Realized Copula Models

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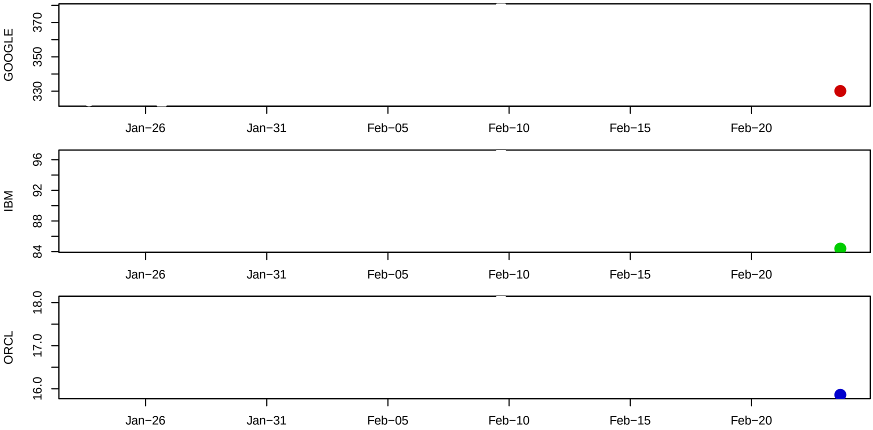
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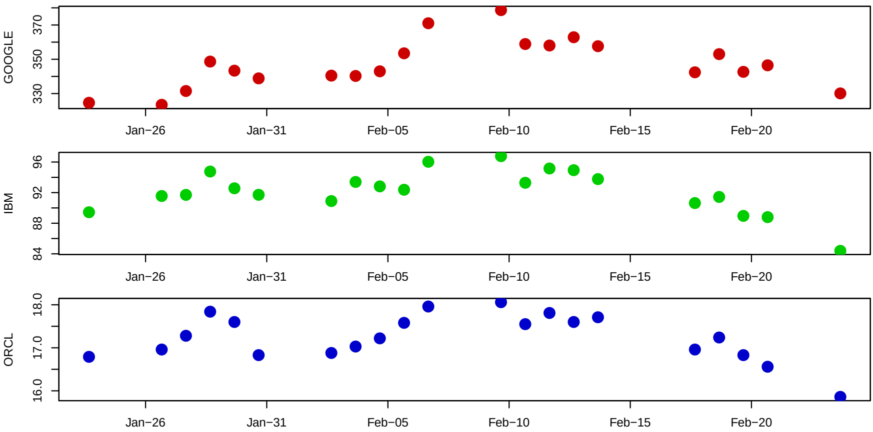
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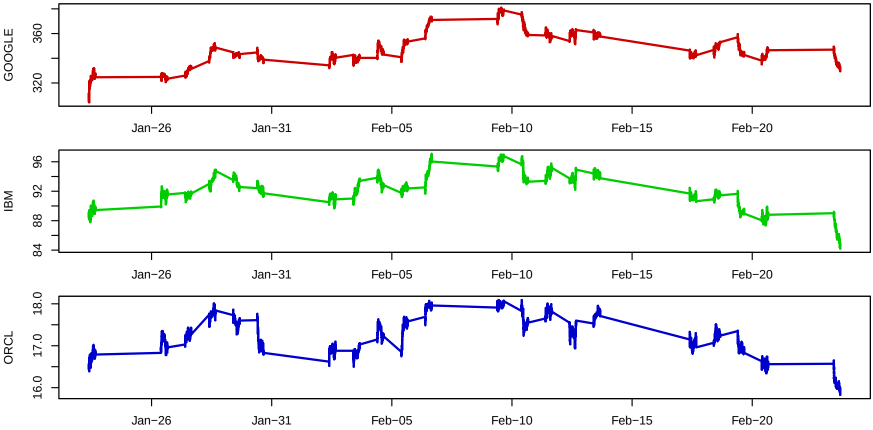
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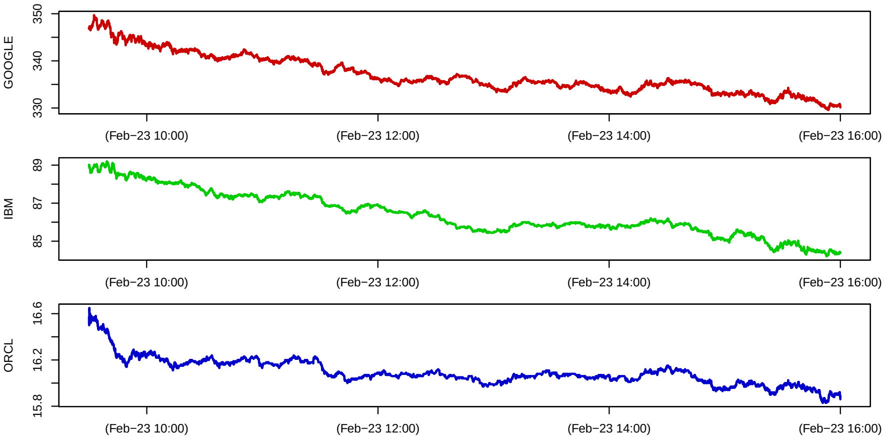

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Realized Variance of Google-IBM-Oracle

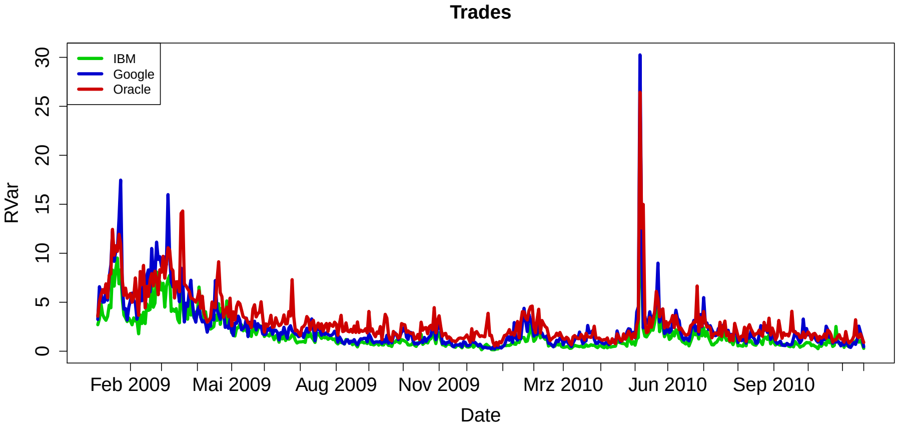


Figure 1: Realized kernel (variance) of Google-IBM-Oracle.

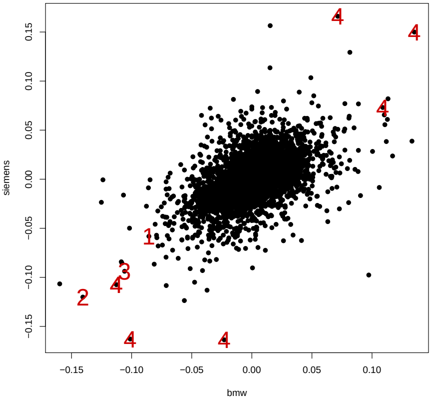
RV: Exploiting intra-day information

Literature of the past 10yrs on high-frequency data shows:

- daily realized (co)variance (RV, RCov) computed from intra-day data serves as an accurate measures of conditional (co)variance of daily returns;
- no specific model is needed (like GARCH);
- can treat an inherently latent variable like an observed one;
- shows excellent forecasting performance.

Heavily discussed in *derivatives pricing, portfolio optimization, risk-management, and volatility forecasting.*

Dependency



1. 19.10.1987
   Black Monday
2. 16.10.1989
   Berlin Wall
3. 19.08.1991
   Kremlin
4. 17.03.2008, 19.09.2008, 10.10.2008, 13.10.2008, 15.10.2008, 29.10.2008
   Crisis

Copulae

Copulae is a convenient tool to capture nonlinear dependence.

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Multivariate RCov models have an underlying *Gaussian* structure.

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How can we suitably combine intra-day RCov information into a *non-Gaussian* model framework?

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How can we suitably combine intra-day RCov information into a *non-Gaussian* model framework?

realized copula (RCop)

Outline

1. Motivation ✓
2. Copula and realized copula
3. Benchmark models
4. Empirical Part
5. References

Copulae

A **copula** is a multivariate distribution with all univariate margins being $U(0,1)$.

Theorem (Sklar, 1959)

Let $X_1, \ldots, X_d$ be random variables with marginal distribution functions $F_1, \ldots, F_d$ and joint distribution function F. Then there exists a d-dimensional copula $C : [0,1]^d \to [0,1]$ such that $\forall x_1, \ldots, x_d \in \overline{\mathbb{R}} = [-\infty, \infty]$

$$F(x_1, \ldots, x_d) = C\{F_1(x_1), \ldots, F_d(x_d)\}$$

Archimedean copula $C : [0,1]^d \rightarrow [0,1]$ defined as

$$C(u_1, \ldots, u_d) = \phi\{\phi^{-1}(u_1) + \cdots + \phi^{-1}(u_d)\}, \qquad (1)$$

where $\phi : [0,\infty) \rightarrow [0,1]$ is strictly decreasing with $\phi(0) = 1$, $\phi(\infty) = 0$ and ϕ^{-1} its (pseudo)inverse.

Example

$$\begin{aligned} \phi_{Gumbel}(u,\theta) &= \exp\{-u^{1/\theta}\}, \text{ where } 1 \leq \theta < \infty \\ \phi_{Clayton}(u,\theta) &= (\theta u + 1)^{-1/\theta}, \text{ where } \theta \in [-1,\infty)\backslash\{0\} \end{aligned}$$

Rotated copula as an example of a non-Archimedean copula:

$$C_{rot}(u_1, u_2) = C(1 - u_1, 1 - u_2) + u_1 + u_2 - 1,$$

which in term of copula density is given through $c_{rot}(u_1, \ldots, u_d) = c(1 - u_1, \ldots, 1 - u_d)$

Realized Copula, I

Lemma (Hoeffding)

Suppose there are two random variables X_i and X_j with marginal distributions F_i and F_j and joint distribution F_{ij} and finite second moments

$$\begin{aligned} \sigma_{ij}(\theta) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left\{F_{i,j}(x_i, x_j, \theta) - F_i(x_i)F_j(x_j)\right\} dx_i dx_j \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[C_\theta\{F_i(x_i), F_j(x_j)\} - F_i(x_i)F_j(x_j)\right] dx_i dx_j \ . \end{aligned}$$

Realized Copula, II

For the notion of *realized copula*, we define θ implicitly through

$$\begin{aligned} h_{ij,t} &= \mathrm{f}_{ij}(\theta_t) \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[C_{\theta_t}\{F_{i,t}(x_i), F_{j,t}(x_j)\} - F_{i,t}(x_i)F_{j,t}(x_j)\right] dx_i dx_j \end{aligned}$$

where $h_{ij,t}$ denotes an element of the RCov matrix measured at day t.

This moment condition, together with the assumptions on the copula and the marginal distributions, identifies the ex-post daily distribution as materialized in RCov.

Method-of-moments estimator, I

Let $d = 2$, with one off-diagonal element $h_{12,t}$ in the RCov. An estimate of θ_t is given by

$$\widehat{\theta}_t^{\mathsf{MM}} = \mathrm{f}_{12}^{-1}(h_{12,t}).$$

Similar to method-of-moments approaches where the copula parameter of an Archimedean copula is estimated from Kendall's tau (Genest and Rivest, 1993).

Method-of-moments estimator, II

For $d > 2$, define

$$\mathrm{g}_{ij}(\theta) = h_{ij,t} - \mathrm{f}_{ij}(\theta),$$

where $i < j$ and $i, j = 1, \ldots, d$.

Stacking all g_{ij} into a vector $\mathbf{g}$ of size $d(d-1)/2$, we define the estimator as

$$\widehat{\theta}_t^{\mathsf{MM}} = \arg\min_{\theta}\; \mathbf{g}^\top(\theta)\boldsymbol{\Omega}\mathbf{g}(\theta)\;,$$

with $\boldsymbol{\Omega}$ denoting a $d(d-1)/2$-dimensional pd weight matrix. A conventional choice would be the unit matrix $\mathbf{I}_{d(d-1)/2}$.

Ad hoc estimator

Under Gaussianity, Kendall's τ is $\tau^{G}_{ij,t} = \frac{2}{\pi} \arcsin \rho_{ij,t}$, and generally, for general Archimedean copulae (Genest and Rivest, 1993):

$$\tau \equiv \mathrm{f}_\tau(\theta) = 4 \int_0^1 {\phi_\theta}^{-1}(v)/({\phi_\theta}^{-1})'(v)\, dv + 1 \, .$$

family	ϕ_θ	f_τ
Gumbel	$\exp\{-x^{1/\theta}\}$	$1 - 1/\theta$
Clayton	$(\theta x + 1)^{-1/\theta}$	$\theta/(2+\theta)$

We define an ad-hoc estimator by

$$\widehat{\theta}_t^{\text{ad hoc}} = \frac{2}{d(d-1)} \sum_{i<j} \mathrm{f}_\tau^{-1}(\widehat{\tau}^{G}_{ij,t}) \, .$$

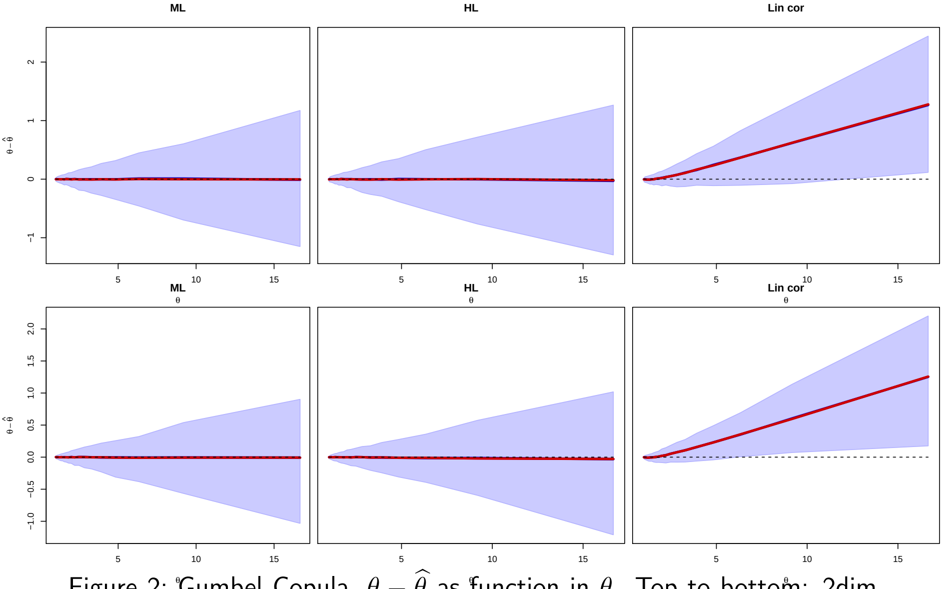


Figure 2: Gumbel Copula, $\theta - \widehat{\theta}$ as function in θ. Top to bottom: 2dim, 3dim. $n = 1000$, $N = 1000$. Shaded area is the simulation based 95% interval.

Forecasting framework for RCop

Let $P_t = (P_{1t}, \dots, P_{dt})^\top$ and $r_t = P_t - P_{t-1},\ t = 1, \dots, T$ be daily log-prices and their log-returns with

$$r_{t+1} \sim F_{r_{t+1}|\mathcal{F}_t}(\widehat{H}_{t+1|t})$$

where $\widehat{H}_{t+1|t}$ is an $\mathcal{F}_t$-measurable forecast of the RC matrix of r_t and

$$F_{r_{t+1}|\mathcal{F}_t}(\widehat{H}_{t+1|t}) = C_{\widehat{\theta}_{t+1|t}}\{F_{1,t}(\widehat{h}_{1,t+1|t}), \dots, F_{d,t}(\widehat{h}_{d,t+1|t})\}$$

As reported in Andersen et al. (2001) returns standardized by ex post RV are close to standard normal, we thus assume that

$$F_{j,t}(\widehat{h}_{j,t+1|t}) = \boldsymbol{N}(0, \widehat{h}_{j,t+1|t})$$

Forecasting framework

Consider the following multivariate forecasting rule:

$$\begin{pmatrix} \log \widehat{h}_{1,t+1|t} \\ \vdots \\ \log \widehat{h}_{d,t+1|t} \\ \widehat{\theta}_{t+1|t} \end{pmatrix} = \boldsymbol{E}_t \begin{pmatrix} \log h_{1,t+1} \\ \vdots \\ \log h_{d,t+1} \\ \theta_{t+1} \end{pmatrix} = \begin{pmatrix} \beta_0^1 + \beta_{\mathrm{D}}^1 \log h_t^{\mathrm{D}} + \beta_{\mathrm{W}}^1 \log h_t^{\mathrm{W}} + \beta_{\mathrm{M}}^1 \log h_t^{\mathrm{M}} \\ \vdots \\ \beta_0^d + \beta_{\mathrm{D}}^d \log h_t^{\mathrm{D}} + \beta_{\mathrm{W}}^d \log h_t^{\mathrm{W}} + \beta_{\mathrm{M}}^d \log h_t^{\mathrm{M}} \\ \alpha_0 + \alpha_{\mathrm{D}} \theta_t^{\mathrm{D}} + \alpha_{\mathrm{W}} \theta_t^{\mathrm{W}} + \alpha_{\mathrm{M}} \theta_t^{\mathrm{M}} \end{pmatrix},$$

where $x_t^{\mathrm{D}} = x_t$ are daily, $x_t^{\mathrm{W}} = \frac{1}{5}\sum_{i=0}^{4} x_{t-i}$ weekly, and $x_t^{\mathrm{M}} = \frac{1}{21}\sum_{i=0}^{20} x_{t-i}$ monthly averages of past realizations of x_t.

Borrowed from the heterogenous autoregressive model (HAR) of Corsi (2009); extended here to the copula parameter.

Empirical application

Compare one day ahead VaR forecasting performance of RCop against a number of standard benchmark models:

- models based on daily data
  - naive rolling window
  - local adaptive estimation
  - dynamic copula model
- models based on intra-day data (RV models)
  - Logm-model
  - Cholesky factorization

Rolling window and adaptive estimation

Naive approach:

- estimate copula parameter on a fixed rolling window

LCP:

- adaptively estimate largest interval where homogeneity hypothesis is accepted
- *Local Change Point* detection (LCP): sequentially test whether θ_t is constant (i.e. $\theta_t = \theta$) within some interval I (local parametric assumption).

Dynamic Copula Model (Patton, 2004):

- $(\varepsilon_{1,t}, \ldots, \varepsilon_{d,t}) \sim C_{\theta_t}\{F_{1,t}(\varepsilon_{1,t}), \ldots, F_{d,t}(\varepsilon_{d,t})\}$, and

$$\theta_t = \Lambda\left(\sum_{i=0}^{d} \gamma_i \mu_{i,t}\right)$$

DCC Model (Engle, 2002):

- $(\varepsilon_{1,t}, \ldots, \varepsilon_{d,t})^\top \sim C_{\nu,R_t}\{F_{1,t}(\varepsilon_{1,t}), \ldots, F_{d,t}(\varepsilon_{d,t})\}$,
  where $\varepsilon_{i,t} = r_{i,t}/\sqrt{h_{i,t}}$, $i = 1, \ldots, d$.

GAS Model (Creal, Koopman, Lucas, 2013):

- $\varepsilon_{i,t} = r_{i,t}/\sqrt{h_{i,t}}$ and $u_{i,t} = F_{i,t}(\varepsilon_{i,t})$, $i, \ldots, d$. Set $u_t = (u_{i,t}, \ldots, u_{d,t})^\top$.

$$\Lambda(\theta_t) = \omega + \beta\Lambda(\theta_{t-1}) + \alpha s_{t-1} \ ,$$

where $$s_{t-1} = S_{t-1}\nabla_{t-1} \text{ and } \nabla_{t-1} = \frac{\partial \log c(u_{t-1};\theta_{t-1})}{\partial \theta_{t-1}} \ ,$$

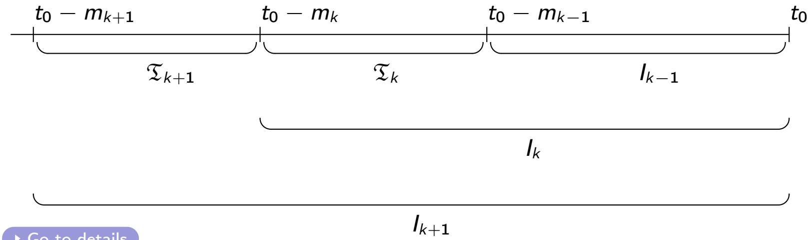
and $\Lambda(\cdot)$ is a monotonic mapping.

GRAS Model (De Lira Salvatierra and Patton, 2013):

- $\Lambda(\theta_t) = \omega + \beta\Lambda(\theta_{t-1}) + \alpha s_{t-1} + \gamma RM_{t-1}$,
  where $RM_t = \frac{2}{d(d-1)} \sum_{i=1}^{d} \sum_{j>i}^{d} \widehat{\rho}_{t,ij}$.

Local Change Point Detection

1. define the family of nested intervals
   $I_0 \subset I_1 \subset I_2 \subset \ldots \subset I_K = I_{K+1}$ with length m_k as
   $I_k = [t_0 - m_k, t_0]$
2. define $\mathfrak{T}_k = [t_0 - m_k, t_0 - m_{k-1}]$



Go to details

Data used in this study

- $d = 3$
- daily (Yahoo Finance) and tick trades (LOBSTER) prices for the two portfolios
  - IBM, Google, Oracle;
  - IBM, Pfizer, Exxon
- timespan = [01.03.2006 – 31.05.2013] for tick data and [12.03.2002 – 31.05.2013] days for daily data
- cleaning high-frequency data as in BNHLS (2008): *9:45-16:00, one stock exchange, multiple quotes or trades with same time stamp, negative spread, etc.*
- Rotated Gumbel and Clayton copulae.

Realized Variance of Google-IBM-Oracle

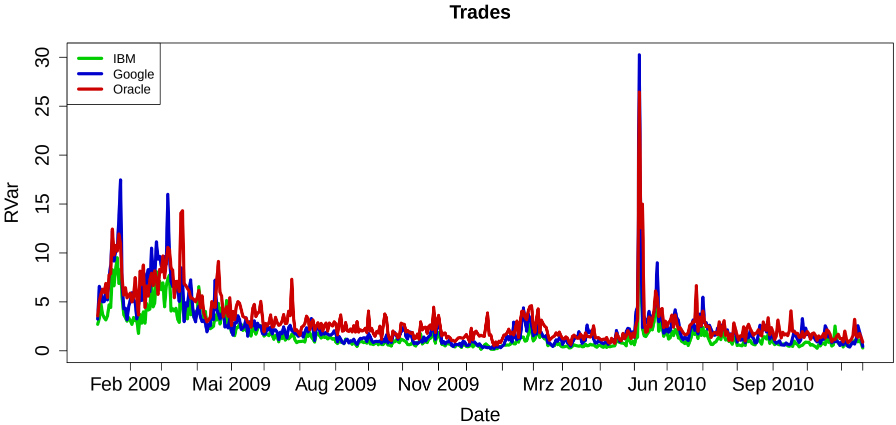


Figure 3: Realized kernel (variance) of Google-IBM-Oracle.

Realized Covariance of Google-IBM-Oracle

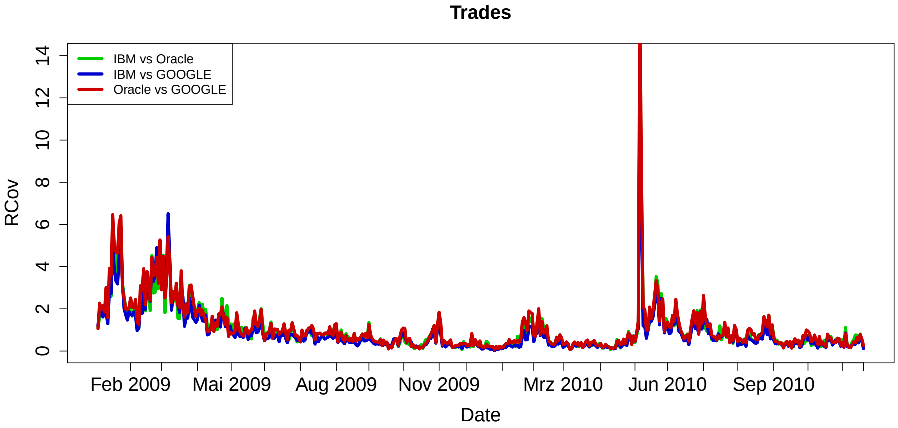


Figure 4: Realized Kernel (covariance) of Google-IBM-Oracle.

Realized Correlation of Google-IBM-Oracle

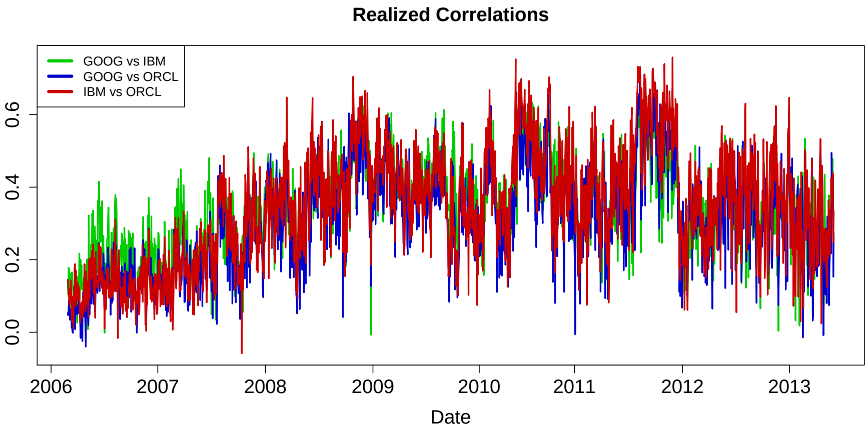


Figure 5: Realized Kernel (correlation) of Google-IBM-Oracle.

Realized Correlation of IBM-Pfizer-Exxon

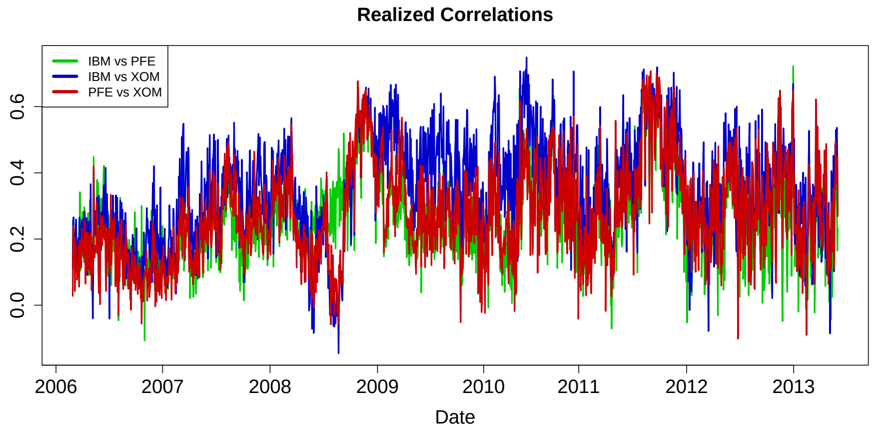


Figure 6: Realized Kernel (correlation) of IBM-Pfizer-Exxon.

Descriptive Statistics

	Min.	Median	Mean	Max.	Std.
RV(Google) $\times 10^5$	2.81	21.17	38.35	929.70	59.52
RV(IBM) $\times 10^5$	1.48	10.07	21.37	1144.00	47.77
RV(Oracle) $\times 10^5$	3.96	23.19	35.39	996.60	49.82
RCov(Google,IBM) $\times 10^5$	-0.13	4.61	11.13	490.40	25.31
RCov(Google,Oracle) $\times 10^5$	-1.97	6.09	13.25	488.20	27.18
RCov(IBM,Oracle) $\times 10^5$	-1.18	4.66	11.21	511.70	25.86
RV(IBM) $\times 10^5$	1.40	9.94	21.40	1141.00	48.70
RV(Pfizer) $\times 10^5$	1.91	13.99	23.65	894.90	37.42
RV(Exxon) $\times 10^5$	1.61	14.19	27.86	2233.00	72.62
RCov(IBM,Pfizer) $\times 10^5$	-2.29	2.82	7.67	449.10	20.61
RCov(IBM,Exxon) $\times 10^5$	-3.54	3.76	10.21	648.40	27.77
RCov(Pfizer,Exxon) $\times 10^5$	-1.96	3.18	9.03	552.60	24.74

Table 1: Descriptive statistics of the realized kernels (Var and Cov).

LCP for Google-IBM-Oracle

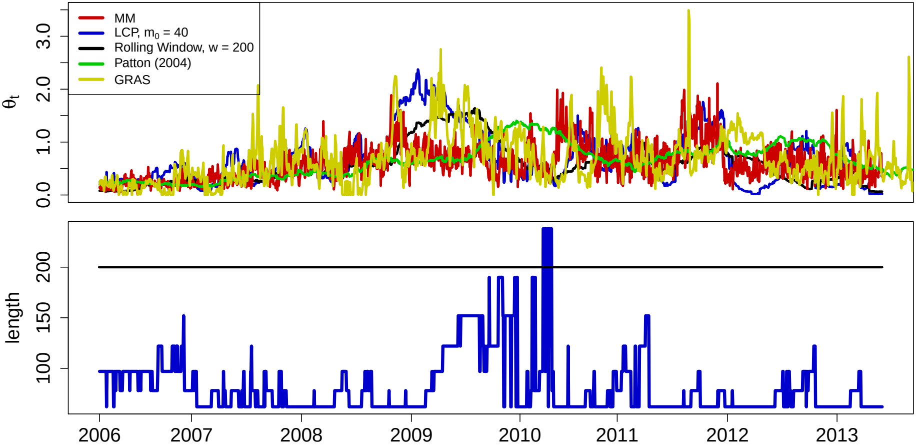


Figure 7: Clayton copulae for Google-IBM-Oracle portfolio.

LCP for IBM-Pfizer-Exxon

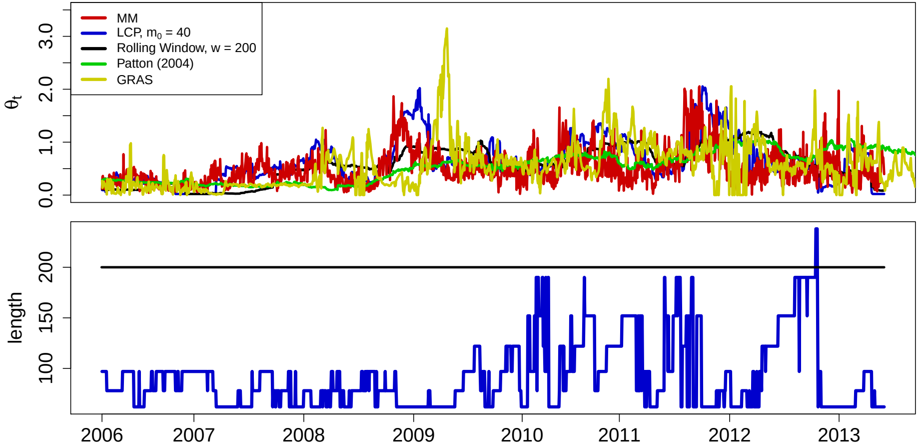


Figure 8: Clayton copula for IBM-Pfizer-Exxon portfolio.

Gaussian models

Recent suggestions in the multivariate RV literature: the matrix-log model (Bauer and Vorkink, 2010) and the Cholesky factorization (Chiriac and Voev, 2011).

For the logm-model, apply the logm to the RV matrix

$$A_t = \mathrm{logm}(H_t)$$

and apply the vech-operator

$$a_t = \mathrm{vech}(A_t)$$

which yields a $d(d+1)/2$ vector a_t.

To this vector the same HAR-forecasting rule is applied. Predictions $\widehat{a}_{t+1|t}$ are converted to positive-definite predicted covariance matrices by applying the reverse vech-operator and the matrix exponential:

$$\widehat{H}_{t+1|t} = \mathrm{expm}(\widehat{A}_{t+1|t})\,.$$

Likewise, for the Cholesky decomposition, find a matrix A such that

$$H = AA^\top\,.$$

For predicitons, use a HAR model on the vector obtained from the vech-operation, and convert predicted Cholesky factors back:

$$\widehat{H}_{t+1|t} = \widehat{A}_{t+1|t}\widehat{A}_{t+1|t}^\top\,.$$

Overview on models

- 2 DCC models (Normals and GED margins)
- 2 Gaussian models (Bauer and Vorkink (2011), and Chiriac and Voev (2010))
- 2 Copulae (Clayton and Rotated Gumbel)
    - 3 Daily Models (Rolling Window, LCP, Patton (2004))
    - 2 GAS models (GAS and GRAS)
    - 2 Margins (Normal and GED) for
        - HAR
        - Bauer and Vorkink (2011)
        - Chiriac and Voev (2010)

In total: $2 + 2 + 2 \times (3 + 2 + 2 \times 3) = 26$ models

Value at Risk (VaR), I

Let $a = \{a_1, \ldots, a_d\}$, $a_i \in \mathbb{Z}$ be the portfolio. The value V_t of a is given by

$$V_t = \sum_{j=1}^{d} a_j S_{j,t}$$

and the *profit and loss (P&L) function* of the portfolio

$$L_{t+1} = (V_{t+1} - V_t) = \sum_{j=1}^{d} a_j S_{j,t} \left\{ \exp(X_{j,t+1}) - 1 \right\},$$

where $w_j = a_{j,t} S_{j,t} / \sum_{i=1}^{d} (a_{i,t} S_{i,t})$ and $w_i = 1/d,\ 1, \ldots, d$.

VaR, II

The distribution function of L is given by

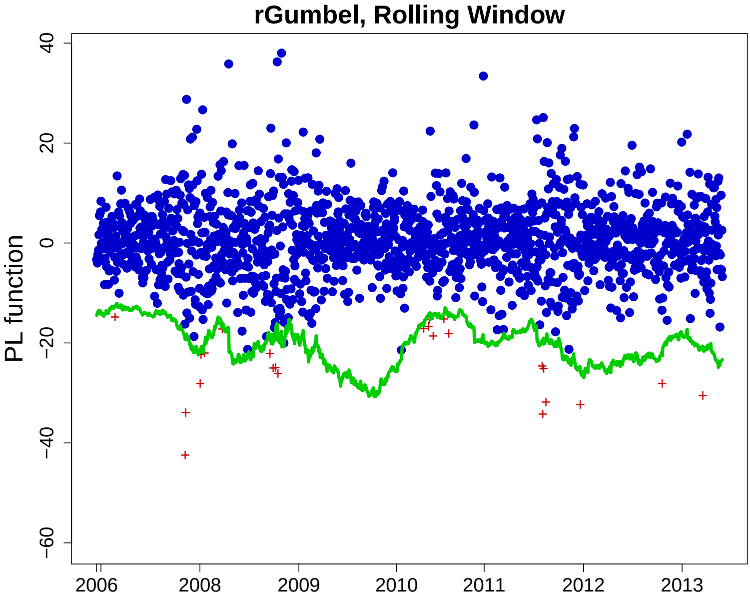
$$F_L(x) = P(L \leq x).$$

The *Value-at-Risk* at level α from w is defined as the α-quantile from F_L:

$$\mathrm{VaR}(\alpha) = F_L^{-1}(\alpha).$$

Backtesting: estimated values of the VaR are compared with the true $\{l_t\}$ of the function L_t, an *exceedance* occurring for each l_t smaller than $\widehat{VaR}_t(\alpha)$. The *exceedances ratio* $\widehat{\alpha}$ is given by:

$$\widehat{\alpha} = \frac{1}{T} \sum_{t=r}^{T} \mathbf{I}\{l_t < \widehat{VaR}_t(\alpha)\}.$$


rGumbel, Rolling Window
PL function
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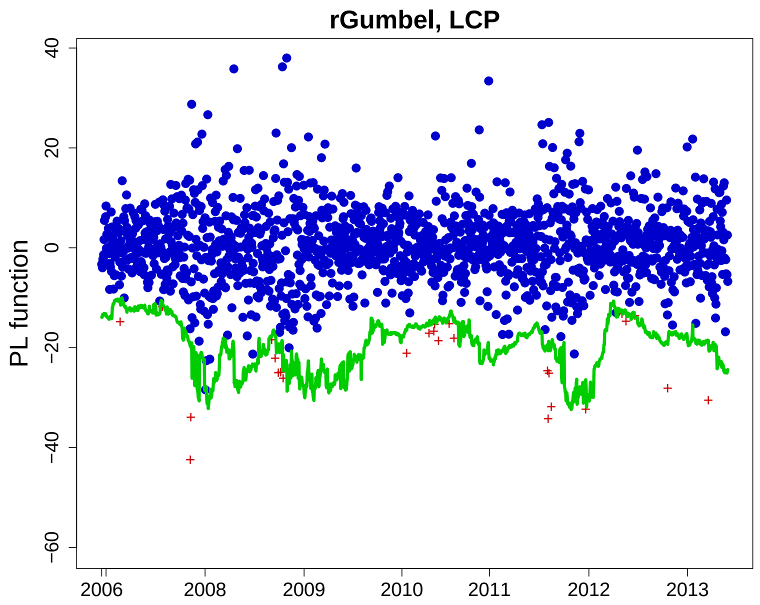
rGumbel, LCP
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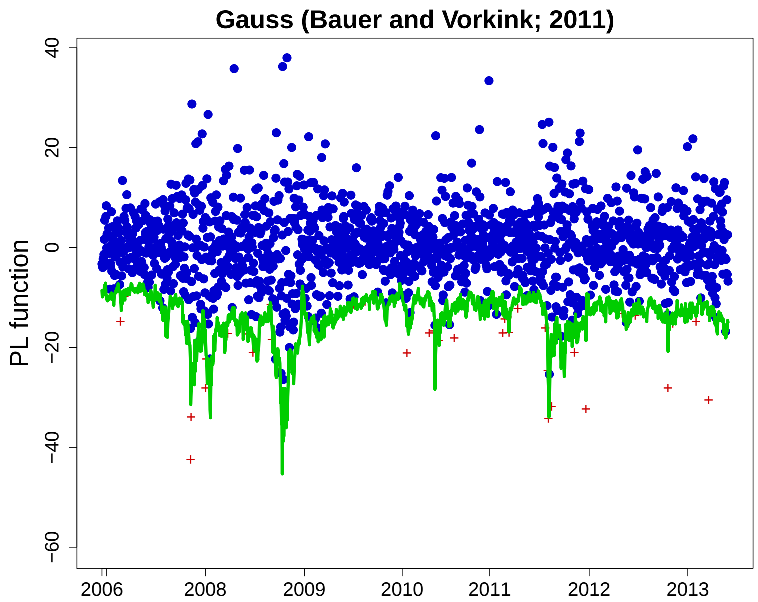
Gauss (Bauer and Vorkink; 2011)
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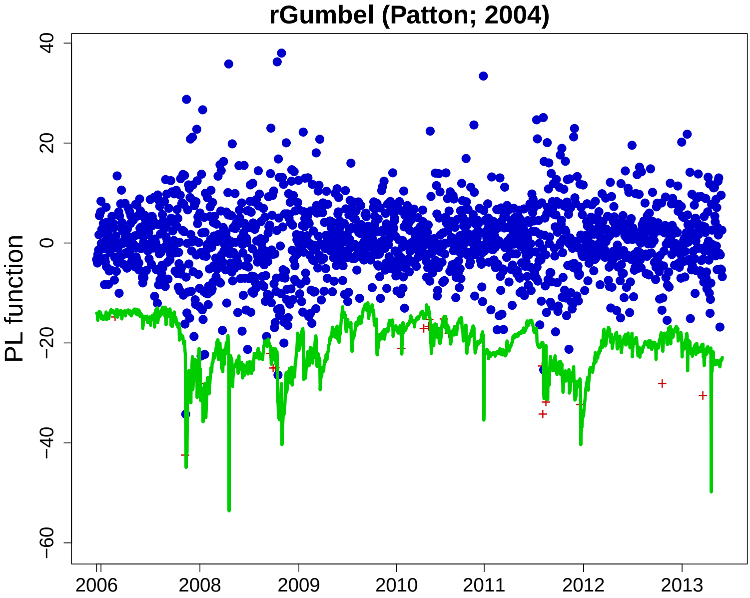
rGumbel (Patton; 2004)
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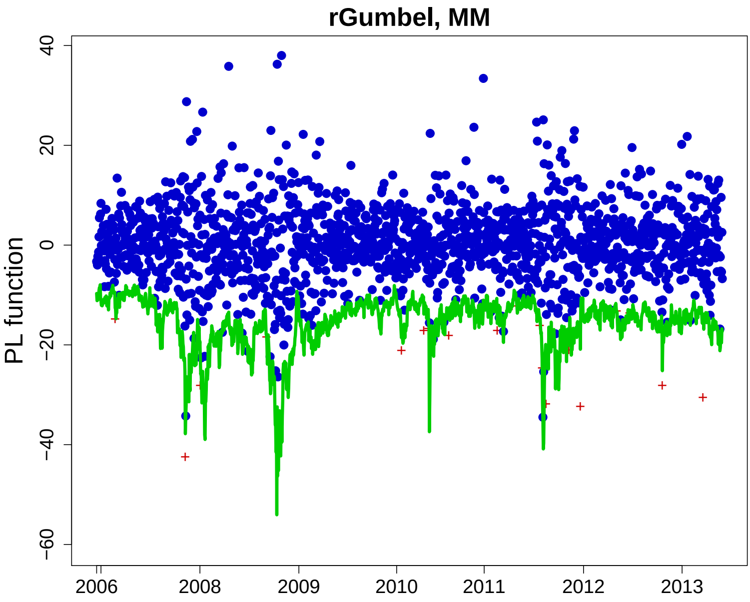

rGumbel, MM
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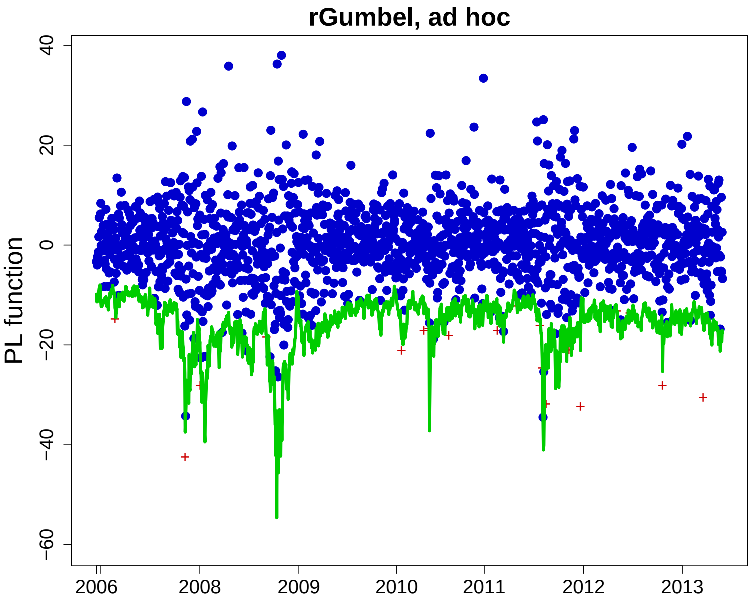

rGumbel, ad hoc
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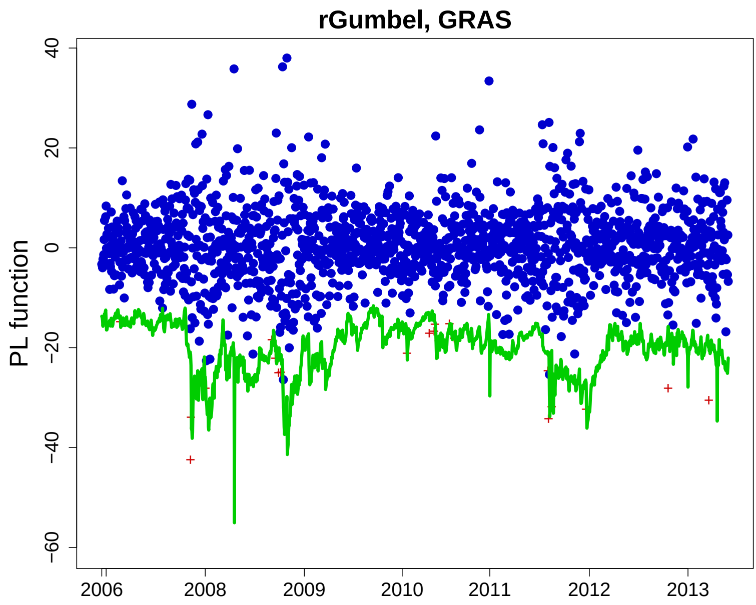
rGumbel, GRAS
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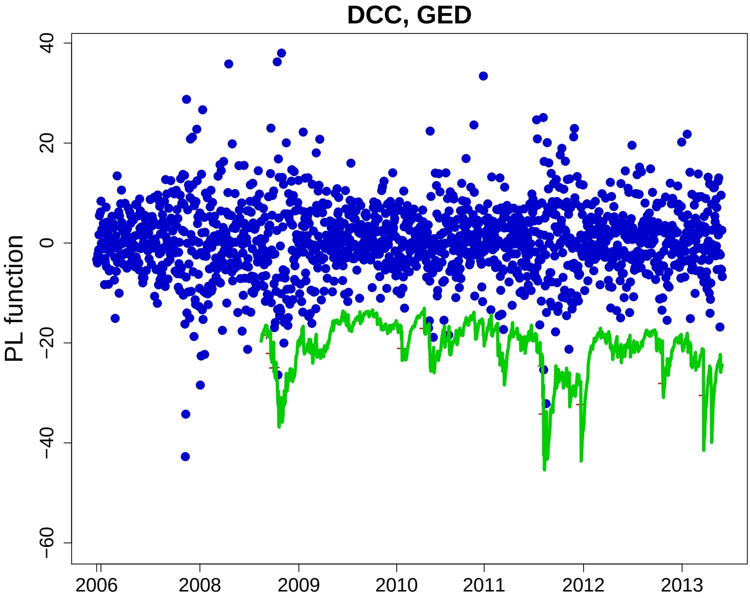
−20
−40
−60
2006
2008
2009
2010
2011
2012
2013


DCC, GED
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2012
2013

VaR Performance for Google-IBM-Oracle

Model	**0.01**	**0.05**	**0.10**
roll. window, Cl.	0.0151 (0.057)	0.0466 (0.530)	0.0957 (0.567)
LCP, Cl.	0.0164 (0.019)	0.0453 (0.387)	0.0913 (0.242)
RCop, HAR, Cl.	0.0139 (0.145)	0.0504 (0.945)	0.1045 (0.550)
RCop, BaVo, Cl.	0.0132 (0.219)	0.0497 (0.963)	0.1058 (0.445)
RCop, ChVo, Cl.	0.0088 (0.628)	0.0485 (0.781)	0.1001 (0.987)
RCop, HAR, GED, Cl.	0.0132 (0.219)	0.0497 (0.963)	0.1058 (0.445)
RCop, BaVo, GED, Cl.	0.0132 (0.219)	0.0497 (0.963)	0.1058 (0.445)
RCop, ChVo, GED, Cl.	0.0088 (0.628)	0.0485 (0.781)	0.0995 (0.947)
Patton, GED, Cl.	0.0120 (0.445)	0.0460 (0.455)	0.0976 (0.750)
GAS, GED, Cl.	0.0145 (0.092)	0.0466 (0.530)	0.1058 (0.445)
GRAS, GED, Cl.	0.0139 (0.145)	0.0497 (0.963)	0.1026 (0.726)
roll. window, rG.	0.0151 (0.057)	0.0416 (0.112)	0.0850 (0.042)
LCP, rG.	0.0164 (0.019)	0.0409 (0.087)	0.0869 (0.076)
RCop, HAR, rG.	0.0120 (0.445)	0.0510 (0.854)	0.1102 (0.182)
RCop, BaVo, rG.	0.0132 (0.219)	0.0523 (0.681)	0.1108 (0.157)
RCop, ChVo, rG.	0.0101 (0.976)	0.0497 (0.963)	0.1058 (0.445)
RCop, HAR, GED, rG.	0.0126 (0.318)	0.0504 (0.945)	0.1102 (0.182)
RCop, BaVo, GED, rG.	0.0132 (0.219)	0.0516 (0.766)	0.1096 (0.210)
RCop, ChVo, GED, rG.	0.0094 (0.823)	0.0504 (0.945)	0.1058 (0.445)
Patton, GED, rG.	0.0120 (0.445)	0.0435 (0.221)	0.0945 (0.458)
GAS, GED, rG.	0.0132 (0.219)	0.0447 (0.325)	0.0970 (0.687)
GRAS, GED, rG.	0.0132 (0.219)	0.0472 (0.609)	0.0932 (0.361)
DCC, t-copula	0.0120 (0.447)	0.0418 (0.125)	0.0751 (0.001)
DCC, GED, t-copula	0.0102 (0.921)	0.0393 (0.042)	0.0743 (0.000)
RV, BaVo	0.0239 (0.000)	0.0655 (0.007)	0.1108 (0.157)
RV, ChVo	0.0170 (0.011)	0.0586 (0.127)	0.1077 (0.313)

Table 2: VaR performance ($\widehat{\alpha}$) for the Google-IBM-Oracle portfolio. p-values of the Kupiec test in brackets.

VaR Performance for IBM-Pfizer-Exxon

Model	0.01	0.05	0.10
roll. window, Cl.	0.0252 (0.000)	0.0554 (0.330)	0.0945 (0.458)
LCP, Cl.	0.0202 (0.000)	0.0586 (0.127)	0.0938 (0.408)
RCop, HAR, Cl.	0.0094 (0.823)	0.0605 (0.064)	0.1089 (0.241)
RCop, BaVo, Cl.	0.0094 (0.823)	0.0605 (0.064)	0.1058 (0.445)
RCop, ChVo, Cl.	0.0082 (0.453)	0.0529 (0.600)	0.0970 (0.687)
RCop, HAR, GED, Cl.	0.0107 (0.780)	0.0592 (0.102)	0.1071 (0.354)
RCop, BaVo, GED, Cl.	0.0088 (0.628)	0.0598 (0.081)	0.1058 (0.445)
RCop, ChVo, GED, Cl.	0.0088 (0.628)	0.0535 (0.524)	0.0976 (0.750)
Patton, GED, Cl.	0.0183 (0.003)	0.0586 (0.127)	0.0995 (0.947)
GAS, GED, Cl.	0.0151 (0.057)	0.0592 (0.102)	0.1052 (0.496)
GRAS, GED, Cl.	0.0145 (0.092)	0.0611 (0.050)	0.1058 (0.445)
roll. window, rG.	0.0202 (0.000)	0.0516 (0.766)	0.0806 (0.008)
LCP, rG.	0.0170 (0.011)	0.0516 (0.766)	0.0856 (0.051)
RCop, HAR, rG.	0.0088 (0.628)	0.0611 (0.050)	0.1096 (0.210)
RCop, BaVo, rG.	0.0088 (0.628)	0.0642 (0.013)	0.1102 (0.182)
RCop, ChVo, rG.	0.0069 (0.193)	0.0567 (0.232)	0.1039 (0.606)
RCop, HAR, GED, rG.	0.0082 (0.453)	0.0617 (0.039)	0.1096 (0.210)
RCop, BaVo, GED, rG.	0.0094 (0.823)	0.0642 (0.013)	0.1102 (0.182)
RCop, ChVo, GED, rG.	0.0069 (0.193)	0.0579 (0.157)	0.1039 (0.606)
Patton, GED, rG.	0.0151 (0.057)	0.0573 (0.191)	0.0957 (0.567)
GAS, GED, rG.	0.0151 (0.057)	0.0611 (0.050)	0.1045 (0.550)
GRAS, GED, rG.	0.0170 (0.011)	0.0598 (0.081)	0.1026 (0.726)
DCC, t-copula	0.0195 (0.001)	0.0542 (0.453)	0.0932 (0.361)
DCC, GED, t-copula	0.0088 (0.628)	0.0491 (0.872)	0.0913 (0.242)
RV, BaVo	0.0239 (0.000)	0.0712 (0.000)	0.1108 (0.157)
RV, ChVo	0.0189 (0.002)	0.0668 (0.003)	0.1033 (0.665)

Table 3: VaR performance ($\widehat{\alpha}$) for the IBM-Pfizer-Exxon portfolio. p-values of the Kupiec test in brackets.

Conclusions

- We introduce the notion of realized copula.
- We suggest a forecasting framework for RCop and thus extend the literature on multivariate RCov models.
- Empirically, we find that model relying on daily data are too inert for good forecasts.
- Standard RCov model are more adaptive, but are dominated by copula models.
- RCop unites both advantages and shows nice forecasting performance.

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Appendix

- Realized kernels
- ML estimation
- Details on LCP
- Kupiec (1995) test

Basis

Let $Y = (Y_1, \ldots, Y_d)^\top$ be a d-dim efficient *(log)price* process

$$dY_t = \mu_t dt + \sigma_t dW_t$$

The market microstructure effect is modeled through an additive component

$$\begin{aligned} P_{jt} &= Y_{jt} + U_{jt}, \text{ with } \boldsymbol{E}(U_{jt}) = 0 \\ \sum_h |h\Omega_h| &< \infty, \text{ where } \Omega_{jh} = Cov(U_{jt}, U_{j,t-h}). \end{aligned}$$

Usual aim: Estimate the *quadratic variation* of Y, i.e. $[Y] = \int_0^1 \Sigma_u du$, with $\Sigma = \sigma\sigma^\top$.

Naive Estimator (realized co/variance)

Synchronization – *last traded*: for time t, the log-price for asset j is given by P_{j,t^*} with $t^* = \max\{t_{j,i} | t_{j,i} \leq t, \ \forall i = 1, \ldots, N_j\}$.
$M = M(m)$ number of subintervals of length m (in seconds)

$$
\begin{aligned}
\mathrm{RC}_{t_1,m,j_1,j_2}(P) &= \sum_{i=1}^{M} (P_{j_1,t_i} - P_{j_1,t_{i-1}})(P_{j_2,t_i} - P_{j_2,t_{i-1}}), \\
\mathrm{RC}_{t_1,m}(P) &= \{\mathrm{RC}_{m,j_1,j_2}\}_{j_1,j_2}, \ \text{for } j_1, j_2 = 1, \ldots, d
\end{aligned}
$$

Realized Kernels, BNHLS (2011, JoE)

Synchronization – *refresh time sampling*

$$
\begin{aligned}
\tau_1 &= \max\{t_{1,1}, \ldots, t_{d,1}\} \\
\tau_{i+1} &= \arg\min\{t_{j,k_j} | t_{j,k_j} > \tau_i,\ \forall j \in 1 \ldots d\}
\end{aligned}
$$

Leads to new high-frequency vector of returns $p_i = P_{\tau_i} - P_{\tau_{i-1}}$, where $i = 1, \ldots, n$ and n is the of refresh time observations.

Realized Kernels

The multivariate realized kernel is defined as

$$K(P) = \sum_{h=-H}^{H} k\left(\frac{|h|}{H+1}\right)\Gamma_h,$$

with Γ_h being a matrix of autocovariances given by

$$\Gamma_h = \begin{cases} \sum_{j=|h|+1}^{n} p_j p_{j-h}^{\top}, & h \geq 0 \\ \sum_{j=|h|+1}^{n} p_{j-h} p_j^{\top}, & h < 0 \end{cases},$$

and $k(x)$ being a weight function of the *Parzen kernel*, defined through

$$k(x) = \begin{cases} 1 - 6x^2 + 6x^3 & 0 \leq x \leq 1/2 \\ 2(1-x)^3 & 1/2 \leq x \leq 1 \\ 0 & x > 1 \end{cases}.$$

Realized Kernels

The multivariate bandwidth parameter

$$H = \left[d^{-1} \sum_{j=1}^{d} H_j \right]$$

where $H_j,\ j = 1, \ldots, d$ is chosen by *mean squared error* criteria as

$$H_j = c^* \xi_j^{4/5} n^{3/5}$$

with $c^* = \left\{ k''(0)^2 / \int_0^1 k(x)^2 dx \right\}^{1/5}$, which is equal to $c^* = 3.511678$ for Parzen kernel.
$\xi^2 = \omega / \sqrt{IQ}$ denotes the *noise-to-signal ratio*, where ω^2 is the *measure of microstructural noise variance* and IQ is *the integrated quarticity* as defined in Barndorff-Nielsen and Shephard (2002).

ML estimation of copula parameters

For a sample of observations $\{x_t\}_{t=1}^{\prime}$ and $\vartheta = (\delta_1, \ldots, \delta_d; \theta) \in \mathbb{R}^{d+1}$ the likelihood function is

$$L(\vartheta; x_1, \ldots, x_T) = \prod_{t=1}^{T} f(x_{1,t}, \ldots, x_{d,t}; \delta_1, \ldots, \delta_d; \theta)$$

and the corresponding log-likelihood function

$$\begin{aligned} \ell(\vartheta; x_1, \ldots, x_T) &= \sum_{t=1}^{T} \log c\{F_{X_1}(x_{1,t}, \delta_1), \ldots, F_{X_d}(x_{d,t}, \delta_d); \theta\} \\ &+ \sum_{t=1}^{T} \sum_{j=1}^{d} \log f_j(x_{j,t}, \delta_j) \end{aligned}$$

"*Oracle*" choice: largest interval $I = [t_0 - m_{k^*}, t_0]$ where the small modelling bias condition (SMB)

$$\triangle_I(\theta) = \sum_{t \in I} \mathcal{K}\{C(\cdot;\theta_t), C(\cdot;\theta)\} \leq \triangle.$$

holds for some $\triangle \geq 0$. m_{k^*} is the ideal scale, θ is ideally estimated from $I = [t_0 - m_{k^*}, t_0]$ and $\mathcal{K}(\cdot,\cdot)$ is the *Kullback-Leibler* divergence

$$\mathcal{K}\{C(\cdot;\theta_t), C(\cdot;\theta)\} = \boldsymbol{E}_{\theta_t} \log \frac{c(\cdot;\theta_t)}{c(\cdot;\theta)}$$

Under the SMB condition on I_{k^*} and assuming that $\max_{k \le k^*} \boldsymbol{E}_{\theta_t} |\mathcal{L}(\widetilde{\theta}_k) - \mathcal{L}(\theta)|^r \le \mathcal{R}_r(\theta_t)$, we obtain

$$\boldsymbol{E}_{\theta_t} \log \left\{ 1 + \frac{|\mathcal{L}(\widetilde{\theta}_{\widehat{k}}) - \mathcal{L}(\theta)|^r}{\mathcal{R}_r(\theta)} \right\} \le 1 + \triangle,$$

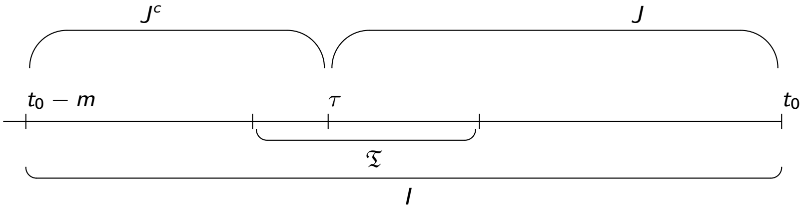
$$\boldsymbol{E}_{\theta_t} \log \left\{ 1 + \frac{|\mathcal{L}(\widetilde{\theta}_{\widehat{k}}) - \mathcal{L}(\widehat{\theta}_{\widehat{k}})|^r}{\mathcal{R}_r(\theta)} \right\} \le 1 + \triangle,$$

where $\widehat{a}_I$ is an adaptive estimator on I and $\widetilde{a}_I$ is any other parametric estimator on I.

Test of homogeneity

Interval $I = [t_0 - m, t_0], \mathfrak{T} \subset I$

$$
\begin{aligned}
H_0 &: \quad \forall \tau \in \mathfrak{T},\ \theta_t = \theta, \forall t \in J = [\tau, t_0], \forall t \in J^c = I \setminus J \\
H_1 &: \quad \exists \tau \in \mathfrak{T},\ \theta_t = \theta_1;\ \forall t \in J,\ \theta_t = \theta_2 \neq \theta_1;\ \forall t \in J^c
\end{aligned}
$$



Test of homogeneity

Likelihood ratio test statistic for fixed change point location:

$$\begin{aligned} T_{I,\tau} &= \max_{\theta_1,\theta_2}\{L_J(\theta_1) + L_{J^c}(\theta_2)\} - \max_{\theta} L_I(\theta) \\ &= ML_J + ML_{J^c} - ML_I \end{aligned}$$

Test statistic for unknown change point location:

$$T_I = \max_{\tau \in \mathfrak{T}_I} T_{I,\tau}$$

Reject H_0 if for a critical value ζ_I

$$T_I > \zeta_I$$

Selection of I_k and ζ_k

- set of numbers m_k defining the length of I_k and $\mathfrak{T}_k$ are in the form of a geometric grid
- $m_k = [m_0 c^k]$ for $k = 1, 2, \ldots, K,\ m_0 \in \{20,\ 40\},\ c = 1.25$ and $K = 10$, where $[x]$ means the integer part of x
- $I_k = [t_0 - m_k, t_0]$ and $\mathfrak{T}_k = [t_0 - m_k, t_0 - m_{k-1}]$ for $k = 1, 2, \ldots, K$

(Mystery Parameters)

Sequential choice of ζ_k

- after k steps there are two cases: change point at some step $\ell \le k$ or no change points.
- let $\mathcal{B}_\ell$ be the event meaning the rejection at step ℓ

$$\mathcal{B}_\ell = \{\, T_1 \le \zeta_1, \ldots, T_{\ell-1} \le \zeta_{\ell-1}, T_\ell > \zeta_\ell \},$$

  and $(\widehat{\theta}_k) = (\widetilde{\theta}_{\ell-1})$ on $\mathcal{B}_\ell$ for $\ell = 1, \ldots, k$.
- we find sequentially such a minimal value of ζ_ℓ that ensures the inequality

$$\max_{k=l,\ldots,K} \boldsymbol{E}_{\theta^*} [|\mathcal{L}(\widetilde{\theta}_k) - \mathcal{L}(\widetilde{\theta}_{\ell-1})|^r \mathbf{I}(\mathcal{B}_\ell)] \le \rho \mathcal{R}_r(\theta^*) k/(K-1)$$

return to LCP

Kupiec (1995) test

LR test based on the binomial model.
$H_0 : \widehat{\alpha} = \alpha$ with test statistic

$$LR_{uc} = 2 \log \frac{\widehat{\alpha}^N (1 - \widehat{\alpha})^{T-N}}{\alpha^N (1 - \alpha)^{T-N}}$$