Valuation of Collateralized Debt Obligations with Hierarchical Archimedean Copulae

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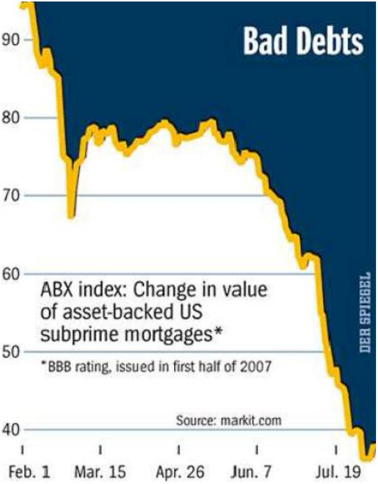
Collateralized Debt Obligation

Triggered the financial crisis.



CDO Dynamics

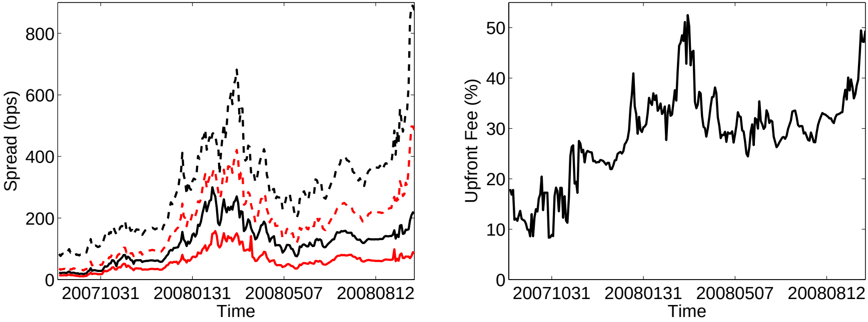


Figure 1: Spreads of iTraxx tranches, Series 8, maturity 5 years, data from 20070920-20081022. Left panel: mezzanine junior (dashed black), mezzanine (dashed red), senior (solid black), super senior (solid red). Right panel: upfront fee of the equity tranche.

Dependence Matters!

The normal world is not enough.

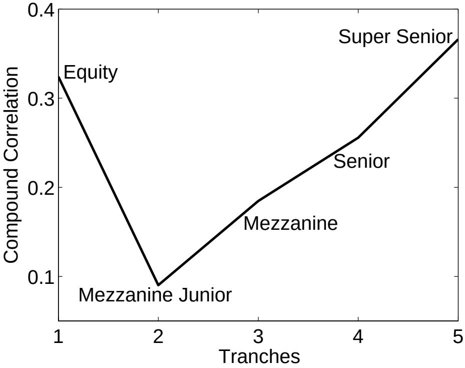


Figure 2: Implied correlation smile in the homogeneous large pool Gaussian copula model. Data from 20071022.

Research Goals

- How to model the smiles?
- How to catch the dependency?
- How to go beyond the normal world?

Outline

1. Motivation ✓
2. CDOs
3. HAC Models
4. Empirical Study
5. Conclusions

Risk Transfer

Tranching

Tranche name	Attachment points (%) Lower l	Upper u
Super Super Senior	22	100
Super Senior	12	22
Senior	9	12
Mezzanine	6	9
Mezzanine Junior	3	6
Equity	0	3

Table 1: iTraxx tranches.

iTraxx Europe

- A static portfolio of 125 equally weighted CDS on European entities;
- Sectors: Consumer (30), Financial (25), TMT (20), Industrials (20), Energy (20), Auto (10);
- New series of iTraxx Europe issued every 6 months (March and September) and the underlying reference entities are reconstituted;
- Maturities: 3Y, 5Y, 7Y, 10Y.

Default

Consider a CDO with a maturity of T years, J tranches, and a pool of d entities. Define a loss variable of i-th obligor until $t \in [t_0, T]$ as

$$I_i(t) = \mathbf{1}(\tau_i < t), \; i = 1, \dots, d,$$

where τ_i is a time to default variable

$$\begin{aligned} F_i(t) &= \mathrm{P}(\tau_i \le t) \\ &= 1 - \exp\left\{ -\int_{t_0}^{t} \lambda_i(u) du \right\} \end{aligned}$$

and λ_i is a deterministic intensity function.

Portfolio Loss

The proportion of defaulted entities in the portfolio at time t is given by

$$\tilde{L}(t) = \frac{1}{d}\sum_{i=1}^{d} I_i(t), \quad t \in [t_0, T].$$

The portfolio loss at time t is defined as

$$L(t) = \mathrm{LGD}\tilde{L}(t),$$

where LGD is a common loss given default.

Tranche Loss

The tranche loss at time t is defined as

$$L_j(t) = \frac{1}{u_j - l_j}\{L^u(t, u_j) - L^u(t, l_j)\},$$

where

$$L^u(t, x) = \min\{L(t), x\} \quad \text{for } x \in [0, 1].$$

The outstanding notional of the tranche j is given by

$$\Gamma_j(t) = \frac{1}{u_j - l_j}\{\Gamma^u(t, u_j) - \Gamma^u(t, l_j)\},$$

where

$$\Gamma^u(t, x) = x - L^u(t, x) \quad \text{for } x \in [0, 1].$$

Valuation of CDO

1. Premium leg

$$\mathrm{PL}_j(t_0) = \sum_{t=t_1}^{T} \beta(t_0, t) s_j(t_0) \Delta t \, \mathrm{E}\{\Gamma_j(t)\}$$

2. Default leg

$$\mathrm{DL}_j(t_0) = \sum_{t=t_1}^{T} \beta(t_0, t) \, \mathrm{E}\{L_j(t) - L_j(t - \Delta t)\}$$

This leads to:

$$s_j(t_0) = \frac{\sum_{t=t_1}^{T} \beta(t_0, t) \, \mathrm{E}\{L_j(t) - L_j(t - \Delta t)\}}{\sum_{t=t_1}^{T} \beta(t_0, t) \Delta t \, \mathrm{E}\{\Gamma_j(t)\}}.$$

Equity Tranche

The equity tranche is quoted in two parts:

1. an upfront fee α payed at t_0,
2. a running spread of 500 BPs.

The premium leg is calculated as

$$\mathsf{PL}_1(t_0) = \alpha(t_0) + \sum_{t=t_1}^{T} \beta(t_0, t) \cdot 500 \cdot \Delta t \, \mathsf{E}\{\Gamma_1(t)\}.$$

The upfront payment given in percent is equal

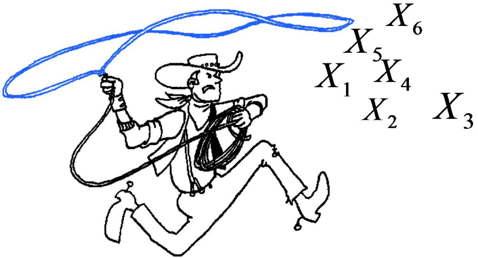
$$\alpha(t_0) = 100 \sum_{t=t_0}^{T} \left(\beta(t, t_0) \left[\mathsf{E}\{L_1(t) - L_1(t - \Delta t)\} - 0.05 \Delta t \, \mathsf{E}\{\Gamma_1(t)\}\right]\right).$$

Copula

For a distribution function F with marginals $F_{X_1} \ldots, F_{X_d}$. There exists a copula $C : [0,1]^d \to [0,1]$, such that

$$F(x_1, \ldots, x_d) = C\{F_{X_1}(x_1), \ldots, F_{X_d}(x_d)\}$$

for all $x_i \in \overline{\mathbb{R}}$, $i = 1, \ldots, d$.



Copula for CDOs

The vector of default times $(\tau_1, \ldots, \tau_d)^\top$ has a (risk-neutral) joint cdf

$$F(t_1, \ldots, t_d) = \mathrm{P}(\tau_1 \le t_1, \ldots, \tau_d \le t_d) \quad \text{for all} \quad (t_1, \ldots, t_d)^\top \in \mathbb{R}_+^d,$$

where $\tau_i \sim F_i$. From the Sklar theorem, there exists a copula such that

$$F(t_1, \ldots, t_d) = C\{F_1(t_1), \ldots, F_d(t_d)\}$$

and determines the default dependency of the credits.

Monte Carlo Simulation Approach

Define a trigger variable as

$$U_i = \bar{p}_i(\tau_i) \sim \mathrm{U}[0,1], \quad i = 1, \dots, d.$$

The ith obligor survives until $t < T$ if and only if

$$\begin{aligned} \tau_i &\geq t \\ \text{or } U_i &\leq \bar{p}_i(t). \end{aligned}$$

The joint and marginal distributions of the triggers satisfy:

$$\begin{aligned} C\{\bar{p}_1(t), \dots, \bar{p}_d(t)\} &= \mathrm{P}\{U_1 \leq \bar{p}_1(t), \dots, U_d \leq \bar{p}_d(t)\}, \\ \mathrm{P}\{U_i \leq \bar{p}_i(t)\} &= \bar{p}_i(t). \end{aligned}$$

Monte Carlo Simulation Approach

The time to default variable

$$\tau_i = \inf\{t \geq t_0 : \bar{p}_i(t) \leq U_i\}$$

is calculated as

$$\tau_i = \bar{p}_i^{-1}(U_i).$$

Assuming constant intensities compute

$$\tau_i = -(\log U_i)/\lambda_i.$$

Industry Sectors

iTraxx pool industry sectors: Consumer (30), Financial (25), TMT (20), Industrials (20), Energy (20), Auto (10).

A cluster analysis applied to the daily log-returns of the CDS spreads (20071022-20081128) returns six clusters: 38, 30, 17, 16, 14, 10.

The ratio between correctly assigned CDS to their industries and the total number of entities is 0.704.

Goal: Construct a joint distribution of the default times that imposes different intra- and inter-industry dependencies.

Hierarchical Archimedean Copulae

- A Hierarchical Archimedean Copula (HAC) is a generalisation of a multivariate Archimedean copula.
- HACs join bivariate or higher-dimensional Archimedean copulae by an another Archimedean copula which allows for non-exchangeable dependency structures.
- HACs are useful when the composition of data is known and where each level of the hierarchy has a natural interpretation.

HAC Model

Partially nested hierarchical Archimedean copula

$$C(u_1,\dots,u_d;\theta) = C_1\{\, C_2(u_1,\dots,u_{m_1};\theta_2), C_2(u_{m_1+1},\dots,u_{m_1+m_2};\theta_2),\\ \dots, C_2(u_{m_1+\dots+m_5+1},\dots,u_d;\theta_2);\theta_1\},$$

where $\theta = (\theta_1,\theta_2)^\top$ and m_k, $k = 1$, ..., 6, indicates a number of the companies in kth industry sector.

- C_2 models the dependency in the industry sector,
- C_1 models the dependency between the industry sectors.

HAC Model

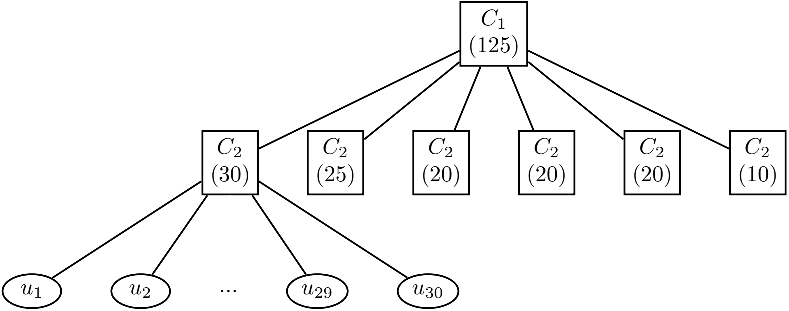


Figure 3: Partially nested 125-dimensional HAC.

Construct HACs composed of Gumbel copulae.

Gaussian Model with Industry Sectors

$$\Sigma = \left(\begin{array}{ccccc}
\boxed{\begin{matrix}1 & \cdots & \rho_2\\ & \ddots & \\ \rho_2 & \cdots & 1\end{matrix}} & \begin{matrix}\rho_1 & \cdots & \cdots\\ \vdots & & \\ \rho_1 & & \end{matrix} & & & \begin{matrix}\cdots & \cdots & \rho_1\\ & & \vdots\\ & & \vdots\end{matrix}\\
\begin{matrix}\rho_1 & \cdots & \rho_1\\ \vdots & & \\ \vdots & & \end{matrix} & \boxed{\begin{matrix}1 & \cdots & \rho_2\\ & \ddots & \\ \rho_2 & \cdots & 1\end{matrix}} & & & \\
 & & \ddots & & \\
 & & & \boxed{\begin{matrix}1 & \cdots & \rho_2\\ & \ddots & \\ \rho_2 & \cdots & 1\end{matrix}} & \begin{matrix} & & \vdots\\ & & \vdots\\ \rho_1 & \cdots & \rho_1\end{matrix}\\
\begin{matrix}\vdots & & \\ \vdots & & \\ \rho_1 & \cdots & \cdots\end{matrix} & & & \begin{matrix} & & \rho_1\\ & & \vdots\\ \cdots & \cdots & \rho_1\end{matrix} & \boxed{\begin{matrix}1 & \cdots & \rho_2\\ & \ddots & \\ \rho_2 & \cdots & 1\end{matrix}}
\end{array}\right)$$

Loss Given Default

- Spreads are calculated on the fraction of the losses that cannot be recovered.
- Altman et al. (2005) and Hamilton et al. (2004) show that the LGD is stochastic and positively correlated with the default probabilities.
- Krekel (2008), Bennani & Maetz (2009) and other studies show that a stochastic LGD fattens the right tail of the portfolio loss distribution.

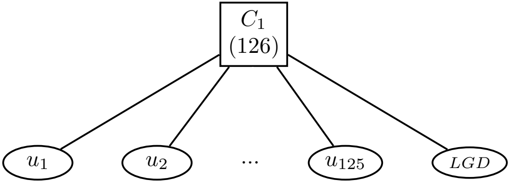
LGD Modelling

Consider one and two-factor models with random

1. Common LGD,
2. Sectorial LGDs.

Assume that LGD $\sim$ U$[0, 1]$.

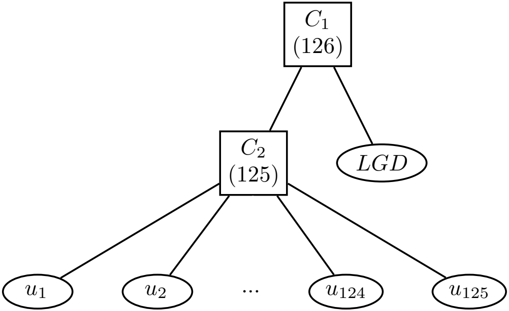
One-factor Model with a Random Bottom-Level LGD



$$C(u_1, \ldots, u_d, u_{d+1}; \theta_1) = C_1(u_1, \ldots, u_d, u_{d+1}; \theta_1),$$

where u_{d+1} is the common LGD variable.

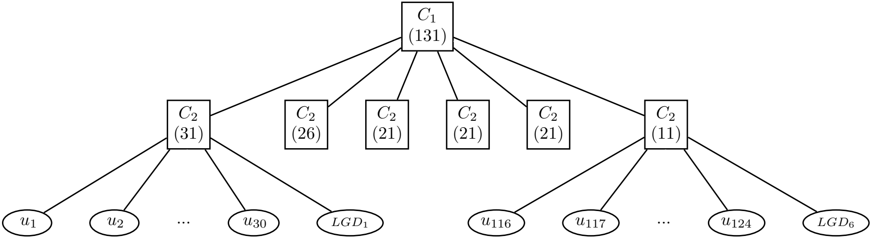
One-factor Model with a Random Top-Level LGD



$$C(u_1, \dots, u_d, u_{d+1}; \theta) = C_1\{u_{d+1}, C_2(u_1, \dots, u_d; \theta_2); \theta_1\},$$

where $\theta = (\theta_1, \theta_2)^\top$ and u_{d+1} is the common LGD variable.

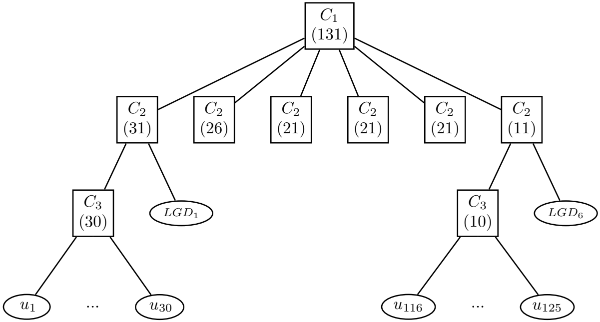
Two-factor Model with Random Bottom-Level LGDs



$$
\begin{aligned}
C(u_1,\dots,u_{d+6};\theta) = C_1\{\, & C_2(u_1,\dots,u_{m_1},u_{d+1};\theta_2),\\
& C_2(u_{m_1+1},\dots,u_{m_1+m_2},u_{d+2};\theta_2),\dots,\\
& C_2(u_{m_1+\dots+m_5+1},\dots,u_d,u_{d+6};\theta_2)\},
\end{aligned}
$$

where $\theta = (\theta_1,\theta_2)^\top$ and u_{d+1}, ..., u_{d+6} are sectorial LGD variables.

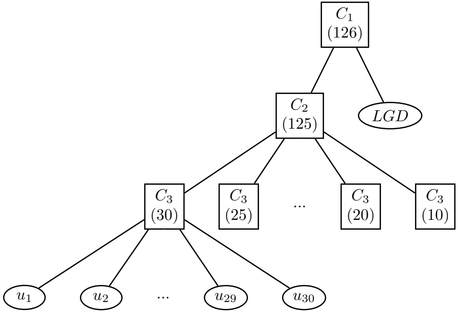
Two-factor Model with Random Middle-Level LGDs



$$
\begin{aligned}
C(u_1,\ldots,u_{d+6};\theta) = C_1[\, & C_2\{C_3(u_1,\ldots,u_{m_1};\theta_3),u_d;\theta_2\}, \\
& C_2\{C_3(u_{m_1+1},\ldots,u_{m_1+m_2};\theta_3),u_{d+1};\theta_2\},\ldots, \\
& C_2\{C_3(u_{m_1+\ldots+m_5+1},\ldots,u_d;\theta_3),u_{d+6};\theta_2\}],
\end{aligned}
$$

where $\theta = (\theta_1,\theta_2,\theta_3)^\top$ and u_{d+1}, ..., u_{d+6} are sectorial LGDs.

Two-factor Model with a Random Top-Level LGD



$$\begin{aligned} C(u_1,\ldots,u_d,u_{d+1};\theta) = C_1[\, & u_{d+1}, C_2\{C_3(u_1,\ldots,u_{m_1};\theta_3),\\ & C_3(u_{m_1+1},\ldots,u_{m_1+m_2};\theta_3),\\ & \ldots, C_3(u_{m_1+\ldots+m_5+1},\ldots,u_d;\theta_3);\theta_2\};\theta_1], \end{aligned}$$

where $\theta = (\theta_1,\theta_2,\theta_3)^\top$ and u_{d+1} is the common LGD variable.

Data

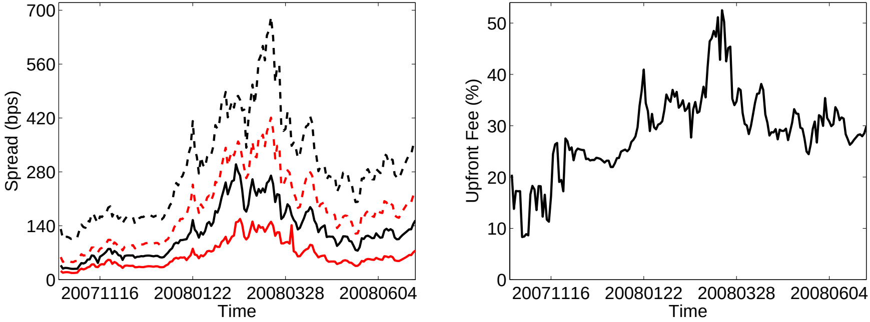


Figure 4: Spreads of iTraxx tranches, Series 8, maturity 5 years, data from 20070920-20081022. Left panel: mezzanine junior (dashed black), mezzanine (dashed red), senior (solid black), super senior (solid red). Right panel: upfront fee of the equity tranche.

Calibration

Look for the parameters θ of Gaussian and HAC copulae that provide the best fit to market data.

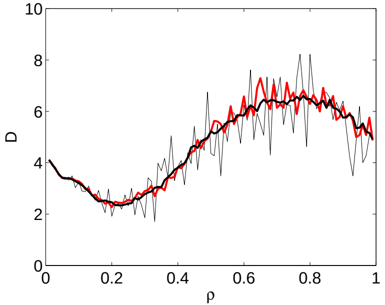
Minimize relative deviations from the market spreads s_j^m

$$D(t_0) \stackrel{\text{def}}{=} \min_{\theta} \sum_{j=1}^{J} \frac{|s_j^c(t_0;\theta) - s_j^m(t_0;\theta)|}{s_j^m(t_0;\theta)}, \tag{1}$$

where s_j^c is a calculated spread of the tranche $j = 1, \dots, 5$.

s_j^c computed from generated $N = 10^6$ times Monte Carlo vectors.
Constant LGD $= 0.6$.

Calibration of Gaussian Models with the Constant LGD



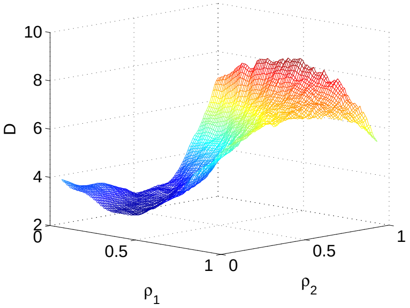


Figure 5: Measure D as a function of the correlation ρ for the number of simulations N: 10^3 (thin black), 10^4 (thick red), 10^5 (thick black) (left) and as a function of ρ_1 and ρ_2 for $N = 10^4$ (right). Data from 20071022.

Intensity Parameters

Assume constant but heterogeneous intensities λ_i, $i = 1, \ldots, 125$.
Calibrate λ_i from the historical spreads of 125 CDS.

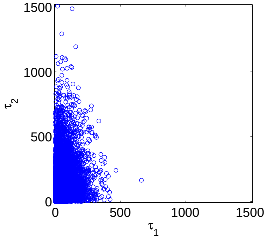


Figure 6: Default times of Kingfisher and Deutsche Bank based on the samples generated from Gumbel copula with the constant LGD and the calibrated parameters.
Data from 20071022.

Measure D for the Gumbel Copula Models

Date	constant LGD		random LGD				
			bottom level	top level	bottom level	middle level	top level
	1-fac.	2-factor	1-factor		2-factor		
20071022	0.2987	0.2168	0.1098	*0.0444*	0.1606	0.1448	0.0497
20071025	0.3605	0.3481	0.2795	*0.0878*	0.2554	0.2644	0.1538
20071116	1.3065	1.2876	1.3147	0.5704	1.0313	*0.1581*	0.6029
20071205	1.1457	1.1153	1.1897	0.5060	0.8672	*0.2286*	0.5359
20080110	1.0741	1.0581	1.0719	*0.1805*	0.8289	0.2665	0.1999
20080227*	1.8529	1.8037	2.1974	*0.5997*	1.8177	0.8902	0.6282
20080313*	1.8892	1.8858	2.7821	0.7743	2.4859	1.9754	*0.7726*
20080404	1.8309	1.8236	1.7136	0.6294	1.4580	*0.2871*	0.8069
20080423	2.0350	2.0279	1.5179	0.8769	1.2842	*0.3564*	0.8850
20080529	1.6679	1.6361	1.4661	0.7067	1.1636	*0.3799*	0.9525
20080630	1.8005	1.7821	2.2130	0.5664	1.8131	*0.3279*	0.5420

Measure D for the Gaussian Copula Models

Date	constant LGD		random LGD				
			bottom level	top level	bottom level	middle level	top level
	1-fac.	2-factor	1-factor		2-factor		
20071022	2.2971	2.2767	2.2717	1.5921	2.2543	2.0973	1.7256
20071025	2.7750	2.7136	2.4977	1.8167	2.5436	2.4087	1.8872
20071116	3.9309	3.9106	3.4996	2.6108	3.5777	3.4831	2.6652
20071205	4.1684	3.9603	3.5419	2.5980	3.5349	3.6268	2.6743
20080110	3.5166	3.5113	3.0834	1.8181	3.1189	3.1049	1.8287
20080227*	2.0559	3.0295	2.4398	2.3170	4.4093	4.1220	2.3754
20080313*	2.3416	2.3483	2.9543	1.8788	5.0172	4.7947	2.4136
20080404	4.1553	4.1141	3.5936	2.2642	3.6229	3.4107	2.3086
20080423	4.4135	4.3862	3.3379	1.9269	3.3241	3.1535	2.0040
20080529	4.1966	4.1749	3.3901	2.1980	3.5004	3.3503	2.6803
20080630	4.1478	4.0947	4.2423	3.0212	4.2249	3.3551	2.9458

Calibrated Parameters for the Gumbel Models

Date	constant LGD			random LGD										
				bottom level	top level		bottom level		middle level			top level		
	1-fac.	2-factor		1-factor			2-factor							
	θ_1	θ_1	θ_2	θ_1	θ_1	θ_2	θ_1	θ_2	θ_1	θ_2	θ_3	θ_1	θ_2	θ_3
20071022	1.1448	1.1340	1.1405	1.0733	1.0626	1.0866	1.0698	1.0809	1.0655	1.0683	1.1062	1.0572	1.0920	1.0933
	(0.030)	(0.016)	(0.024)	(0.010)	(0.027)	(0.060)	(0.040)	(0.044)	(0.024)	(0.027)	(0.029)	(0.040)	(0.020)	(0.028)
20071025	1.1160	1.1108	1.1119	1.0638	1.0320	1.1018	1.0610	1.0691	1.0642	1.0646	1.0652	1.0121	1.1130	1.1271
	(0.042)	(0.020)	(0.035)	(0.005)	(0.026)	(0.061)	(0.043)	(0.049)	(0.026)	(0.029)	(0.029)	(0.040)	(0.022)	(0.028)
20071116	1.1523	1.1637	1.1645	1.0843	1.0008	1.2117	1.0860	1.0861	1.1381	1.1420	1.1530	1.0013	1.2040	1.2086
	(0.059)	(0.014)	(0.020)	(0.009)	(0.016)	(0.049)	(0.033)	(0.035)	(0.017)	(0.020)	(0.024)	(0.039)	(0.020)	(0.027)
20071205	1.1337	1.1339	1.1342	1.0776	1.0014	1.1807	1.0792	1.0801	1.1194	1.1523	1.1533	1.0048	1.1709	1.1721
	(0.034)	(0.016)	(0.025)	(0.010)	(0.020)	(0.040)	(0.036)	(0.038)	(0.017)	(0.021)	(0.027)	(0.040)	(0.020)	(0.027)
20080110	1.1877	1.1838	1.1887	1.1038	1.0009	1.2708	1.1074	1.1079	1.1160	1.3038	1.3100	1.0028	1.2633	1.2705
	(0.033)	(0.014)	(0.020)	(0.009)	(0.019)	(0.035)	(0.024)	(0.024)	(0.015)	(0.020)	(0.022)	(0.035)	(0.022)	(0.027)
20080227	1.6654	1.6429	1.6484	1.1210	1.0005	1.4440	1.1258	1.1268	1.0121	1.5348	1.5975	1.0015	1.4208	1.4324
	(0.116)	(0.049)	(0.052)	(0.040)	(0.012)	(0.065)	(0.015)	(0.016)	(0.014)	(0.090)	(0.114)	(0.034)	(0.033)	(0.035)
20080313	1.5704	1.5647	1.5684	1.2510	1.0009	1.4056	1.1607	1.1613	1.0019	1.5858	1.6209	1.0008	1.4373	1.4384
	(0.086)	(0.167)	(0.185)	(0.054)	(0.011)	(0.062)	(0.017)	(0.017)	(0.014)	(0.101)	(0.122)	(0.037)	(0.035)	(0.038)
20080404	1.1854	1.1842	1.1871	1.1062	1.0033	1.3160	1.1128	1.1130	1.0569	1.5244	1.5745	1.0021	1.2167	1.3065
	(0.042)	(0.020)	(0.022)	(0.020)	(0.017)	(0.055)	(0.021)	(0.021)	(0.015)	(0.058)	(0.075)	(0.038)	(0.024)	(0.031)
20080423	1.1749	1.1735	1.1798	1.1102	1.0020	1.2833	1.1164	1.1175	1.0445	1.4365	1.4380	1.0008	1.3096	1.3191
	(0.037)	(0.017)	(0.021)	(0.016)	(0.017)	(0.040)	(0.021)	(0.021)	(0.016)	(0.041)	(0.052)	(0.037)	(0.023)	(0.031)
20080529	1.1462	1.1478	1.1589	1.0851	1.0008	1.2117	1.0992	1.0999	1.0570	1.3370	1.3574	1.0004	1.2979	1.3228
	(0.022)	(0.017)	(0.022)	(0.008)	(0.017)	(0.042)	(0.027)	(0.028)	(0.014)	(0.025)	(0.030)	(0.038)	(0.022)	(0.030)
20080630	1.1843	1.1849	1.1904	1.0815	1.0033	1.3230	1.1036	1.1039	1.0463	1.5931	1.6571	1.0022	1.3127	1.3190
	(0.053)	(0.024)	(0.026)	(0.036)	(0.016)	(0.072)	(0.020)	(0.021)	(0.015)	(0.067)	(0.084)	(0.037)	(0.024)	(0.031)

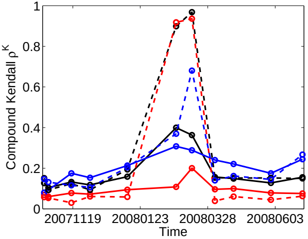
Table 1: Parameters and corresponding empirical standard deviations (in brackets).

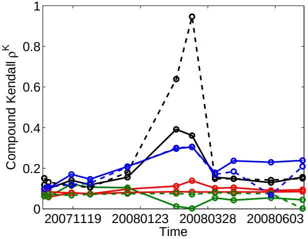
Calibrated Parameters for the Gaussian Models

Date	constant LGD			random LGD										
				bottom level	top level		bottom level		middle level			top level		
	1-fac.	2-factor		1-factor			2-factor							
	θ_1	θ_1	θ_2	θ_1	θ_1	θ_2	θ_1	θ_2	θ_1	θ_2	θ_3	θ_1	θ_2	θ_3
20071022	0.2355	0.2348	0.2396	0.0899	0.0028	0.2279	0.1324	0.1325	0.1164	0.1173	0.3649	0.0062	0.1528	0.1605
	(0.017)	(0.014)	(0.023)	(0.018)	(0.005)	(0.011)	(0.007)	(0.020)	(0.004)	(0.014)	(0.013)	(0.004)	(0.007)	(0.010)
20071025	0.1459	0.2048	0.2275	0.0839	0.0088	0.2045	0.1303	0.1312	0.1132	0.1142	0.3576	0.0093	0.1684	0.1787
	(0.032)	(0.010)	(0.013)	(0.011)	(0.005)	(0.011)	(0.007)	(0.018)	(0.004)	(0.013)	(0.012)	(0.003)	(0.007)	(0.009)
20071116	0.1930	0.1794	0.2514	0.0465	0.0088	0.1808	0.1220	0.1231	0.1043	0.1061	0.3779	0.0101	0.1860	0.2755
	(0.047)	(0.004)	(0.008)	(0.022)	(0.008)	(0.012)	(0.008)	(0.025)	(0.004)	(0.016)	(0.015)	(0.004)	(0.006)	(0.010)
20071205	0.1459	0.1635	0.2246	0.0952	0.0023	0.1646	0.1157	0.1211	0.1122	0.1131	0.3336	0.0039	0.1972	0.2262
	(0.042)	(0.004)	(0.009)	(0.020)	(0.008)	(0.011)	(0.007)	(0.025)	(0.004)	(0.016)	(0.015)	(0.003)	(0.006)	(0.010)
20080110	0.2997	0.2785	0.4015	0.0916	0.0010	0.3149	0.1251	0.1252	0.1168	0.1173	0.3410	0.0042	0.3176	0.3254
	(0.088)	(0.008)	(0.013)	(0.011)	(0.013)	(0.017)	(0.008)	(0.027)	(0.006)	(0.020)	(0.017)	(0.005)	(0.007)	(0.011)
20080227	0.9874	0.8433	0.9985	0.9917	0.0027	0.5484	0.1318	0.1318	0.1184	0.1185	0.3423	0.0010	0.4537	0.4965
	(0.069)	(0.005)	(0.010)	(0.032)	(0.014)	(0.025)	(0.013)	(0.028)	(0.009)	(0.030)	(0.026)	(0.005)	(0.007)	(0.010)
20080313	0.9988	0.9964	0.9997	0.9950	0.0139	0.8765	0.1298	0.1299	0.1116	0.1133	0.3517	0.0017	0.4609	0.5063
	(0.069)	(0.005)	(0.010)	(0.032)	(0.014)	(0.025)	(0.013)	(0.028)	(0.009)	(0.030)	(0.026)	(0.005)	(0.007)	(0.010)
20080404	0.2382	0.2299	0.2501	0.0607	0.0064	0.2177	0.1299	0.1300	0.1259	0.1269	0.3266	0.0031	0.2624	0.3502
	(0.029)	(0.005)	(0.009)	(0.024)	(0.010)	(0.013)	(0.011)	(0.027)	(0.007)	(0.029)	(0.023)	(0.005)	(0.007)	(0.011)
20080423	0.2361	0.2300	0.2416	0.0943	0.0035	0.2514	0.1298	0.1299	0.1193	0.1209	0.3408	0.0010	0.2845	0.3583
	(0.024)	(0.016)	(0.018)	(0.022)	(0.007)	(0.010)	(0.010)	(0.028)	(0.007)	(0.026)	(0.022)	(0.004)	(0.007)	(0.010)
20080529	0.2363	0.2207	0.3168	0.0698	0.0013	0.2254	0.1298	0.1299	0.1177	0.1198	0.3448	0.0097	0.1061	0.1587
	(0.032)	(0.007)	(0.012)	(0.020)	(0.007)	(0.009)	(0.010)	(0.027)	(0.007)	(0.023)	(0.020)	(0.004)	(0.006)	(0.009)
20080630	0.2326	0.2326	0.2453	0.0978	0.0051	0.4074	0.1330	0.1333	0.0019	0.0035	0.6979	0.0029	0.3222	0.3444
	(0.043)	(0.005)	(0.009)	(0.021)	(0.010)	(0.013)	(0.010)	(0.027)	(0.006)	(0.023)	(0.021)	(0.005)	(0.007)	(0.010)

Table 2: Parameters and corresponding empirical standard deviations (in brackets).

Implied Parameters Over Time





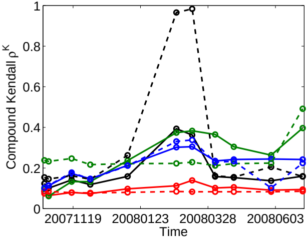


Figure 7: Gumbel (solid) and Gaussian (dashed) copula models. One-factor models (upper left), two-factor models with dependence between (upper right) and inside (lower) the sectors. **Constant LGD**, bottom-level random LGDs, middle-level random LGDs, top-level random LGD.

Boxplots of Parameters

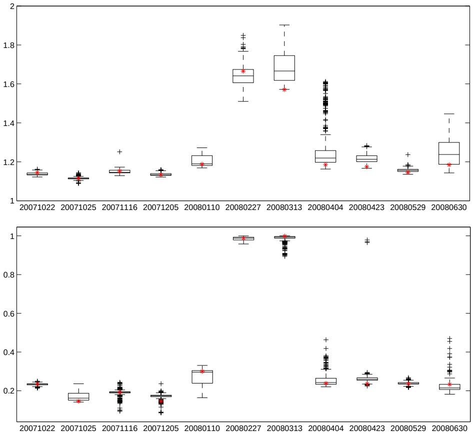


Figure 8: One-factor Gumbel (top) and Gaussian (bottom) copula model with the constant LGD with $N = 10^5$. The final values obtained with $N = 10^6$ simulations are marked with stars.

Implied Parameters Curves

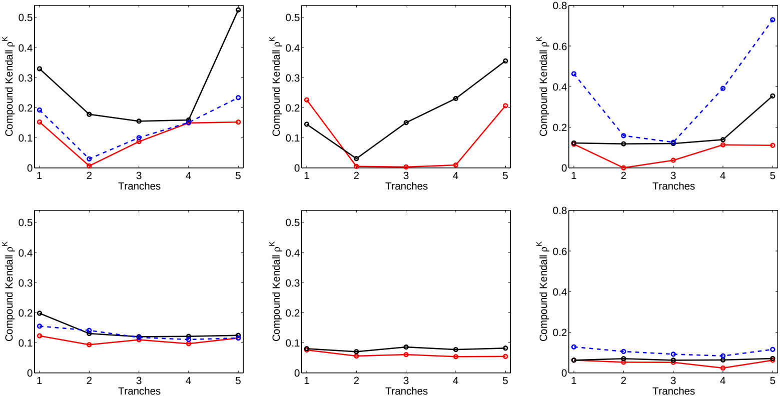


Figure 9: Gaussian (top) & Gumbel (bottom). Left: one and two-factor models with the constant LGD. Middle: one-factor models with the random top-level LGD. Right: two-factor models with the random middle-level LGDs. One-parameter models ρ_1^K, two and three-parameter models ρ_1^K, ρ_2^K, ρ_3^K.

Conclusions

- Introduced CDO pricing models that use HACs to describe the dependency structure between default times and LGDs.
- The models specify distinct dependencies within the industry sector and between them and incorporate either a common random LGD or sectorial random LGDs.
- Default intensities are calibrated to historical spreads of 125 CDS.
- The one-factor Gumbel copula model with the random top-level LGD and the two-factor Gumbel copula model with the random middle-level LGDs reveal the best fit to the market spreads.
- Gumbel copula models flatten the implied parameter curves.
- Calibration of HAC models requires enormous computing power.

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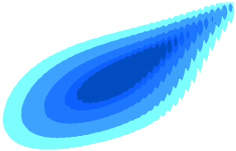

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