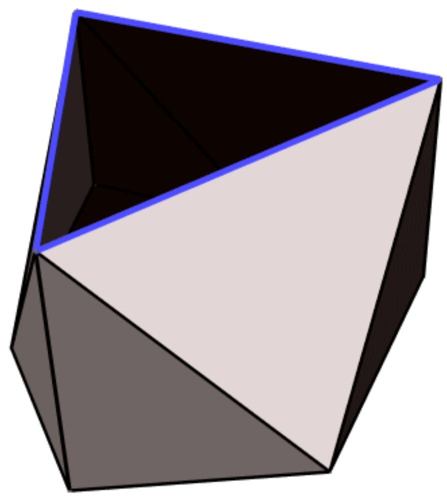
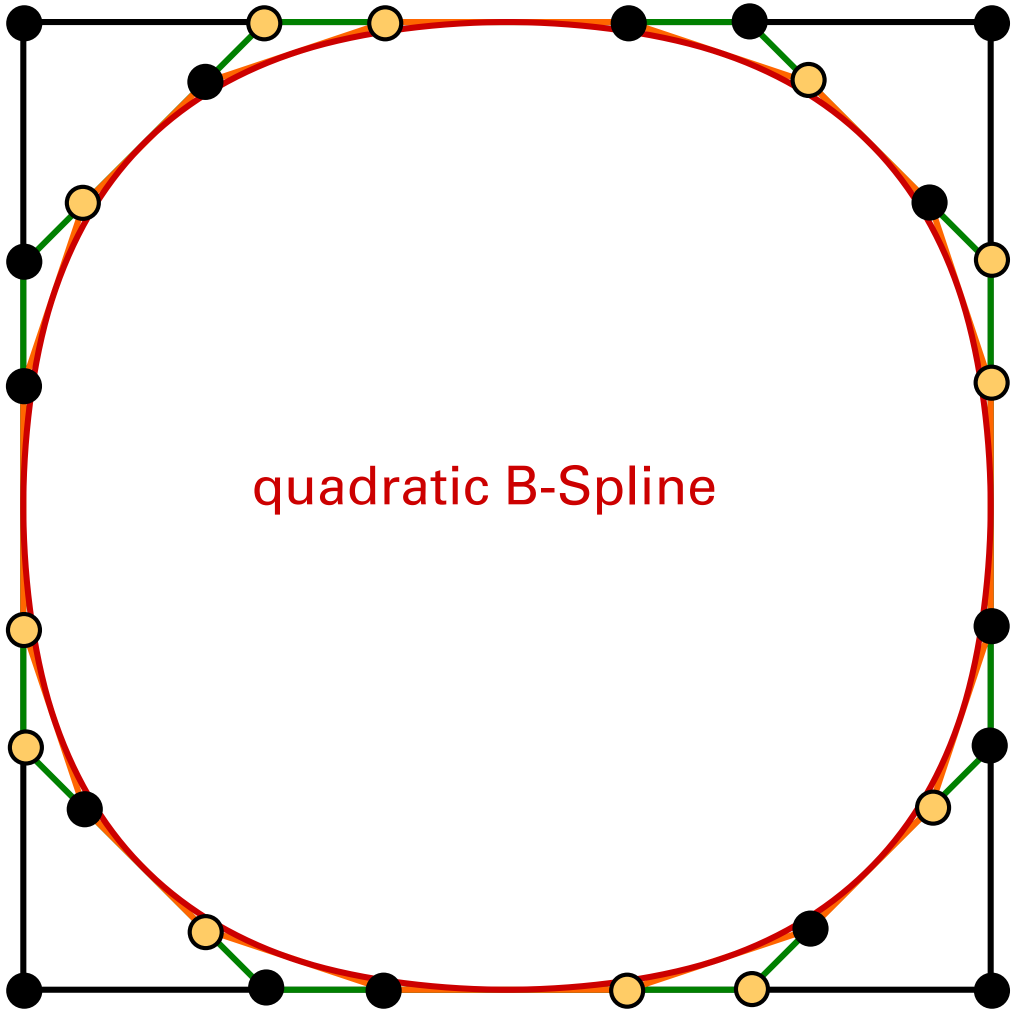
Subdivision Curves



References

- SIGGRAPH 2000 Kurs
- Warren, Weimer: Subdivision Methods for Geometric Design: a constructive approach, Morgan Kaufmann 2002
- Malcom Sabin: Continuity Analysis of Subdivision Curves

Curves – Corner Cutting



Curves – Vocabulary

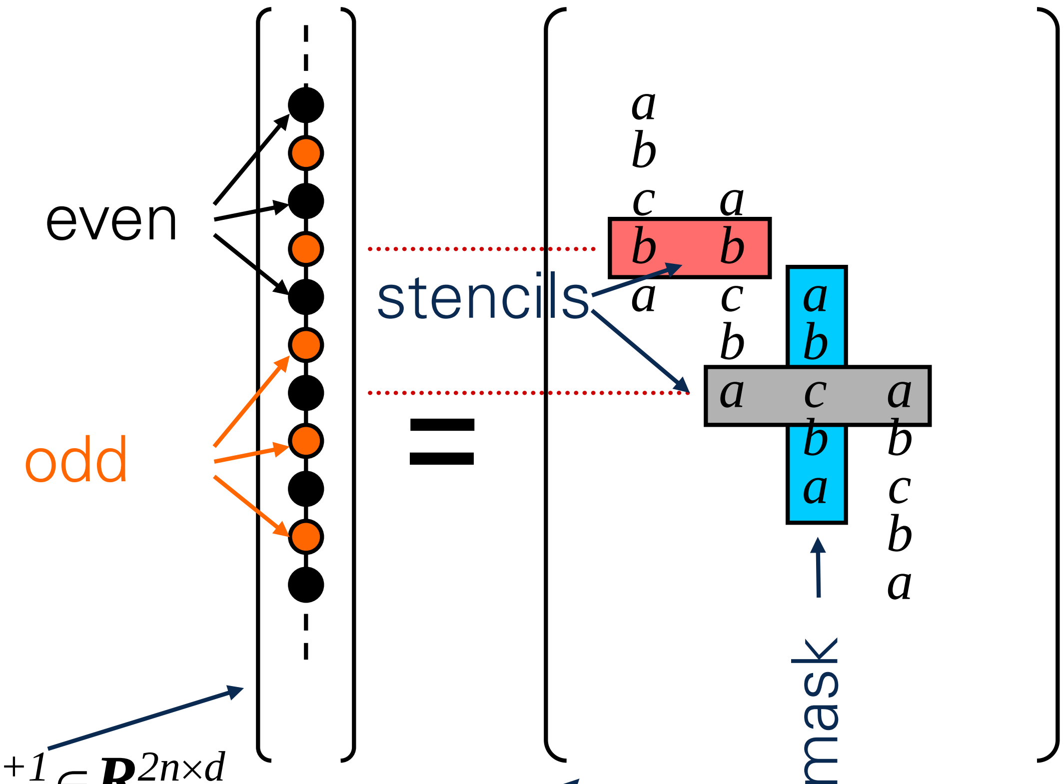
- **subdivision scheme:** rule that maps polygon to a polygon with more points
- **subdivision step:** one application of the subdivision rule
- **arity a:** fraction of number of points after subdivision step divided by number of points before, often $a=2$
- **stationary schemes:** subdivision rule is independent of location in polygon and independent of point position
- **even points:** old points that existed before a subdivision step and are still existing afterwards
- **odd points:** points inserted by a subdivision step
- **linear:** rule is affine combination of even points
- **Interpolating vs. approximating:** rule preserves / does not preserve positions of even points

quiz

Curves – matrix notation

- linear, stationary schemata

$P^{j+1} = S^j P^j$



$P^{j+1} \in R^{2n \times d}$

subdivision matrix $S^j \in R^{2n \times n}$

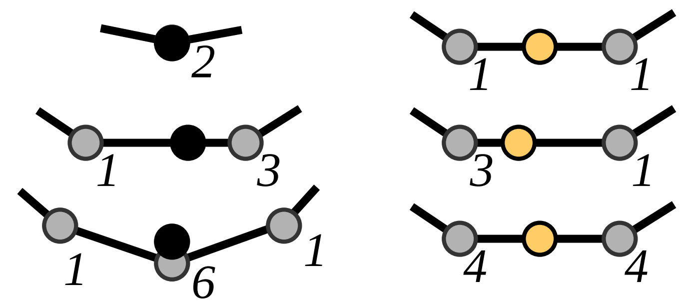
point matrix $P^j \in R^{n \times d}$

each stencil must sum to 1

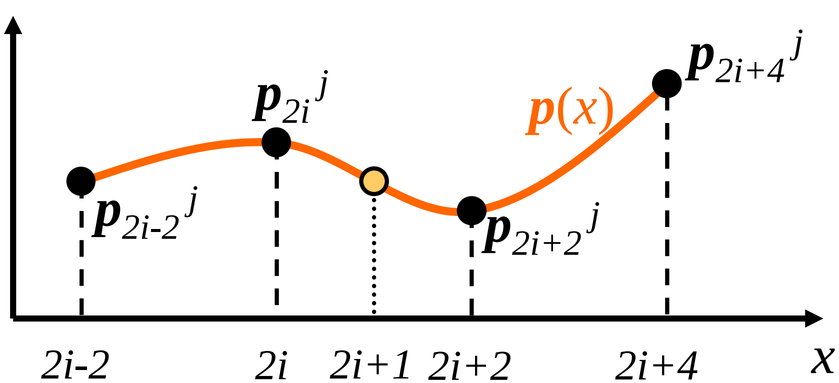
Curves – Schemata

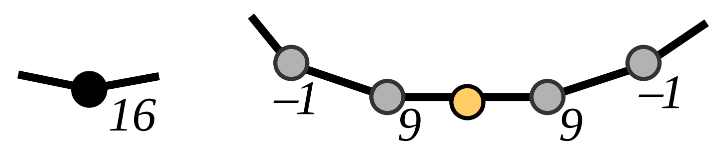
- approximating spline-schemata

Mask	Factor	[old], [new]	stencil
$\{1\ 2\ 1\}$	$\frac{1}{2}$	$[2], [1\ 1]$	
$\{1\ 3\ 3\ 1\}$	$\frac{1}{4}$	$[1\ 3], [3\ 1]$	
$\{1\ 4\ 6\ 4\ 1\}$	$\frac{1}{8}$	$[1\ 6\ 1], [4\ 4]$	



- interpolating: 4-point scheme





mask: $\{-1\ \ 0\ \ 9\ \ 16\ \ 9\ \ 0\ \ -1\}$

Curves – Analysis

Eigenvalue Analysis

- Examines the part of the subdivision matrix that corresponds to the mask of a vertex.
- Applies eigenvalue analysis to it
- Returns a necessary continuity condition: if the curve is C^k, then the condition holds
- provides stencils for the limit position and limit tangents

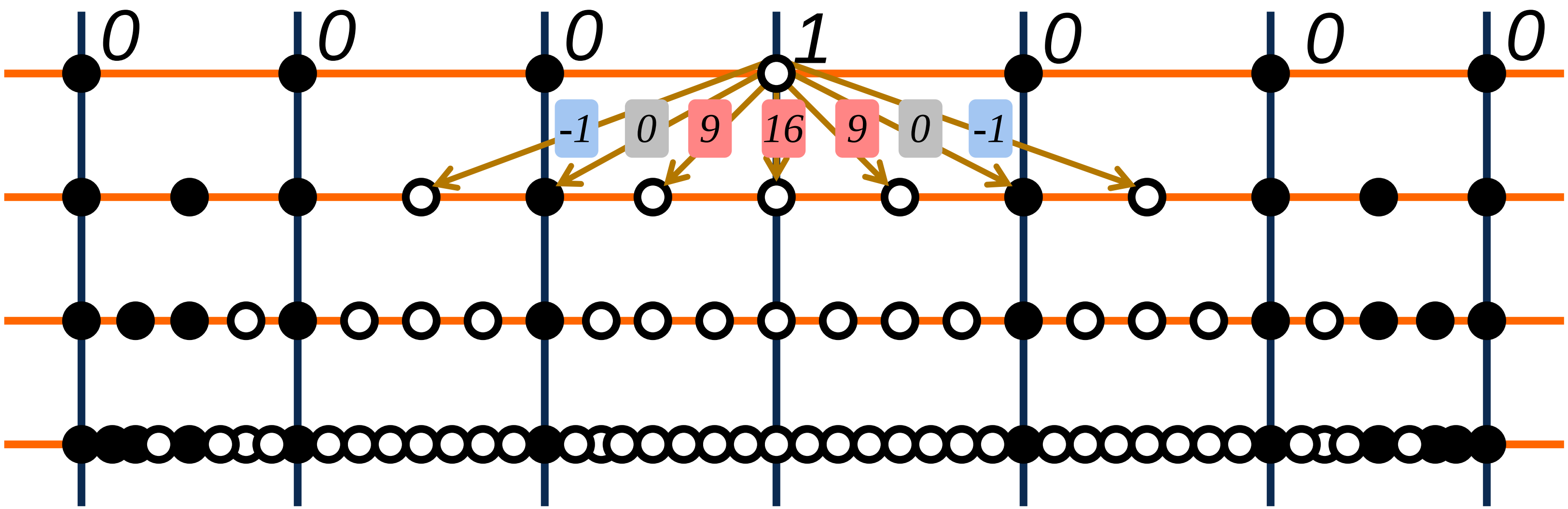
z-Transformation

- maps the coordinate vectors to polynomials
- Allows you to analyze the difference schemes that correspond to the derivatives.
- Returns a sufficient continuity condition: if condition holds, then the curve is C^k

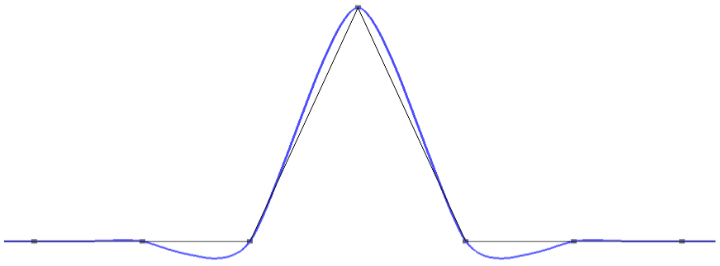
Computergraphik und Visualisierung

From mask follows influence region of point

Motivation with 4-point scheme $\{ -1 \;\; 0 \;\; 9 \;\; 16 \;\; 9 \;\; 0 \; -1 \}$



➔ For a mask of size m and an arity of a each point influences $(m-1) / a$ intervals to left and right

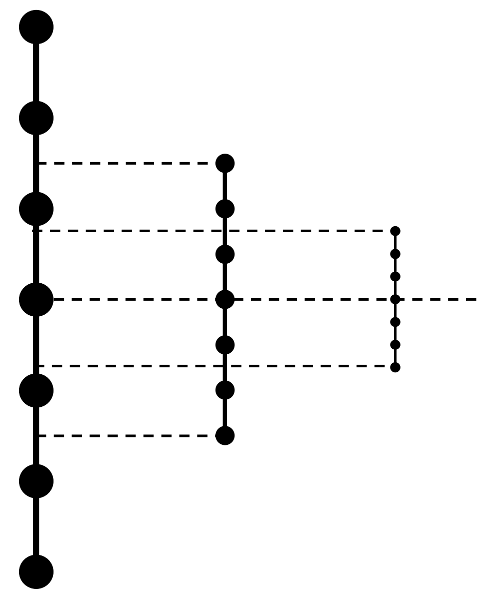


Infinitely many subdivision of $(0,\ldots,0,1,0,\ldots,0)$ yields base function

Curves – Eigenvalue Analysis

Local Subdivision Matrix

$$\begin{pmatrix} \underline{\mathbf{p}}_{-3}^{j+1} \\ \underline{\mathbf{p}}_{-2}^{j+1} \\ \underline{\mathbf{p}}_{-1}^{j+1} \\ \underline{\mathbf{p}}_{0}^{j+1} \\ \underline{\mathbf{p}}_{1}^{j+1} \\ \underline{\mathbf{p}}_{2}^{j+1} \\ \underline{\mathbf{p}}_{3}^{j+1} \end{pmatrix} = \underbrace{\frac{1}{16}\begin{pmatrix} -1 & 9 & 9 & -1 & 0 & 0 & 0 \\ 0 & 0 & 16 & 0 & 0 & 0 & 0 \\ 0 & -1 & 9 & 9 & -1 & 0 & 0 \\ 0 & 0 & 0 & 16 & 0 & 0 & 0 \\ 0 & 0 & -1 & 9 & 9 & -1 & 0 \\ 0 & 0 & 0 & 0 & 16 & 0 & 0 \\ 0 & 0 & 0 & -1 & 9 & 9 & -1 \end{pmatrix}}_{S_l} \begin{pmatrix} \underline{\mathbf{p}}_{-3}^{j} \\ \underline{\mathbf{p}}_{-2}^{j} \\ \underline{\mathbf{p}}_{-1}^{j} \\ \underline{\mathbf{p}}_{0}^{j} \\ \underline{\mathbf{p}}_{1}^{j} \\ \underline{\mathbf{p}}_{2}^{j} \\ \underline{\mathbf{p}}_{3}^{j} \end{pmatrix}$$



$$\mathbf{P}_l^{\,j} \xrightarrow{S_l} \mathbf{P}_l^{\,j+1} \xrightarrow{S_l} \mathbf{P}_l^{\,j+2}$$

Analysis applies only to countable many vertex positions, not to overcountable many positions in between

Curves – Eigenvalue Analysis

Analysis of local subdivision matrix:

- $\boldsymbol{e}$ is Eigenvector of $\boldsymbol{S}_l$ for Eigenvalue λ, iff $\boldsymbol{S}_l \boldsymbol{e} = \lambda \boldsymbol{e}$.
- **assumption:**
  $n \times n$-matrix $\boldsymbol{S}_l$ has basis of Eigenvectors $\boldsymbol{e}_0, \ldots, \boldsymbol{e}_{n-1}$ with Eigenvalues
  $$1 = \lambda_0 > \lambda_1 > \ldots > \lambda_{n-1}$$
- example: cubic B-splines ($n=5$)

$$\boldsymbol{S}_l = \frac{1}{8}\begin{pmatrix} 1 & 6 & 1 & 0 & 0 \\ 0 & 4 & 4 & 0 & 0 \\ 0 & 1 & 6 & 1 & 0 \\ 0 & 0 & 4 & 4 & 0 \\ 0 & 0 & 1 & 6 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \boldsymbol{e}_0 & \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 & \boldsymbol{e}_4 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 & 1 & 0 \\ 1 & -\frac{1}{2} & \frac{2}{11} & 0 & 0 \\ 1 & 0 & -\frac{1}{11} & 0 & 0 \\ 1 & \frac{1}{2} & \frac{2}{11} & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \lambda_0 & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{8} \end{pmatrix}$$

Curves – Eigenvalue Analysis

- If local subdivision matrix $\boldsymbol{S}_l$ has real basis of Eigenvectors $\boldsymbol{e}_i$ stored column-wise in matrix $\boldsymbol{E}$ and if the corresponding Eigenvalues λ_i are placed in the diagonal matrix $\boldsymbol{\Lambda}$, then the rows $\boldsymbol{e}_i'$ of $\boldsymbol{E}' = \boldsymbol{E}^{-1}$ are left Eigen-vectors of $\boldsymbol{S}_l$ with $\boldsymbol{e}_i'\boldsymbol{S}_l = \lambda_i \boldsymbol{e}_i'$ and:
$$\boldsymbol{S}_l = \boldsymbol{E}\boldsymbol{\Lambda}\boldsymbol{E}'$$
- j applications of $\boldsymbol{S}_l$ maps original point matrix $\boldsymbol{P}^0$ to by factor of 2^j shrunk neighborhood with point matrix $\boldsymbol{P}^j$:
$$\boldsymbol{P}^j = \boldsymbol{S}_l^j \boldsymbol{P}^0 = \boldsymbol{E}\boldsymbol{\Lambda}^j\boldsymbol{E}'\boldsymbol{P}^0$$
- For $j \mapsto \infty$ neighborhood gets shrunk to point with limit position, such that matrix $\boldsymbol{P}^\infty$ contains limit position in all rows. Furthermore, all entries in $\boldsymbol{\Lambda}^\infty$ are zero except for top left entry being $1 = 1^\infty$.
- Therefore, $\boldsymbol{e}_0'$ defines stencil to compute limit position

Curves – Eigenvalue Analysis

Analysis of local subdivision matrix:

- j subdivisions yield $\boldsymbol{P}^j = \boldsymbol{S}_l{}^j \boldsymbol{P}^0$ with an environment shrinked by a factor of 2^j.
- Consider the Eigenvalue decomposition

stencil for limit position

$$\boldsymbol{S}_l = \boldsymbol{E}\Lambda\boldsymbol{E}^{-1} = \begin{pmatrix} 1 & & \\ \vdots & \cdots & \boldsymbol{e}_{n-1} \\ 1 & & \end{pmatrix} \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & \lambda_{n-1} \end{pmatrix} \begin{pmatrix} \boldsymbol{e}'_0 \\ \vdots \\ \boldsymbol{e}'_{n-1} \end{pmatrix} \begin{pmatrix} \boldsymbol{P}^j \end{pmatrix}$$

- with $\boldsymbol{e}_j{}' \boldsymbol{e}_i = \delta_{ij}$ and $\lambda_0 \geq \lambda_1 \geq \ldots \geq \lambda_{n-1}$
- Each row of $\boldsymbol{S}_l$ (stencil) sums to 1
  $\Rightarrow \lambda_0 = 1$ and $\boldsymbol{e}_0 = (1, 1, 1, \ldots, 1)^t$

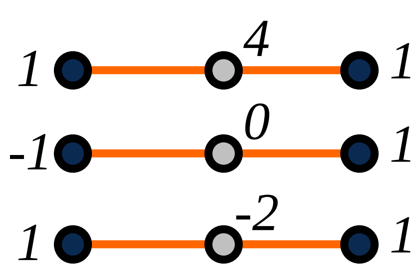
Curves – Eigenvalue Analysis

- example:
  - mask $\{1,4,6,4,1\}$
- Eigenvalue decomposition: →

$$S_l = \frac{1}{8}\begin{pmatrix} 1 & 6 & 1 & 0 & 0 \\ 0 & 4 & 4 & 0 & 0 \\ 0 & 1 & 6 & 1 & 0 \\ 0 & 0 & 4 & 4 & 0 \\ 0 & 0 & 1 & 6 & 1 \end{pmatrix}$$

$$E = \begin{pmatrix} 1 & -1 & 1 & 1 & 0 \\ 1 & -\frac{1}{2} & \frac{2}{11} & 0 & 0 \\ 1 & 0 & -\frac{1}{11} & 0 & 0 \\ 1 & \frac{1}{2} & \frac{2}{11} & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 \end{pmatrix} \quad \Lambda = \begin{pmatrix} 1 & & & & \\ & \frac{1}{2} & & & \\ & & \frac{1}{4} & & \\ & & & \frac{1}{8} & \\ & & & & \frac{1}{8} \end{pmatrix} \quad E^{-1} = \begin{pmatrix} 0 & \frac{1}{6} & \frac{4}{6} & \frac{1}{6} & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & \frac{11}{6} & \frac{-22}{6} & \frac{11}{6} & 0 \\ 1 & -3 & 3 & -1 & 0 \\ 0 & -1 & 3 & -3 & 1 \end{pmatrix}$$

- Resulting stencils
  - limit position
  - tangent at limit curve
  - curvature vector



Curves – Eigenvalue Analysis

convergence and continuity criteria

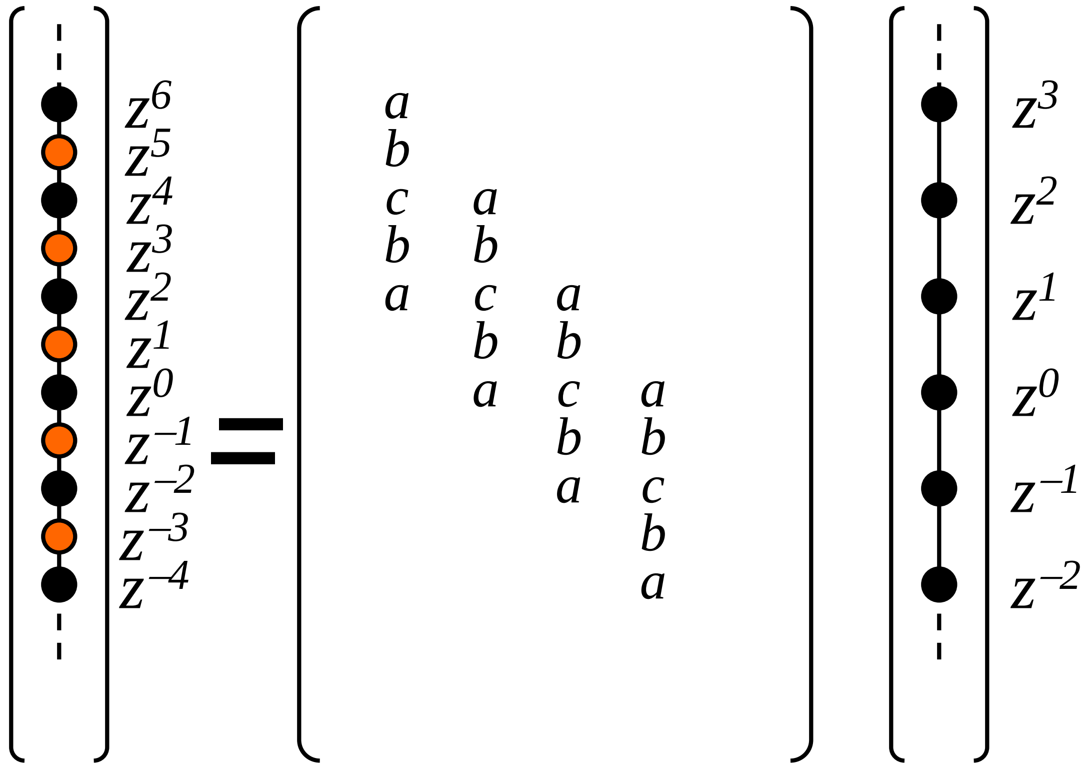
- In case of convergence, then $\lambda_0 = 1 > \lambda_1 \geq \lambda_2 \ldots$
- If limit curve is
  - C^1 continuous then $\lambda_1 > \lambda_2$
  - has unlimited curvature then $\lambda_1^2 < \lambda_2$
  - limited curvature then $\lambda_1^2 = \lambda_2$
  - curvature 0 everywhere then $\lambda_1^2 > \lambda_2$

- necessary continuity condition:
  if limit curve is C^k, then there exists

$$\exists \alpha < 1 : \forall i = 1 \ldots k : \lambda_i = \alpha^i$$

from matrix to Laurent polynomial notation

$p_{j+1}(z) =$



$m(z)\,p_j(z^2)$

$$m(z) = \sum_{i=1}^{m} \boldsymbol{M}_i z^{i-\frac{m+1}{2}}$$

Laurent polynomial

$$p_j(z) = \sum_i \boldsymbol{P}_i^j z^i$$

Curves – z-Transformation

$$m(z) = az^{-2} + bz^{-1} + c + dz + ez^{2}$$

$$p_j(z) = x_{-2}z^{-2} + x_{-1}z^{-1} + x_0 + x_1 z + x_2 z^{2}$$

$$p_j(z^2) = x_{-2}z^{-4} + x_{-1}z^{-2} + x_0 + x_1 z^{2} + x_2 z^{4}$$

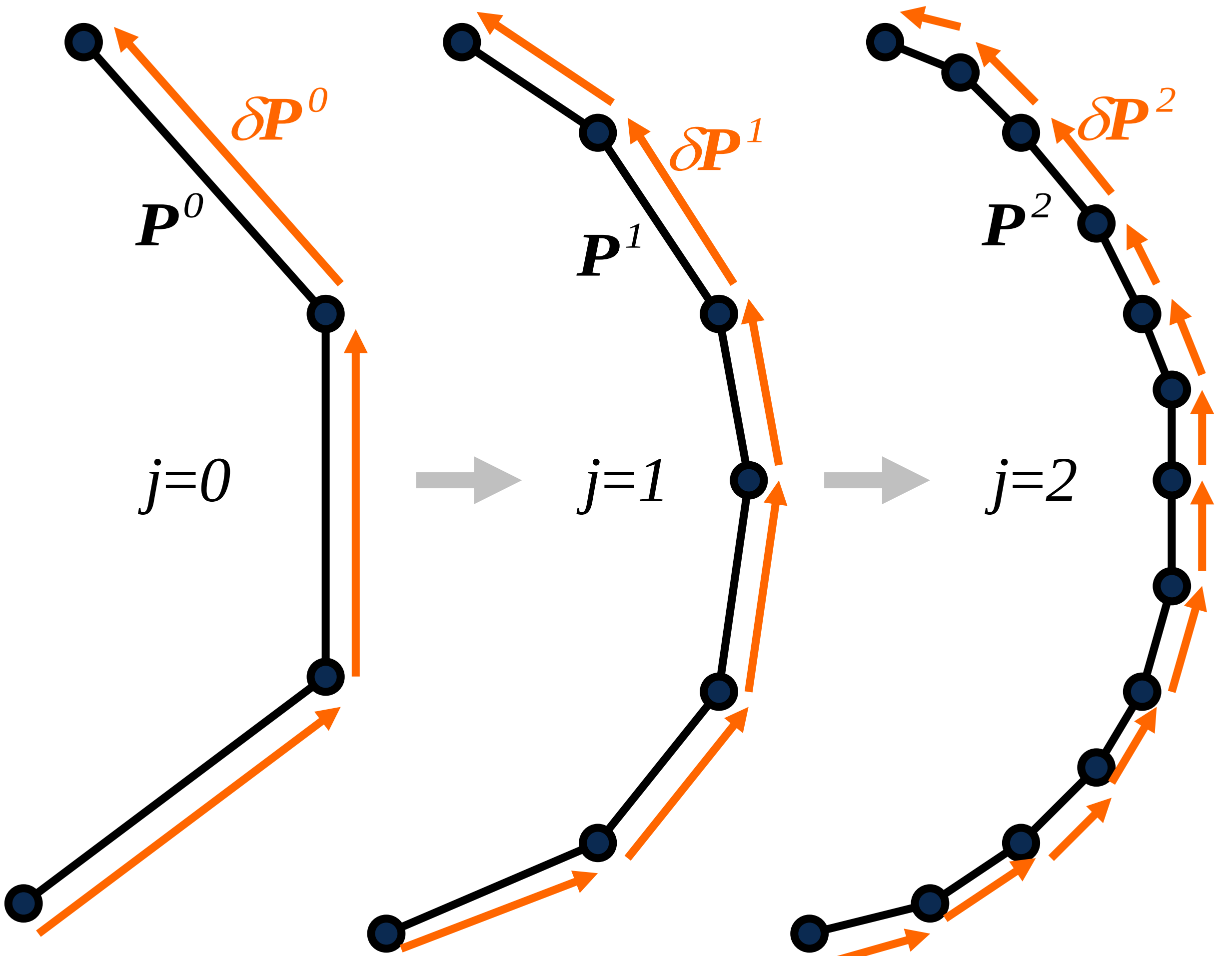
$$p_j(z^2)m(z) = \left(x_{-2}z^{-4} + x_{-1}z^{-2} + x_0 + x_1 z^{2} + x_2 z^{4}\right)m(z)$$

$$\begin{aligned}
= {} & ax_{-2}z^{-6} + ax_{-1}z^{-4} + ax_0 z^{-2} + ax_1 + ax_2 z^{2} + {} && az^{-2}\\
& bx_{-2}z^{-5} + bx_{-1}z^{-3} + bx_0 z^{-1} + bx_1 z + bx_2 z^{3} + {} && bz^{-1}\\
& cx_{-2}z^{-4} + cx_{-1}z^{-2} + cx_0 + cx_1 z^{2} + cx_2 z^{4} + {} && \cdot\; c\\
& dx_{-2}z^{-3} + dx_{-1}z^{-1} + dx_0 z + dx_1 z^{3} + dx_2 z^{5} + {} && dz\\
& ex_{-2}z^{-2} + ex_{-1} + ex_0 z^{2} + ex_1 z^{4} + ex_2 z^{6} && ez^{2}
\end{aligned}$$

$$= \ldots + \left(ax_0 + cx_{-1} + ex_{-2}\right)z^{-2} + \left(bx_0 + dx_{-1}\right)z^{-1} +$$
$$\left(ax_1 + cx_0 + ex_{-1}\right) + \left(bx_1 + dx_0\right)z + \left(ax_2 + cx_1 + ex_0\right)z^{2} + \ldots$$

Curves – z-Transformation

- motivation for convergence criterium



- Differences must converge to zero
- This can be proven, if the absolute value of the difference vectors is limited by c^j with $c<1$
- In this case one calls the difference scheme contractive

Difference Scheme

- multiplication with z shifts all points by one
- Laurent-polynom for difference of neighboring points computes to

$$\delta p_j(z) = (z-1)p_j(z) \qquad (1)$$

- We look for the mask δm of the difference scheme, for which holds: $$\delta p_{j+1}(z) = \delta m(z)\cdot \delta p_j(z^2) \qquad (2)$$
- derivation from m with $$p_{j+1}(z) = m(z)\cdot p_j(z^2) \qquad (3)$$

$$\delta p_{j+1}(z) \overset{(2)}{=} \delta m(z)\cdot \delta p_j(z^2) \overset{(1)}{=} \delta m(z)(z^2-1)p_j(z^2) = \delta m(z)(z+1)(z-1)p_j(z^2)$$

$$\delta p_{j+1}(z) \overset{(1)}{=} (z-1)p_{j+1}(z) \overset{(3)}{=} (z-1)\cdot m(z)\cdot p_j(z^2) = m(z)\cdot(z-1)\cdot p_j(z^2)$$

- mask of difference scheme computes to $$\delta m(z) = \frac{m(z)}{z+1}$$

Curves – z-Transformation

Sufficient Continuity Condition

- a mask is called contractive, if the absolute values of even and odd elements, i.e. of the stencils, sum to less than 1.
- The limit curve is C^k-continuous, if the $k+1$-th difference scheme multiplied by 2^k is contractive, i.e. $m(z)$ can be divided by $(z+1)^{k+1}$ and $2^k \cdot \delta^{k+1} m(z)$ is contractive.

corner cutting example: $m(z) = \frac{1}{4}z^2 + \frac{3}{4}z^1 + \frac{3}{4}z^0 + \frac{1}{4}z^{-1}$

$$\delta m(z) = \left(\frac{1}{4}z^2 + \frac{3}{4}z^1 + \frac{3}{4}z^0 + \frac{1}{4}z^{-1}\right) : (z+1) = \frac{1}{4}z^1 + \frac{1}{2}z^0 + \frac{1}{4}z^{-1}$$

$$\frac{1}{4}z^2 + \frac{1}{4}z^1$$

$$\frac{1}{2}z^1 + \frac{3}{4}z^1 + \frac{1}{4}z^{-1}$$

$$\frac{1}{2}z^1 + \frac{1}{2}z^0$$

$$\frac{1}{4}z^0 + \frac{1}{4}z^{-1}$$

C^0 check:

$2^0 \cdot \delta m(z)$ contractive?

$$\frac{1}{4} + \frac{1}{4} = \frac{1}{2} < 1 \checkmark$$

$$\frac{1}{2} < 1 \checkmark$$

$$\delta^2 m(z) = \left(\frac{1}{4}z^1 + \frac{1}{2}z^0 + \frac{1}{4}z^{-1}\right) : (z+1) = \frac{1}{4}z^0 + \frac{1}{4}z^{-1}$$

$$\frac{1}{4}z^1 + \frac{1}{4}z^0$$

$$\frac{1}{4}z^0 + \frac{1}{4}z^{-1}$$

C^1 check:

$2^1 \cdot \delta^2 m(z)$ contractive?

$$2 \cdot \frac{1}{4} = \frac{1}{2} < 1 \checkmark$$