# Solving ill-posed linear inverse problems without regularization 

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Motivation<br>Theory: Structure-Interference-Method (SIM)<br>Example: The Ipswich Data<br>Conclusions

Analysis of electromagnetic scattering is a common inverse problem.

The challenge is to derive information of the scattering body from measurements of the scattered field (superposed by the incident field).

We assume that the forward problem can be solved, i.e. the scattered fields can be calculated if the properties of the scattering body are known.

In many cases, the inverse problem is ill-posed: there is no unique solution - the scattering of different bodies fits the measured results sufficiently

Strictly: solving an ill-posed inverse problem $=$ inverting a singular matrix

The art is to obtain a physically meaningful solution anyway.

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2D Target Oriented Coordinate System


All angles are referenced to the $x$ axis
$\alpha=$ angle of incidence
$\phi_{\mathrm{s}}=$ receiver direction (known as phi_scat in data files)
$\phi_{\mathrm{i}}=$ illuminating plane wave direction or view
(note: conventional bistatic angle equals $\phi_{s}-\alpha$ )

$$
\begin{aligned}
u\left(\phi_{i}, \phi_{S}\right) & =\int_{R O I} K\left(\phi_{i}, \phi_{s}, \vec{x}\right) m(\vec{x}) d \vec{x} \\
m(\vec{x}) & =1-n(\vec{x}) \quad \text { Contrast } \\
n(\vec{x}) & =\frac{1}{\varepsilon_{0}}\left[\varepsilon(\vec{x})+\frac{j \sigma(\vec{x})}{2 \pi f}\right]
\end{aligned}
$$

Representation on a grid:

$$
m(\vec{x})=\sum c_{i} \varphi_{i} \quad \text { test functions } \varphi_{i}
$$

Optimization (Least Squares):

$$
\left\|T \vec{c}-\vec{u}_{\text {exp }}\right\|^{2}=\operatorname{Min}
$$

$\vec{c}$ is the unknown parameter vector
$\mathcal{A}(\vec{c})>0$ and $\mathcal{B}(\vec{c})>0$ two positive functionals
Both minimazation problems min $(\mathcal{A}(\vec{c}))$ and $\min (\mathcal{B}(\vec{c}))$ can be solved with redards to $\vec{c}$, but generally with different solutions $\vec{c}$.

In contrast, the optimization problem $\min (\mathcal{A}(\vec{c}))$ with constraint $\mathcal{B}(\vec{c})=b$ has a unique solution given by the minimization of the Lagrange function

$$
\mathcal{L}(\vec{c}, \lambda)=\mathcal{A}(\vec{c})+\lambda \mathcal{B}(\vec{c})
$$

Nice for mathematicans!

In real life: $b$ is unknown, how to select $\mathcal{B} \rightarrow$ still an infinite number of solutions

Selection of $\mathcal{B}$ (e.g. smoothness) and $\lambda \rightarrow$ a priori information is being used

Can we do it without regularization? Yes! (And it is simple.)
The idea of the Structure-Interference Method (SIM):

> A physical meaningful solution has to be be independent of the discretization (grid) used for its representation. Structures in a solution that depend on the actual grid are not relavant.

## Algorithm:

- use random gids to calculate solutions that fit the data (easy)
- average all single solution to a global solution (easy)

Restiction: linear problem (average must be a solution as well)

Why does it work: common structures are intensified, structures depending on the grid annihilate each other

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1 solution


## 10 solutions



## 100 solutions





Figure 7. IPS007: Two penetrable cardboard tubes, filled with air. The height is 96.5 cm , the thickness of each tube is 0.2 cm , the diameter of the inner tube is 7.6 cm , and the diameter of


Figure 8. IPS008: Two penetrable cardboard tubes, with the same dimensions as IPS007. However, the inner tube is filled with salt, and the outer tube is fllled with sand.
[McGahan, Kleinmann: IEEE Ant. \& Prop. Mag., 39(2), 1997]

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Method for solving linear ill-posed inverse problems without regularization

## Advantages:

- no a priori assumptions needed
- 'black-box' algorithm, i.e. no parameter to control the algorithm
- very stable due to averaging


## Drawbacks:

- only for linear problems (average of solutions has to be solution)
- computation time is longer, because $N$ solutions have to be computed (but it can be done in parallel)

Conclusion: It's simple. Give it a try!

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