CDO Surfaces Dynamics

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**Motivation**

**iTraxx over Time**

Figure 1: Spreads of iTraxx tranches, Series 5, maturity 5 (left) and 10 (right) years, data from 20060407-20081103. Tranches: 1, 2, 3, 4, 5.
**iTraxx Spread Surface**

Figure 2: Spreads of tranches of all series observed on 20080909 (left) and 20090119 (right).
Research Goals

- Modelling the dynamics of CDO surfaces
  - spread surfaces
  - base correlation surfaces
- Applications in trading
Dynamic Semiparametric Factor Model

Applications:

Outline

1. Motivation ✓
2. CDOs
3. DSFM
4. Empirical Study
5. Applications
6. Conclusions
Risk Transfer

22% → 12% → 9% → 6% → 3%
iTraxx Europe

- A static portfolio of 125 equally weighted CDS on European entities;
- Sectors: Consumer (30), Financial (25), TMT (20), Industrials (20), Energy (20), Auto (10);
- New series of iTraxx Europe issued every 6 months (March and September) and the underlying reference entities are reconstituted;
- Maturities: 3Y, 5Y, 7Y, 10Y.
Large Pool Gaussian Copula Model

Default times are modelled from the Gaussian vector \((X_1, \ldots, X_d)^\top\):

\[
X_i = \sqrt{\rho} Y + \sqrt{1 - \rho} Z_i,
\]

where \(Y\) (systematic risk factor), \(\{Z_i\}_{i=1}^d\) (idiosyncratic risk factors) are i.i.d. \(N(0, 1)\). Assume that:

- obligors have the same default probability \(p\) and LGD,
- one dependence parameter \(\rho\),
- \(d\) is large.

The cdf of the portfolio loss equals

\[
P(\tilde{L} \leq x) = \Phi \left\{ \frac{\sqrt{1 - \rho} \Phi^{-1}(x) - \Phi^{-1}(p)}{\sqrt{\rho}} \right\}.
\]
Correlation’s Types

Compound correlation $\rho(l_j, u_j), j = 1, \ldots, J$.

Figure 3: Implied correlation smile in the Gaussian one factor model, 20071022.
Correlation’s Types

Base correlation (BC) $\rho(0, u_j)$, $j = 1, \ldots, J$.

Represent the expected loss $E\{L_{(l_j,u_j)}\}$ as a difference:

$$E\{L_{(l_j,u_j)}\} = E_{\rho(0,u_j)}\{L_{(0,u_j)}\} - E_{\rho(0,l_j)}\{L_{(0,l_j)}\}, \; j = 2, \ldots, J.$$ 

of the expected losses of two fictive tranches $(0, u_j)$ and $(0, l_j)$.

**Bootstrapping process:** $E\{L_{(0,3\%)}\}$ is traded on the market,

$$E\{L_{(3\%,6\%)}\} = E_{\rho(0,6\%)}\{L_{(0,6\%)}\} - E_{\rho(0,3\%)}\{L_{(0,3\%)}\},$$

$$E\{L_{(6\%,9\%)}\} = E_{\rho(0,9\%)}\{L_{(0,9\%)}\} - E_{\rho(0,6\%)}\{L_{(0,6\%)}\}, \ldots$$
Figure 4: BC of iTraxx tranches, Series 5, maturity 5 (left) and 10 (right) years, data from 20060510-20081023. Tranches: 1, 2, 3, 4, 5.
Base Correlation Surfaces

Figure 5: Implied base correlations on day 20080909 (left) and 20090119 (right).
Dynamic Semiparametric Factor Model

\[
Y_{t,k} = m_0(X_{t,k}) + \sum_{l=1}^{L} Z_{t,l} m_l(X_{t,k}) + \varepsilon_{t,k} = Z_t^\top A\psi(X_{t,k}) + \varepsilon_{t,k}
\]

- \( Y_{t,k} \): log-spreads and Z-transformed BC on day \( t \), \( t = 1, \ldots, T \)
- \( k \): intra-day numbering of BCs on day \( t \), \( k = 1, \ldots, K_t \)
- \( X_{t,k} \): two-dimensional vector of the tranche seniority and the time-to-maturity
- \( m_l \): factor functions, time invariant, nonparametric estimation
- \( Z_{t,l} \): time series, \( l = 0, \ldots, L \), dynamic behavior
- \( \psi(X_{t,k}) \): tensor B-spline basis
- \( A \): coefficient matrix
Estimation

Using an iterative algorithm:

\[
(\hat{Z}_t, \hat{A}) = \arg \min_{Z_t, A} \sum_{t=1}^{T} \sum_{k=1}^{K_t} \left\{ Y_{t,k} - Z_t^T A \psi(X_{t,k}) \right\}^2
\]

Selection of \( L \), the numbers of spline knots \( R_1, R_2 \) and the orders of splines \( k_1, k_2 \) by maximising the explained variance criterion:

\[
EV(L, R_1, r_1, R_2, r_2) = 1 - \frac{\sum_{t=1}^{T} \sum_{k=1}^{K_t} \left\{ Y_{t,k} - \sum_{l=1}^{L} Z_{t,l} m_l(X_{t,k}) \right\}^2}{\sum_{t=1}^{T} \sum_{k=1}^{K_t} \left\{ Y_{t,j} - \tilde{m}_0(X_{t,k}) \right\}^2},
\]

where \( \tilde{m}_0 \) is an empirical mean surface.
DSFM without the Mean Factor

Reduce the number of factors estimated in the iterative algorithm by first subtracting the empirical mean $\tilde{m}_0$ and then fitting the DSFM:

$$Y_{t,k} = \tilde{m}_0(X_{t,k}) + \sum_{l=1}^{L} Z_{t,l} m_l(X_{t,k}) + \varepsilon_{t,k} = \tilde{m}_0(X_{t,k}) + Z_t^T A\psi(X_{t,k}) + \varepsilon_{t,k},$$

where $m_l$ are new factor functions, $l = 1, \ldots, L$. 

CDO Surfaces Dynamics
Data

- Series 2-10
- Maturities 5, 7, 10Y
- 1004 days between 20050330-20090202
- 49,502 data points

<table>
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<tr>
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<th>3Y</th>
<th>5Y</th>
<th>7Y</th>
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<td>1478</td>
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<td>2006</td>
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<td>All</td>
<td>488</td>
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<td>15803</td>
<td>16840</td>
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</table>

Table 1: Number of observed values of iTraxx tranches in the period 20050330-20090202.
DSFM for Z-transformed-BC

Figure 6: Proportion of the explained variance as a function of $R_2$ (up left) with $r_2 = 2$, as a function of $r_2$ (up right) with $R_2 = 10$, as a function of $L$ (down) for $L = 1, L = 2, L = 3$, $r_1 = 2$ and $R_1 = 5$. 
DSFM w/o Mean F. for Z-transformed-BC

Figure 7: Estimated factor functions and loadings ($\hat{Z}_{t,1}$, $\hat{Z}_{t,2}$).
DSFM Estimation Results

For DSFM for both data types

- $\hat{Z}_{t,1}$ is a slope-curvature factor
- $\hat{Z}_{t,2}$ is a shift factor

<table>
<thead>
<tr>
<th>Model</th>
<th>Log-Spr</th>
<th>Z-BC</th>
</tr>
</thead>
<tbody>
<tr>
<td>DSFM</td>
<td>0.016</td>
<td>0.004</td>
</tr>
<tr>
<td>DSFM w/o mean f.</td>
<td>0.045</td>
<td>0.006</td>
</tr>
</tbody>
</table>

Table 2: Mean squared error of the in-sample fit.
DSFM without the mean factor Fit

Figure 8: In-sample fit of the models to data on 20080909 and 20090119.
Curve Trades

So, how can I make money with this?

Combine tranches of different time to maturity, see Felsenheimer et al. (2004) and Kakodkar et al. (2006):

- Flattener – sell protection on a long-term tranche, buy protection on a short-term tranche
  Example: sell protection on 10Y 3-6% and buy on 5Y 6-9%
  Outlook: bullish long-term, bearish short-term

- Steepener – opposite trade
Curve Trades

Figure 9: Mechanism of a flattener and a steepener strategy. **Current spread curve**, expectation of the future spread curve, indication of the direction of change.

CDO Surfaces Dynamics
JP Morgan Trading Loss, May 2012

J.P. Morgan’s flattener – bought 5Y CDX IG 9 index, sold 10Y CDX IG 9 index in a 3:1 ratio. The final loss reached $6.2 billion.
Flattener

Sell protection at $s_1(t_0)$ for the period $[t_0, T_1]$ and buy protection at $s_2(t_0)$ for $[t_0, T_2]$, $T_1 > T_2$. At $t_0$ for $\ell = 1, 2$:

$$\text{MTM}_\ell(t_0) = \sum_{t=t_1}^{T_\ell} \beta(t_0, t) \left[ s_\ell(t_0) \Delta t \mathbb{E}\{F_\ell(t)\} - \mathbb{E}\{L_\ell(t) - L_\ell(t - \Delta t)\} \right] = 0.$$ 

At $\tilde{t} > t_0$, the market quotes $s_\ell(\tilde{t})$ and

$$\text{MTM}_\ell(\tilde{t}) = \left\{ s_\ell(t_0) - s_\ell(\tilde{t}) \right\} \sum_{t=\tilde{t}_1}^{T_\ell} \beta(\tilde{t}, t) \Delta t \mathbb{E}\{F_\ell(t)\}.$$
Curve Trade

- A positive MTM means a positive value to the protection seller.
- If the protection seller closes the position at time $\tilde{t}$, then receives from the protection buyer $\text{MTM}_\ell(\tilde{t})$.
- Flattener-trader aims to maximize the total MTM value
  \[
  \text{PL}(\tilde{t}) = \text{MTM}_1(\tilde{t}) - \text{MTM}_2(\tilde{t}).
  \]
Risk in Curve Trades

- If one buys 5Y 6-9% and sells 10Y 6-9%, then the trade is hedged for default until the maturity of the 5Y tranche. Defaults that emerge from 10Y 6-9% are covered by 5Y 6-9% till it expires.
- Series differ in the composition of the collateral.
- If one buys 5Y 6-9% and sells 10Y 3-6%, then these tranches provide protection of different portion of portfolio risk. If there is any default in 10Y 3-6%, then we must deliver a payment obligation and incur a loss.
Empirical Study

Idea
- Use DSFM to forecast spread and BC surfaces
- Calculate forecasted MTM surfaces
- Recover those tranches that maximise P&L

Remarks
- Because of many missing data and short data histories, the standard econometric methods cannot be used for the forecasting.
- Consider trades that generate no or a positive carry – the spread of the long tranche doesn’t exceed the spread of the short tranche.
- Do not account for default payments (no data of historical defaults in iTraxx), do not account for the positive carry.
Forecasting with DSFM in Rolling Windows

Let $Y_t$ be log-spreads or $Z$-transformed-BC.

- Consider a rolling window of $w = 250$.
- Estimate the DSFM$s$ using $\{Y_\nu\}_{\nu=t-w+1}^t$ for $t = w, \ldots, T - h$.
- As a result, we get $T - w + 1$ times $\hat{m} = (\hat{m}_0, \ldots, \hat{m}_L)^\top$ and $\hat{Z}_t = (\hat{Z}_{t,0}, \ldots, \hat{Z}_{t,L})^\top$ of length $w$.
- Compute $h$-day forecast of the factor loadings using VAR.
- Due to the fixed issuing scheme, $X_{t+h,k}$ is not forecasted.
- Calculate the forecast $\hat{Y}_{t+h}$ from the forecast $\hat{Z}_{t+h}$.
- Transform $\hat{Y}_{t+h}$ suitably to get $\hat{s}(t + h)$ or $\hat{\rho}(t + h)$.
Forecasting MTM Surfaces

For predicted \( \{ \hat{s}_k(t), \hat{\rho}_k(t) \} \), \( t = w + h, \ldots, T \), \( k = 1, \ldots, K_t \), compute \( \hat{\text{MTM}}_k(t) \), where the initial spread \( s_k(t_0) \) is observed on \( t_0 = t - h \).

Figure 10: MTM surfaces on 20080909 (left) and 20090119 (right) calculated using one-day spread and BC predictions obtained with the DSFM.
Transaction Costs

Calculate the ask (bid) spread by increasing (reducing) the observed spread by the following percentage:

<table>
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<tr>
<th>Maturity</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
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<tr>
<td>5Y</td>
<td>1.88</td>
<td>1.78</td>
<td>2.52</td>
<td>3.77</td>
<td>6.28</td>
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<td>7Y</td>
<td>1.49</td>
<td>1.65</td>
<td>2.31</td>
<td>2.97</td>
<td>4.87</td>
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<td>10Y</td>
<td>1.41</td>
<td>1.66</td>
<td>1.83</td>
<td>2.52</td>
<td>4.09</td>
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</table>

Table 3: Average bid-ask spread excess over the mid spread as a percentage of the mid spread for Series 8 during the period 20070920-20090202.
Trading Strategies

Construct a curve trade

1. Fit and forecast the DSFM models to spreads and BC.
2. Calculate $h$-day forecasts of the MTM surfaces.
3. Recover which two tranches optimize a given strategy.

Strategies – restrict the choice to a flattener (or a steepener) with

1. a fixed tranche and fixed maturities,
2. a fixed tranche and all maturities,
3. all tranches and fixed maturities,
4. all tranches and all maturities (no restrictions),

or allow to combine flatteners and steepeners.
Backtesting

- Consider the time horizons $h = 1, 5, 20$ days.
- For the tranches that optimize a given strategy, check the corresponding historical market spreads, calculate the resulting MTM values, and the realised P&L.
## Mean of Daily Gains in Percent

<table>
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<tr>
<th>Strategy</th>
<th>1 day</th>
<th>1 week</th>
<th>1 month</th>
<th>1 day</th>
<th>1 week</th>
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<td>LZ</td>
<td>Z</td>
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<td>0.29</td>
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<td>0.33</td>
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<td>0.13</td>
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</tr>
</tbody>
</table>

Table 4: Calculations based on predictions of log-spreads and Z-transformed BCs marked as LZ; based only on Z-transformed BCs marked as Z.
Investor’s Strategy

Follow a certain strategy over a year and constantly rebalance the portfolio. At $t_0$ enter an optimal (according to the DSFM) curve trade for $h$-day horizon. At $t_0 + h$ chose:

1. keep the current position for the next $h$-days,
2. close the current position and enter a new one.

Assume a margin of 10% of your notional. Every time the position is closed, add to the margin the realized P&L. If margin $\leq 0$, quit the trade.
Investor’s Strategy

Figure 11: Daily cumulated P&L over one year 20070614–20080529. Rebalancing after: 1 day (upper left), 1 week (upper right), 1 month (lower). Calculations based on the DSFM predictions of log-spreads and Z-transformed BCs.
Conclusions

- Investigated evolution over time of tranche spread surfaces and base correlation surfaces using the DSFM.
- Empirical study is conducted using an extensive data set of 49,502 observations of iTraxx Europe tranches in 2005-2009.
- Proposed a modification to the classic DSFM.
- Both DSFMs successfully reproduce the dynamics in data.
- Used DSFM in constructing the curve trades.
- Analysed the performance of 43 strategies that combine different positions, tranches, and maturities.
- Backtesting showed high daily gains of the resulting curve trades.
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CDO Surfaces Dynamics

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Default

Consider a CDO with a maturity of $T$ years, $J$ tranches, and a pool of $d$ entities. Define a loss variable of $i$-th obligor until $t \in [t_0, T]$ as

$$ l_i(t) = 1(\tau_i < t), \ i = 1, \ldots, d, $$

where $\tau_i$ is a time to default variable

$$ F_i(t) = P(\tau_i \leq t) $$

$$ = 1 - \exp \left\{ - \int_{t_0}^{t} \lambda_i(u) du \right\} $$

and $\lambda_i$ is a deterministic intensity function.
Portfolio Loss

The proportion of defaulted entities in the portfolio at time $t$ is given by

$$\tilde{L}(t) = \frac{1}{d} \sum_{i=1}^{d} l_i(t), \quad t \in [t_0, T].$$

The portfolio loss at time $t$ is defined as

$$L(t) = \text{LGD} \tilde{L}(t),$$

where LGD is a common loss given default.
Tranche Loss

The tranche loss at time \( t \) is defined as

\[
L_j(t) = \frac{1}{u_j - l_j} \{ L^u(t, u_j) - L^u(t, l_j) \},
\]

where

\[
L^u(t, x) = \min \{ L(t), x \} \quad \text{for } x \in [0, 1].
\]

The outstanding notional of the tranche \( j \) is given by

\[
\Gamma_j(t) = \frac{1}{u_j - l_j} \{ \Gamma^u(t, u_j) - \Gamma^u(t, l_j) \},
\]

where

\[
\Gamma^u(t, x) = x - L^u(t, x) \quad \text{for } x \in [0, 1].
\]
Valuation of CDO

1. Premium leg

\[ PL_j(t_0) = \sum_{t=t_1}^{T} \beta(t_0, t) s_j(t_0) \Delta t \, \mathbb{E}\{\Gamma_j(t)\} \]

2. Default leg

\[ DL_j(t_0) = \sum_{t=t_1}^{T} \beta(t_0, t) \, \mathbb{E}\{L_j(t) - L_j(t - \Delta t)\} \]

This leads to:

\[ s_j(t_0) = \frac{\sum_{t=t_1}^{T} \beta(t_0, t) \, \mathbb{E}\{L_j(t) - L_j(t - \Delta t)\}}{\sum_{t=t_1}^{T} \beta(t_0, t) \Delta t \, \mathbb{E}\{\Gamma_j(t)\}}. \]
Equity Tranche

The equity tranche is quoted in two parts:
1. an upfront fee $\alpha$ payed at $t_0$,
2. a running spread of 500 BPs.

The premium leg is calculated as

$$PL_1(t_0) = \alpha(t_0) + \sum_{t=t_1}^{T} \beta(t_0, t) \cdot 500 \cdot \Delta t \mathbb{E}\{\Gamma_1(t)\}.$$ 

The upfront payment given in percent is equal

$$\alpha(t_0) = 100 \sum_{t=t_0}^{T} (\beta(t, t_0) [\mathbb{E}\{L_1(t) - L_1(t - \Delta t)\} - 0.05\Delta t \mathbb{E}\{\Gamma_1(t)\}]).$$
Copula

For a distribution function $F$ with marginals $F_{X_1}, \ldots, F_{X_d}$. There exists a copula $C : [0, 1]^d \to [0, 1]$, such that

$$F(x_1, \ldots, x_d) = C\{F_{X_1}(x_1), \ldots, F_{X_d}(x_d)\}$$

for all $x_i \in \mathbb{R}$, $i = 1, \ldots, d$. 
Copula for CDOs

The vector of default times \((\tau_1, \ldots, \tau_d)^\top\) has a (risk-neutral) joint cdf
\[
F(t_1, \ldots, t_d) = P(\tau_1 \leq t_1, \ldots, \tau_d \leq t_d) \quad \text{for all } (t_1, \ldots, t_d)^\top \in \mathbb{R}_+^d,
\]
where \(\tau_i \sim F_i\). From the Sklar theorem, there exists a copula such that
\[
F(t_1, \ldots, t_d) = C\{F_1(t_1), \ldots, F_d(t_d)\}
\]
and determines the default dependency of the credits.
Monte Carlo Simulation Approach

Define a trigger variable as

\[ U_i = \bar{p}_i(\tau_i) \sim U[0, 1], \quad i = 1, \ldots, d. \]

The \( i \)th obligor survives until \( t < T \) if and only if

\[ \tau_i \geq t \]

or \( U_i \leq \bar{p}_i(t) \).

The joint and marginal distributions of the triggers satisfy:

\[
C\{\bar{p}_1(t), \ldots, \bar{p}_d(t)\} = P\{U_1 \leq \bar{p}_1(t), \ldots, U_d \leq \bar{p}_d(t)\},
\]

\[
P\{U_i \leq \bar{p}_i(t)\} = \bar{p}_i(t).
\]
Monte Carlo Simulation Approach

The time to default variable

\[ \tau_i = \inf \{ t \geq t_0 : \bar{p}_i(t) \leq U_i \} \]

is calculated as

\[ \tau_i = \bar{p}_i^{-1}(U_i). \]

Assuming constant intensities compute

\[ \tau_i = -(\log U_i)/\lambda_i. \]
Large Pool Approach for Linear Factor Models

Default times are calculated from a vector \((X_1, \ldots, X_d)^\top\)

\[ X_i = \sqrt{\rho} Y + \sqrt{1 - \rho} Z_i, \]

where \(Y\) (systematic risk factor), \(\{Z_i\}_{i=1}^d\) (idiosyncratic risk factors) are i.i.d. Assume that

- obligors have the same default probability \(p\) and LGD,
- one dependence parameter \(\rho\),
- \(d\) is large.
Large Pool Approximation

Computations are simplified significantly when the portfolio loss distribution is approximated:

\[
P(L \leq x) = 1 - F_Y \left\{ \frac{F_X^{-1}(p) - \sqrt{\rho}F_Z^{-1}(x)}{\sqrt{1-\rho}} \right\},
\]

where \( X_i \sim F_X, Z_i \sim F_Z, Y \sim F_Y \).
Gaussian Copula Model

The factors $Y$ and $\{Z_i\}_{i=1}^d$ are i.i.d. $N(0, 1)$. Thus, $X_i \sim N(0, 1)$
The cdf of the portfolio loss equals

$$P(\tilde{L} \leq x) = \Phi \left\{ \frac{\sqrt{1 - \rho} \Phi^{-1}(x) - \Phi^{-1}(p)}{\sqrt{\rho}} \right\}.$$

Default times are given by $\tau_i = F_i^{-1}\{\Phi(X_i)\}$. 
NIG Model

Factors:

\[ Y \sim \text{NIG}\left(\alpha, \beta, -\frac{\beta \gamma^2}{\alpha^2}, \frac{\gamma^3}{\alpha^2}\right), \quad \gamma = \sqrt{\alpha^2 - \beta^2}, \]

\[ Z_i \sim \text{NIG}\left(\frac{\sqrt{1 - \rho}}{\sqrt{\rho}} \alpha, \frac{\sqrt{1 - \rho}}{\sqrt{\rho}} \beta, -\frac{\sqrt{1 - \rho}}{\sqrt{\rho}} \frac{\beta \gamma^2}{\alpha^2}, \frac{\sqrt{1 - \rho}}{\sqrt{\rho}} \frac{\gamma^3}{\alpha^2}\right). \]

Because of the stability under convolution

\[ X_i \sim \text{NIG}\left(\frac{\alpha}{\sqrt{\rho}}, \frac{\beta}{\sqrt{\rho}}, -\frac{1}{\sqrt{\rho}} \frac{\beta \gamma^2}{\alpha^2}, \frac{1}{\sqrt{\rho}} \frac{\gamma^3}{\alpha^2}\right) = \text{NIG}(1/\sqrt{\rho}). \]

Default times are given by \( \tau_i = F_i^{-1}\{\text{NIG}(1/\sqrt{\rho})(X_i)\} \).
Double-\( t \) Model

Define

\[
X_i = \sqrt{\rho} \sqrt{\frac{\nu_Y - 2}{\nu_Y}} Y + \sqrt{1 - \rho} \sqrt{\frac{\nu_Z - 2}{\nu_Z}} Z_i, \quad i = 1, \ldots, d,
\]

where \( Y \) and \( Z_i \) are \( t \) distributed with \( \nu_Y \) and \( \nu_Z \) DoF respectively.

The \( t \) distribution is not stable under convolution: \( X_i \) are not \( t \) distributed and the copula is not a \( t \) copula, \( X_i \sim F_X \) has to be computed numerically. Default times are computed as

\[
\tau_i = F_i^{-1}\{F_X(X_i)\}.
\]
Large Pool Approach for Archimedean Copulae

$d$-dimensional Archimedean copula $C : [0, 1]^d \rightarrow [0, 1]$ is

$$C(u_1, \ldots, u_d) = \phi\{\phi^{-1}(u_1) + \cdots + \phi^{-1}(u_d)\}, \quad u_1, \ldots, u_d \in [0, 1],$$

where $\phi \in \{ \phi : [0; \infty) \rightarrow [0, 1] \mid \phi(0) = 1, \phi(\infty) = 0; (-1)^j \phi^{(j)} \geq 0; j = 1, \ldots, \infty \}$ is a copula generator.

Each $\phi$ is a Laplace transform of a cdf of a positive random variable $Y \sim F_Y$

$$\phi(t) = \int_0^\infty e^{-tw} dF_Y(w), \quad t \geq 0.$$
Large Pool Approach for Archimedean Copulae

If $X_i, i = 1, \ldots, d$, i.i.d. $U[0,1]$ and $Y$’s Laplace transform is $\phi$, then the Archimedean Copula $C$ is a joint cdf of $U_i = \phi \left( -\frac{\log X_i}{Y} \right)$.

Conditional on the realisation of $Y$, $U_i$ are independent.

The large pool approximation of the loss distribution is

$$P(\tilde{L} \leq x) = F_Y \left\{ -\frac{\log(1 - x)}{\phi^{-1}(\bar{p})} \right\}.$$

For the Gumbel copula

$$C(u_1, \ldots, u_d; \theta) = \exp \left[ -\left\{ \sum_{i=1}^{d} (-\log u_i)^\theta \right\}^{\theta^{-1}} \right],$$

$F_Y$ is an $\alpha$-stable distribution with $\alpha = 1/\theta$. 
Correlation’s Types

Compound correlation $\rho(l_j, u_j)$, $j = 1, \ldots, J$.

Figure 12: Implied correlation smile in the Gaussian one factor model, 20071022.
Correlation’s Types

Base correlation (BC) $\rho(0, u_j)$, $j = 1, \ldots, J$.

Represent the expected loss $E\{L(l_j, u_j)\}$ as a difference:

$$E\{L(l_j, u_j)\} = E_{\rho(0, u_j)}\{L(0, u_j)\} - E_{\rho(0, l_j)}\{L(0, l_j)\}, \ j = 2, \ldots, J.$$ of the expected losses of two fictive tranches $(0, u_j)$ and $(0, l_j)$.

**Bootstrapping process:** $E\{L(0,3\%)\}$ is traded on the market,

$$E\{L(3\%, 6\%)\} = E_{\rho(0, 6\%)}\{L(0, 6\%)\} - E_{\rho(0, 3\%)}\{L(0, 3\%)\},$$

$$E\{L(6\%, 9\%)\} = E_{\rho(0, 9\%)}\{L(0, 9\%)\} - E_{\rho(0, 6\%)}\{L(0, 6\%)\}, \ldots$$
Appendix A. CDO Modelling Introduction

**Base Correlations**

![Graph](image)

Figure 13: Expected loss of the equity tranche calculated using the Gaussian copula model with a one-year default probability computed from the iTraxx index Series 8 with 5 years maturity (left) and the base correlation smile (right) on 20071022.
DSFM for Log-Spreads

Figure 14: Estimated factor functions and loadings ($\hat{Z}_{t,1}, \hat{Z}_{t,2}$).
DSFM without the Mean Factor for Log-Spreads

Figure 15: Estimated factor functions and loadings ($\hat{Z}_{t,1}, \hat{Z}_{t,2}$).
Appendix B

DSFM for Z-transformed-BC

Figure 16: Estimated factor functions and loadings ($\hat{Z}_{t,1}$, $\hat{Z}_{t,2}$).
Investor’s Strategy

Figure 17: Combined flatteners and steepeners from all tranches and all maturities. Closing profits after one year. Rebalancing after: 1 day (upper left), 1 week (upper right), 1 month (lower). Calculations based on the DSFM predictions of log-spreads and Z-transformed BCs.