\lambda}_t$ of daily average temperatures in Asia. All coefficients are nonzero at 1% significance level. Data source: Bloomberg.

return

AR(p) $\rightarrow$ CAR(p)

City	ADF	KPSS	AR(3)			CAR(3)				
	$\hat{\tau}$	$\hat{k}$	β_1	β_2	β_3	α_1	α_2	α_3	$\tilde{\lambda}_1$	$\tilde{\lambda}_{2,3}$
Portland	-45.13+	0.05*	0.86	-0.22	0.08	2.13	1.48	0.26	-0.27	-0.93
Atlanta	-55.55+	0.21***	0.96	-0.38	0.13	2.03	1.46	0.28	-0.30	-0.86
New York	-56.88+	0.08*	0.76	-0.23	0.11	2.23	1.69	0.34	-0.32	-0.95
Houston	-38.17+	0.05*	0.90	-0.39	0.15	2.09	1.57	0.33	-0.33	-0.87
Berlin	-40.94+	0.13**	0.91	-0.20	0.07	2.08	1.37	0.20	-0.21	-0.93
Essen	-23.87+	0.11*	0.93	-0.21	0.11	2.06	1.34	0.16	-0.16	-0.95
Tokyo	-25.93+	0.06*	0.64	-0.07	0.06	2.35	1.79	0.37	-0.33	-1.01
Osaka	-18.65+	0.09*	0.73	-0.14	0.06	2.26	1.68	0.34	-0.33	-0.96
Beijing	-30.75+	0.16***	0.72	-0.07	0.05	2.27	1.63	0.29	-0.27	-1.00
Kaohsiung	-37.96+	0.05*	0.73	-0.08	0.04	2.26	1.60	0.29	-0.45	-0.92
Taipei	-32.82+	0.09*	0.79	-0.22	0.06	2.20	1.63	0.36	-0.40	-0.90

Table 7: ADF and KPSS-Statistics, coefficients of $AR(3)$, $CAR(3)$ and eigenvalues $\lambda_{1,2,3}$, for the daily average temperatures time series. + 0.01 critical values, * 0.1 critical value, **0.05 critical value (0.14), ***0.01 critical value. Historical data: 19470101-20091210.

return to AR