CG3

Introduction to Physically Based Simulation

Content

Introduction to Physically Based Simulation

Introduction to Physically Based Simulation

HARMONIC OSCILLATOR

Preliminaries
Newton's laws of motion

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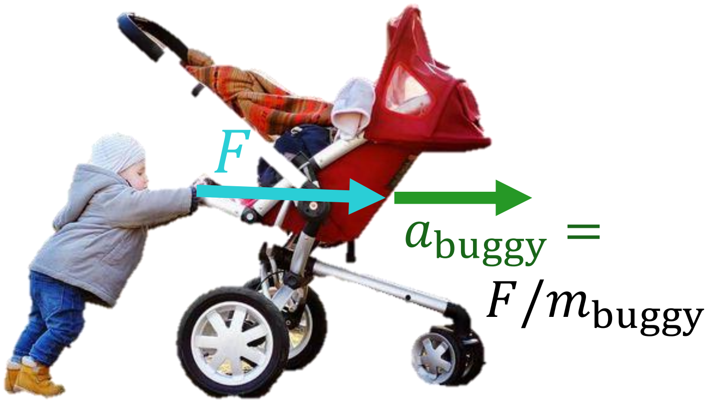
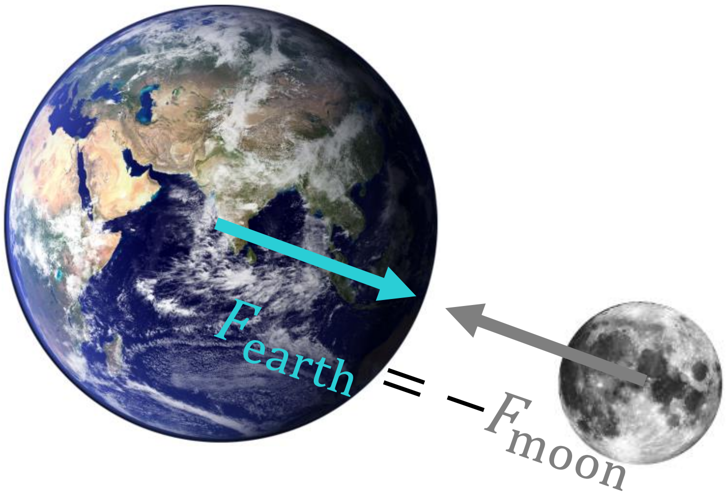
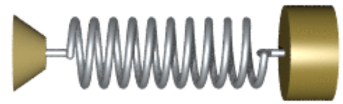
- **First law:** In an inertial frame of reference, an object either remains at rest or continues to move at a constant velocity, unless acted upon by a force.
- **Second law:** In an inertial frame of reference, the vector sum of the forces F on an object with constant mass m is equal to m multiplied by the acceleration a of the object: $F = ma$.
- **Third law:** When one body exerts a force on a second body, the second body simultaneously exerts a force equal in magnitude and opposite in direction on the first body.





Harmonic Oscillator
Analytic Solution



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- Hook's law: $F_H = -kx$ (spring force: F_H; stiffness constant of spring: k)
- 2nd Newton's law: $F = F_H = ma = m\ddot{x}$, with $\ddot{x} = \frac{d^2x}{dt^2}$
- eq. of motion: $m\ddot{x} = -kx$, with $x(0) = x_0$, $\dot{x}(0) = \dot{x}_0$
- solution to initial value problem:

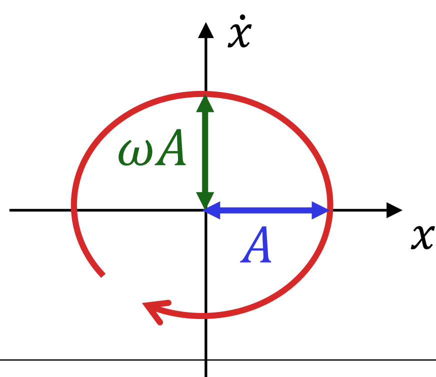
$$x(t) = A \cdot \cos(\omega t + \phi) \quad \text{with } \omega = \sqrt{\frac{k}{m}}, \quad [A, \phi](x_0, \dot{x}_0)$$

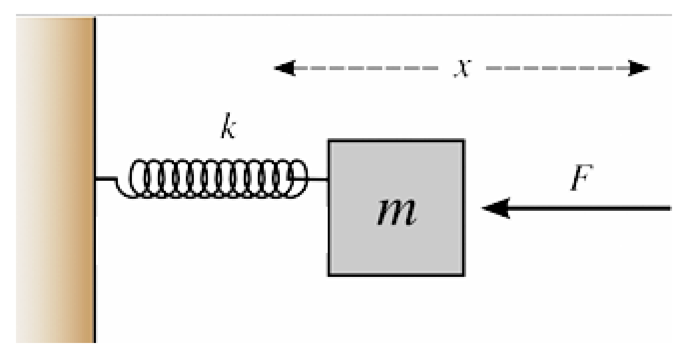
(angular frequency: ω; phase: ϕ)

$$\dot{x}(t) = -\omega A \cdot \sin(\omega t + \phi)$$

$$\ddot{x}(t) = -\omega^2 A \cdot \cos(\omega t + \phi)$$

- state space diagram:





Harmonic Oscillator Numeric Solution

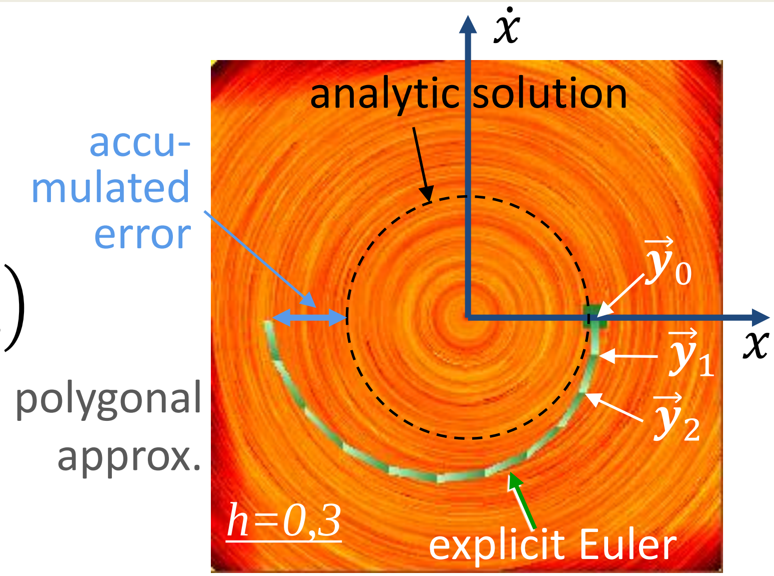
- combine location and velocity to state vector $\vec{y} = \begin{pmatrix} x \\ v \equiv \dot{x} \end{pmatrix}$
- rewrite equations of motion

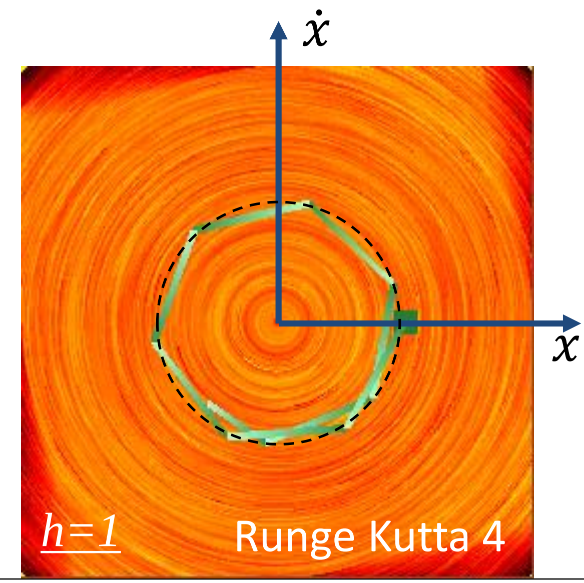
$$m\ddot{x} = -kx \Rightarrow \dot{\vec{y}}_i = \begin{pmatrix} \dot{x} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix} \begin{pmatrix} x \\ v \end{pmatrix}$$

- start with initial state $\vec{y}_0 = \begin{pmatrix} x_0 \\ v_0 \end{pmatrix}$
- discretize state space trajectory with polygon through $\vec{y}_{i=0\ldots n}$
- simplest approach is explicit Euler:

$$\vec{y}_{i+1} = \vec{y}_i + h\dot{\vec{y}}_i$$

  with step width h.
- very small step widths necessary for explicit Euler
- use higher order methods





Introduction to Physically Based Simulation

PHYSICAL QUANTITIES

Physical Quantities
Physical Constants

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- A physical constant is a physical quantity that is generally believed to be both universal in nature and constant in time.
- The concrete value of a physical constant depends on the chosen units.
- In SI units the most fundamental constants are
  - speed of light in vacuum c 299 792 458 m·s^{-1}
  - Newtonian constant of gravitation γ 6.674 30(15)×10^{-11}m^3·kg^{-1}·s^{-2}
  - Planck constant h 6.626 070 15 × 10^{-34} J·s

uncertainty in last two digits

- The appearance of our universe strongly depends on the values of these constants
- Anthropic principle: the values of the constants must be like that as we do observe them as living beings.

Physical Quantities
Physical Units

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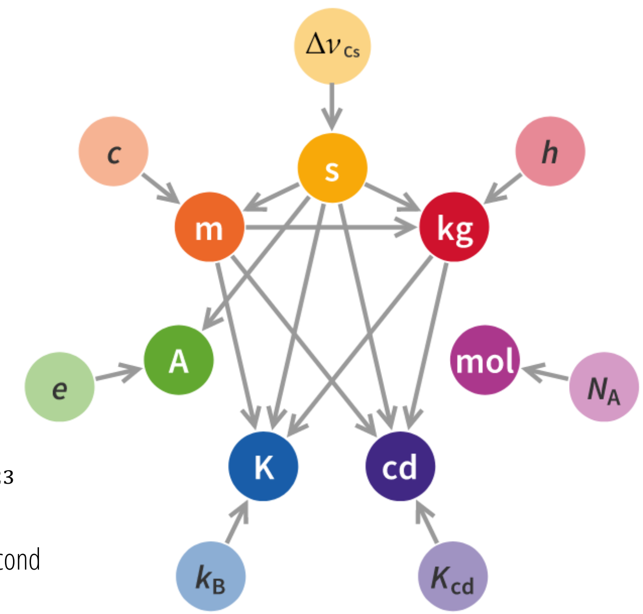
- A physical quantity is always accompanied by a physical unit

$$l = 1\text{m}, v = 5\frac{\text{m}}{\text{s}}, m = 2\text{kg}, \ldots$$

- Système International d'unités defines the 7 base units
  - Kelvin (temperature), 1/273.16 of temperature of the triple point of water
  - second (time), 9192631770 cycles of a Caesium atomic clock
  - meter (distance), path length travelled by light in a vacuum in 1/299792458 second
  - kilogram (mass), mass of Big K, till Nov 2018, now from $h = 6.62607015 \cdot 10^{-34}\ \frac{\text{kg}\cdot m}{s^2}$
  - candela (luminous intensity),
  - mole (amount of substance) number $N_A = 6.02214076 \times 10^{23}$ of atoms in 12g ^{12}C
  - Ampere (electric current) electric current carried by $1/e$ electrons per second with $e = 1.602176634 \cdot 10^{-19}\text{C}$



Dependence of 7 SI base unit definitions on physical constants with fixed numerical values and on other base units

- derived units can be written with a scale n and 7 exponents in terms of a base unit:

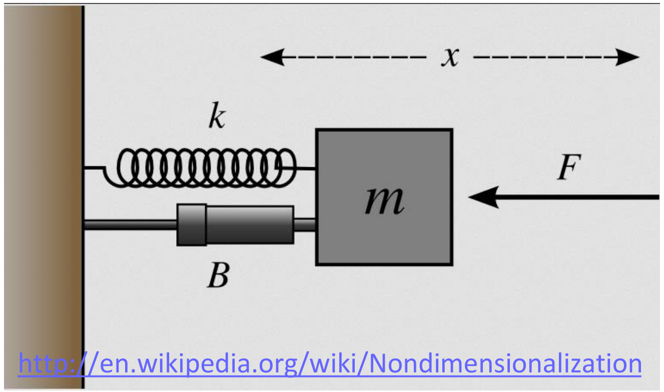
$$10^n \cdot \text{m}^\alpha \cdot \text{kg}^\beta \cdot s^\gamma \cdot \text{A}^\delta \cdot \text{K}^\varepsilon \cdot \text{mol}^\zeta \cdot \text{cd}^\eta$$

$$Ohm: \Omega = \text{m}^2 \cdot \text{kg}^1 \cdot s^{-3} \cdot \text{A}^{-2}$$

Physical Quantities
Physical Units

- Physical units help to validate physical formulas and to derive units of physical constants:
  - one cannot add quantities of different units
  - left and right side of an equality must have the same unit
  - The arguments of functions like sin, cos, exp, … cannot have a unit (also steradian is not allowed here)

Example: damped harmonic oszillator



- $m\frac{d^2x}{dt^2} + B\frac{dx}{dt} + kx = F(t)$
- What unit has damping constant?

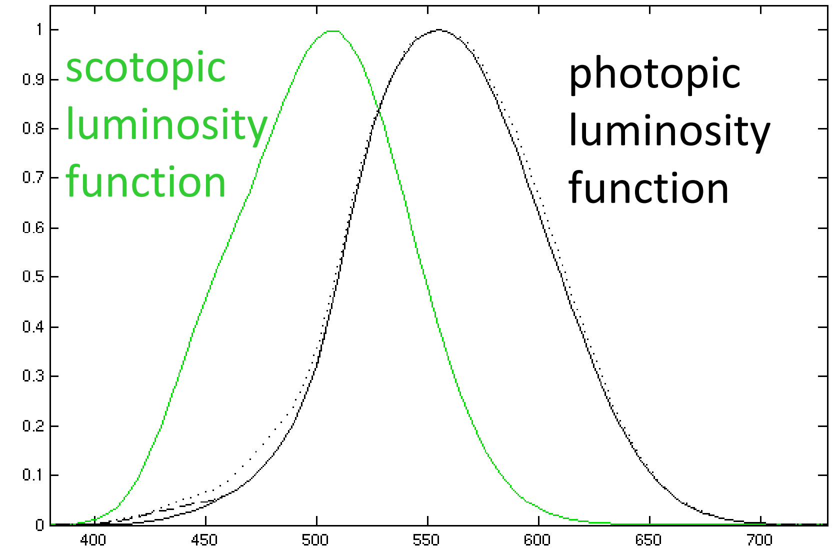
$$[B] = [F(t)]/\left[\frac{dx}{dt}\right]$$

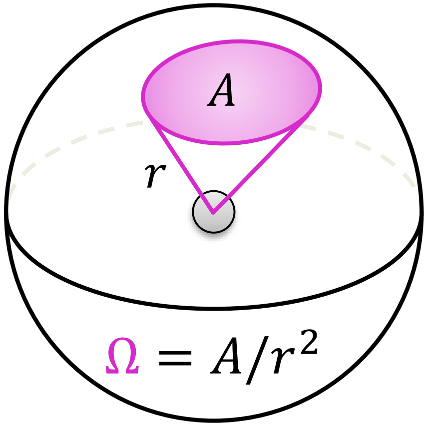
$$= \mathrm{kg}\cdot\mathrm{m}\cdot\mathrm{s}^{-2}/\mathrm{m}\cdot\mathrm{s}^{-1} = \mathrm{kg}\cdot\mathrm{s}^{-1}$$

x = displacement from equilibrium [m]
t = time [s]
F = external force [$\mathrm{kg}\cdot\mathrm{m}\cdot\mathrm{s}^{-2}$]
m = mass of the block [kg]
B = damping constant
k = force constant of spring [$\mathrm{kg}\cdot\mathrm{s}^{-2}$]

Physical Quantities
Candela and Luminous Intensity

- origin: 1 candela measures how bright the human eye perceives a wax candle.
- quantity: luminous intensity
- Brightness perception of human eye increases with light power and depends on wavelength through the CIE standardized photopic luminosity function $\bar{y}(\lambda)$
- To relate light intensity (emitting from a point) to power [Watt] one integrates intensity over direction
- Solid angle measures directions by their covered area on the unit sphere and is measured in steradian (sr)

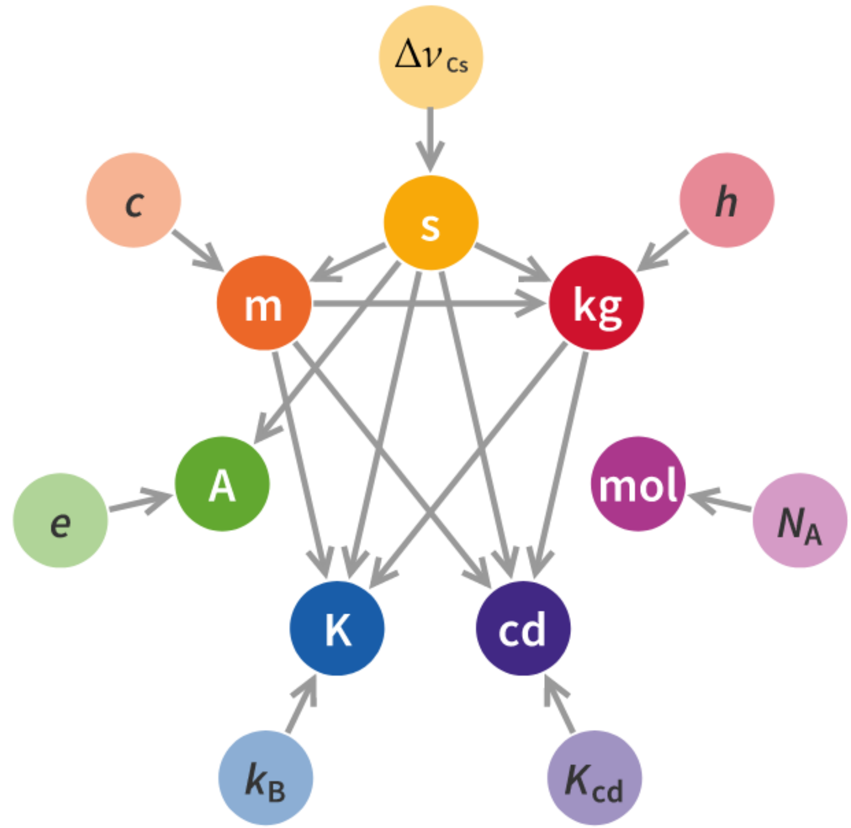




solid angle Ω of area A measured in steradian sr on sphere with radius r

Physical Quantities
Candela and Luminous Intensity

- **Def.:** 1 candela in given direction is luminous intensity of light source emitting 1/683 Watt of monochromatic green ($K_{\mathrm{cd}} = 540 \times 10^{12}\mathrm{Hz}$) light per sr.
- Its definition depends on Watt, Hertz and steradian and therefore on m, s, and kg:
  - hertz: $\mathrm{Hz} = \mathrm{s}^{-1}$
  - watt: $\mathrm{W} = \mathrm{kg} \cdot \mathrm{m}^2 \cdot \mathrm{s}^{-3}$
  - steradian: $\mathrm{sr} = \mathrm{m}^2 \cdot \mathrm{m}^{-2}$
- For steradian all base units cancel out, but we still write it as sr to distinguish from a pure scalar.



Physical Quantities
Standard and Non-Standard Units

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For physically based graphics (excluding molecular and electro dynamics) the following standard and non-standard units are important

Second	s	duration, time		s
Meter	m	length, position, size		m
Kilogram	kg	mass		kg
Kelvin	K	temperature		K
Candela	cd	luminous intensity		cd
Steradian	sr	solid angle		$m^2 \cdot m^{-2}$
Lumen	lm	luminous flux		$cd \cdot sr$
Lux	lx	illuminance	lm/m^2 =	$cd \cdot sr \cdot m^{-2}$
Hertz	Hz	frequency		s^{-1}
Newton	N	force		$kg \cdot m \cdot s^{-2}$
Pascal	Pa	pressure	N/m^2 =	$kg \cdot m^{-1} \cdot s^{-2}$
Joule	J	energy, work, heat	$N \cdot m$ =	$kg \cdot m^2 \cdot s^{-2}$
Watt	W	power, radiant flux	J/s =	$kg \cdot m^2 \cdot s^{-3}$

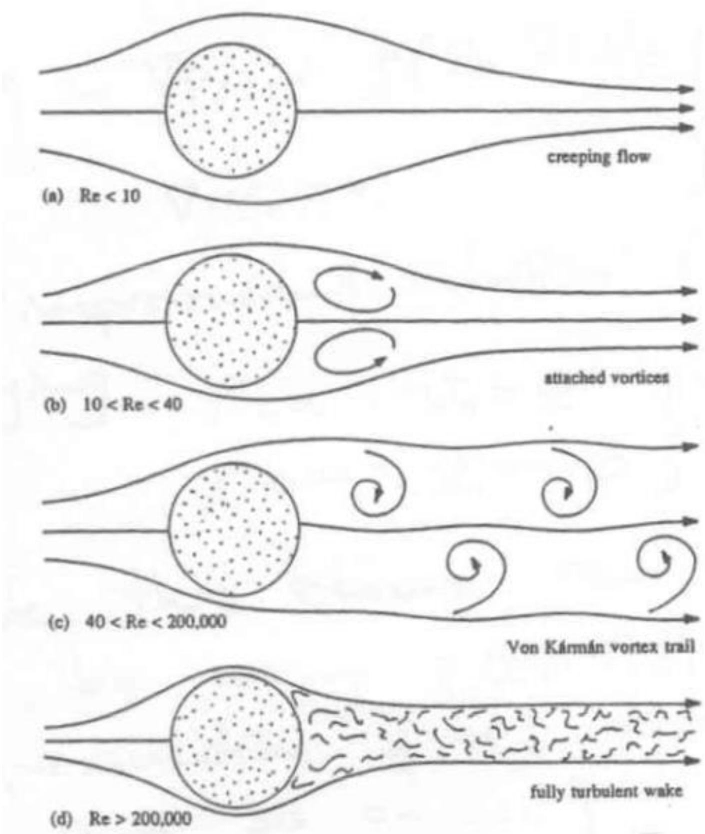
Physical Quantities Nondimensionalization

- If the physical laws of a system are known one can get rid of all units by a simple variable substitution.
- The remaining constants are dimensionless and parameterize different behaviors of the system.

Example:

- dimensionless Navier Stokes Equations of incompressible fluids are parametrized over Reynolds number Re:

$$\left.\frac{\partial u_i}{\partial t}+\sum_j u_j\frac{\partial u_i}{\partial x_j}=-\frac{\partial p}{\partial x_i}+\frac{1}{\mathrm{Re}}\sum_j\frac{\partial^2 u_i}{\partial x_j^2}+f\right|_{i\in\{1,2,3\}} \quad \text{and} \quad \sum_j\frac{\partial u_j}{\partial x_j}=0$$



Patterns in fluid flow around a cylinder as a function of the Reynolds number.

Image Source: http://www.hitech-projects.com/euprojects/artic/index/Low%20Reynolds%20number%20flows.pdf

Nondimensionalization

Example: damped harmonic oszillator

$$m\frac{d^2x}{dt^2} + B\frac{dx}{dt} + kx = AF(t)$$

- Substitutions: $\tau = \frac{t}{t_c}$ and $\chi = \frac{x}{x_c}$

$$m\frac{x_c}{t_c^2}\frac{d^2\chi}{d\tau^2} + B\frac{x_c}{t_c}\frac{d\chi}{d\tau} + kx_c\chi = AF(\tau t_c)$$

- Division of constant from highest derivative order term:

$$\frac{d^2\chi}{d\tau^2} + t_c\frac{B}{m}\frac{d\chi}{d\tau} + t_c^2\frac{k}{m}\chi = \frac{t_c^2}{mx_c}Af(\tau)$$

- Define constant $t_c = \sqrt{\frac{m}{k}}$ and $x_c = \frac{A}{k}$

$$\frac{d^2\chi}{d\tau^2} + 2\zeta\frac{d\chi}{d\tau} + \chi = f(\tau),$$ with damping ratio $2\zeta = \frac{B}{\sqrt{mk}}$

- unit free damping ratio defines system behavior

Physical Quantities
Summary

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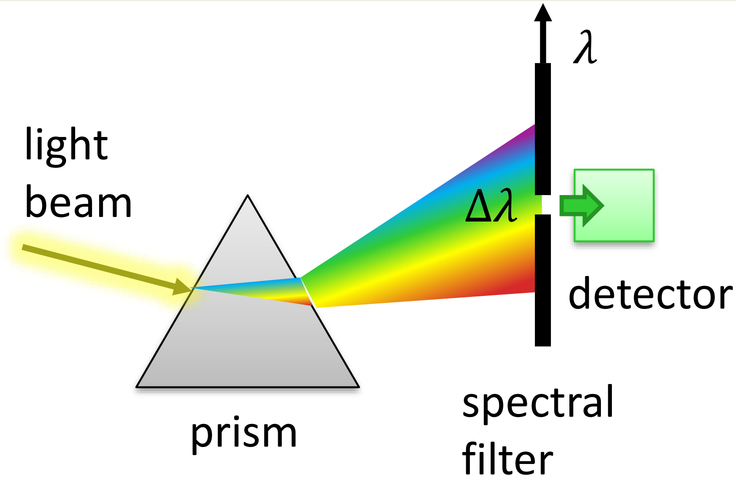
- Physical quantities carry units that help to validate formula and to interpret constants
- SI defines 7 base units and expresses all other units as powers of these
- One of these 7 is Candela, which is used for photogram-metric measurements of visible light intensity relative to the spectral sensitivity of the human eye.
- A few physical constants define the physics in our universe that would look very different if their values would change slightly
- For a given physical system nondimensionalization allows to describe the system with unit free variables and a few characteristic unit free parameters

Introduction to Physically Based Simulation

APPLIED ANALYSIS

Applied Analysis
One Independent Variable

- The specialization of a quantity with respect to some independent variable is mathematically describes as a derivative
- From a physical point of view one designs a filter that restricts the measurement process to a small interval of the independent variable
- We need specialization with respect to time to learn about the time evolution
  - power is work done per time
  - velocity is path length change per time
  - acceleration is velocity change per time
- Other important independent variables are location, direction and wavelength
  - radiant power per wavelength is spectral power



power: $P = \frac{dW}{dt}$

velocity: $v = \frac{ds}{dt}$

acceleration: $a = \frac{dv}{dt} = \frac{d^2 s}{dt^2}$

spectral power: $\Phi_\lambda = \frac{d\Phi}{d\lambda}$

Density

- triple derivative of mass m with respect to volume V yields density ρ
- unit of density is kg/m^3
- triple integration of density gives back mass

$$\rho = \frac{dm}{dV} = \frac{d^3 m(x,y,z)}{dxdydz}$$

$$m = \iiint_V \rho(x,y,z)dxdydz$$

Applied Analysis
Field vs Particles

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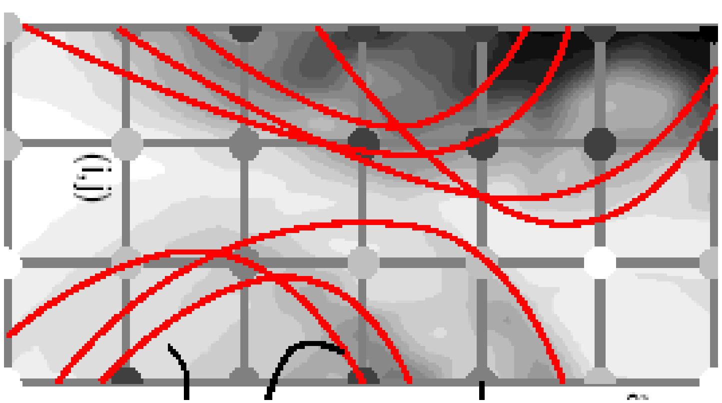
Eulerian View

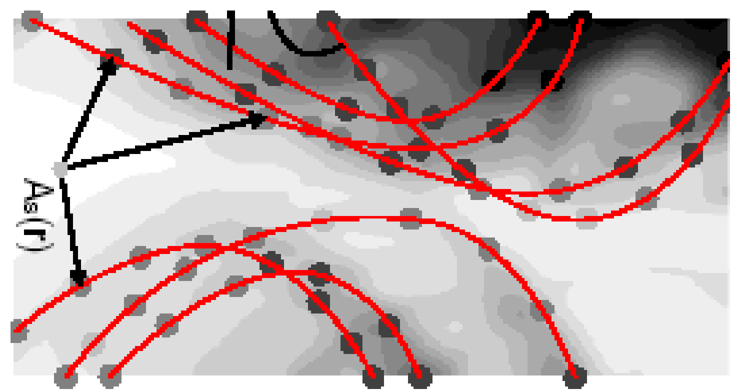
- Describe physics as fields (flow field, irradiance fields, …) over space
- For simulation fields are often discretized over grids or meshes and finite difference or finite element methods are applied.



fields are easy to implement

Lagragian View

- Describe physics in form of particles that move in space
- During particle simulations, particles are the discretization unit and typically do not represent single physical particles (photons, molecules, …) but bundles of them



particles are conceptually simple

Field to Particles and Back

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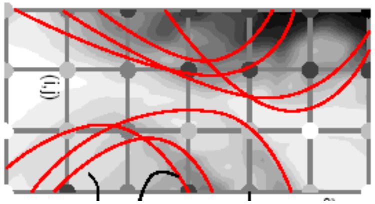


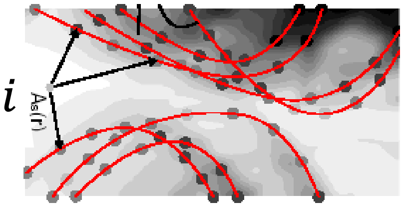
ρ_{ijk}

specialize

m

specialize & discretize

p_i



$\rho(x,y,z) = \partial_x \partial_y \partial_z m$

discretize

ρ_{ijk}

density values per grid location

$P = \{p_i = (x_i, y_i, z_i, m_i)\}_i$

set of particles

- Conversion from field to particles is done by interpreting density as particle probability and sampling

$\rho(x,y,z)$ — sampling → P

- Conversion back to [discretized] fields through density estimation and reconstruction

$\rho(x,y,z)$ ← reconstruction — ρ_{ijk} ← density estimation — P

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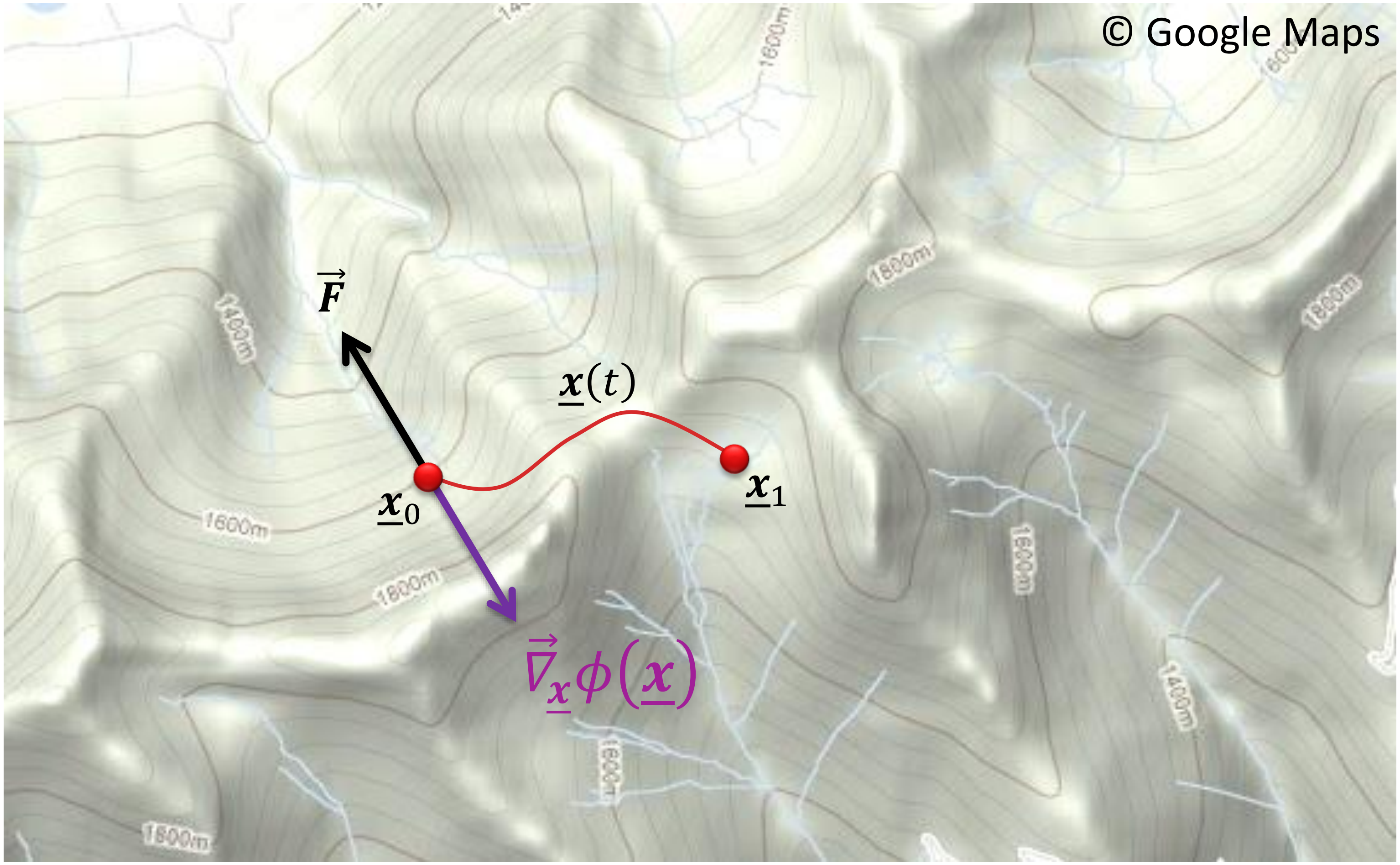


gradient is direction of steepest ascent, force points in opposite direction

Example: Potential energy for near-Earth gravity $\phi_g(\underline{x}) = m \cdot g \cdot h(\underline{x})$

Conservative Force, Potential Energy

- negated gradient of potential energy ϕ with respect to location $\underline{\boldsymbol{x}}$ yields conservative force $\vec{\boldsymbol{F}}$. Unit: $N = J/m$
- work W done by force from integration along path $\underline{\boldsymbol{x}}(t)$
- gradient theorem states that work done by force is potential difference of path end points
- no work done nor necessary for cyclic paths where $\underline{\boldsymbol{x}}_0 = \underline{\boldsymbol{x}}_1$
- force is conservative, iff curl of force vanishes everywhere

$$\vec{\boldsymbol{F}} = -\vec{\nabla}_{\underline{\boldsymbol{x}}}\phi(\underline{\boldsymbol{x}})$$

$$W = \int_{\underline{\boldsymbol{x}}_0}^{\underline{\boldsymbol{x}}_1} \vec{\boldsymbol{F}} d\underline{\boldsymbol{x}} = \int_{t_0}^{t_1} \langle \vec{\boldsymbol{F}}, \vec{\dot{\boldsymbol{x}}} \rangle dt$$

$$W = -\int_{\underline{\boldsymbol{x}}_0}^{\underline{\boldsymbol{x}}_1} \vec{\nabla}_{\underline{\boldsymbol{x}}}\phi(\underline{\boldsymbol{x}}) d\underline{\boldsymbol{x}}$$

$$= -\phi(\underline{\boldsymbol{x}})\Big|_{\underline{\boldsymbol{x}}_0}^{\underline{\boldsymbol{x}}_1} = \phi(\underline{\boldsymbol{x}}_0) - \phi(\underline{\boldsymbol{x}}_1)$$

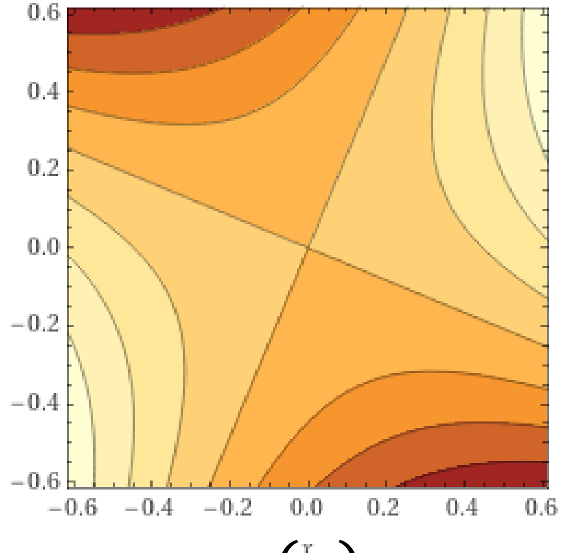
$$\vec{\nabla}_{\underline{\boldsymbol{x}}} \times \vec{\boldsymbol{F}} = \begin{pmatrix} \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \\ \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \\ \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \end{pmatrix} = \vec{\boldsymbol{0}}$$

Example 1 – Conservative Force

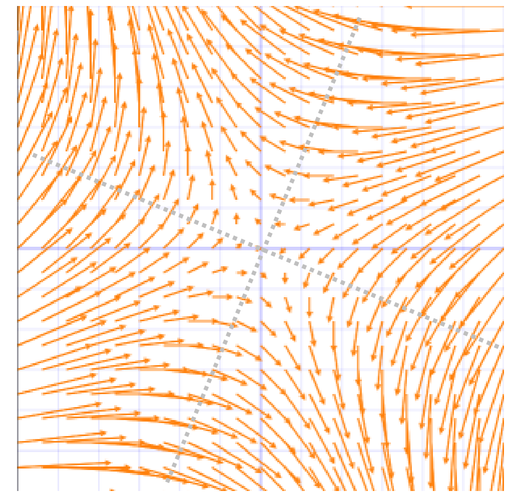
$$\phi(\underline{\boldsymbol{x}}) = x^2 + 2xy - y^2$$

$$F_x = -\partial_x \phi = -2x - 2y$$
$$F_y = -\partial_y \phi = -2x + 2y$$

$$\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{F}}\Big|_z = \partial_x F_y - \partial_y F_x$$
$$= -2 - (-2) = 0$$



$\phi(\underline{\boldsymbol{x}})$



$$\vec{\boldsymbol{F}} = -2\begin{pmatrix} x + y \\ x - y \end{pmatrix}$$

$$-\int F_x\, dx = x^2 + xy + C_x$$
$$\parallel$$
$$-\int F_y\, dy = -y^2 + xy + C_y$$

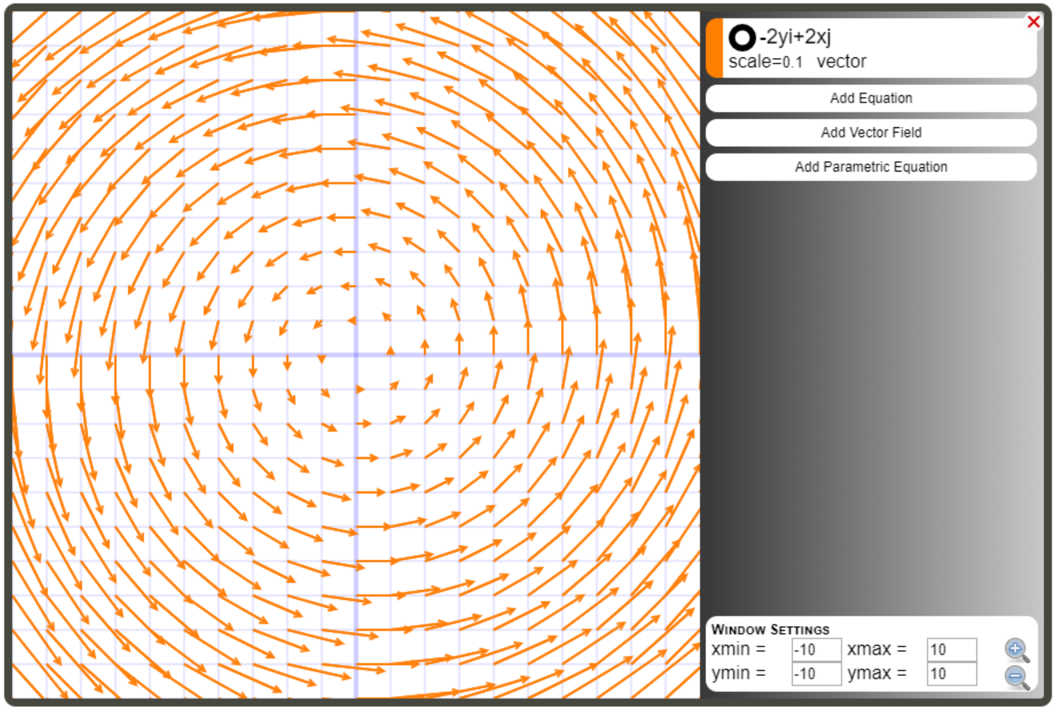
$$\Rightarrow \quad C_x = -y^2 + C$$
$$C_y = x^2 + C$$

$$\phi(\underline{\boldsymbol{x}}) = x^2 + 2xy - y^2 + C$$

Example 2 – Non Conservative Force

$$\vec{F} = 2\begin{pmatrix}-y\\x\end{pmatrix}$$

$$\vec{\nabla} \times \vec{F}\Big|_z = \partial_x F_y - \partial_y F_x$$
$$= 2 - (-2) = 4 \neq 0$$



$$-\int F_x\, dx = 2xy + C_x$$

$$\parallel$$

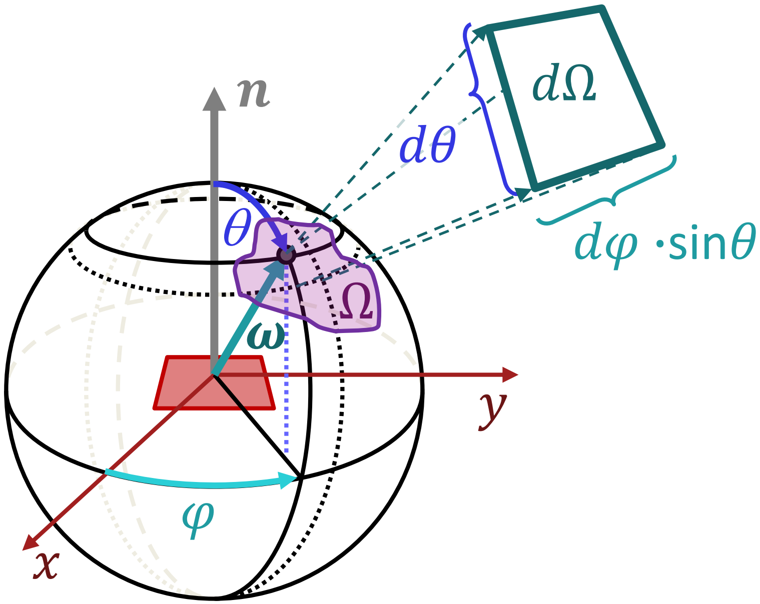
$$-\int F_y\, dy = -2xy + C_y$$

⇒

$$C_x = ?$$
$$C_y = ?$$

Directions and Solid Angle

- Solid angle is used to measure a set of directions $\boldsymbol{\omega}$ repre-sented as unit vectors
- Typically $\boldsymbol{\omega}$ is parameterized over the *unit sphere* in spherical coordinates φ, θ
- *This parametrization is relative to local surface normal* $\boldsymbol{n}$
- solid angle Ω is measured in area covered on unit sphere
- nonstandard unit: *sr (steradian)*
- integration of solid angle yields double integral over φ and θ



$$\boldsymbol{\omega} = \begin{pmatrix} \cos\varphi\sin\theta \\ \sin\varphi\sin\theta \\ \cos\theta \end{pmatrix}, \quad \begin{matrix} \varphi \in [-\pi, \pi] \\ \theta \in [0, \pi] \end{matrix}$$

$$d\Omega = d\varphi \cdot \sin\theta d\theta$$

Applied Analysis
Directions and Solid Angle

- The solid angle corresponding to all directions is 4π
- The solid angle of a hemisphere (directions to the outside at surface point), is therefore 2π
- One just needs to integrate 1 over spherical coordinates to show that:

$$\Omega^{\text{all}} = \iint_{\Omega^{\text{all}}} 1\, d\Omega = \int_0^{\pi} \left(\int_{-\pi}^{\pi} 1 d\varphi \right) \cdot \sin\theta d\theta = 2\pi \int_0^{\pi} \sin\theta d\theta$$

$$= 2\pi[-\cos\theta]_0^{\pi} = 2\pi\big(1-(-1)\big) = 4\pi$$

Applied Analysis
Summary

- a lot of physical quantities are derivatives of others
- the variables with respect to which the derivation is applied, add their units to the denominator
- with respect to location one can do
  - triple derivatives yielding again a scalar density or
  - gradients that result in vector valued quantities like forces
- integration of conservative forces along paths can be computed from differences in potential energy
- not all force fields can be integrated, only the ones where the curl vector vanishes
- when integrating directions in spherical coordinates an additional $\sin\theta$ is needed

Introduction to Physically Based Simulation

MINIMIZATION PRINCIPLE

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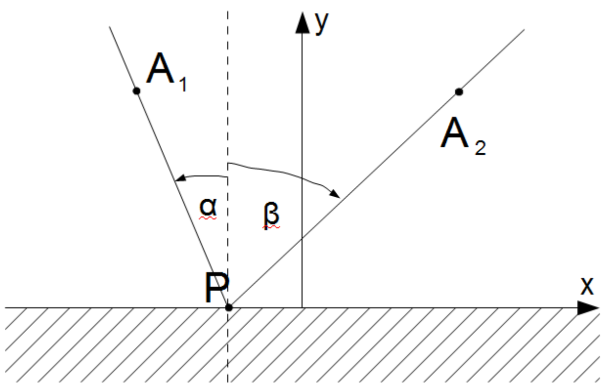
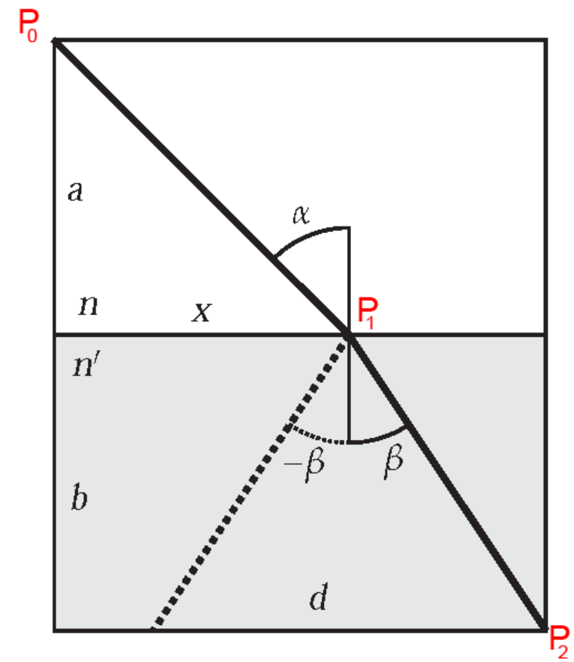
Light travels along the shortest path with respect to time

- from this the laws for reflection and refraction follow

http://de.wikipedia.org/wiki/Fermatsches_Prinzip



$$\alpha = \beta$$



$$\frac{c_2}{c_1} = \frac{\sin\beta}{\sin\alpha}$$

Minimization Principle
Principle of Least / Stationary Action

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- mechanical systems can be completely described through the scalar Lagrangian L that depends on the time dependent state vector $\vec{\boldsymbol{y}}(t)$ of the system and potentially on time t:

$$L(\vec{\boldsymbol{y}}(t), t) = T - V$$

- with the kinetic energy T and the potential energy V.
- the state vector $\vec{\boldsymbol{y}}$ contains all object positions and velocities
- the action S of the system is defined as the functional that maps the time evolution of the system state to a scalar:

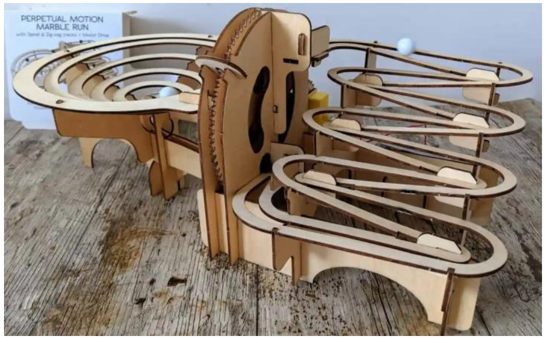
$$S[L](t_1, t_2) = \int_{t_1}^{t_2} L(\vec{\boldsymbol{y}}(t), t)dt$$

The path $\vec{\boldsymbol{y}}(t)$ taken by the system between times t_1 and t_2 is the one for which the action is stationary (no change) to first order.

Minimization Principle
Principle of Least / Stationary Action

- With variational calculus the Euler Lagrange Equations can be derived from the principle of stationary action:

$$\forall i: \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i}$$



- here i enumerates the generalized positions q_i and generalized velocities $\dot{q}_i$.

Example: harmonic oscillator

- $L = T - V = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$
- $\frac{\partial L}{\partial \dot{x}} = m\dot{x}\ ,\ \ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m\ddot{x}\ ,\ \ \frac{\partial L}{\partial x} = -kx \Longrightarrow m\ddot{x} = -kx$

Minimization Principle
Summary

- the dynamics of physical systems can be formulated as minimization problem
- examples:
  - Fermat's Principle (shortest paths)
  - Least Action Principle
- if minimization is over functions, one needs variational calculus
- from Least Action Principle one can derive the Euler Lagrange Equations that generalize equations of motions

Introduction to Physically Based Simulation

SYMMETRIES AND CONSERVED QUANTITIES

Symmetries & Conserved Quantities Noether Theorem

Computergraphik und Visualisierung

Any differentiable symmetry of the action of a physical system has a corresponding conservation law.

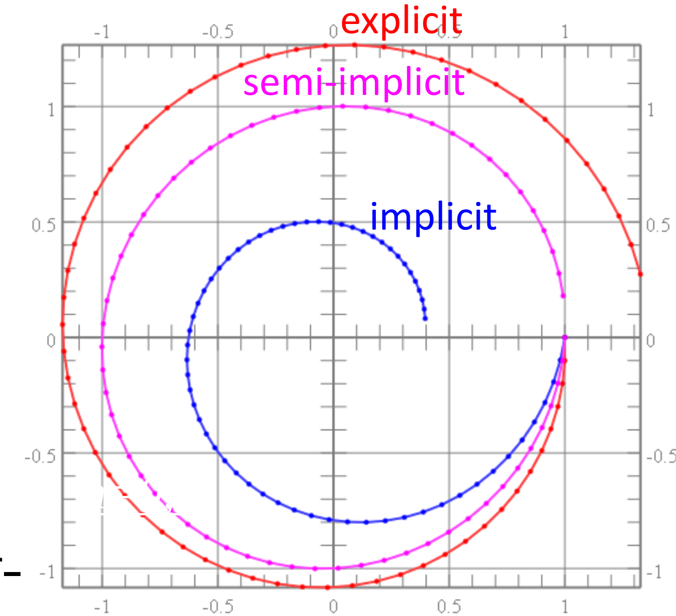
Examples

- time symmetry: As laws of physics / experiments do not depend on when they are done, energy is conserved
- location symmetry: As laws of physics / experiments do not depend on where they are done, linear momentum (mass times linear velocity) is conserved
- orientation symmetry: As laws of physics / experiments do not depend on their spatial orientation, angular momentum (inertia tensor times angular velocity) is conserved

Symmetries & Conserved Quantities
Symplectic Numerical Integration

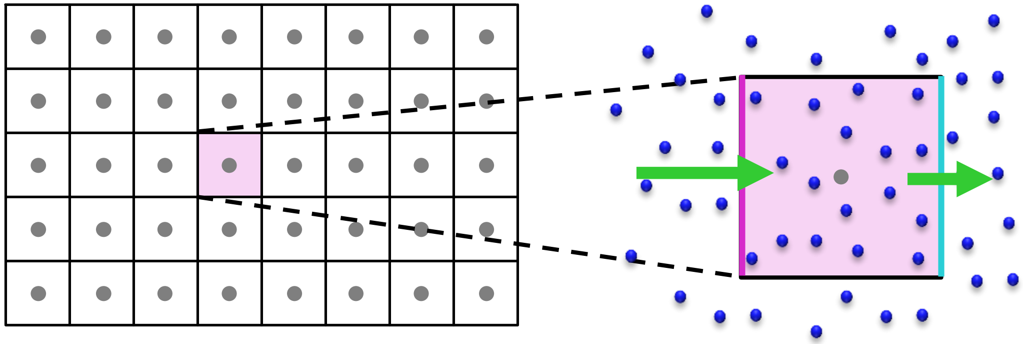
Computergraphik und Visualisierung

- Explicit numerical integration techniques like the explicit Euler add energy to the system (system becomes instable)
- Implicit integration techniques are stable but unnaturally damp the system and remove energy
- Symplectic integrators conserve energy as good as possible but are not stable for stiff systems.



$\dot{x} = f(v,t)$ $\dot{v} = g(x,t)$	$x_{i+1} = x_i + h \cdot f(v_i, t)$ $v_{i+1} = v_i + h \cdot g(x_i, t)$	$x_{i+1} = x_i + h \cdot f(v_{i+1}, t)$ $v_{i+1} = v_i + h \cdot g(x_{i+1}, t)$	$v_{i+1} = v_i + h \cdot g(x_i, t)$ $x_{i+1} = x_i + h \cdot f(v_{i+1}, t)$
system of diff. equa.	explicit Euler	implicit Euler	semi-implicit / symplectic Euler

Example: mass preservation



Lagragian view

- each particle p_i carries mass m_i; automatic preservation if particles persist

Eulerian view

- mass is represented as density field $\rho(\underline{\boldsymbol{x}})$ over grid
- cell volume: $dV = dx^3$
- imagine mass is split into equal sized particles of density $\pi(\underline{\boldsymbol{x}})$ and mass m
- mass density: $\rho = m \cdot \underbrace{\pi \cdot dV}_{\substack{nr.part.\\ in\ cell}} / dV$
- each particle travels with velocity $\vec{\boldsymbol{u}}(\underline{\boldsymbol{x}})$

How does particle movement change ρ over time step dt?

- let us restrict motion to x dir. and examine one cell
- traveled distance: $u_x \cdot dt$
- particle in/outflow(left/right): $dx^2 \cdot (\pi \cdot u_x)(x \mp dx/2) \cdot dt$
- change in mass density: $\frac{\partial \rho}{\partial t} = -m \cdot \frac{\partial(\pi \cdot u_x)}{\partial x} = -\frac{\partial(\rho \cdot u_x)}{\partial x}$

- as we have the same in/outflows along the other coordinate directions, we get:

$$\frac{\partial\rho}{\partial t}(\underline{\boldsymbol{x}}) + \frac{\partial\left(\rho(\underline{\boldsymbol{x}})\cdot u_x(\underline{\boldsymbol{x}})\right)}{\partial x} + \frac{\partial\left(\rho(\underline{\boldsymbol{x}})\cdot u_y(\underline{\boldsymbol{x}})\right)}{\partial y} + \frac{\partial\left(\rho(\underline{\boldsymbol{x}})\cdot u_z(\underline{\boldsymbol{x}})\right)}{\partial z} = 0$$

- introducing the mass flux vector $\vec{\boldsymbol{j}} = \rho\cdot\vec{\boldsymbol{u}}$ this simplifies:

$$\frac{\partial\rho}{\partial t}(\underline{\boldsymbol{x}}) + \frac{\partial j_x(\underline{\boldsymbol{x}})}{\partial x} + \frac{\partial j_y(\underline{\boldsymbol{x}})}{\partial y} + \frac{\partial j_z(\underline{\boldsymbol{x}})}{\partial z} = 0$$

- the formula simplifies further by introducing the divergence operator $\mathrm{div}\,\vec{\boldsymbol{v}} = \partial_x v_x + \partial_y v_y + \partial_z v_z$:

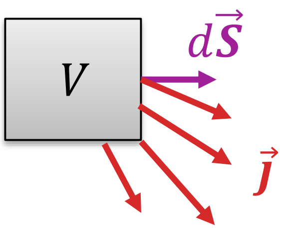
$$\partial_t\rho + \mathrm{div}(\vec{\boldsymbol{j}}) = \partial_t\rho + \mathrm{div}(\rho\vec{\boldsymbol{u}}) = 0$$

- Finally we introduce a volumetric mass source $\sigma(\underline{\boldsymbol{x}})$ yielding the mass continuity equation:

$$\partial_t\rho + \mathrm{div}(\rho\vec{\boldsymbol{u}}) = \sigma$$

Symmetries & Conserved Quantities
Continuity equation

- The continuity equation can be constructed for any other quantity $q(\underline{\boldsymbol{x}})$ carried with fluid particles like electric charge by exchanging the symbol ρ with q.
- An alternative derivation from density $\rho(\underline{\boldsymbol{x}})$, velocity $\vec{\boldsymbol{u}}(\underline{\boldsymbol{x}})$, flux $\vec{\boldsymbol{j}}(\underline{\boldsymbol{x}})$ and source $\sigma(\underline{\boldsymbol{x}})$ fields results from an integral formulation, that keeps book on all changes in q:



$$\frac{d\rho}{dt} + \oiint_S \vec{j} \cdot d\vec{\boldsymbol{S}} = \iiint_V \sigma dV$$

$d\vec{\boldsymbol{S}}$…infinitesimal surface area times surface normal

- The differential form is given again as:

$$\frac{\partial\rho}{\partial t} + div\vec{\boldsymbol{j}} = \sigma$$

- For conserved quantities $\sigma \equiv 0$.

Symmetries & Conserved Quantities Summary

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- It is important to be aware of quantities that are conserved in physical systems like energy
- These are due to spatial and temporal symmetries in the laws of physics
- Symplectic numerical integration methods target for energy preservation
- Continuity equations describe temporal changes of physical quantities inside of fluids or fields. For conserved quantities they do not have a source term.