Valuation of Collateralized Debt Obligations with Hierarchical Archimedean Copulae

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Motivation

Collateralized Debt Obligation

Triggered the financial crisis.

CDO and HAC
**CDO Dynamics**

Figure 1: Spreads of iTraxx tranches, Series 8, maturity 5 years, data from 20070920-20081022. Left panel: mezzanine junior (dashed black), mezzanine (dashed red), senior (solid black), super senior (solid red). Right panel: upfront fee of the equity tranche.
**Dependence Matters!**

The normal world is not enough.

![Graph Illustrating Correlation Smile](image)

Figure 2: Implied correlation smile in the homogeneous large pool Gaussian copula model. Data from 20071022.
Research Goals

- How to model the smiles?
- How to catch the dependency?
- How to go beyond the normal world?
Outline

1. Motivation ✓
2. CDOs
3. HAC Models
4. Empirical Study
5. Conclusions
Risk Transfer

CDO and HAC
## Tranching

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<th>Attachment points (%)</th>
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*Table 1: iTraxx tranches.*
iTraxx Europe

- A static portfolio of 125 equally weighted CDS on European entities;
- Sectors: Consumer (30), Financial (25), TMT (20), Industrials (20), Energy (20), Auto (10);
- New series of iTraxx Europe issued every 6 months (March and September) and the underlying reference entities are reconstituted;
- Maturities: 3Y, 5Y, 7Y, 10Y.
Default

Consider a CDO with a maturity of $T$ years, $J$ tranches, and a pool of $d$ entities. Define a loss variable of $i$-th obligor until $t \in [t_0, T]$ as

$$l_i(t) = 1(\tau_i < t), \; i = 1, \ldots, d,$$

where $\tau_i$ is a time to default variable

$$F_i(t) = P(\tau_i \leq t)$$

$$= 1 - \exp \left\{ - \int_{t_0}^{t} \lambda_i(u) du \right\}$$

and $\lambda_i$ is a deterministic intensity function.
Portfolio Loss

The proportion of defaulted entities in the portfolio at time $t$ is given by

$$\tilde{L}(t) = \frac{1}{d} \sum_{i=1}^{d} l_i(t), \quad t \in [t_0, T].$$

The portfolio loss at time $t$ is defined as

$$L(t) = \text{LGD} \tilde{L}(t),$$

where LGD is a common loss given default.
Tranche Loss

The tranche loss at time $t$ is defined as

$$L_j(t) = \frac{1}{u_j - l_j} \{ L^u(t, u_j) - L^u(t, l_j) \},$$

where

$$L^u(t, x) = \min\{ L(t), x \} \text{ for } x \in [0, 1].$$

The outstanding notional of the tranche $j$ is given by

$$\Gamma_j(t) = \frac{1}{u_j - l_j} \{ \Gamma^u(t, u_j) - \Gamma^u(t, l_j) \},$$

where

$$\Gamma^u(t, x) = x - L^u(t, x) \text{ for } x \in [0, 1].$$
Valuation of CDO

1. Premium leg

\[ PL_j(t_0) = \sum_{t=t_1}^{T} \beta(t_0, t)s_j(t_0) \Delta t \mathbb{E}\{\Gamma_j(t)\} \]

2. Default leg

\[ DL_j(t_0) = \sum_{t=t_1}^{T} \beta(t_0, t) \mathbb{E}\{L_j(t) - L_j(t - \Delta t)\} \]

This leads to:

\[ s_j(t_0) = \frac{\sum_{t=t_1}^{T} \beta(t_0, t) \mathbb{E}\{L_j(t) - L_j(t - \Delta t)\}}{\sum_{t=t_1}^{T} \beta(t_0, t) \Delta t \mathbb{E}\{\Gamma_j(t)\}}. \]
Equity Tranche

The equity tranche is quoted in two parts:

1. an upfront fee $\alpha$ payed at $t_0$,
2. a running spread of 500 BPs.

The premium leg is calculated as

$$ PL_1(t_0) = \alpha(t_0) + \sum_{t=t_1}^{T} \beta(t_0, t) \cdot 500 \cdot \Delta t \, E\{\Gamma_1(t)\}. $$

The upfront payment given in percent is equal

$$ \alpha(t_0) = 100 \sum_{t=t_0}^{T} (\beta(t, t_0) [E\{L_1(t) - L_1(t-\Delta t)\} - 0.05\Delta t \, E\{\Gamma_1(t)\}]). $$
Copula

For a distribution function $F$ with marginals $F_{X_1}, \ldots, F_{X_d}$. There exists a copula $C : [0, 1]^d \to [0, 1]$, such that

$$F(x_1, \ldots, x_d) = C\{F_{X_1}(x_1), \ldots, F_{X_d}(x_d)\}$$

for all $x_i \in \mathbb{R}$, $i = 1, \ldots, d$. 
Copula for CDOs

The vector of default times \((\tau_1, \ldots, \tau_d)^\top\) has a (risk-neutral) joint cdf

\[
F(t_1, \ldots, t_d) = P(\tau_1 \leq t_1, \ldots, \tau_d \leq t_d) \quad \text{for all} \quad (t_1, \ldots, t_d)^\top \in \mathbb{R}_+^d,
\]

where \(\tau_i \sim F_i\). From the Sklar theorem, there exists a copula such that

\[
F(t_1, \ldots, t_d) = C\{F_1(t_1), \ldots, F_d(t_d)\}
\]

and determines the default dependency of the credits.
Monte Carlo Simulation Approach

Define a trigger variable as

\[ U_i = \bar{p}_i(\tau_i) \sim U[0, 1], \quad i = 1, \ldots, d. \]

The \textit{i}th obligor survives until \( t < T \) if and only if

\[ \tau_i \geq t \quad \text{or} \quad U_i \leq \bar{p}_i(t). \]

The joint and marginal distributions of the triggers satisfy:

\[ C\{\bar{p}_1(t), \ldots, \bar{p}_d(t)\} = P\{U_1 \leq \bar{p}_1(t), \ldots, U_d \leq \bar{p}_d(t)\}, \]

\[ P\{U_i \leq \bar{p}_i(t)\} = \bar{p}_i(t). \]
Monte Carlo Simulation Approach

The time to default variable

$$\tau_i = \inf \{ t \geq t_0 : \bar{p}_i(t) \leq U_i \}$$

is calculated as

$$\tau_i = \bar{p}_i^{-1}(U_i).$$

Assuming constant intensities compute

$$\tau_i = -(\log U_i)/\lambda_i.$$
Industry Sectors

iTraxx pool industry sectors: Consumer (30), Financial (25), TMT (20), Industrials (20), Energy (20), Auto (10).

A cluster analysis applied to the daily log-returns of the CDS spreads (20071022-20081128) returns six clusters: 38, 30, 17, 16, 14, 10.

The ratio between correctly assigned CDS to their industries and the total number of entities is 0.704.

**Goal:** Construct a joint distribution of the default times that imposes different intra- and inter-industry dependencies.
Hierarchical Archimedean Copulae

- A Hierarchical Archimedean Copula (HAC) is a generalisation of a multivariate Archimedean copula.
- HACs join bivariate or higher-dimensional Archimedean copulae by another Archimedean copula which allows for non-exchangeable dependency structures.
- HACs are useful when the composition of data is known and where each level of the hierarchy has a natural interpretation.
HAC Model

Partially nested hierarchical Archimedean copula

\[
C(u_1, \ldots, u_d; \theta) = C_1\{ C_2(u_1, \ldots, u_{m_1}; \theta_2), C_2(u_{m_1+1}, \ldots, u_{m_1+m_2}; \theta_2), \\
\quad \ldots, C_2(u_{m_1+\ldots+m_5+1}, \ldots, u_d; \theta_2); \theta_1\},
\]

where \( \theta = (\theta_1, \theta_2)^\top \) and \( m_k, k = 1, \ldots, 6 \), indicates a number of the companies in \( k \)th industry sector.

- \( C_2 \) models the dependency in the industry sector,
- \( C_1 \) models the dependency between the industry sectors.
Construct HACs composed of Gumbel copulae.

Figure 3: Partially nested 125-dimensional HAC.
Gaussian Model with Industry Sectors

\[
\Sigma = \begin{pmatrix}
1 & \ldots & \rho_2 \\
\vdots & \ddots & \vdots \\
\rho_2 & \ldots & 1 \\
\rho_1 & \ldots & \rho_1 \\
\vdots & \ddots & \vdots \\
\vdots & \ddots & \vdots \\
\rho_1 & \ldots & \rho_1 \\
\end{pmatrix}
\]
Loss Given Default

- Spreads are calculated on the fraction of the losses that cannot be recovered.

- Altman et al. (2005) and Hamilton et al. (2004) show that the LGD is stochastic and positively correlated with the default probabilities.

- Krekel (2008), Bennani & Maetz (2009) and other studies show that a stochastic LGD fattens the right tail of the portfolio loss distribution.
LGD Modelling

Consider one and two-factor models with random

1. Common LGD,

2. Sectorial LGDs.

Assume that LGD $\sim U[0, 1]$. 

One-factor Model with a Random Bottom-Level LGD

\[ C(u_1, \ldots, u_d, u_{d+1}; \theta_1) = C_1(u_1, \ldots, u_d, u_{d+1}; \theta_1), \]

where \( u_{d+1} \) is the common LGD variable.
One-factor Model with a Random Top-Level LGD

\[ C(u_1, \ldots, u_d, u_{d+1}; \theta) = C_1\{u_{d+1}, C_2(u_1, \ldots, u_d; \theta_2); \theta_1\}, \]

where \( \theta = (\theta_1, \theta_2)^T \) and \( u_{d+1} \) is the common LGD variable.
Two-factor Model with Random Bottom-Level LGDs

\[ C(u_1, \ldots, u_{d+6}; \theta) = C_1 \{ \begin{array}{c} C_2(u_1, \ldots, u_{m_1}, u_{d+1}; \theta_2), \\ C_2(u_{m_1+1}, \ldots, u_{m_1+m_2}, u_{d+2}; \theta_2), \ldots, \\ C_2(u_{m_1+\ldots+m_5+1}, \ldots, u_d, u_{d+6}; \theta_2) \end{array} \}, \]

where \( \theta = (\theta_1, \theta_2)^T \) and \( u_{d+1}, \ldots, u_{d+6} \) are sectorial LGD variables.
Two-factor Model with Random Middle-Level LGDs

\[
C(u_1, \ldots, u_{d+6}; \theta) = C_1 \left[ \begin{array}{c}
C_2 \{ C_3(u_1, \ldots, u_m; \theta_3), u_d; \theta_2 \}, \\
C_2 \{ C_3(u_{m+1}, \ldots, u_{m+m_2}; \theta_3), u_{d+1}; \theta_2 \}, \\
\vdots \\
C_2 \{ C_3(u_{m+\ldots+m_5+1}, \ldots, u_d; \theta_3), u_{d+6}; \theta_2 \}\end{array} \right],
\]

where \( \theta = (\theta_1, \theta_2, \theta_3)^\top \) and \( u_{d+1}, \ldots, u_{d+6} \) are sectorial LGDs.
Two-factor Model with a Random Top-Level LGD

\[ C(u_1, \ldots, u_d, u_{d+1}; \theta) = C_1[u_{d+1}, C_2\{C_3(u_1, \ldots, u_{m_1}; \theta_3), C_3(u_{m_1+1}, \ldots, u_{m_1+m_2}; \theta_3), \ldots, C_3(u_{m_1+\ldots+m_5+1}, \ldots, u_d; \theta_3); \theta_2\}; \theta_1] \]

where \( \theta = (\theta_1, \theta_2, \theta_3)^T \) and \( u_{d+1} \) is the common LGD variable.
Data

Figure 4: Spreads of iTraxx tranches, Series 8, maturity 5 years, data from 20070920-20081022. Left panel: mezzanine junior (dashed black), mezzanine (dashed red), senior (solid black), super senior (solid red). Right panel: upfront fee of the equity tranche.

CDO and HAC
Calibration

Look for the parameters $\theta$ of Gaussian and HAC copulae that provide the best fit to market data.

Minimize relative deviations from the market spreads $s_j^m$

$$D(t_0) \overset{\text{def}}{=} \min_{\theta} \sum_{j=1}^{J} \frac{|s_j^c(t_0; \theta) - s_j^m(t_0; \theta)|}{s_j^m(t_0; \theta)},$$

(1)

where $s_j^c$ is a calculated spread of the tranche $j = 1, \ldots, 5$.

$s_j^c$ computed from generated $N = 10^6$ times Monte Carlo vectors. Constant LGD = 0.6.
Calibration of Gaussian Models with the Constant LGD

Figure 5: Measure $D$ as a function of the correlation $\rho$ for the number of simulations $N$: $10^3$ (thin black), $10^4$ (thick red), $10^5$ (thick black) (left) and as a function of $\rho_1$ and $\rho_2$ for $N = 10^4$ (right). Data from 20071022.
Intensity Parameters

Assume constant but heterogeneous intensities $\lambda_i$, $i = 1, \ldots, 125$. Calibrate $\lambda_i$ from the historical spreads of 125 CDS.

Figure 6: Default times of Kingfisher and Deutsche Bank based on the samples generated from Gumbel copula with the constant LGD and the calibrated parameters. Data from 20071022.
Measure $D$ for the Gumbel Copula Models

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CDO and HAC
Measure $D$ for the Gaussian Copula Models

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CDO and HAC
## Calibrated Parameters for the Gumbel Models

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Table 1: Parameters and corresponding empirical standard deviations (in brackets).
### Calibrated Parameters for the Gaussian Models

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Table 2: Parameters and corresponding empirical standard deviations (in brackets).

CDO and HAC
**Implied Parameters Over Time**

Figure 7: Gumbel (solid) and Gaussian (dashed) copula models. One-factor models (upper left), two-factor models with dependence between (upper right) and inside (lower) the sectors. **Constant LGD**, bottom-level random LGDs, middle-level random LGDs, top-level random LGD.
Boxplots of Parameters

Figure 8: One-factor Gumbel (top) and Gaussian (bottom) copula model with the constant LGD with $N = 10^5$. The final values obtained with $N = 10^6$ simulations are marked with stars.
Implied Parameters Curves

Figure 9: Gaussian (top) & Gumbel (bottom). Left: one and two-factor models with the constant LGD. Middle: one-factor models with the random top-level LGD. Right: two-factor models with the random middle-level LGDs. One-parameter models $\rho^K_1$, two and three-parameter models $\rho^K_1, \rho^K_2, \rho^K_3$. CDO and HAC
Conclusions

- Introduced CDO pricing models that use HACs to describe the dependency structure between default times and LGDs.
- The models specify distinct dependencies within the industry sector and between them and incorporate either a common random LGD or sectorial random LGDs.
- Default intensities are calibrated to historical spreads of 125 CDS.
- The one-factor Gumbel copula model with the random top-level LGD and the two-factor Gumbel copula model with the random middle-level LGDs reveal the best fit to the market spreads.
- Gumbel copula models flatten the implied parameter curves.
- Calibration of HAC models requires enormous computing power.
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