TECHNISCHE UNIVERSITÄT DRESDEN

Performance Analysis of Computer Systems

Introduction to Queuing Theory

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Summary of Previous Lecture

Performance Simulation and Prediction

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Discrete-Event Simulation

- Events have a time value (timestamp) at which they are be processed
- Common overall structure:
  - Event scheduler
  - Global time variable
  - Event-processing routines
  - Event-generation mechanisms

Discrete-Event Simulation (cont.)

- Event Scheduler:
  - Maintains a list of pending events in global time order
  - successive takes the event with the smallest time and executes the event processing routine for this event
  - Inserts newly generated events into the event list
  - Updates the global time
- Global time update mechanisms
  - Fixed-increment (unit time): scheduler increments global time by a small amount and than executes any events which should occur at this time value
  - Event-driven: the earliest event sets the global time to the time at which this event should be processed

Discrete-Event Simulation (cont.)

- Event generation
- Execution-Driven
  - Executes a given program/benchmark
  - Similar to an emulator
  - Needs very detailed simulator
- Distribution-Driven
  - Events are generated by the simulator itself with the help of random numbers

Parallel Discrete-Event Simulation

- Motivation
  - In large simulation models independent events could be processed concurrently
  - Use this to parallelize the simulator
  - Problem: How to determine if events can be processed concurrently?

Parallel Discrete-Event Simulation (cont.)

- Prerequisite
  - Partition the state variables into disjoint sets
    - These are called "Logical Processes" (LP)
    - LPs communicating with *time stamped messages*
  - Shared state variables can be either emulated with an additional logical process, or
  - replicated into all LPs with a synchronization algorithm
- Causality Errors
  - Events need to be processed in non-decreasing timestamp order
  - Resulting errors are called *causality errors*

Types of PDES

- Conservative Approaches
  - Strictly *avoids* causality errors
  - Need to determine when it is safe to process an event
- Optimistic Approaches
  - *Detect* and *recover* from causality errors
  - Speculative executes events

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ZIH

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If the facts don't fit the theory, change the facts.

Albert Einstein

Outline

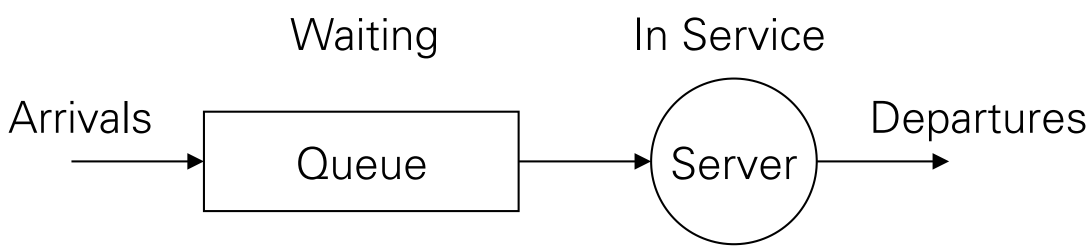
- Motivation
- Queuing/Kendall notation
- Queuing in daily life
- Exponential distribution and its memoryless property
- Little's law
- Stochastic processes, birth-death process
- M|M|1 queuing model

Motivation

- **Sharing of system resources** in computer systems:
  - CPU, Disk, Network, etc.
- Generally, only one job can use the resource at any time
- All other jobs using the same resource **wait in queues**
- **Queuing** or **queuing theory** helps in determining the time that jobs spend in various system queues.
- These times can be combined to predict the response time of jobs

Queuing Notation

- Imagine yourself at a supermarket checkout
- The checkout has a number of open cash points
- Usually, the cash points are busy and arriving customers have to wait
- In queuing theory terms you would be called "**customer**" or "**job**"
- In order to analyze such systems, the following system characteristics should be specified:
  1. Arrival Process
  2. Service Time Distribution
  3. Number of Servers
  4. System Capacity
  5. Population Size
  6. Service Discipline



Queuing Notation

1. **Arrival Process (Ankunftsprozess)**
   - If customers arrive at $t_1, t_2, \ldots, t_j$, the random variables $\tau_j = t_j - t_{j-1}$ are called **interarrival times (Zwischenankunftszeiten)**.
   - General assumption: The τ_j form a sequence of **i**ndependent and **i**dentically **d**istributed (**IID**) random variables
   - Most common arrival process is the **Poisson Process** which has exponentially distributed inter-arrival times
   - Erlang- and hyper-exponential distributions are also used
2. **Service Time Distribution (Antwortzeitverteilung)**
   - The time a customer spends at the service station e.g. the cash points
   - This time is called the **service time (Antwortzeit)**
   - Commonly assumed to be IID random variables
   - Exponential distribution is often used

Queuing Notation

3. Number of Servers (Anzahl der Bedienstationen)

- The number of service providing entities available to customers
- If in the same queuing system, servers are assumed to be:
  - Identical
  - Available to all customers

5. System Capacity

- The maximum number of customers who can stay in the system
- In most systems the capacity is finite
- However, if the number is large, infinite capacity is often assumed for simplicity
- The number includes both waiting and served customers

Queuing Notation

5. **Population Size**
   - The total number of serviced customers
   - In most real systems the population is finite
   - If this size is large, once again, the size is assumed infinite for simplicity reasons
6. **Service Discipline** or **Scheduling**
   - The order in which customers are served:
     - First come first served (**FCFS**)
     - Last come first served (**LCFS**) with or without preemption (**PR**)
     - Round Robin (**RR**) with fixed size quantum
     - Shortest processing time **(SPT)**
     - **RANDOM**
     - System with fixed delay, e.g. satellite link
     - Prioritized scheduling (**PRIO**)

Kendall Notation

- These six parameters need to be specified in order to define a single **queuing station**
- To compactly describe the queuing station in an unambigous way, the so called **Kendall Notation** is often used:

  Arrivals | Services | Servers | Capacity | Population | Scheduling

  - Arrivals ➡ customer arrival process
  - Services ➡ customer service requirements
  - Servers ➡ number of service providing entities
  - Capacity ➡ maximum number of customers in queuing station
  - Population ➡ size of the customer population
  - Scheduling ➡ employed scheduling strategy
- Population and Scheduling are often omitted i.e. assumed to be infinitely and FCFS

Kendall Notation

- The specific values of the parameters, especially **Arrivals** and **Services**, are diverse. Some commonly used once are:
  - ***M*** (Markovian or Memory-less): whenever the interarrival or service times are (negative) exponentially distributed
  - ***G*** (General): whenever the times involved may be arbitrarily distributed
  - ***D*** (Deterministic): whenever the times involved are constant
  - E_r (*r*-stage Erlang): whenever the times involved are distributed according to an *r*-stage Erlang distribution
  - H_r: whenever the times involved are distributed according to an *r*-stage hyper-exponential distribution

Kendall Notation - Example

- **M|G|2|8||LCFS** denotes a queuing station with:
  - Negative exponentially distributed interarrival times
  - Generally distributed service times
  - 2 service providing entities
  - Maximal 8 customers present
  - No limitation on the total customer population
  - LCFS scheduling strategy
- Simple queuing stations as above can be used for many queuing phenomena in computer and communication systems
- However, just a single queue with single service entity is considered, only allowing performance evaluations of parts of a complex system
- Examples: Analysis of network access mechanisms, simple transmission links, or various disk and CPU scheduling mechanisms

Queuing in Daily Life

- Coin-operated coffee machines
  - Service time, i.e., the time for preparing the coffee, is deterministic
  - Waiting time occurs due to the stochastic in the arrival process
  - Kendall notation: G|D|1
- Visiting a doctor with appointment
  - Arrival times of patients is deterministic (if their appointments are accurate)
  - However, one often experiences long waiting times due to the stochastic service times (time the doctor talks to or examines patients)
  - Kendall notation: D|G|1
- Visiting a doctor without appointment
  - Things become get even worse during “walk-in” consulting hours
  - Both arrival and service process obeys only general characteristics and the perceived waiting time increases
  - Kendall notation: G|G|1

Exponential (Markov) Distribution

- The (negative) exponential distribution is used extensively in queuing models
- It is the only continuous distribution with the so-called **memoryless property** which strongly simplifies the analysis:
  - *Remembering the time since the last event does not help in predicting the time till the next event!*
- Commonly used to model random durations, e.g.:
  - Duration of a phone call, Time between two phone calls
  - Duration of services, reparations, maintenance
  - Lifetime of radiactive atoms
  - Lifetime of parts, machines, technical equipment (without decline!)

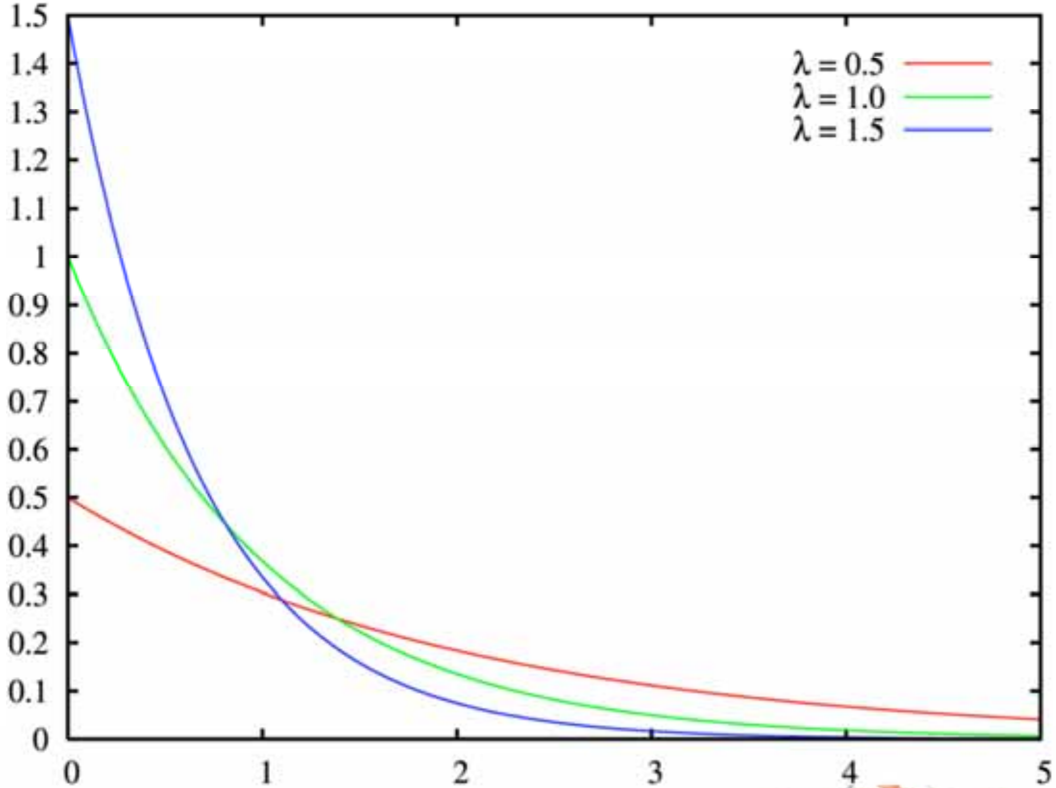
Exponential Distribution

Probability density function (Dichtefunktion), short: pdf

$$f(x;\lambda) = \begin{cases} \lambda e^{-\lambda x} & , \quad x \geq 0, \\ 0 & , \quad x < 0. \end{cases}$$

- Supported on interval $[0,\infty)$
- $\lambda > 0$ is a parameter of the distribution
- Often called **rate parameter**
- Probability of continuous random variable X:

$$\mathrm{P}(a \leq X \leq b) = \int_a^b f(x)\mathrm{d}x$$



Exponential Distribution

Cumulative distribution function (Verteilungsfunktion), short CDF

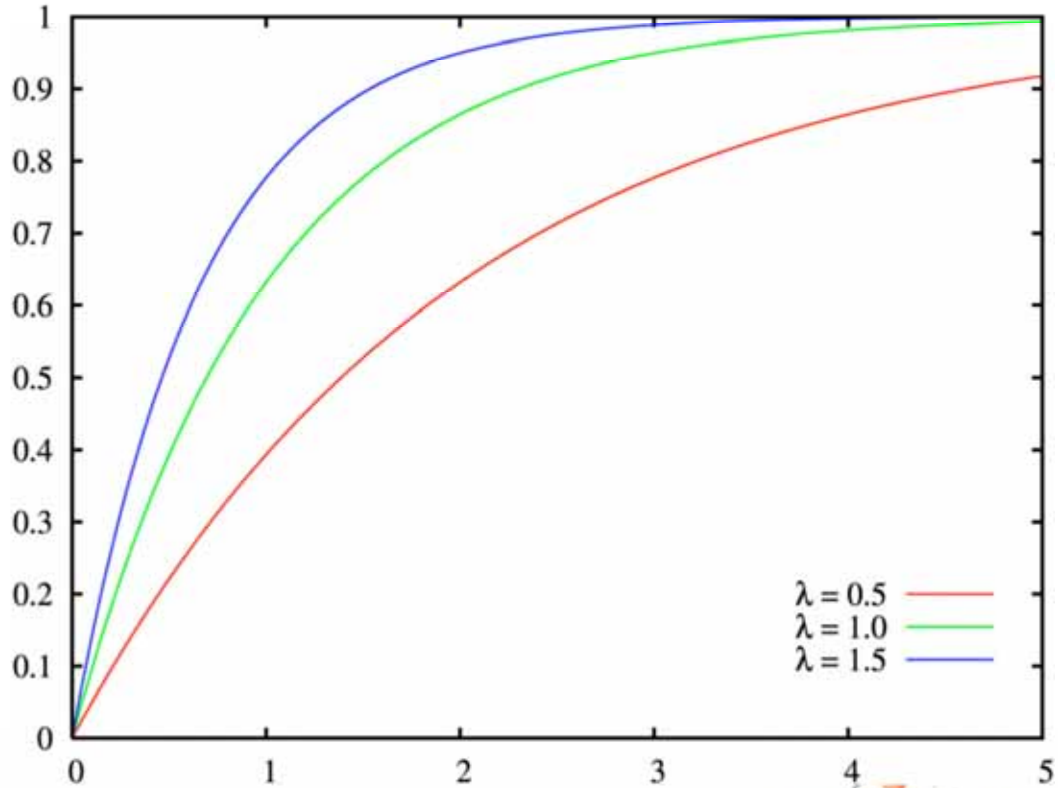
$$F(x,\lambda) = \begin{cases} 1 - e^{-\lambda x} & , \quad x \geq 0, \\ 0 & , \quad x < 0. \end{cases}$$

- Mean:

$$\mathrm{E}[X] = \frac{1}{\lambda}$$

- Variance:

$$\mathrm{Var}[X] = \frac{1}{\lambda^2}$$



Memoryless Property

- Stated earlier: Remembering the time since the last event does not help in predicting the time till the next event!
- Probability distribution of an exponentially distributed event *T* to occur within time *t:*

$$F(T) = P(T < t) = 1 - e^{-\lambda t}, t \geq 0$$

- We see an arrival event and start the clock at $t = 0$. The mean time to the next arrival event is $1/\lambda$.
- Suppose we do not see an arrival event until $t = x$. The distribution of the time remaining until the next arrival is:

$$\begin{aligned} P(T < x + t \mid T > x) &= \frac{P(x < T < x + t)}{P(T > x)} \\ &= \frac{P(T < x + t) - P(T < x)}{P(T > x)} \\ &= \frac{(1 - e^{-\lambda(x+t)}) - (1 - e^{-\lambda x})}{e^{-\lambda t}} \\ &= 1 - e^{-\lambda t} \end{aligned}$$

Memoryless Property

- A random variable T is said to be memoryless if:

$$P(T < x + t \mid T > x) = P(T < t) \quad \forall\ x, t \geq 0$$

- Example:
  - Give a real-life example whose lifetime can be modeled by a variable T such that $P(T > s + t \mid T > s)$ goes down as s goes up
  - Bus with exponentially distributed arrival times and $\lambda=2$ per hour
    - Average waiting time?
    - Expected waiting time when already waiting for 15 minutes?

Little's Law

- Named after John Little (MIT) who proved the law in 1961
- One of the most general laws in performance analysis
- Can be applied almost unconditionally to all queuing models and at many levels of abstraction
- Interesting point of notice: Long used before actually proved

- Little's Law basically **relates** the **average number of jobs** *N* in queuing station **to** the **average number of arrivals** per time unit λ and the **average time** *R* spent in the queuing station

$$N = \lambda R$$

Little's Law - Understanding

- Consider a queuing station as a black box
- On average λ jobs arrive per time unit
- Upon its arrival, a job is either served or has to wait
- Denote *E*[*R*] (**r**esidence time or **r**esponse time) as the average time spend in the queuing system
- Denote average number of jobs in the queuing system as *E*[*N*]
- Observe a single marked job which enters the system at $t=t_i$ leaves at $t=t_o$.
- On average $t_o - t_i$ will be equal to *E*[*R*]
- While this particular job passes the system, other jobs have arrived
- Since on average *E*[*R*] time units elapsed, their average number is λ × *E*[*R*]
- This number must be equal to the previously defined *E*[*N*] as every job could be the marked job. Thus:

$$E[N] = \lambda E[R]$$

Little's Law - Remarks

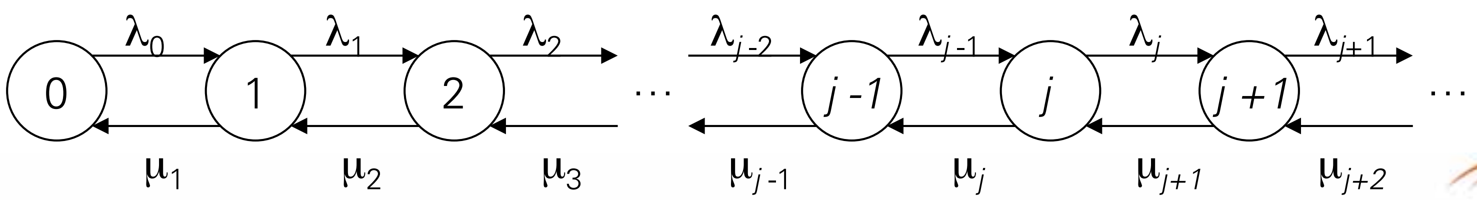
- We assumed that the queue throughput T equals the arrival rate λ
- Always the case if system is not overloaded and infinite buffers
- Otherwise customers will get lost and $E[N] = T\,E[R]$
- The relationship expressed by Little's law is valid independently of the form of the involved distributions
- This law is valid independently of the scheduling discipline and the number of servers
- $E[N]$ is easy to obtain and measures like $E[R]$ can be derived from it
- Applies also to networks of queuing stations

Stochastic Processes

- Analytical modeling uses several random variables but also **stochastic processes** which are sequences of random variables
- Collection of random variables $\{ X(t) \mid t \in T \}$, indexed by the parameter t *(usually time)* which can take values of set T
- Values that $X(t)$ assumes are called **states**. All possible states are called **state space *I***.
- The state space and the time parameter can be discrete or continuous
- Discrete-state stochastic processes are also called **chain**, often with $I = \{ 0,1,2,\ldots\}$
- Famous representatives: Markov Process, Birth-Death Process, and Poisson Process (form a hierarchy)

Birth-Death Process

- Future states of the process are independent of the past and depend only on the present
- Special case of the continuous time Markov chain
- State transitions are restricted to neighboring states
- States are represented by integers. State *n* can only change to state *n+1* or state *n-1*
- Example: The number of jobs in a queue with a single server and individual arrivals (no bulk arrivals)
- An arrival to the queue (birth) causes the state to change by +1. A departure (death) causes the state to change by -1
- Below: State transition diagram with *n* states, arrival rates λ_n and service rates μ_n. Arrival times and service times are exponentially distributed



Birth-Death Process

- The **steady-state probability** p_n of a birth-death process being in state n is given by the following theorem:

$$p_n = p_0 \frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n}$$

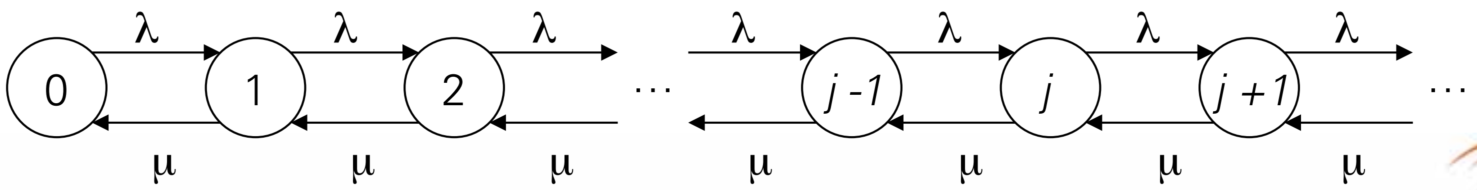
$$= p_0 \prod_{j=0}^{n-1} \frac{\lambda_j}{\mu_{j+1}}, \qquad n = 1, 2, \ldots, \infty$$

- p_0 is the probability of being in the **zero state**
- Can be proven (see book)
- Now that we have an expression for state probabilities we are able to analyze queues in the form of M/M/m/B/K
- Based on the state probabilities we can compute many other performance measures

M|M|1 Queuing Model

- **Most commonly used** type of queue
- Can be used to model single-processor system or individual devices in a computer system
- Interarrival and service times are exponentially distributed, one server
- No buffer or population size limits, FCFS service discipline
- Analysis: We need the **mean arrival rate λ** and the **mean service rate μ**
- State transition similar to birth-death process with $\lambda_n = \lambda$ *and* $\mu_n = \mu$
- The probability of n jobs in the system becomes:

$$p_n = \left(\frac{\lambda}{\mu}\right)^n p_0, \qquad n = 1, 2, \ldots, \infty$$



M|M|1 Queuing Model

$$p_n = \left(\frac{\lambda}{\mu}\right)^n p_0, \qquad n = 1, 2, \ldots, \infty$$

- The quantity $\lambda/\mu = \rho$ is called **traffic intensity**
- Thus $p_n = \rho^n p_0$
- All probabilities should add to 1. Knowing this we can derive an equation for the probability of zero jobs (p_0) in the systems:

$$p_0 = \frac{1}{1 + \rho + \rho^2 + \ldots + \rho\infty} = 1 - \rho$$

- Substituting p_0 in p_n leads to:

$$p_n = (1 - \rho)\rho^n, \qquad n = 0, 1, 2, \ldots, \infty$$

- Based on this expression, many other properties can be derived
- Utilization of the server: $U = 1 - p_0 = \rho$
- The mean number of jobs in the system:

$$E[n] = \sum_{n=1}^{\infty} n p_n = \sum_{n=1}^{\infty} n(1 - \rho)\rho^n = \frac{\rho}{1 - \rho}$$

M|M|1 Queuing Model

- The probability of n or more jobs in the system is:

$$P(\geq n \text{ jobs in the system}) = \sum_{j=n}^{\infty} p_j = \sum_{j=n}^{\infty} (1-\rho)\rho^j = \rho^n$$

- Using Little's law we can compute the mean response time:

$$E[n] = \lambda E[r]$$

$$E[r] = \frac{E[n]}{\lambda} = \left(\frac{\rho}{1-\rho}\right)\frac{1}{\lambda} = \frac{1/\mu}{1-\rho}$$

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Thank You!

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