Pricing Tranches of Credit Default Swap Index: A Mixed Copula Approach

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CDS Index - Credit Default Swap Index

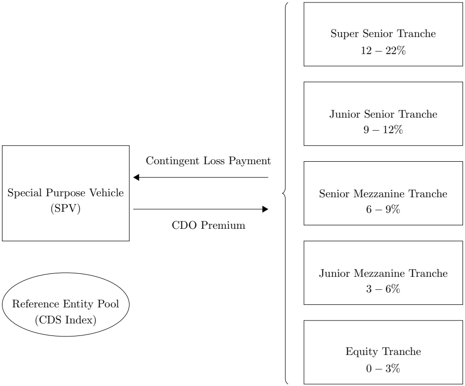


Figure 1: The structure of CDS Index.

Advantages of CDS Index

- Credit risk hedging or investment of a basket of credit entities.
- Standardised credit security over the counter.
- More liquid trading than the single-name CDS and CDO.

Yearly Issuance of CDOs

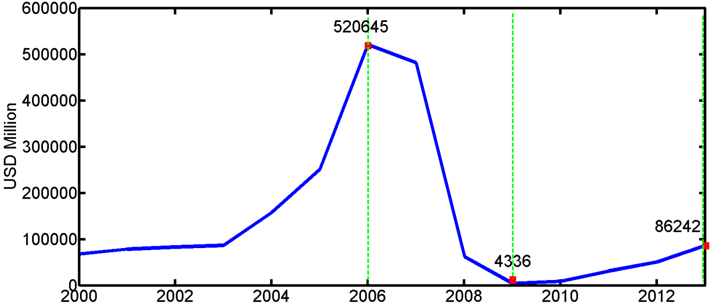


Figure 2: Yearly issuance of CDOs. Data: SIFMA.

Quarterly Issuance of CDOs

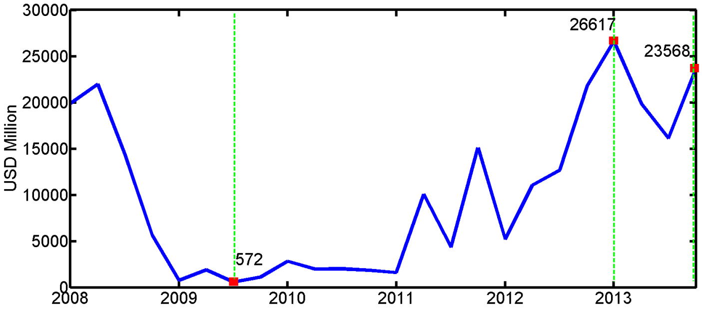


Figure 3: Quarterly issuance of CDOs. Data: SIFMA.

Problems in Pricing Study

- The standard pricing model in industry is the Gaussian copula model introduced in Li (1999) and Li (2000).
- It has drawbacks in destitution of modeling the heterogeneous dependence and the asymmetrical tail-dependence of multivariate defaults.

Correlation Smile by Li's Model

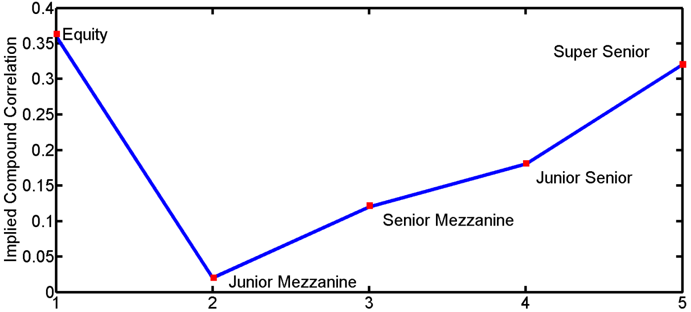


Figure 4: Implied correlation of iTraxx Europe Series 8 on 20071102.

Highlights of Our Study

- First time employing the mixed copula in the pricing of CDS index tranches.
- Extensive comparison of pricing performance between 21 mixed copula based pricing models against 22 benchmark models.

Outline

1. Motivation ✓
2. CDS Index Pricing
3. Copula
4. Empirics
5. Conclusion

An Example of CDS Index Tranches

Interval	0-3%	3-6%	6-9%	9-12%	12-22%
spreads 5y	16.67	106.42	45.95	28.00	17.50
spreads 7y	26.43	166.72	80.16	45.90	29.58

Table 1: Spreads of tranches of the iTraxx Europe Series 8 on 2007-10-23 with maturities of 5 years and 7 years sourced from Bloomberg.

Marginal Default

Let $\tau_k, k = 1, \dots, d$ be the random variable of the default time for the k-th entity in the reference pool, then the CDF of τ_k is defined as follows,

$$\begin{aligned} F_k(z) &= \mathrm{P}(\tau_k \leq z), \\ &= 1 - \exp\left[- \int_0^z h_k(s) \mathrm{d}s \right], \end{aligned}$$

where $z \in [0, T]$ and $h_k(s)$ is the intensity function.

Single Entity Loss and Portfolio Loss

For the default counting, Λ_{k,t_j} is defined as the single default variable of the k-th entity at the point t_j as follows,

$$\Lambda_{k,t_j} = \mathbf{1}_{\{\tau_k \le t_j\}},\ k = 1, \ldots, d.$$

Then the portfolio loss process can be given,

$$L_{t_j} = \frac{1}{d} \sum_{k=1}^{d} (1 - R) \Lambda_{k,t_j},\ j = 1, \ldots, J,$$

where $1 - R$ is the constant loss given default (LGD).

Tranche Loss and Outstanding Notional

The q-th tranche loss L_{q,t_j} at the time point t_j,

$$L_{q,t_j} = \begin{cases} 0 & \text{if } L_{t_j} \leq A_q, \\ L_{t_j} - A_q & \text{if } A_q \leq L_{t_j} \leq D_q, \\ D_q - A_q & \text{if } L_{t_j} > D_q, \end{cases}$$

where A_q and D_q, $q = 1, \ldots, Q$ are correspondingly the attachment and detachment point of the q-th tranche.
The outstanding notional P_{q,t_j} can be represented as follows,

$$P_{q,t_j} = D_q - A_q - L_{q,t_j}.$$

Default Leg and Premium Leg

The default leg DL_q of the q-th tranche can be given as follows,

$$DL_q = \mathbb{E}\left\{ \sum_{j=1}^{J} Y_{t_j} N(L_{q,t_j} - L_{q,t_{j-1}}) \right\},$$

where Y_{t_j} is the discount factor and N is the notional of the portfolio.

The non-equity tranche premium leg PL_q can be given as follows,

$$PL_q = \mathbb{E}\left\{ \sum_{j=1}^{J} Y_{t_j} S_q(t_j - t_{j-1})(P_{q,t_j} + P_{q,t_{j-1}})N/2 \right\},\ q \geq 2.$$

Tranche Spread

The main idea of CDO pricing is to imply tranche spreads under the following equation,

$$PL_q = DL_q, \tag{1}$$

where PL_q and DL_q are respectively the premium leg and the default leg of the q-th tranche.
The non-equity q-th ($q \geq 2$) tranche spread can be given,

$$S_q = \frac{\mathbb{E}\left\{\sum_{j=1}^{J} Y_{t_j}(L_{q,t_j} - L_{q,t_{j-1}})\right\}}{\mathbb{E}\left\{\sum_{j=1}^{J} Y_{t_j}(t_j - t_{j-1})(P_{q,t_j} + P_{q,t_{j-1}})/2\right\}}. \tag{2}$$

CDO and Copula

TORONTO STAR

WEDNESDAY, MARCH 18, 2009 • thestar.com • RAIN. HIGH 11C (WEATHER MAP ON S12)

TERRIBLE TWIST FOR FILM ROYALTY

Actress Natasha Richardson, part of a cinema dynasty and wife of a superstar, fell while skiing and seemed just fine. Then things went horribly wrong

NEWS — A3, PROFILE — E1

Meet the Canadian whose big idea felled Wall Street

Math whiz proposed applying this statistical formula to credit risk and financial meltdown ensued

Machine's 'mistake' kills $43M casino win

Slots player says no di... to consolation dinner, calls his lawyer instead

Figure 5: $\Pr[\tau_A < 1, \tau_B < 1] = \Phi_2(\Phi^{-1}(F_A(1)), \Phi^{-1}(F_B(1)), \gamma)$.

Source: TORONTO STAR, 18.03.2009.

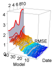
Copula

Theorem (Sklar's Theorem)

Given a d-dimensional joint CDF F such that $F(x_1, \dots, x_d) = \mathbb{P}(X_1 \le x_1, \dots, X_d \le x_d)$ of a random vector $(X_1, X_2, \dots, X_d)^\top$ with margins $F_k(x) = \mathbb{P}(X_k \le x)$, there exists a d-dimensional copula C such that

$$F(x_1, \dots, x_d) = C\{F_1(x_1), \dots, F_d(x_d)\}.$$

The copula C is unique if every $F_k,\ k = 1, 2, \dots, d$, is continuous, otherwise C is uniquely defined on $\prod_{k=1}^{d} Range(F_k)$.



Copula and CDO Pricing

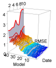
Let the random variable τ_k as

$$\tau_k = \inf\left\{ z \mid U_k \geq \exp\left(-\int_0^z h_k(s)\mathrm{d}s \right) \right\},\ k = 1, \ldots, d,$$

and the random vector $(\tau_1, \ldots, \tau_d)^\top \sim F(z_1, \ldots, z_d)$, then the joint default CDF of d entities at the respective time points $(z_1, \ldots, z_d)^\top$ can be given as follows,

$$\begin{aligned} F(z_1, \ldots, z_d) &= \mathbb{P}(\tau_1 \leq z_1, \ldots, \tau_d \leq z_d), \\ &= C\left\{F_1(z_1), \ldots, F_d(z_d)\right\}, \end{aligned}$$

where $F_k(z) = 1 - \exp\left\{ -\int_0^z h_k(s)\mathrm{d}s \right\}$ and $h_k(s)$ is the intensity function.



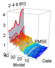
Copula and CDO Pricing

For simplicity we set here the intensity function $h_k(s)$ as a constant such that $h_k(s) = h$, therefore we have

$$\begin{aligned} F_k(z) &= 1 - \exp\left\{ -\int_0^z h_k(s)\mathrm{d}s \right\}, \\ &= 1 - \exp\left(-hz\right). \end{aligned} \tag{3}$$

Then under $\tau_k \sim F_k$ it can be obtained that

$$\exp(-h\tau_k) \sim \mathcal{U}[0,1]. \tag{4}$$



Copula and CDO Pricing

Let us define a random vector $(U_1, \ldots, U_d)^\top$, where

$$U_k = \exp(-h\tau_k), \;\; k = 1, \ldots, d. \tag{5}$$

As it has been given that $U_k \sim \mathcal{U}[0,1]$, therefore the joint CDF of $(U_1, \ldots, U_d)^\top$ can be given as follows,

$$\mathbb{P}(U_1 \leq u_1, \ldots, U_d \leq u_d) \;\; = \;\; C\{\exp(-hz_1), \ldots, \exp(-hz_d)\} \tag{6}$$

Algorithm of Sampling of Joint Default Times

- **Step 1**: Sample $(u_1^m, \ldots, u_d^m)^\top$ using $(U_1, \ldots, U_d)^\top$, where $m = 1, \ldots, M$ is the runs of Monte Carlo simulation and $(U_1, \ldots, U_d)^\top \sim C(u_1, \ldots, u_d)$.
- **Step 2**: Compute $(\tau_1, \ldots, \tau_d)^\top = (\frac{-\log U_1}{h}, \ldots, \frac{-\log U_d}{h})^\top$ according to $U_k = \exp(-h\tau_k)$.
- **Step 3**: Obtain $(z_1^m, \ldots, z_d^m)^\top = (\frac{-\log u_1^m}{h}, \ldots, \frac{-\log u_d^m}{h})^\top$.

Pricing CDO with Empirical Samples

1. Calculate the q-th tranche loss at the point t_j, $j \in \{0, \ldots, J\}$,

$$\hat{\mathbb{E}}[L_{q,t_j}] = \frac{1}{M} \sum_{m=1}^{M} \left[\min\{\max\{L_{t_j} - A_q, 0\}, D_q - A_q\} \right]. \tag{7}$$

2. Calculate the q-th non equity tranche spread, $q \in \{2, \ldots, Q\}$,

$$\hat{S}_{q \geq 2} = \frac{\hat{\mathbb{E}}\left\{\sum_{j=1}^{J} Y_{t_j}(L_{q,t_j} - L_{q,t_{j-1}})\right\}}{\hat{\mathbb{E}}\left\{\sum_{j=1}^{J} Y_{t_j}(t_j - t_{j-1})(P_{q,t_j} + P_{q,t_{j-1}})/2\right\}}. \tag{8}$$

Gaussian Copula

Definition (Gaussian Copula)

For a d-dimensional uniform vector $u = (u_1, \dots, u_d) \in [0,1]^d$ the Gaussian copula can be represented as follows,

$$C_{gs}(u; \rho) = \Phi_d \left\{ \Phi^{-1}(u_1), \dots, \Phi^{-1}(u_d); \rho \right\},$$

where ρ is a $(d \times d)$ correlation matrix, Φ_d is a d-dimensional standard normal distribution function and Φ is a one dimensional standard normal distribution function.

Model 1 (Li 1999, 2000):

$$C(u_1, \dots, u_d; \theta) \quad = \quad \Phi_d \left\{ \Phi^{-1}(u_1), \dots, \Phi^{-1}(u_d); \rho \right\}.$$

Exchangeable Gaussian Copula

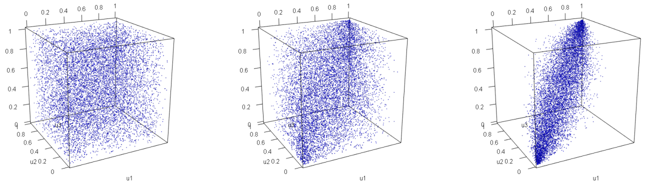


Figure 6: 10000 Monte Carlo simulations for 3-dimensional Gaussian copula $C_{gs}(u_1, u_2, u_3; \rho_{\mathcal{P}})$ with an exchangeable correlation matrix, where from left to right $\rho_{\mathcal{P}} = 0.1, 0.5, 0.9$.

Exchangeable Gaussian Copula

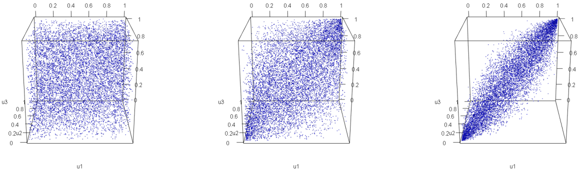


Figure 7: 10000 Monte Carlo simulations for 3-dimensional Gaussian copula $C_{gs}(u_1, u_2, u_3; \rho_{\mathcal{P}})$ with an exchangeable correlation matrix, where from left to right $\rho_{\mathcal{P}} = 0.1, 0.5, 0.9$.

Gaussian Copula

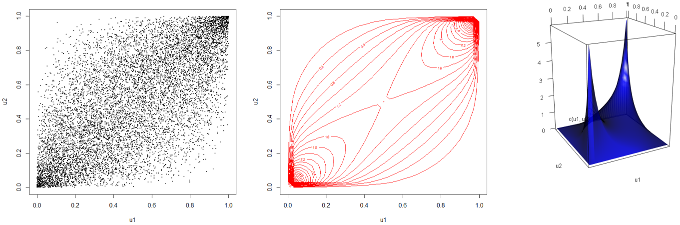
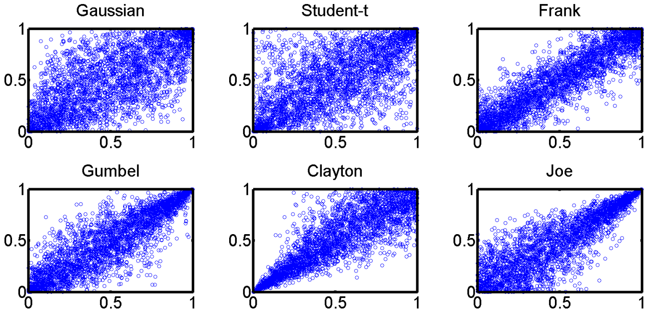


Figure 8: Plot left 1 is a 2-dimensional scatter plot drawing from Gaussian copula $C_{gs}(u_1, u_2; \rho_{\mathcal{P}})$. Plot left 2 is a 2-dimensional contour plot of copula density. Plot left 3 is a 2-dimensional copula density plot. The Gaussian copula is employed here with $\rho_{\mathcal{P}} = 0.7$.

Scatterplot of 2-dimensional Copulas



Mixed Copula

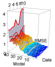
Definition (Mixed Copula)

Let $w_l \in [0,1]$, $l \in \{1,2\}$ be the weight for the lth component copula, then the mixed copula model can be given as follows,

$$\begin{aligned} & C_{e1-e2}(u_1,\dots,u_d;\theta) \\ = \; & w_1 C_{e1}(u_1,\dots,u_d;\theta_1) + w_2 C_{e2}(u_1,\dots,u_d;\theta_2), \end{aligned}$$

where the component copulas $e1$ and $e2 \in \{ga, t, fr, cl, gu, jo\}$ and parameters θ_1 and θ_2 belong correspondingly to the component $e1$ and $e2$.

Papers: Hu (2006), Wang (2008), Cai et al. (2009).



Examples of Mixed Copula Models

Model 24 :

$$\begin{aligned} & C_{ga-t}(u_1, \ldots, u_d; \theta) \\ = \quad & w_1 C_t(u_1, \ldots, u_d; \theta_1) + w_2 C_{ga}(u_1, \ldots, u_d; \theta_2). \end{aligned}$$

Model 35 :

$$\begin{aligned} & C_{fr-cl}(u_1, \ldots, u_d; \theta) \\ = \quad & w_1 C_{fr}(u_1, \ldots, u_d; \theta_1) + w_2 C_{cl}(u_1, \ldots, u_d; \theta_2). \end{aligned}$$

Model 42 :

$$\begin{aligned} & C_{gu-jo}(u_1, \ldots, u_d; \theta) \\ = \quad & w_1 C_{gu}(u_1, \ldots, u_d; \theta_1) + w_2 C_{jo}(u_1, \ldots, u_d; \theta_2). \end{aligned}$$

Data Set: CDX NA IG Series 19

- CDX NA IG Series 19 from the Bloomberg .
- 125 names distributed in 5 sectors including Industry (23, 18%), Consumer (44, 35%), Energy (16, 13%), Finance (19, 15%) and TMT (23, 18%).
- Maturity of 5 years with the period of 20120920-20171220.
- 10 evaluation dates: 20140601, 20140703, 20140815, 20140923, 20141011, 20141117, 20141201, 20150107, 20150210, 20150315.
- 4 standard tranches with intervals $[0\%, 3\%)$, $[3\%, 7\%)$, $[7\%, 15\%)$, $[15\%, 100\%]$.

Calibration

$$RMSE = \sqrt{\frac{1}{Q}\sum_{q=1}^{Q}\left(\widehat{S}_q^{CDO} - S_q^{Market}\right)^2}.$$

$\widehat{S}_q^{CDO}$: the q-th estimated tranche spread;

S_q^{Market}: the market tranche spread;

$M = 10000$, the runs of the Monte Carlo simulation;

$R = 0.4$, the constant recovery rate.

Table of Employed 43 Models

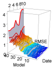
Model	Notation	Model	Notation	Model	Notation	Model	Notation
1	C_{ga}	12	C_{t4}	23	C_{ga-ga}	34	C_{fr-fr}
2	C_{t}	13	C_{t5}	24	C_{ga-t}	35	C_{fr-cl}
3	C_{ga1}	14	C_{t6}	25	C_{ga-fr}	36	C_{fr-gu}
4	C_{ga2}	15	C_{fr}	26	C_{ga-cl}	37	C_{fr-jo}
5	C_{ga3}	16	C_{cl}	27	C_{ga-gu}	38	C_{cl-cl}
6	C_{ga4}	17	C_{gu}	28	C_{ga-jo}	39	C_{cl-gu}
7	C_{ga5}	18	C_{jo}	29	C_{t-t}	40	C_{cl-jo}
8	C_{ga6}	19	C_{ng2}	30	C_{t-fr}	41	C_{gu-gu}
9	C_{t1}	20	C_{ng3}	31	C_{t-cl}	42	C_{gu-jo}
10	C_{t2}	21	C_{ng4}	32	C_{t-gu}	43	C_{jo-jo}
11	C_{t3}	22	C_{ng5}	33	C_{t-jo}		

Table 2: Abbr.: *ga*: Gaussian, *t*: Student-*t*, *fr*: Frank, *cl*: Clayton, *gu*: Gumbel, *jo*: Joe, $\{gai\}_{i=1}^{6}$: Gaussian with the correlation matrix $\{R_{gai}\}_{i=1}^{6}$, $\{tj\}_{j=1}^{6}$: Student-*t* with the same correlation matrix structure as $\{gai\}_{i=1}^{6}$, *ng*: HAC with the Gumbel generator function.

Table of Ranking of Mean of RMSEs

Rank	Notation	Rank	Notation	Rank	Notation	Rank	Notation
1	C_{cl-jo}	12	C_{ga-ga}	23	C_{jo}	34	C_{t4}
2	C_{fr-gu}	13	C_{cl-cl}	24	C_{ng3}	35	C_{ga2}
3	C_{cl-gu}	14	C_{fr-cl}	25	C_{ng2}	36	C_{t1}
4	C_{t-cl}	15	C_{t-gu}	26	C_{fr-fr}	37	C_{ga4}
5	C_{ga-jo}	16	C_{gu-gu}	27	C_{t}	38	C_{ga1}
6	C_{fr-jo}	17	C_{t-jo}	28	C_{ga6}	39	C_{t4}
7	C_{t-fr}	18	C_{ng5}	29	C_{t6}	40	C_{fr}
8	C_{ga-cl}	19	C_{ng4}	30	C_{ga5}	41	C_{t-t}
9	C_{ga-t}	20	C_{gu-jo}	31	C_{t5}	42	C_{cl}
10	C_{ga-gu}	21	C_{gu}	32	C_{t2}	43	C_{ga}
11	C_{ga-fr}	22	C_{jo-jo}	33	C_{ga3}		

Table 3: Abbr.: *ga*: Gaussian, *t*: Student-*t*, *fr*: Frank, *cl*: Clayton, *gu*: Gumbel, *jo*: Joe, $\{gai\}_{i=1}^{6}$: Gaussian with the correlation matrix $\{R_{gai}\}_{i=1}^{6}$, $\{tj\}_{j=1}^{6}$: Student-*t* with the same correlation matrix structure as $\{gai\}_{i=1}^{6}$, *ng*: HAC with the Gumbel generator function.



RMSE Comparison of Models

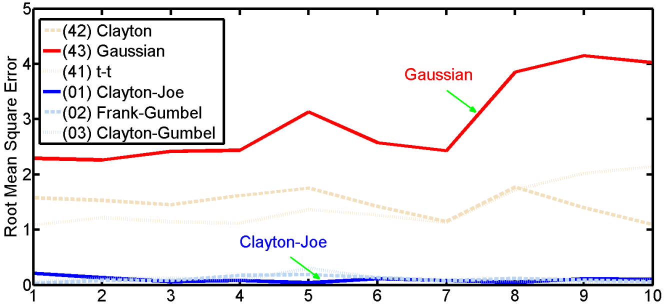


Figure 9: RMSEs of 3 best models and 3 worst models at 10 date points

RMSE Comparison of Models

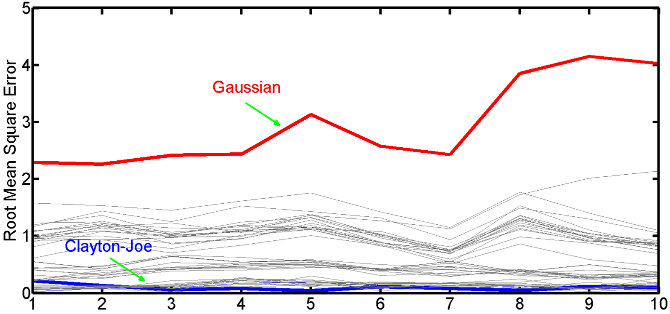


Figure 10: RMSEs of 43 models at 10 date points.

RMSEs of 43 Models in 3D

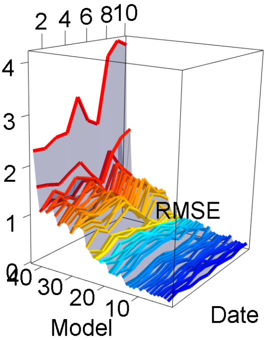


Figure 11: 3D illustration of RMSEs of 43 models at 10 date dates.

RMSEs of 43 Models in 3D

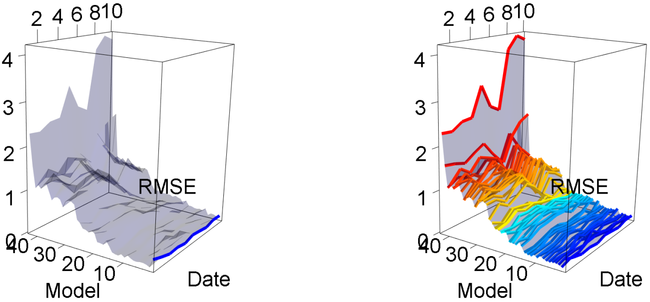
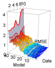


Figure 12: 3D illustration of RMSEs of 43 models at 10 date dates.



Conclusion

- Apply 21 mixed copulae in CDO pricing compared with 14 elliptical copulae, 5 exchangeable Archimedean copulae, 3 HACs.
- Mixed copula models outperformed benchmark models based on the mean of 10 RMSEs.
- Asymmetrical tail-dependence matters. Mixed copulae which own at least one component copula coming from the Gumbel, Joe or Clayton copula, show top performance.

Conclusion

- In elliptical family, the Student-t copula models performed similar to the Gaussian copula models.
- In exchangeable Archimedean family, the right tail-dependent copula models such as the Joe and the Gumbel outperform the Frank and the Clayton copula models.
- The mixed family performs best, and Archimedean and HAC families are in the middle. Most elliptical members belong to the worst ten models.

For Further Reading

B. Choroś-Tomczyk, W. Härdle and O. Okhrin
Valuation of Collateralized Debt Obligations with Hierarchical Archimedean Copulae
Journal of Empirical Finance, 2013

Z. Cai and X. Wang
Selection of Mixed Copula Model via Penalized Likelihood
Journal of the American Statistical Association, 2014

H. Joe
Dependence Modeling with Copulas
Chapman & Hall/CRC, 2014

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Chair of Econometrics and Statistics, esp Transportation
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