Properties of Hierarchical Archimedean Copulas

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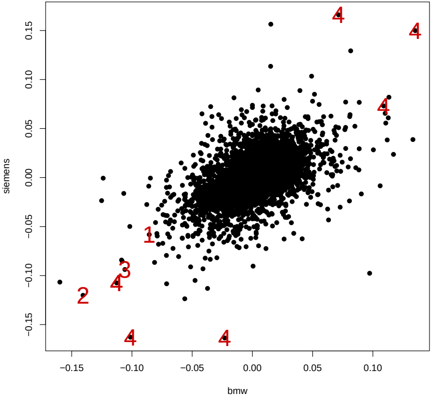
"Extreme, **synchronized rises and falls** in financial markets occur infrequently but they do occur. The problem with the models is that they did not assign a high enough chance of occurrence to the scenario in which **many things go wrong at the same time - the "perfect storm" scenario**"

(Business Week, September 1998)

Correlation



1. 19.10.1987
   Black Monday
2. 16.10.1989
   Berlin Wall
3. 19.08.1991
   Kremlin
4. 17.03.2008, 19.09.2008, 10.10.2008, 13.10.2008, 15.10.2008, 29.10.2008
   Crisis

Outline

Archimedean Copula

A **copula** is a multivariate distribution with all univariate margins being $U(0,1)$.

Multivariate Archimedean copula $C : [0,1]^d \to [0,1]$ defined as

$$C(u_1, \dots, u_d) = \phi\{\phi^{-1}(u_1) + \cdots + \phi^{-1}(u_d)\}, \qquad (1)$$

where $\phi : [0,\infty) \to [0,1]$ is continuous and strictly decreasing with $\phi(0) = 1$, $\phi(\infty) = 0$ and ϕ^{-1} its pseudo-inverse.

Example

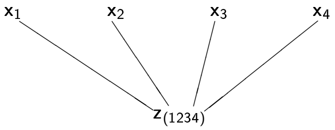
$$\begin{aligned} \phi_{Gumbel}(u,\theta) &= \exp\{-u^{1/\theta}\}, \text{ where } 1 \leq \theta < \infty \\ \phi_{Clayton}(u,\theta) &= (\theta u + 1)^{-1/\theta}, \text{ where } \theta \in [-1,\infty)\backslash\{0\} \end{aligned}$$

Disadvantages: too restrictive, single parameter, exchangeable

Hierarchical Archimedean Copulas

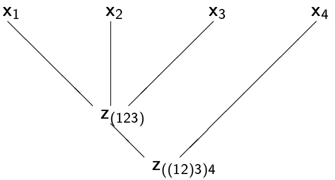
Simple AC with s=(1234)

$$C(u_1, u_2, u_3, u_4) = C_1(u_1, u_2, u_3, u_4)$$



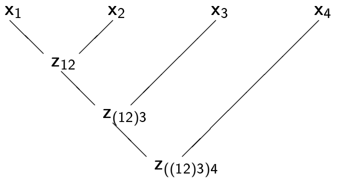
AC with s=((123)4)

$$C(u_1, u_2, u_3, u_4) = C_1\{C_2(u_1, u_2, u_3), u_4\}$$



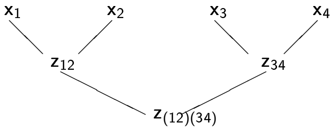
Fully nested AC with s=(((12)3)4)

$$C(u_1, u_2, u_3, u_4) = C_1[C_2\{C_3(u_1, u_2), u_3\}, u_4]$$



Partially Nested AC with s=((12)(34))

$$C(u_1, u_2, u_3, u_4) = C_1\{C_2(u_1, u_2), C_3(u_3, u_4)\}$$



Hierarchical Archimedean Copula

Advantages of HAC:

- flexibility and wide range of dependencies:
  for $d = 10$ more than $2.8 \cdot 10^8$ structures
- dimension reduction:
  $d - 1$ parameters to be estimated
- subcopulas are also HAC

Theoretical motivation

Let M be the cdf of a positive random variable and ϕ denotes its Laplace transform, i.e. $\phi(t) = \int_0^\infty e^{-tw} dM(w)$. For an arbitrary cdf F there exists a unique cdf G, such that

$$F(x) = \int_0^\infty G^\alpha(x) dM(\alpha) = \phi\{-\ln G(x)\}.$$

Now consider a d-variate cumulative distribution function F with margins $F_1, \ldots, F_d$. Then it holds for $G_j = \exp\{-\phi^{-1}(F_j)\}$ that

$$\int_0^\infty G_1^\alpha(x_1) \cdot \cdots \cdot G_d^\alpha(x_d) dM(\alpha) = \phi\left\{-\sum_{i=1}^d \ln G_i(x_i)\right\} = \phi\left[\sum_{i=1}^d \phi^{-1}\{F_i(x_i)\}\right].$$

$$C(u_1, \ldots, u_d) =$$

$$\int_0^\infty \ldots \int_0^\infty G_1^{\alpha_1}(u_1) G_2^{\alpha_1}(u_2) dM_1(\alpha_1, \alpha_2)\; G_3^{\alpha_2}(u_3) dM_2(\alpha_2, \alpha_3) \ldots G_d^{\alpha_{d-1}}(u_d) dM_{d-1}(\alpha_{d-1}).$$

Recovering the structure (theory)

To guarantee that C is a HAC we assume that $\phi_{d-i}^{-1} \circ \phi_{d-j} \in \mathcal{L}^*$, $i < j$ with

$$\mathcal{L}^* = \{\omega : [0,\infty) \to [0,\infty) \,|\, \omega(0) = 0,\ \omega(\infty) = \infty,\ (-1)^{j-1}\omega^{(j)} \geq 0,\ j \geq 1\}.$$

$\Rrightarrow$ for most of the generator functions the parameters should decrease from the lowest level to the highest

Theorem

Let F be an arbitrary multivariate distribution function based on HAC. Then F can be uniquely recovered from the marginal distribution functions and all bivariate copula functions.

$$C(u_1, \dots, u_6) = C_1[C_2(u_1, u_2), C_3\{u_3, C_4(u_4, u_5), u_6\}].$$

The bivariate marginal distributions are then given by

$$\begin{array}{lll}
(U_1, U_2) \sim C_2(\cdot,\cdot), & (U_2, U_3) \sim C_1(\cdot,\cdot), & (U_3, U_5) \sim C_3(\cdot,\cdot), \\
(U_1, U_3) \sim C_1(\cdot,\cdot), & (U_2, U_4) \sim C_1(\cdot,\cdot), & (U_3, U_6) \sim C_3(\cdot,\cdot), \\
(U_1, U_4) \sim C_1(\cdot,\cdot), & (U_2, U_5) \sim C_1(\cdot,\cdot), & (U_4, U_5) \sim C_4(\cdot,\cdot), \\
(U_1, U_5) \sim C_1(\cdot,\cdot), & (U_2, U_6) \sim C_1(\cdot,\cdot), & (U_4, U_6) \sim C_3(\cdot,\cdot), \\
(U_1, U_6) \sim C_1(\cdot,\cdot), & (U_3, U_4) \sim C_3(\cdot,\cdot), & (U_5, U_6) \sim C_3(\cdot,\cdot).
\end{array}$$

$$\mathcal{C}_2\{\mathcal{N}(C)\} = \{C_1(\cdot,\cdot), C_2(\cdot,\cdot), C_3(\cdot,\cdot), C_4(\cdot,\cdot)\}.$$

- each variable belongs to at least one bivariate margin C_1 $\rightsquigarrow$ the distribution of $u_1, \dots, u_6$ has C_1 at the top level.
- C_3 covers the largest set of variables u_3, u_4, u_5, u_6 $\rightsquigarrow$ C_3 is at the top level of the subcopula containing u_3, u_4, u_5, u_6.

$$U_1, \dots, U_6 \sim C_1\{u_1, u_2, C_3(u_3, u_4, u_5, u_6)\}.$$

- C_2 and C_4 and they join u_1, u_2 and u_4, u_5 respectively.

$$(U_1, \dots, U_6) \sim C_1[C_2(u_1, u_2), C_3\{u_3, C_4(u_4, u_5), u_6\}]$$

Recovering the structure (practice)

$(12) \rightsquigarrow \widehat{\theta}_{12}$ $\color{red}{(13)} \rightsquigarrow \widehat{\theta}_{13}$ $(14) \rightsquigarrow \widehat{\theta}_{14}$ $(23) \rightsquigarrow \widehat{\theta}_{23}$ $(24) \rightsquigarrow \widehat{\theta}_{24}$ $(34) \rightsquigarrow \widehat{\theta}_{34}$ ——— $(123) \rightsquigarrow \widehat{\theta}_{123}$ $(124) \rightsquigarrow \widehat{\theta}_{124}$ $(234) \rightsquigarrow \widehat{\theta}_{234}$ $(134) \rightsquigarrow \widehat{\theta}_{134}$ $(1234) \rightsquigarrow \widehat{\theta}_{1234}$	best fit (13)	$\rightsquigarrow$	$z_{(13),i} = \widehat{C}\{\widehat{F}_1(x_{1i}), \widehat{F}_3(x_{3i})\}$ $(13)2 \rightsquigarrow \widehat{\theta}_{(13)2}$ $\color{red}{(13)4} \rightsquigarrow \widehat{\theta}_{(13)4}$ $24 \rightsquigarrow \widehat{\theta}_{24}$ ——— $(13)24 \rightsquigarrow \widehat{\theta}_{(13)24}$	best fit ((13)4)	$\rightsquigarrow$	$z_{((13)4),i} = \widehat{C}\{z_{(13)i}, \widehat{F}_4(x_{4i})\}$ $\color{red}{((13)4)2} \rightsquigarrow \widehat{\theta}_{((13)4)2}$

Estimation: multistage MLE with nonparametric and parametric margins

Criteria for grouping: goodness-of-fit tests, parameter-based method, etc.

Estimation Issues - Multistage Estimation

$$\left(\frac{\partial \mathcal{L}_1}{\partial \boldsymbol{\theta}_1^\top}, \dots, \frac{\partial \mathcal{L}_p}{\partial \boldsymbol{\theta}_p^\top}\right)^\top = \mathbf{0},$$

$$\begin{aligned} \text{where} \quad \mathcal{L}_j &= \sum_{i=1}^{n} l_j(\mathbf{X}_i) \\ l_j(\mathbf{X}_i) &= \log\left(c(\{\phi_\ell, \boldsymbol{\theta}_\ell\}_{\ell=1,\dots,j};\, s_j)\left[\{\check{F}_m(x_{mi})\}_{m\in s_j}\right]\right) \\ &\text{for } j = 1, \dots, p. \end{aligned}$$

Theorem

Under regularity conditions, estimator $\widehat{\boldsymbol{\theta}}$ is consistent and

$$n^{\frac{1}{2}}(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}) \overset{a}{\sim} N(\mathbf{0}, \mathbf{B}^{-1}\Sigma\mathbf{B}^{-1})$$

Criteria for grouping

Alternatives:

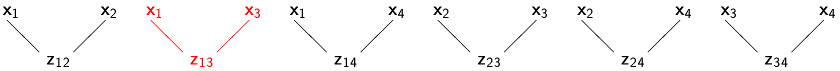
- goodness-of-fit tests $\rightsquigarrow$ to be discussed
    - dimension dependent
    - KS type tests are difficult to implement
    - possible choice $\rightsquigarrow$ Chen et al. (2004, WP of LSE), Fermanian (2005, JMA)
- distance measures
    - dimension dependent
- parameter-based methods

  Note that, if the true structure is (123) then

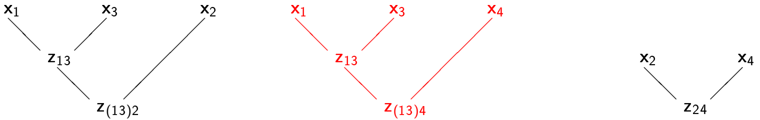
  $$\theta_{(12)} = \theta_{(13)} = \theta_{(23)} = \theta_{(123)}.$$

    - heurithtic methods
    - test-based methods
- tests on exchangeability

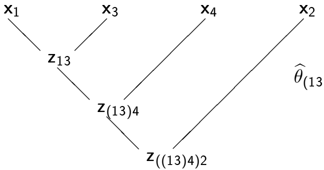
Recovering the structure (practice)



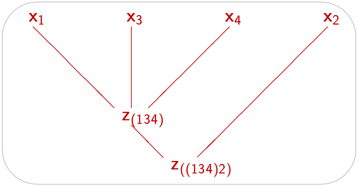
$$\max\{\widehat{\theta}_{12},\widehat{\theta}_{13},\widehat{\theta}_{14},\widehat{\theta}_{23},\widehat{\theta}_{24},\widehat{\theta}_{34}\}=\widehat{\theta}_{13}\quad\Rightarrow$$



$$\max\{\widehat{\theta}_{(13)2},\widehat{\theta}_{(13)4},\widehat{\theta}_{24}\}=\widehat{\theta}_{(13)4}\quad\Rightarrow$$



$$\widehat{\theta}_{(13)}\approx\widehat{\theta}_{((13)4)}\Rightarrow$$



Missspesification

Let $H(x_1, \dots, x_k)$ – true df with density h. Since H is unknown we specify $F(x_1, \dots, x_k, \boldsymbol{\eta})$ with density f.

- F is correctly specified:
  $\exists \boldsymbol{\eta}_0 : F(x_1, \dots, x_k, \boldsymbol{\eta}_0) = H(x_1, \dots, x_k),\ \forall (x_1, \dots, x_k)$ then $\widehat{\boldsymbol{\eta}}$ is consistent for $\boldsymbol{\eta}_0$.
- F is not correctly specified:
  $\nexists \boldsymbol{\eta}_0 : F(x_1, \dots, x_k, \boldsymbol{\eta}_0) = H(x_1, \dots, x_k),\ \forall (x_1, \dots, x_k)$, then $\widehat{\boldsymbol{\eta}}$ is an estimator for $\boldsymbol{\eta}_*$ which minimizes the Kullback–Leibler divergence between f and h as

$$\mathcal{K}(h, f, \boldsymbol{\eta}) = \boldsymbol{E}_h\{\log[h(x_1, \dots, x_k)/f(x_1, \dots, x_k, \boldsymbol{\eta})]\},$$

Distribution of HAC

Let $V = C\{F_1(X_1), \dots, F_d(X_d)\}$ and let $K(t)$ denote the distribution function (K-distribution) of the random variable V.

We consider a HAC of the form $C_1\{u_1, C_2(u_2, \dots, u_d)\}$.

Theorem

Let $U_1 \sim U[0,1]$, $V_2 \sim K_2$ and let $P(U_1 \leq x, V_2 \leq y) = C_1\{x, K_2(y)\}$ with $C_1(a,b) = \phi\left\{\phi^{-1}(a) + \phi^{-1}(b)\right\}$ for $a, b \in [0,1]$. Under certain regularity conditions the distribution function K_1 of the random variable $V_1 = C_1(U_1, V_2)$ is given by

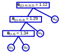
$$
\begin{aligned}
K_1(t) &= t - \int_0^{\phi^{-1}(t)} \phi'\left\{\phi^{-1}(t) + \phi^{-1} \circ K_2 \circ \phi(u) - u\right\} du \\
& \quad \text{for} \quad t \in [0,1].
\end{aligned}
$$

Let us consider 3dim fully nested HAC with Gumbel generator

$$\begin{aligned}
\phi_\theta(t) &= \exp(-t^{1/\theta}), \\
\phi_\theta^{-1}(t) &= \{-\log(t)\}^\theta, \\
\phi_\theta'(t) &= -\frac{1}{\theta}\exp(-t^{1/\theta})t^{-1+1/\theta}.
\end{aligned}$$

Following Genest and Rivest (1993), K for the simple 2-dim Archimedean copula with generator ϕ is given by $K(t) = t - \phi^{-1}(t)\phi'\{\phi^{-1}(t)\}$. Thus

$$K_2(t,\theta) = t - \frac{t}{\theta}\log(t)$$



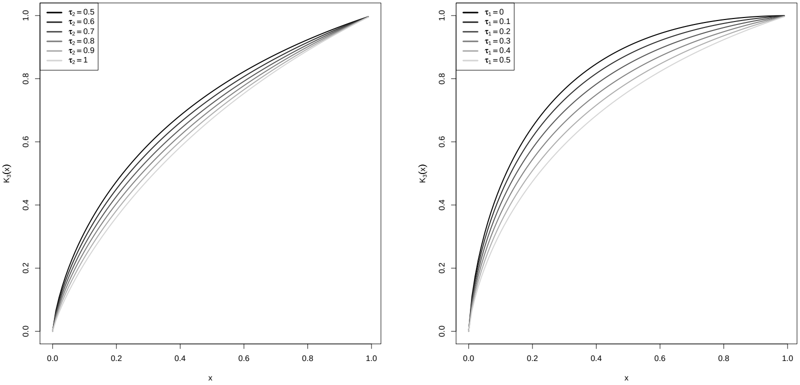


Figure 1: K distribution for three-dimensional HAC with Gumbel generators

Next consider $V_3 = C_3(V_4, V_5)$ with $V_4 = C_4(U_1, \dots, U_\ell)$ and $V_5 = C_5(U_{\ell+1}, \dots, U_d)$.

Theorem

Let $V_4 \sim K_4$ and $V_5 \sim K_5$ and $P(V_4 \leq x, V_5 \leq y) = C_3\{K_4(x), K_5(y)\}$ with $C_3(a, b) = \phi\left\{\phi^{-1}(a) + \phi^{-1}(b)\right\}$ for $a, b \in [0, 1]$. Under certain regularity conditions the distribution function K_3 of the random variable $V_3 = C_3(V_4, V_5)$ is given by

$$
\begin{aligned}
K_3(t) &= K_4(t) - \\
&- \int_0^{\phi^{-1}(t)} \phi'\left[\phi^{-1} \circ K_5 \circ \phi(u)\right. \\
&\quad \left. + \phi^{-1} \circ K_4 \circ \phi\{\phi^{-1}(t) - u\}\right] \, d\phi^{-1} \circ K_4 \circ \phi(u)
\end{aligned}
$$

for $t \in [0, 1]$.

Dependence orderings

C' is more **concordant** than C if

$$C \prec_c C' \Leftrightarrow C(\mathbf{x}) \leq C'(\mathbf{x}) \text{ and } \overline{C}(\mathbf{x}) \leq \overline{C'}(\mathbf{x}) \; \forall \mathbf{x} \in [0;1]^d.$$

where $\overline{C}(u,v) = u + v - 1 + C(1-u, 1-v)$

Theorem

If two hierarchical Archimedean copulas $C^1 = C^1_{\boldsymbol{\phi}_1}(u_1, \ldots, u_d)$ *and* $C^2 = C^2_{\boldsymbol{\phi}_2}(u_1, \ldots, u_d)$ *differ only by the generator functions on the level* r *as* $\boldsymbol{\phi}_1 = (\phi_1, \ldots, \phi_{r-1}, \phi, \phi_{r+1}, \ldots, \phi_p)$ *and* $\boldsymbol{\phi}_2 = (\phi_1, \ldots, \phi_{r-1}, \phi^*, \phi_{r+1}, \ldots, \phi_p)$ *with* $\phi^{-1} \circ \phi^* \in \mathcal{L}^*$, *then* $C^1 \prec_c C^2$.

Theorem

(Deheuvels (1978)) Let $\{X_{1i}, \ldots, X_{di}\}_{i=1,\ldots,n}$ *be a sequence of the random vectors with the distribution function* F*, marginal distributions* $F_1, \ldots, F_d$ *and copula* C*. Let also* $M_j^{(n)} = \max_{1 \le i \le n} X_{ji}$, $j = 1, \ldots, d$ *be the componentwise maxima. Then*

$$\lim_{n \to \infty} P\left\{ \frac{M_1^{(n)} - a_{1n}}{b_{1n}} \le x_1, \ldots, \frac{M_d^{(n)} - a_{dn}}{b_{dn}} \le x_d \right\} = F^*(x_1, \ldots, x_d),$$

$$\forall (x_1, \ldots, x_d) \in \mathbb{R}^d$$

with $b_{jn} > 0,\ j = 1, \ldots, d,\ n \ge 1$ *if and only if*

1. *for all* $j = 1, \ldots, d$ *there exist some constants* a_{jn} *and* b_{jn} *and a non-degenerating limit distribution* F_j^* *such that*

$$\lim_{n \to \infty} P\left\{ \frac{M_j^{(n)} - a_{jn}}{b_{jn}} \le x_j \right\} = F_j^*(x_j), \quad \forall x_j \in \mathbb{R};$$

2. *there exists a copula* C^* *such that*

$$C^*(u_1, \ldots, u_d) = \lim_{n \to \infty} C^n(u_1^{1/n}, \ldots, u_d^{1/n}).$$

Let F_{ds} be the class of d dimensional HAC with structure s.

Theorem

If $C \in F_{ds_1},\ C^* \in F_{ds_2}, \forall \varphi_\theta \in \mathcal{N}(C), \partial[\varphi_\ell^{-1}(t)/(\varphi_\ell^{-1})'(t)]/\partial t\big|_{t=1}$ *exists and is equal to* $1/\theta$ *and* $C \in MDA(C^*)$ *and* $C \in MDA(C^*)$ *then* $s_1 = s_2, \forall \phi_\theta \in \boldsymbol{N}(C^*),\ \phi_\theta(x) = \exp\{-x^{1/\theta}\}$.

If the multivariate HAC C (under some minor condition) belongs to the domain of attraction of the HAC C^*. The extreme value HAC C^* has the same structure as the given copula C, with generators on all levels of the hierarchy being Gumbel generators, but with probably other parameters.

Tail dependency

The upper and lower tail indices of two random variables $X_1 \sim F_1$ and $X_2 \sim F_2$ are given by

$$\lambda_U = \lim_{u \to 1^-} P\{X_2 > F_2^{-1}(u) \mid X_1 > F_1^{-1}(u)\} = \lim_{u \to 1^-} \frac{\overline{C}(u,u)}{1-u}$$

$$\lambda_L = \lim_{u \to 0^+} P\{X_2 \leq F_2^{-1}(u) \mid X_1 \leq F_1^{-1}(u)\} = \lim_{u \to 0^+} \frac{C(u,u)}{u}.$$

Theorem (Nelsen (1997))

For a bivariate Archimedean copula with the generator ϕ it holds

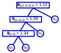
$$\lambda_U = 2 - \lim_{u \to 1^-} \frac{1-\phi\{2\phi^{-1}(u)\}}{1-u} = 2 - \lim_{w \to 0^+} \frac{1-\phi(2w)}{1-\phi(w)},$$

$$\lambda_L = \lim_{u \to 0^+} \frac{\phi\{2\phi^{-1}(u)\}}{u} = \lim_{w \to \infty} \frac{\phi(2w)}{\phi(w)}.$$

A function $\phi : (0,\infty) \to (0,\infty)$ is **regularly varying at infinity with tail index** $\lambda \in \boldsymbol{R}$ (written $RV_\lambda(\infty)$) if $\lim_{w\to\infty} \frac{\phi(tw)}{\phi(w)} = t^\lambda$ for all $t > 0$. $\phi \in RV_{-\infty}(\infty)$ if

$$\lim_{w\to\infty} \frac{\phi(tw)}{\phi(w)} = \begin{cases} \infty & \text{if} \quad t < 1 \\ 1 & \text{if} \quad t = 1 \\ 0 & \text{if} \quad t > 1 \end{cases} .$$

It holds for $\lambda \geq 0$ that if $\phi \in RV_{-\lambda}(\infty)$ then $\phi^{-1} \in RV_{-1/\lambda}(0)$. The function ϕ^{-1} is **regularly varying at zero with the tail index** γ, if $\lim_{w\to 0^+} \frac{\phi^{-1}(1-tw)}{\phi^{-1}(1-w)} = t^\gamma$. For the direct function $\lim_{w\to 0^+} \frac{1-\phi(tw)}{1-\phi(w)} = t^{1/\gamma}$.



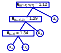
$$\lim_{u\to 0^+} P\{X_i \le F_i^{-1}(u_i u) \text{ for } i \notin \mathcal{S} \subset \mathcal{K} = \{1,\dots,k\} \mid X_j \le F_j^{-1}(u_j u) \text{ for } j \in \mathcal{S}\}$$

$$\lim_{u\to 0^+} P\{X_i > F_i^{-1}(1-u_i u) \text{ for } i \notin \mathcal{S} \subset \mathcal{K} = \{1,\dots,k\} \mid X_j > F_j^{-1}(1-u_j u) \text{ for } j \in \mathcal{S}\}.$$

The above limits can be established via the limits

$$\lambda_L(u_1,\dots,u_k) = \lim_{u\to 0^+} \frac{1}{u} C(u_1 u,\dots,u_k u) \qquad \text{and}$$

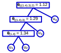
$$\begin{aligned}\lambda_U(u_1,\dots,u_k) &= \lim_{u\to 0^+} \frac{1}{u} \overline{C}(1-u_1 u,\dots,1-u_k u) \\ &= \lim_{u\to 0^+} \sum_{s_1\in\mathcal{K}} (-1)^{|s_1|+1}\{1 - C_{s_1}(1-u_j u, j\in s_1)\}.\end{aligned}$$



Theorem (Lower Tail Dependency)

Assume that the limits $\lim_{u\to 0^+} u^{-1} C_i(uu_{k_{i-1}+1},\dots,uu_{k_i}) = \lambda_{L,i}(u_{k_{i-1}+1},..,u_{k_i})$ *exist for all* $0 < u_{k_{i-1}+1},..,u_{k_i} < 1,\ i = 1,\dots,m$. *Suppose that* $m + k - k_m \geq 2$. *If* ϕ_0^{-1} *is regularly varying at infinity with index* $-\lambda_0 \in [-\infty, 0]$, *then it holds for all* $0 < u_i < 1,\ i = 1,..,m$ *that*

$$\lim_{u\to 0^+} \frac{C(uu_1,\dots,uu_k)}{u} = \begin{cases} \min\{\lambda_{L,1}(u_1,..,u_{k_1}),\dots,\lambda_{L,m}(u_{k_{m-1}+1},..,u_{k_m}), u_{k_m+1},\dots,u_k\} \\ \quad \textit{if} \quad \lambda_0 = \infty, \\ \left(\sum_{i=1}^m \lambda_{L,i}(u_{k_{i-1}+1},..,u_{k_i})^{-\lambda_0} + \sum_{j=k_m+1}^k u_j^{-\lambda_0}\right)^{-1/\lambda_0} \\ \quad \textit{if} \quad 0 < \lambda_0 < \infty, \\ 0 \ \textit{if} \quad \lambda_0 = 0. \end{cases}$$



In the following let

$$
\begin{aligned}
C_j^*(u) &= C_j(u_{k_j-1+1}u, .., u_{k_j}u) \mid_{u_{k_{j-1}+1}=..=u_{k_j}=1}, \\
C^*(u) &= C(u_1 u, .., u_k u) \mid_{u_1=..=u_k=1}, \\
\lambda^*_{L,j}(u, u_{k_{j-1}+1}, .., u_{k_j}) &= C_j(u_{k_{j-1}+1}u, \ldots, u_{k_j}u)/C_j^*(u).
\end{aligned}
$$

Note that $0 \leq \lambda^*_{L,j}(u, u_{k_{j-1}+1}, .., u_{k_j}) \leq 1$. Moreover, if $\lim_{u\to 0^+} u^{-1} C_j(uu_{k_{j-1}+1}, \ldots, uu_{k_j}) = \lambda_{L,j}(u_{k_{j-1}+1}, .., u_{k_j}) > 0$ for all $0 < u_{k_{j-1}+1}, .., u_{k_j} \leq 1$ then

$$
\begin{aligned}
\lambda^*_{L,j}(u_{k_{j-1}+1}, .., u_{k_j}) &= \lim_{u\to 0+} \frac{C_j(u_{k_j-1+1}u, .., u_{k_j}u)/u}{C_j^*(u)/u} \\
&= \frac{\lambda_{L,j}(u_{k_{j-1}+1}, .., u_{k_j})}{\lambda_{L,j}(1, .., 1)}
\end{aligned}
$$

Theorem (Lower Tail Dependency 2)

Assume that the limits

$$\lim_{u\to 0^+} \frac{C_i(uu_{k_{i-1}+1},\dots,uu_{k_i})}{C_i^*(u)} = \lambda^*_{L,i}(u_{k_{i-1}+1},..,u_{k_i})$$

exist for all $0 < u_{k_{i-1}+1},..,u_{k_i} \leq 1$, $i = 1,\dots,m$. *Let* $\phi_0^{-1} \in RV_0(0)$ *and let* $\psi(v) = -\phi_0(v)/\phi_0'(v)$ *be regularly varying at infinity with finite tail index* $\varkappa$ *then* $\varkappa \leq 1$ *and it holds for all* $0 < u_i < 1$, $i = 1,..,m$ *that*

$$\lim_{u\to 0^+} \frac{C(uu_1,\dots,uu_k)}{C^*(u)} =$$

$$\prod_{j=1}^{m}[\lambda^*_{L,j}(u_{k_{j-1}+1},\dots,u_{k_j})]^{(m+k-k_m)^{-\varkappa}} \cdot \prod_{j=k_m+1}^{k} u_j^{(m+k-k_m)^{-\varkappa}}.$$

Theorem (Upper Tail Dependency)

Assume that the limits $\lim_{u \to 0^+} u^{-1}[1 - C_i(1 - uu_{k_{i-1}+1}, \ldots, 1 - uu_{k_i})] = \lambda_{U,i}(u_{k_{i-1}+1}, .., u_{k_i})$ *exist for all* $0 < u_{k_{i-1}+1}, .., u_{k_i} < 1$, $i = 1, \ldots, m$. *Suppose that* $m + k - k_m \geq 2$. *If* $\phi_0^{-1}(1 - w)$ *is regularly varying at zero with index* $-\gamma_0 \in [-\infty, -1]$, *then it holds for all* $0 < u_i < 1$, $i = 1, .., m$ *that*

$$\lim_{u \to 0^+} \frac{1 - C(1 - uu_1, \ldots, 1 - uu_k)}{u} = \begin{cases} \min\{\lambda_{U,1}(u_1, .., u_{k_1}), \ldots, \lambda_{U,m}(u_{k_{m-1}+1}, .., u_{k_m}), u_{k_m+1}, \ldots, u_k\} \\ \qquad \text{if} \quad \gamma_0 = \infty, \\ \left(\sum_{i=1}^m [\lambda_{U,i}(u_{k_{i-1}+1}, .., u_{k_i})]^{\gamma_0} + \sum_{j=k_m+1}^k u_j^{\gamma_0} \right)^{1/\gamma_0} \\ \qquad \text{if} \quad 1 \leq \gamma_0 < \infty, \end{cases}$$

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