20th October 2008, Ostap Okhrin

Complex Numbers

Complex number are introduces in the similar way as the fractions ($\frac{n}{m} \in \mathbb{Q},\ \forall n, m \in \mathbb{Z}$). Let us consider the set of pairs of numbers $n, m \in \mathbb{Z}$ with two operations $\{'+', '\cdot'\}$ on it. This operations fulfills following properties

1. $\forall (n, m), (p, q) \in \mathbb{Q}:\ (n, m) + (p, q) = (nq + pm, mq)$
2. $\forall (n, m), (p, q) \in \mathbb{Q}:\ (n, m) \cdot (p, q) = (np, mq)$

Subtraction are defined based on the multiplication. Two rational numbers are to be equal if $nq = pm$. If the second number in the pair is replaced by the 1, then these pair behaves like the integer numbers

1. $(n, 1) + (p, 1) = (n1 + p1, 1 \cdot 1) = (n + p, 1) = n + p,\ \forall n, p \in \mathbb{Z}$
2. $(n, 1) \cdot (p, 1) = (n \cdot p, 1) = np,\ \forall n, p \in \mathbb{Z}$

This pair are also denoted as $(n, m) = \frac{n}{m} \in \mathbb{Q}$. In the similar way we introduce the complex numbers.

Definition 1. *The set of pair of the real numbers (x, y), $\forall x, y \in \mathbb{R}$ is called the field of complex numbers if there are introduced two operations $\{'+', '\cdot'\}$ with following properties*

1. $\forall (x, y), (u, v) \in \mathbb{C}:\ (x, y) + (u, v) = (x + u, y + v)$
2. $\forall (x, y), (u, v) \in \mathbb{C}:\ (x, y) \cdot (u, v) = (xu - yv, xv + yu)$

Subtraction are defined based on the multiplication. If the second number in the pair is replaced by the 0, then these pair behaves like the real numbers

1. $(x, 0) + (u, 0) = (x + u, 0 + 0) = (x + u, 0) = x + u,\ \forall x, u \in \mathbb{R}$
2. $(x, 0) \cdot (u, 0) = (x \cdot u - 0 \cdot 0, x \cdot 0 + 0 \cdot u) = xu,\ \forall x, u \in \mathbb{R}$

Notice following relationship between real and complex numbers if $x, u, v \in \mathbb{R}$ then $x(u, v) = (u, v)x = (x, 0)(u, v) = (xu, xv)$. Based on this every complex number $z = (x, y),\ x, y \in \mathbb{R}$ could be written in the form

$$z = (x, y) = (x, 0) + (0, y) = (x, 0) + y(0, 1) = x + y(0, 1)$$

When we denote $i = (0, 1)$, then $z = x + iy$. That is classical representation of complex numbers. It is very important to know what this number i means. For that et us look on the i^2

$$i^2 = (0, 1)(0, 1) = (-1, 0) = -1$$

And we write that $z \in \mathbb{C}$. Let us have a look on the properties of the operation we put on that space. Let $z = (x, y) \in \mathbb{C}, w = (u, v) \in \mathbb{C}$ than

$$\begin{aligned}
z = w &\Leftrightarrow x = u \wedge y = v \\
z = 0 &\Leftrightarrow x = 0 \wedge y = 0 \\
z + w &= (x + iy) + (u + iv) = \\
&= (x + u) + i(v + y) = x_1^* + iy_1^* \\
zw &= (x + iy)(u + iv) = \\
&= xu + ivx + iyu + i^2 yv \\
&= (xu - yv) + i(vx + yu) \\
&= x_2^* + iy_2^* \\
\frac{z}{w} &= \frac{x + iy}{u + iv} = \\
&= \frac{x + iy}{u + iv} \cdot \frac{u - iv}{u - iv} \\
&= \frac{(xu + yv) + i(yu - xv)}{u^2 + v^2} \\
&= \frac{xu + yv}{u^2 + v^2} + i\frac{yu + xv}{u^2 + v^2} \\
&= x_3^* + iy_3^*
\end{aligned}$$

the fraction is fine except when $u^2 + v^2 = 0$, that is when $u = v = 0$. So this means, that we may divide by any complex number except $(0, 0)$. From the calculations above, we may see, that the set of complex numbers is closed under the given operations. Neutral complex number under the operation of $'+'$ is zero

$$z + 0 = (x, y) + (0, 0) = (x, y) = z$$

And neutral number under the operation $'\cdot'$ is $(1, 0)$

$$z \cdot 1 = (x, y) \cdot (1, 0) = (x, y) = z$$

And for each number $z \in \mathbb{C}$ exists an inverse for both operations $\{'+',' \cdot'\}$

$$\begin{aligned}
-z &= -(x, y) = (-x, -y) = -x - iy \Rightarrow \\
z + (-z) &= x + iy + (-x - iy) = 0 \\
z^{-1} \cdot z &= 1 \\
z^{-1} &= \frac{1}{x + iy} = \frac{x - iy}{x^2 + y^2} = \frac{x}{x^2 + y^2} + i\frac{-y}{x^2 + y^2}
\end{aligned}$$

The part x in the complex number $z = x + iy$ is called Real part and could be get by $Re(z) = x$. The part y is then called as the imaginary part and $Im(z) = y$. In the lies above we often used complex number $x - iy$ that is based on the $x + iy$. This number is called conjugate to $z = x + iy$ and is denoted by $\overline{z} = x - iy$. The product between the z and $\overline{z}$

$$z\overline{z} = (x + iy)(x - iy) = x^2 + y^2$$

is called the square of the length of the complex number.

$$|z|^2 = z\overline{z} = x^2 + y^2$$

We have taken the notation from the modulus $|\cdot|$, just because $|(x,0)| = \sqrt{x^2} = |x|$. Having notation of the conjugate complex number and the length of the complex number the operation of subtraction is now much more easy

$$\begin{aligned}
\frac{z}{w} &= \frac{xu+yv}{|w|^2} + i\frac{yu-xv}{|w|^2} \\
\frac{1}{z} &= \frac{x}{|z|^2} + i\frac{-y}{|z|^2}
\end{aligned}$$

Length and the conjugate of the complex number have several interesting properties. Assume $z = x + iy \in \mathbb{C}$ and $w = u + iv \in \mathbb{C}$ then holds

$$\begin{aligned}
\overline{(z+w)} &= (x+u) - i(y+v) \\
&= (x-iy) + (u-iv) \\
&= \overline{z} + \overline{w} \\
\overline{zw} &= \overline{(x+iy)\cdot(u+iv)} \\
&= \overline{xu + ixv + iyu - yv} = xu - ixv - iyu - yv \\
&= (x-iy)(u-iv) = \overline{z}\overline{w} \\
|z+w|^2 &= (z+w)\overline{(z+w)} = (z+w)(\overline{z}+\overline{w}) \\
&= z\overline{z} + (w\overline{z} + \overline{w}z) + w\overline{w} \\
&= |z|^2 2Re(w\overline{z}) + |w|^2 \\
&\leq |z|^2 + 2|z||w| + |w|^2 = (|z|+|w|)^2 \Rightarrow \\
|z+w| &\leq |z| + |w|
\end{aligned}$$

As the complex number is represented by the pair of the real numbers it is natural to present the complex number on the Decart plane (Euclidean space), where on the OX axis we present the Real part of the complex number, and on the OY axis the imaginary part. The sum of two complex numbers on the plane is calculated similarly as the sum of the vectors, and is the the last vertex on the parallelogram $(0,0); (x,y); (u,v)$. Geometrically, the conjugate of z is simply the reflection of z on the horizontal axes. The length of the complex number is just a length of the vector $(0,0),(x,y)$.

As we could represent the complex number on the plane, we also can introduce another pair of coordinates: polar system. That is based on the angle of the vector (θ) and on the length (r) of the number. So changed the system $(x,y) \to (r,\theta)$. The θ is called the argument of the z, and of course there are a lot of possibilities of θ, where θ is called the argument, and one writes $Arg(z) = \theta 1$. In this new parametrization the complex number is that written by $z = r(\cos(\theta) + i\sin(\theta))$. The very important formula of this trigonometrical representation of the complex numbers is Moivre-Laplace Formula.

Suppose, that $z = r(\cos(\theta) + i\sin(\theta))$ and $w = s(\cos(\vartheta) + i\sin(\vartheta))$ than

$$\begin{aligned}
zw &= r(\cos(\theta) + i\sin(\theta))s(\cos(\vartheta) + i\sin(\vartheta)) \\
&= rs[(\cos\theta\cos\vartheta - \sin\theta\sin\vartheta) + i(\sin\theta\cos\vartheta + \sin\vartheta\cos\theta)] \\
&= rs(\cos(\theta+\vartheta) + i\sin(\theta+\vartheta))
\end{aligned}$$

That is nice result of the product of two complex numbers. It is also we often in use in the calculation of the powers of the complex numbers

$$z^n = r^n(\cos(n\theta) + i\sin(n\theta)) \tag{1}$$

Graphically the result of the product of two complex numbers is the complex number which length is the product of two lengths and an argument is the sum of those two arguments. The last, but also very important parametrization of the complex numbers is the exponential form.

$$re^{i\theta} = r(\cos\theta + i\sin\theta)$$

This parametrization leads to the same results as trigonometrical form, but in much easier way

$$\begin{aligned} r &= |z| \\ zw &= re^{i\theta}se^{i\vartheta} = rse^{i(\theta+\vartheta)} \\ \frac{1}{z} &= \frac{1}{re^{i\theta}} = \frac{1}{r}e^{-i\theta} \\ \frac{z}{w} &= \frac{re^{i\theta}}{se^{i\vartheta}} = \frac{r}{s}e^{i(\theta-\vartheta)} \end{aligned}$$

Examples

1. get powers of i.

$$\begin{aligned} i^{4n} &= 1 \\ i^{4n+1} &= i \\ i^{4n+2} &= -1 \\ i^{4n+3} &= -i \end{aligned}$$

2. Simplify

$$\begin{aligned} \sqrt{7+24i} &= \sqrt{7+2\cdot3\cdot4\cdot i} = \sqrt{7+2\cdot3\cdot4\cdot i+16-16} \\ &= \sqrt{2\cdot3\cdot4\cdot i+4^2-9} = \sqrt{2\cdot3\cdot4\cdot i+4^2+(3i)^2} \\ &= \sqrt{(3i+4)^2} = 3i+4 \end{aligned}$$

3. Simplify

$$\begin{aligned} \frac{1+i}{7-i} &= \frac{1+i}{7-i}\cdot\frac{7+i}{7+i} = \frac{(1+i)(7+i)}{49-i^2} \\ &= \frac{7+i+7i+i^2}{49+1} = \frac{6+8i}{50} \\ &= \frac{3}{25} + i\frac{4}{25} \end{aligned}$$

Characteristic Functions

Definition 2. *The characteristic function of a random variable X is defined for real t as*

$$\phi(t) = \int_{-\infty}^{\infty} e^{itx}\,dF_X(x) = \int_{-\infty}^{\infty} \cos(tx)\,dF_X(x) + i\int_{-\infty}^{\infty} \sin(tx)\,dF_X(x)$$

All properties for the expectation of the complex variable stays the same as are for the real variable. The characteristic function is the inverse Fourier transform of the distribution function.

Definition 3. *The Fourier Transform is the transformation of the function over orthogonal basis functions (imaginarial exponential of functions). Under the Fourier transform we think of different transformations of infinite dimensional vector from one basis to another*

$$f(x) = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} e^{-itx} F(x)\, dx$$

There is also an Fourier inversion theorem that says

Theorem 1. *Suppose* $g, \phi \in L$ *and*

$$\phi(x) = \int\limits_{-\infty}^{\infty} e^{itx} g(x)\, dx$$

then

$$g(x) = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} e^{-itx} \phi(x)\, dx$$

everywhere.

The characteristic function always exist, because distribution function is always integrable.

Properties of the characteristic function

1. Characteristic function completely characterize the distribution function (uniquely determines). Based on the Fourier inversion theorem

$$f(x) = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} e^{-itx} \phi(x)\, dx$$

2. The characteristic function of $X_1 + X_2$ is the product of characteristic functions $\phi_1(t)$ and $\phi_2(t)$.
   Let $X_1 \sim F_1$ and $X_2 \sim F_2$ from the definition of the distribution function that means that

$$\begin{aligned}
F_1(x) &= P(X_1 \leq x) \\
F_2(x) &= P(X_2 \leq x) \\
F_{X_1+X_2}(s) &= P(X_1 + X_2 \leq s) = \int\limits_{-\infty}^{\infty} F_{s-x_2}\, dF_2(x) \\
&= F_1 * F_2(s) \\
\phi_{F_1 * F_2}(t) &= E(e^{it(x_1+x_2)}) = E(e^{itx_1} e^{itx_2}) = Ee^{itx_1} Ee^{itx_2} \\
&= \phi_{F_1}(t)\phi_{F_2}(t)
\end{aligned}$$

   $F_1 * F_2(s)$ is called convolution of two distribution functions. The convolution look the same for the density functions.

3. ϕ is uniformly continuous

   Definition 4. *A function $f(x)$ is uniformly continuous in A if* $\forall \varepsilon > 0 \exists \delta > 0$

   $$\sup_{\{(x,y)\in A^2 : |x-y|\leq\delta\}} |f(x) - f(y)| < \varepsilon$$

4. $\phi(0) = 1$ and $|\phi(\xi)| \leq 1 \ \forall \xi$

5. $aX + b$ has a characteristic function

   $$E(e^{it(aX+b)}) = Ee^{itaX}Ee^{ib} = e^{ib}\phi(at)$$

   where $\phi(t)$ is the characteristic function of X

Some explicit forms of the characteristic functions

distribution	$f(t)$	$\phi(t)$
Normal	$\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}$	$e^{-\frac{1}{2}t^2}$
Uniformal on $[0; a]$	$\frac{1}{a}$	$\frac{e^{iat-1}}{iat}$
Cauchy	$\frac{1}{\pi}\frac{\xi}{\xi^2+x^2}$	$e^{-\xi\|t\|}$