CDO Surfaces Dynamics

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iTraxx over Time

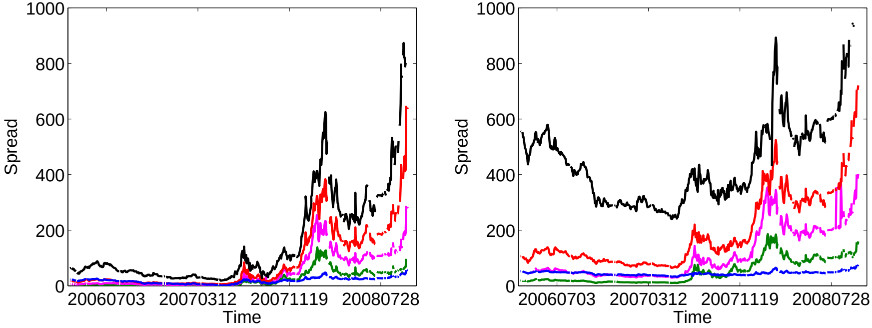


Figure 1: Spreads of iTraxx tranches, Series 5, maturity 5 (left) and 10 (right) years, data from 20060407-20081103. Tranches: 1, **2**, 3, 4, 5.

iTraxx Spread Surface

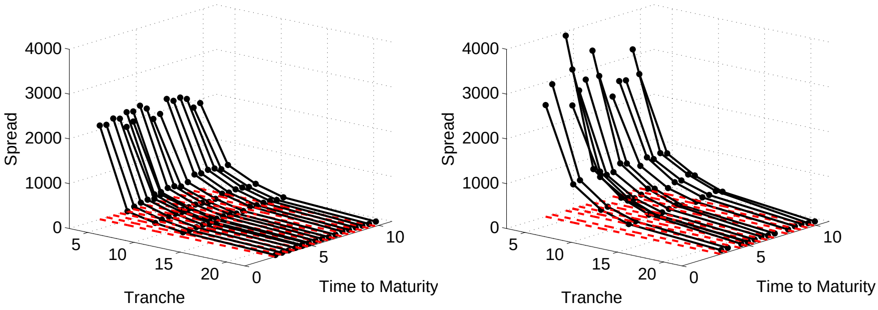


Figure 2: Spreads of tranches of all series observed on 20080909 (left) and 20090119 (right).

Research Goals

- Modelling the dynamics of CDO surfaces
  - spread surfaces
  - base correlation surfaces
- Applications in trading

Dynamic Semiparametric Factor Model

Applications:

1. Implied volatility surfaces in M. R. Fengler, W. Härdle and E. Mammen, *JFE* (2007) and B. Park, E. Mammen, W. Härdle, and S. Borak, *JASA* (2009)
2. Risk neutral densities in E. Giacomini, W. Härdle, and V. Krätschmer, *AStA* (2009)
3. Limit order book in W. Härdle, N., Hautsch, and A. Mihoci, *JEF* (2012)
4. Variance swaps in K. Detlefsen and W. Härdle, *QF* (2013)
5. fMRI images in A. Myšicková, S. Song, P. Majer, P. Mohr, H. Heekeren, W. Härdle, *Psychometrika* (2013)

Outline

1. Motivation ✓
2. CDOs
3. DSFM
4. Empirical Study
5. Applications
6. Conclusions

Risk Transfer

iTraxx Europe

- A static portfolio of 125 equally weighted CDS on European entities;
- Sectors: Consumer (30), Financial (25), TMT (20), Industrials (20), Energy (20), Auto (10);
- New series of iTraxx Europe issued every 6 months (March and September) and the underlying reference entities are reconstituted;
- Maturities: 3Y, 5Y, 7Y, 10Y.

▸ Valuation

Large Pool Gaussian Copula Model

Default times are modelled from the Gaussian vector $(X_1, \dots, X_d)^\top$:

$$X_i = \sqrt{\rho} Y + \sqrt{1-\rho} Z_i,$$

where Y (systematic risk factor), $\{Z_i\}_{i=1}^d$ (idiosyncratic risk factors) are i.i.d. $N(0,1)$. Assume that:

- obligors have the same default probability p and LGD,
- one dependence parameter ρ,
- d is large.

The cdf of the portfolio loss equals

$$\mathrm{P}(\tilde{L} \le x) = \Phi \left\{ \frac{\sqrt{1-\rho}\Phi^{-1}(x) - \Phi^{-1}(p)}{\sqrt{\rho}} \right\}.$$

Correlation's Types

Compound correlation $\rho(l_j, u_j)$, $j = 1, \ldots, J$.

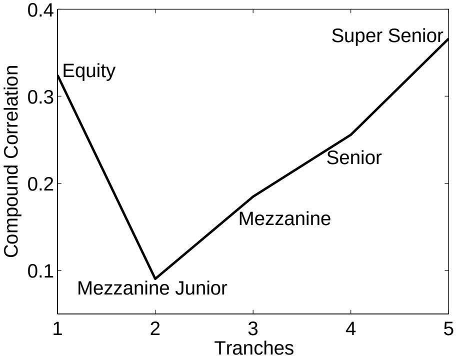


Figure 3: Implied correlation smile in the Gaussian one factor model, 20071022.

Correlation's Types

Base correlation (BC) $\rho(0, u_j)$, $j = 1, \ldots, J$.

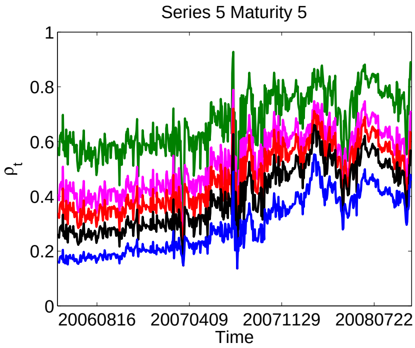
Represent the expected loss $\mathrm{E}\{L_{(l_j,u_j)}\}$ as a difference:

$$\mathrm{E}\{L_{(l_j,u_j)}\} = \mathrm{E}_{\rho(0,u_j)}\{L_{(0,u_j)}\} - \mathrm{E}_{\rho(0,l_j)}\{L_{(0,l_j)}\},\ j = 2, \ldots, J.$$

of the expected losses of two fictive tranches $(0, u_j)$ and $(0, l_j)$.
Bootstrapping process: $\mathrm{E}\{L_{(0,3\%)}\}$ is traded on the market,

$$\begin{aligned} \mathrm{E}\{L_{(3\%,6\%)}\} &= \mathrm{E}_{\rho(0,6\%)}\{L_{(0,6\%)}\} - \mathrm{E}_{\rho(0,3\%)}\{L_{(0,3\%)}\}, \\ \mathrm{E}\{L_{(6\%,9\%)}\} &= \mathrm{E}_{\rho(0,9\%)}\{L_{(0,9\%)}\} - \mathrm{E}_{\rho(0,6\%)}\{L_{(0,6\%)}\}, \ldots \end{aligned}$$

Base Correlations over Time



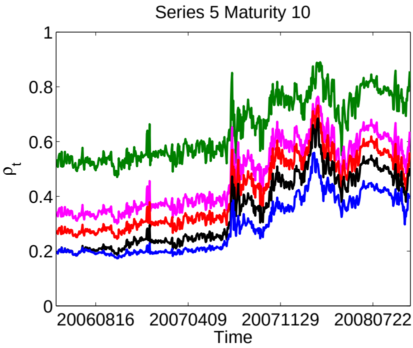


Figure 4: BC of iTraxx tranches, Series 5, maturity 5 (left) and 10 (right) years, data from 20060510-20081023. Tranches: 1, **2**, 3, 4, 5.

Base Correlation Surfaces

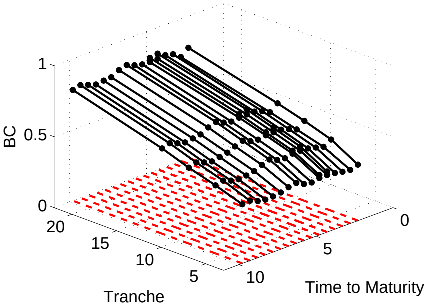


Figure 5: Implied base correlations on day 20080909 (left) and 20090119 (right).

Dynamic Semiparametric Factor Model

$$Y_{t,k} = m_0(X_{t,k}) + \sum_{l=1}^{L} Z_{t,l} m_l(X_{t,k}) + \varepsilon_{t,k} = Z_t^\top A\psi(X_{t,k}) + \varepsilon_{t,k}$$

$Y_{t,k}$	log-spreads and Z-transformed BC on day t, $t = 1, \ldots, T$
k	intra-day numbering of BCs on day t, $k = 1, \ldots, K_t$
$X_{t,k}$	two-dimensional vector of the tranche seniority and the time-to-maturity
m_l	factor functions, time invariant, nonparametric estimation
$Z_{t,l}$	time series, $l = 0, \ldots, L$, dynamic behavior
$\psi(X_{t,k})$	tensor B-spline basis
A	coefficient matrix

Estimation

Using an iterative algorithm:

$$(\widehat{Z}_t, \widehat{A}) = \arg\min_{Z_t, A} \sum_{t=1}^{T} \sum_{k=1}^{K_t} \left\{ Y_{t,k} - Z_t^\top A \psi(X_{t,k}) \right\}^2$$

Selection of L, the numbers of spline knots R_1, R_2 and the orders of splines k_1, k_2 by maximising the explained variance criterion:

$$\mathrm{EV}(L, R_1, r_1, R_2, r_2) = 1 - \frac{\sum_{t=1}^{T} \sum_{k=1}^{K_t} \left\{ Y_{t,k} - \sum_{l=1}^{L} Z_{t,l} m_l(X_{t,k}) \right\}^2}{\sum_{t=1}^{T} \sum_{k=1}^{K_t} \{ Y_{t,j} - \widetilde{m}_0(X_{t,k}) \}^2},$$

where $\widetilde{m}_0$ is an empirical mean surface.

DSFM without the Mean Factor

Reduce the number of factors estimated in the iterative algorithm by first subtracting the empirical mean $\widetilde{m}_0$ and then fitting the DSFM:

$$Y_{t,k} = \widetilde{m}_0(X_{t,k}) + \sum_{l=1}^{L} Z_{t,l} m_l(X_{t,k}) + \varepsilon_{t,k} = \widetilde{m}_0(X_{t,k}) + Z_t^\top A\psi(X_{t,k}) + \varepsilon_{t,k},$$

where m_l are new factor functions, $l = 1, \ldots, L$.

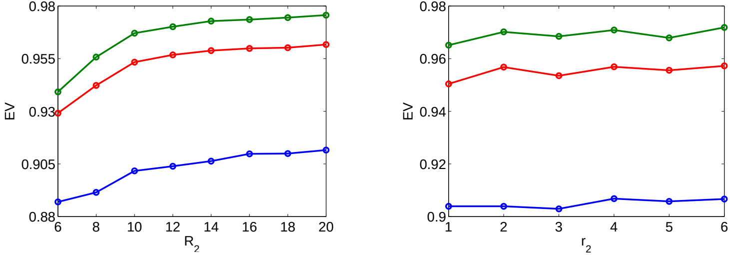
Data

- Series 2-10
- Maturities 5, 7, 10Y
- 1004 days between 20050330-20090202
- 49 502 data points

Year	3Y	5Y	7Y	10Y
2005	0	1478	715	1532
2006	181	3998	3739	4005
2007	75	5155	5170	5172
2008	232	5904	5916	5932
2009	0	260	263	263
All	488	16740	15803	16840

Table 1: Number of observed values of iTraxx tranches in the period 20050330-20090202.

DSFM for Z-transformed-BC



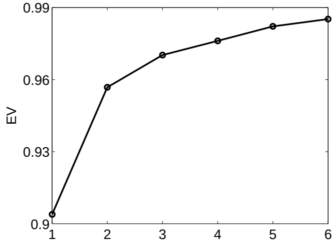


Figure 6: Proportion of the explained variance as a function of R_2 (up left) with $r_2 = 2$, as a function of r_2 (up right) with $R_2 = 10$, as a function of L (down) for $L = 1$, $L = 2$, $L = 3$, $r_1 = 2$ and $R_1 = 5$.

DSFM w/o Mean F. for Z-transformed-BC

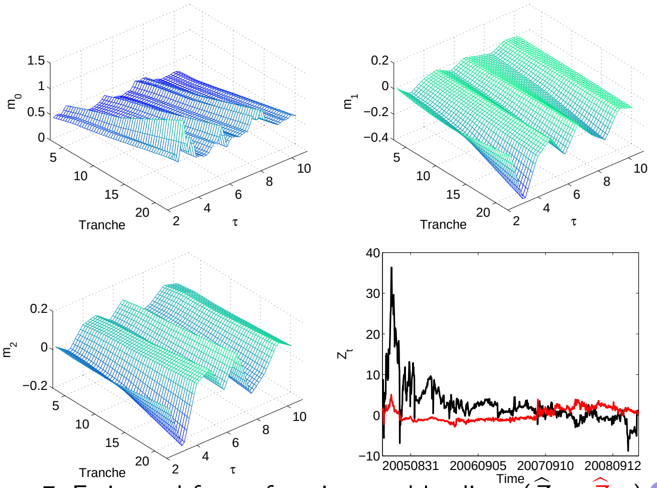


Figure 7: Estimated factor functions and loadings $(\widehat{Z}_{t,1}, \widehat{Z}_{t,2})$. App

DSFM Estimation Results

For DSFM for both data types

- $\widehat{Z}_{t,1}$ is a slope-curvature factor
- $\widehat{Z}_{t,2}$ is a shift factor

Model	Log-Spr	Z-BC
DSFM	0.016	0.004
DSFM w/o mean f.	0.045	0.006

Table 2: Mean squared error of the in-sample fit.

DSFM without the mean factor Fit

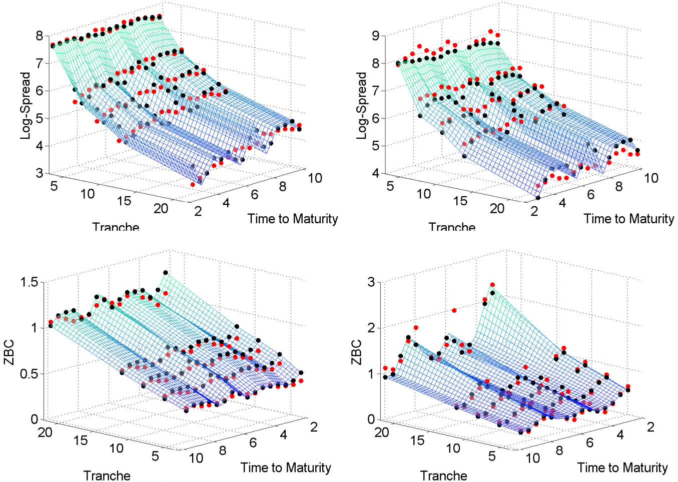


Figure 8: In-sample **fit** of the models to data on 20080909 and 20090119.

Curve Trades

So, how can I make money with this?

Combine tranches of different time to maturity, see Felsenheimer et al. (2004) and Kakodkar et al. (2006):

- Flattener – sell protection on a long-term tranche, buy protection on a short-term tranche
  Example: sell protection on 10Y 3-6% and buy on 5Y 6-9%
  Outlook: bullish long-term, bearish short-term
- Steepener – opposite trade

Curve Trades

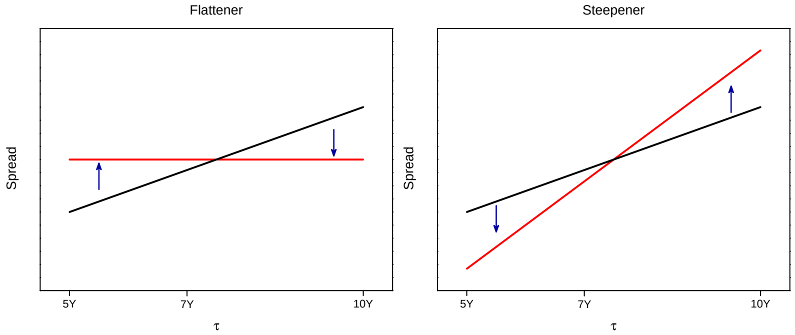
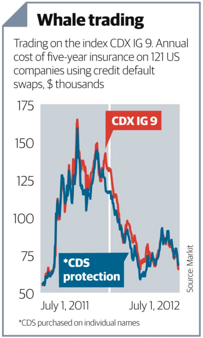


Figure 9: Mechanism of a flattener and a steepener startegy. **Current spread curve**, expectation of the future spread curve, indication of the direction of change.

JP Morgan Trading Loss, May 2012

J.P. Morgan's flattener – bought 5Y CDX IG 9 index, sold 10Y CDX IG 9 index in a 3:1 ratio. The final loss reached $6.2 billion.



Flattener

Sell protection at $s_1(t_0)$ for the period $[t_0, T_1]$ and buy protection at $s_2(t_0)$ for $[t_0, T_2]$, $T_1 > T_2$. At t_0 for $\ell = 1, 2$:

$$\mathrm{MTM}_\ell(t_0) = \sum_{t=t_1}^{T_\ell} \beta(t_0, t)\,[s_\ell(t_0)\Delta t \mathrm{E}\{F_\ell(t)\} - \mathrm{E}\{L_\ell(t) - L_\ell(t-\Delta t)\}] = 0.$$

At $\tilde{t} > t_0$, the market quotes $s_\ell(\tilde{t})$ and

$$\mathrm{MTM}_\ell(\tilde{t}) = \{s_\ell(t_0) - s_\ell(\tilde{t})\} \sum_{t=\tilde{t}_1}^{T_\ell} \beta(\tilde{t}, t)\Delta t \mathrm{E}\{F_\ell(t)\}.$$

Curve Trade

- A positive MTM means a positive value to the protection seller.
- If the protection seller closes the position at time $\tilde{t}$, then receives from the protection buyer $\mathrm{MTM}_\ell(\tilde{t})$.
- Flattener-trader aims to maximize the total MTM value

$$\mathrm{PL}(\tilde{t}) = \mathrm{MTM}_1(\tilde{t}) - \mathrm{MTM}_2(\tilde{t}).$$

Risk in Curve Trades

- If one buys 5Y 6-9% and sells 10Y 6-9%, then the trade is hedged for default until the maturity of the 5Y tranche. Defaults that emerge from 10Y 6-9% are covered by 5Y 6-9% till it expires.
- Series differ in the composition of the collateral.
- If one buys 5Y 6-9% and sells 10Y 3-6%, then these tranches provide protection of different portion of portfolio risk. If there is any default in 10Y 3-6%, then we must deliver a payment obligation and incur a loss.

Empirical Study

Idea

- Use DSFM to forecast spread and BC surfaces
- Calculate forecasted MTM surfaces
- Recover those tranches that maximise P&L

Remarks

- Because of many missing data and short data histories, the standard econometric methods cannot be used for the forecasting.
- Consider trades that generate no or a positive carry – the spread of the long tranche doesn't exceed the spread of the short tranche.
- Do not account for default payments (no data of historical defaults in iTraxx), do not account for the positive carry.

Forecasting with DSFM in Rolling Windows

Let Y_t be log-spreads or Z-transformed-BC.

- Consider a rolling window of $w = 250$.
- Estimate the DSFMs using $\{Y_\nu\}_{\nu=t-w+1}^{t}$ for $t = w, \ldots, T-h$.
- As a result, we get $T - w + 1$ times $\widehat{m} = (\widehat{m}_0, \ldots, \widehat{m}_L)^\top$ and $\widehat{Z}_t = (\widehat{Z}_{t,0}, \ldots, \widehat{Z}_{t,L})^\top$ of length w.
- Compute h-day forecast of the factor loadings using VAR.
- Due to the fixed issuing scheme, $X_{t+h,k}$ is not forecasted.
- Calculate the forecast $\widehat{Y}_{t+h}$ from the forecast $\widehat{Z}_{t+h}$.
- Transform $\widehat{Y}_{t+h}$ suitably to get $\hat{s}(t+h)$ or $\hat{\rho}(t+h)$.

Forecasting MTM Surfaces

For predicted $\{\hat{s}_k(t), \hat{\rho}_k(t)\}$, $t = w + h, \ldots, T$, $k = 1, \ldots, K_t$, compute $\widehat{\mathrm{MTM}}_k(t)$, where the initial spread $s_k(t_0)$ is observed on $t_0 = t - h$.

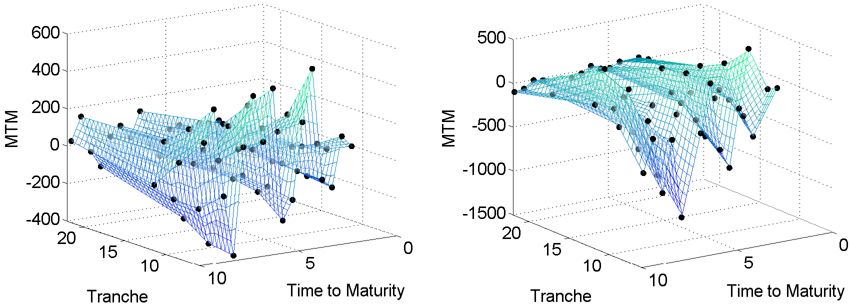


Figure 10: MTM surfaces on 20080909 (left) and 20090119 (right) calculated using one-day spread and BC predictions obtained with the DSFM.

Transaction Costs

Calculate the ask (bid) spread by increasing (reducing) the observed spread by the following percentage:

Maturity	1	2	3	4	5
5Y	1.88	1.78	2.52	3.77	6.28
7Y	1.49	1.65	2.31	2.97	4.87
10Y	1.41	1.66	1.83	2.52	4.09

Table 3: Average bid-ask spread excess over the mid spread as a percentage of the mid spread for Series 8 during the period 20070920-20090202.

Trading Strategies

Construct a curve trade

1. Fit and forecast the DSFM models to spreads and BC.
2. Calculate *h*-day forecasts of the MTM surfaces.
3. Recover which two tranches optimize a given strategy.

Strategies – restrict the choice to a flattener (or a steepener) with

1. a fixed tranche and fixed maturities,
2. a fixed tranche and all maturities,
3. all tranches and fixed maturities,
4. all tranches and all maturities (no restrictions),

or allow to combine flatteners and steepeners.

Backtesting

- Consider the time horizons $h = 1$, 5, 20 days.
- For the tranches that optimize a given strategy, check the corresponding historical market spreads, calculate the resulting MTM values, and the realised P&L.

Mean of Daily Gains in Percent

Strategy	DSFM						DSFM without the mean factor					
	1 day		1 week		1 month		1 day		1 week		1 month	
	LZ	Z	LZ	Z	LZ	Z	LZ	Z	LZ	Z	LZ	Z
FS-AllT-AllM	0.29	0.35	0.10	0.13	0.05	0.04	0.30	0.30	0.11	0.13	0.04	0.03
FS-T2-AllM	0.29	0.33	0.13	0.14	0.06	0.05	0.33	0.28	0.12	0.13	0.05	0.04
FS-T3-AllM	0.19	0.22	0.07	0.08	0.03	0.02	0.18	0.23	0.07	0.07	0.02	0.02
FS-T4-AllM	0.14	0.17	0.04	0.05	0.02	0.01	0.12	0.18	0.04	0.05	0.01	0.01
FS-T5-AllM	0.09	0.11	0.04	0.04	0.02	0.01	0.08	0.11	0.03	0.04	0.01	0.01
F-T2-AllM	0.30	0.34	0.12	0.12	0.06	0.04	0.28	0.32	0.12	0.11	0.05	0.04
F-T3-AllM	0.16	0.20	0.06	0.07	0.02	0.01	0.16	0.20	0.06	0.07	0.02	0.01
F-T4-AllM	0.10	0.15	0.03	0.04	0.01	0.01	0.10	0.15	0.03	0.04	0.01	0.01
F-T5-AllM	0.09	0.10	0.03	0.03	0.01	0.01	0.08	0.10	0.03	0.03	0.01	0.01
S-T2-AllM	0.39	0.43	0.15	0.17	0.07	0.06	0.45	0.46	0.13	0.16	0.05	0.06
S-T3-AllM	0.27	0.31	0.09	0.10	0.04	0.03	0.30	0.35	0.09	0.09	0.02	0.02
S-T4-AllM	0.20	0.25	0.06	0.07	0.03	0.02	0.20	0.24	0.05	0.06	0.01	0.01
S-T5-AllM	0.12	0.15	0.04	0.04	0.02	0.02	0.12	0.16	0.04	0.04	0.01	0.02
F-AllT-105	0.20	0.21	0.07	0.09	0.03	0.02	0.19	0.21	0.06	0.08	0.02	0.02
F-AllT-107	0.22	0.26	0.07	0.08	0.03	0.03	0.25	0.25	0.08	0.08	0.03	0.03
F-AllT-75	0.15	0.15	0.04	0.06	0.01	-0.00	0.14	0.15	0.04	0.05	0.01	0.00
S-AllT-510	0.16	0.17	0.05	0.08	0.02	0.01	0.16	0.18	0.05	0.08	0.01	0.00
S-AllT-710	0.17	0.23	0.05	0.10	0.02	0.03	0.21	0.25	0.07	0.09	0.02	0.02
S-AllT-57	0.11	0.13	0.03	0.03	0.01	-0.01	0.12	0.13	0.03	0.03	0.00	-0.01

Table 4: Calculations based on predictions of log-spreads and Z-transformed BCs marked as LZ; based only on Z-transformed BCs marked as Z.

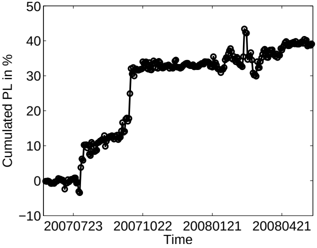
Investor's Strategy

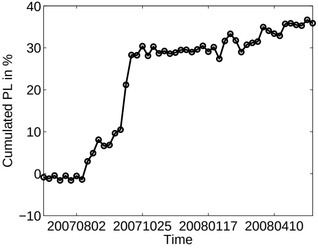
Follow a certain strategy over a year and constantly rebalance the portfolio. At t_0 enter an optimal (according to the DSFM) curve trade for h-day horizon. At $t_0 + h$ chose:

1. keep the current position for the next h-days,
2. close the current position and enter a new one.

Assume a margin of 10% of your notional. Every time the position is closed, add to the margin the realized P&L. If margin ≤ 0, quit the trade.

Investor's Strategy





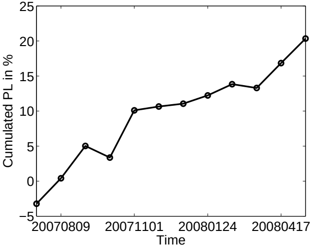


Figure 11: Daily cumulated P&L over one year 20070614–20080529. Rebalancing after: 1 day (upper left), 1 week (upper right), 1 month (lower). Calculations based on the DSFM predictions of log-spreads and Z-transformed BCs. ▸ App

Conclusions

- Investigated evolution over time of tranche spread surfaces and base correlation surfaces using the DSFM.
- Empirical study is conducted using an extensive data set of 49,502 observations of iTraxx Europe tranches in 2005-2009.
- Proposed a modification to the classic DSFM.
- Both DSFMs successfully reproduce the dynamics in data.
- Used DSFM in constructing the curve trades.
- Analysed the performance of 43 strategies that combine different positions, tranches, and maturities.
- Backtesting showed high daily gains of the resulting curve trades.

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Default

Consider a CDO with a maturity of T years, J tranches, and a pool of d entities. Define a loss variable of i-th obligor until $t \in [t_0, T]$ as

$$I_i(t) = \mathbf{1}(\tau_i < t),\ i = 1, \ldots, d,$$

where τ_i is a time to default variable

$$\begin{aligned} F_i(t) &= \mathrm{P}(\tau_i \leq t) \\ &= 1 - \exp\left\{ -\int_{t_0}^{t} \lambda_i(u) du \right\} \end{aligned}$$

and λ_i is a deterministic intensity function. ▸ Talk

Portfolio Loss

The proportion of defaulted entities in the portfolio at time t is given by

$$\tilde{L}(t) = \frac{1}{d} \sum_{i=1}^{d} I_i(t), \quad t \in [t_0, T].$$

The portfolio loss at time t is defined as

$$L(t) = \mathrm{LGD}\tilde{L}(t),$$

where LGD is a common loss given default.

Tranche Loss

The tranche loss at time t is defined as

$$L_j(t) = \frac{1}{u_j - l_j}\{L^u(t, u_j) - L^u(t, l_j)\},$$

where

$$L^u(t, x) = \min\{L(t), x\} \quad \text{for } x \in [0, 1].$$

The outstanding notional of the tranche j is given by

$$\Gamma_j(t) = \frac{1}{u_j - l_j}\{\Gamma^u(t, u_j) - \Gamma^u(t, l_j)\},$$

where

$$\Gamma^u(t, x) = x - L^u(t, x) \quad \text{for } x \in [0, 1].$$

Valuation of CDO

1. Premium leg

$$\mathrm{PL}_j(t_0) = \sum_{t=t_1}^{T} \beta(t_0, t) s_j(t_0) \Delta t \,\mathsf{E}\{\Gamma_j(t)\}$$

2. Default leg

$$\mathrm{DL}_j(t_0) = \sum_{t=t_1}^{T} \beta(t_0, t) \,\mathsf{E}\{L_j(t) - L_j(t - \Delta t)\}$$

This leads to:

$$s_j(t_0) = \frac{\sum_{t=t_1}^{T} \beta(t_0, t) \,\mathsf{E}\{L_j(t) - L_j(t - \Delta t)\}}{\sum_{t=t_1}^{T} \beta(t_0, t) \Delta t \,\mathsf{E}\{\Gamma_j(t)\}}.$$

Equity Tranche

The equity tranche is quoted in two parts:

1. an upfront fee α payed at t_0,
2. a running spread of 500 BPs.

The premium leg is calculated as

$$\mathrm{PL}_1(t_0) = \alpha(t_0) + \sum_{t=t_1}^{T} \beta(t_0, t) \cdot 500 \cdot \Delta t \, \mathsf{E}\{\Gamma_1(t)\}.$$

The upfront payment given in percent is equal

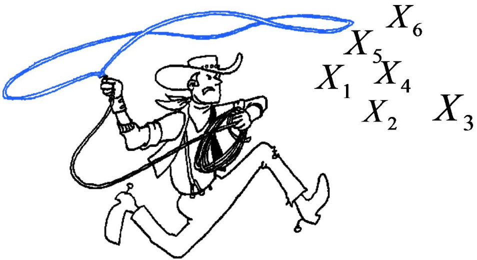
$$\alpha(t_0) = 100 \sum_{t=t_0}^{T} \left(\beta(t, t_0) \left[\mathsf{E}\{L_1(t) - L_1(t - \Delta t)\} - 0.05 \Delta t \, \mathsf{E}\{\Gamma_1(t)\}\right]\right).$$

Copula

For a distribution function F with marginals $F_{X_1} \ldots, F_{X_d}$. There exists a copula $C : [0,1]^d \to [0,1]$, such that

$$F(x_1, \ldots, x_d) = C\{F_{X_1}(x_1), \ldots, F_{X_d}(x_d)\}$$

for all $x_i \in \overline{\mathbb{R}}$, $i = 1, \ldots, d$.



Copula for CDOs

The vector of default times $(\tau_1, \ldots, \tau_d)^\top$ has a (risk-neutral) joint cdf

$$F(t_1, \ldots, t_d) = \mathrm{P}(\tau_1 \leq t_1, \ldots, \tau_d \leq t_d) \quad \text{for all} \quad (t_1, \ldots, t_d)^\top \in \mathbb{R}_+^d,$$

where $\tau_i \sim F_i$. From the Sklar theorem, there exists a copula such that

$$F(t_1, \ldots, t_d) = C\{F_1(t_1), \ldots, F_d(t_d)\}$$

and determines the default dependency of the credits.

Monte Carlo Simulation Approach

Define a trigger variable as

$$U_i = \bar{p}_i(\tau_i) \sim \mathrm{U}[0,1], \quad i = 1, \ldots, d.$$

The ith obligor survives until $t < T$ if and only if

$$\begin{aligned} \tau_i &\geq t \\ \text{or } U_i &\leq \bar{p}_i(t). \end{aligned}$$

The joint and marginal distributions of the triggers satisfy:

$$\begin{aligned} C\{\bar{p}_1(t), \ldots, \bar{p}_d(t)\} &= \mathrm{P}\{U_1 \leq \bar{p}_1(t), \ldots, U_d \leq \bar{p}_d(t)\}, \\ \mathrm{P}\{U_i \leq \bar{p}_i(t)\} &= \bar{p}_i(t). \end{aligned}$$

Monte Carlo Simulation Approach

The time to default variable

$$\tau_i = \inf\{t \geq t_0 : \bar{p}_i(t) \leq U_i\}$$

is calculated as

$$\tau_i = \bar{p}_i^{-1}(U_i).$$

Assuming constant intensities compute

$$\tau_i = -(\log U_i)/\lambda_i.$$

Large Pool Approach for Linear Factor Models

Default times are calculated from a vector $(X_1, \ldots, X_d)^\top$

$$X_i = \sqrt{\rho}Y + \sqrt{1-\rho}Z_i,$$

where Y (systematic risk factor), $\{Z_i\}_{i=1}^d$ (idiosyncratic risk factors) are i.i.d. Assume that

- obligors have the same default probability p and LGD,
- one dependence parameter ρ,
- d is large.

Talk

Large Pool Approximation

Computations are simplified significantly when the portfolio loss distribution is approximated:

$$\mathrm{P}(L \leq x) = 1 - F_Y \left\{ \frac{F_X^{-1}(p) - \sqrt{\rho} F_Z^{-1}(x)}{\sqrt{1-\rho}} \right\},$$

where $X_i \sim F_X$, $Z_i \sim F_Z$, $Y \sim F_Y$.

Gaussian Copula Model

The factors Y and $\{Z_i\}_{i=1}^d$ are i.i.d. $N(0,1)$. Thus, $X_i \sim N(0,1)$
The cdf of the portfolio loss equals

$$\mathrm{P}(\tilde{L} \leq x) = \Phi\left\{\frac{\sqrt{1-\rho}\Phi^{-1}(x) - \Phi^{-1}(p)}{\sqrt{\rho}}\right\}.$$

Default times are given by $\tau_i = F_i^{-1}\{\Phi(X_i)\}$.

NIG Model

Factors:

$$
\begin{aligned}
Y &\sim \text{NIG}\left(\alpha, \beta, -\frac{\beta\gamma^2}{\alpha^2}, \frac{\gamma^3}{\alpha^2}\right), \gamma = \sqrt{\alpha^2 - \beta^2},\\
Z_i &\sim \text{NIG}\left(\frac{\sqrt{1-\rho}}{\sqrt{\rho}}\alpha, \frac{\sqrt{1-\rho}}{\sqrt{\rho}}\beta, -\frac{\sqrt{1-\rho}}{\sqrt{\rho}}\frac{\beta\gamma^2}{\alpha^2}, \frac{\sqrt{1-\rho}}{\sqrt{\rho}}\frac{\gamma^3}{\alpha^2}\right).
\end{aligned}
$$

Because of the stability under convolution

$$
X_i \sim \text{NIG}\left(\frac{\alpha}{\sqrt{\rho}}, \frac{\beta}{\sqrt{\rho}}, -\frac{1}{\sqrt{\rho}}\frac{\beta\gamma^2}{\alpha^2}, \frac{1}{\sqrt{\rho}}\frac{\gamma^3}{\alpha^2}\right) = \text{NIG}_{(1/\sqrt{\rho})}.
$$

Default times are given by $\tau_i = F_i^{-1}\{\text{NIG}_{(1/\sqrt{\rho})}(X_i)\}$.

Double-t Model

Define

$$X_i = \sqrt{\rho}\sqrt{\frac{\nu_Y - 2}{\nu_Y}}Y + \sqrt{1-\rho}\sqrt{\frac{\nu_Z - 2}{\nu_Z}}Z_i,\ i = 1, \ldots, d,$$

where Y and Z_i are t distributed with ν_Y and ν_Z DoF respectively.

The t distribution is not stable under convolution: X_i are not t distributed and the copula is not a t copula, $X_i \sim F_X$ has to be computed numerically. Default times are computed as

$$\tau_i = F_i^{-1}\{F_X(X_i)\}.$$

Large Pool Approach for Archimedean Copulae

d-dimensional Archimedean copula $C : [0,1]^d \rightarrow [0,1]$ is

$$C(u_1, \ldots, u_d) = \phi\{\phi^{-1}(u_1) + \cdots + \phi^{-1}(u_d)\}, \quad u_1, \ldots, u_d \in [0,1],$$

where $\phi \in \{\phi : [0;\infty) \rightarrow [0,1] \,|\, \phi(0) = 1,\, \phi(\infty) = 0;\, (-1)^j \phi^{(j)} \geq 0;\, j = 1, \ldots, \infty\}$ is a copula generator.

Each ϕ is a Laplace transform of a cdf of a positive random variable $Y \sim F_Y$

$$\phi(t) = \int_0^\infty e^{-tw} dF_Y(w), \;\; t \geq 0.$$

Large Pool Approach for Archimedean Copulae

If X_i, $i = 1, \ldots, d$, i.i.d. $U[0,1]$ and Y's Laplace transform is ϕ, then the Archimedean Copula C is a joint cdf of $U_i = \phi\left(-\frac{\log X_i}{Y}\right)$.
Conditional on the realisation of Y, U_i are independent.
The large pool approximation of the loss distribution is

$$\mathrm{P}(\tilde{L} \le x) = F_Y\left\{-\frac{\log(1-x)}{\phi^{-1}(\bar{p})}\right\}.$$

For the Gumbel copula

$$C(u_1, \ldots, u_d; \theta) = \exp\left[-\left\{\sum_{i=1}^{d}(-\log u_i)^{\theta}\right\}^{\theta^{-1}}\right],$$

F_Y is an α-stable distribution with $\alpha = 1/\theta$.

Correlation's Types

Compound correlation $\rho(l_j, u_j)$, $j = 1, \ldots, J$.

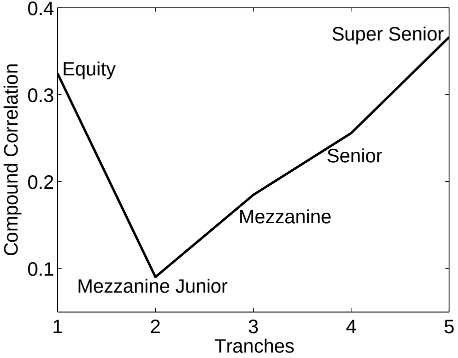


Figure 12: Implied correlation smile in the Gaussian one factor model, 20071022.

Correlation's Types

Base correlation (BC) $\rho(0, u_j)$, $j = 1, \ldots, J$.

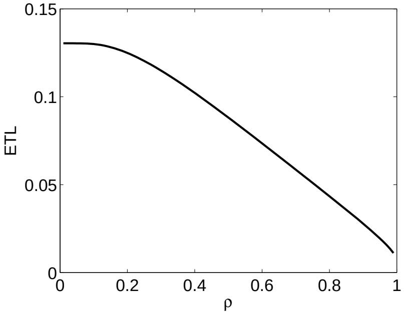
Represent the expected loss $\mathsf{E}\{L_{(l_j,u_j)}\}$ as a difference:

$$\mathsf{E}\{L_{(l_j,u_j)}\} = \mathsf{E}_{\rho(0,u_j)}\{L_{(0,u_j)}\} - \mathsf{E}_{\rho(0,l_j)}\{L_{(0,l_j)}\},\ j = 2, \ldots, J.$$

of the expected losses of two fictive tranches $(0, u_j)$ and $(0, l_j)$.
Bootstrapping process: $\mathsf{E}\{L_{(0,3\%)}\}$ is traded on the market,

$$\begin{aligned} \mathsf{E}\{L_{(3\%,6\%)}\} &= \mathsf{E}_{\rho(0,6\%)}\{L_{(0,6\%)}\} - \mathsf{E}_{\rho(0,3\%)}\{L_{(0,3\%)}\}, \\ \mathsf{E}\{L_{(6\%,9\%)}\} &= \mathsf{E}_{\rho(0,9\%)}\{L_{(0,9\%)}\} - \mathsf{E}_{\rho(0,6\%)}\{L_{(0,6\%)}\}, \ldots \end{aligned}$$

Base Correlations



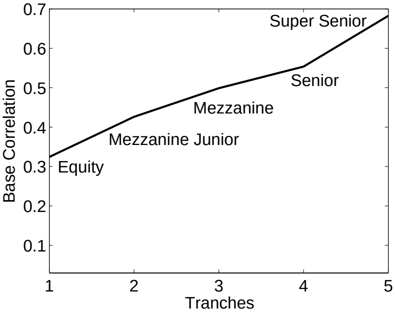


Figure 13: Expected loss of the equity tranche calculated using the Gaussian copula model with a one-year default probability computed from the iTraxx index Series 8 with 5 years maturity (left) and the base correlation smile (right) on 20071022. ▸ Talk

DSFM for Log-Spreads

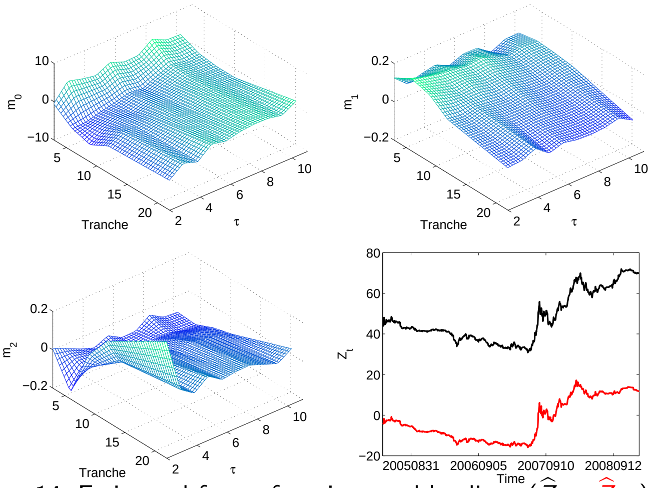


Figure 14: Estimated factor functions and loadings ($\widehat{Z}_{t,1}$, $\widehat{Z}_{t,2}$).

Talk

DSFM without the Mean Factor for Log-Spreads

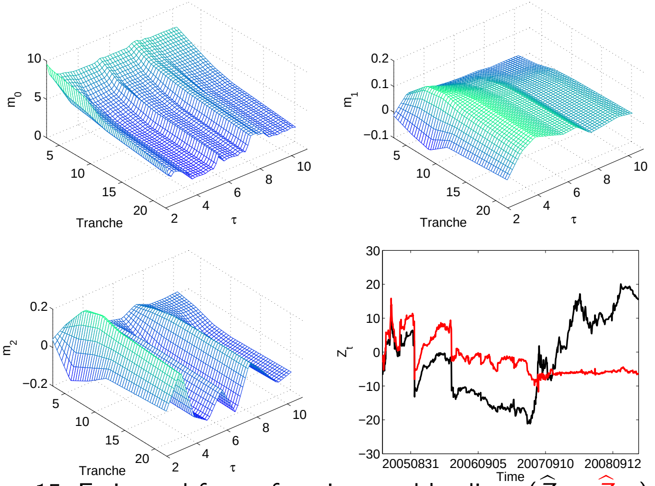


Figure 15: Estimated factor functions and loadings ($\widehat{Z}_{t,1}$, $\widehat{Z}_{t,2}$).

DSFM for Z-transformed-BC

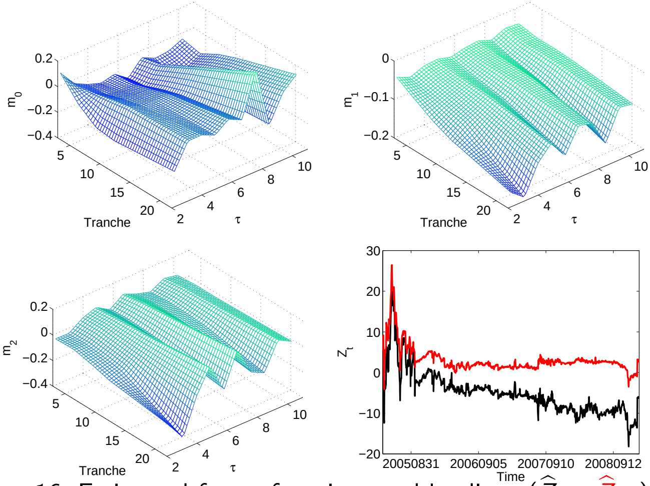
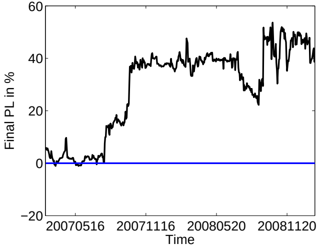
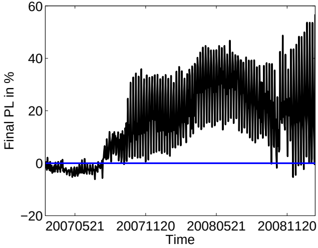


Figure 16: Estimated factor functions and loadings ($\widehat{Z}_{t,1}$, $\widehat{Z}_{t,2}$). Talk

Investor's Strategy





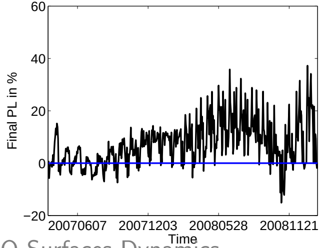


Figure 17: Combined flatteners and steepeners from all tranches and all maturities. Closing profits after one year. Rebalancing after: 1 day (upper left), 1 week (upper right), 1 month (lower). Calculations based on the DSFM predictions of log-spreads and Z-transformed BCs.

Talk