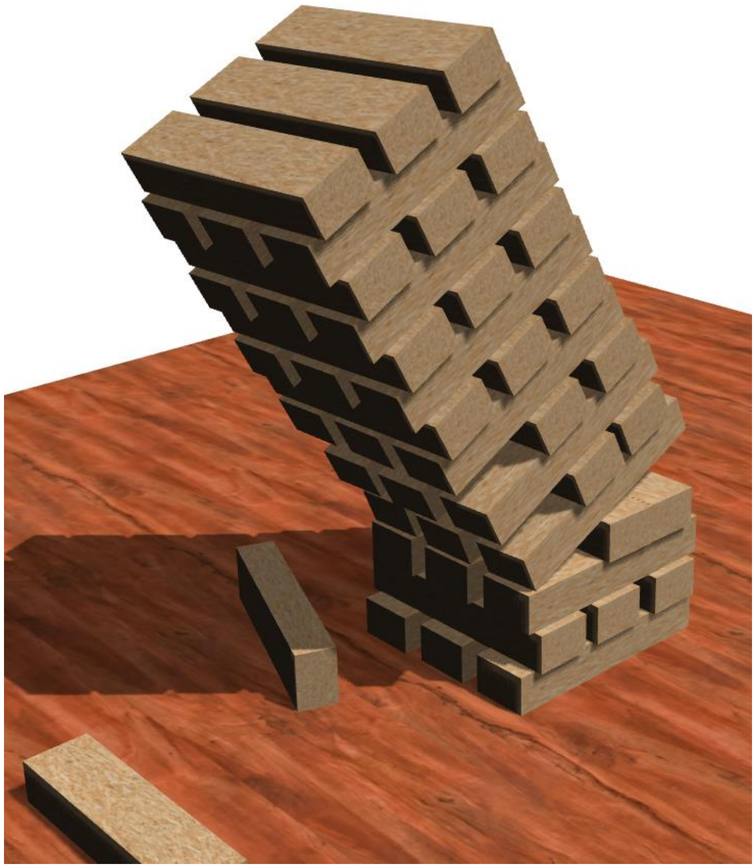
TECHNISCHE UNIVERSITÄT DRESDEN

Computergraphik und Visualisierung

CG3 Part II.5 Collisions



Impacts

Friction

Stacking

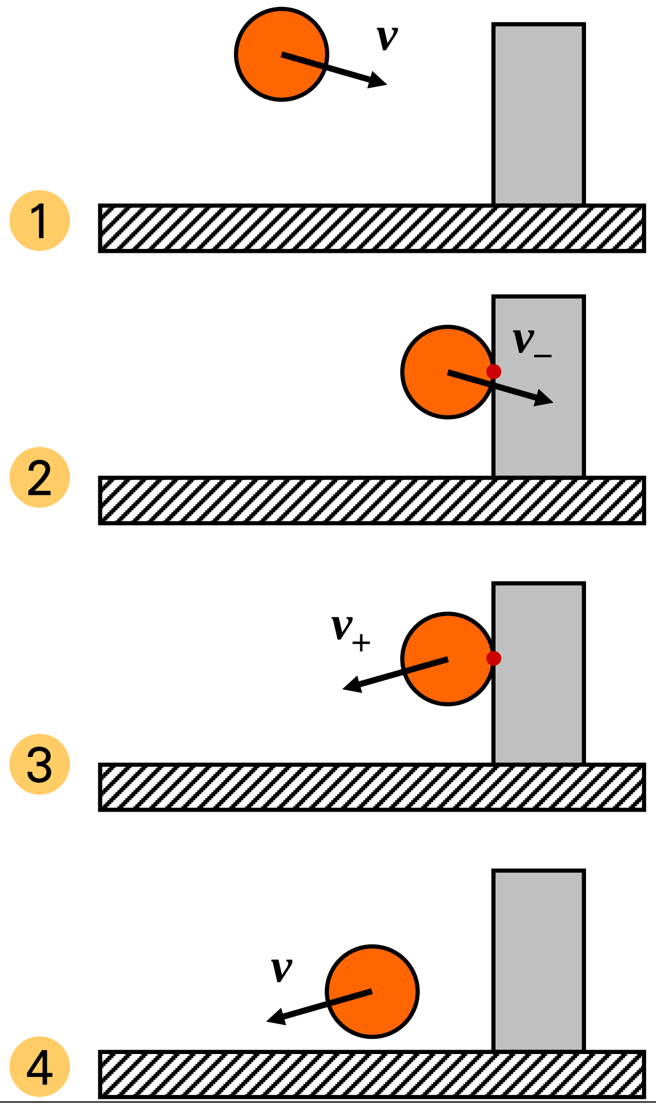
Content

- Introduction
- Impact & Friction
- Contacts
- Collision Handling
  - Single Contact
  - Multiple Contacts
- Sequential Impulses
- Contact Force Approach
- Summary

Introduction

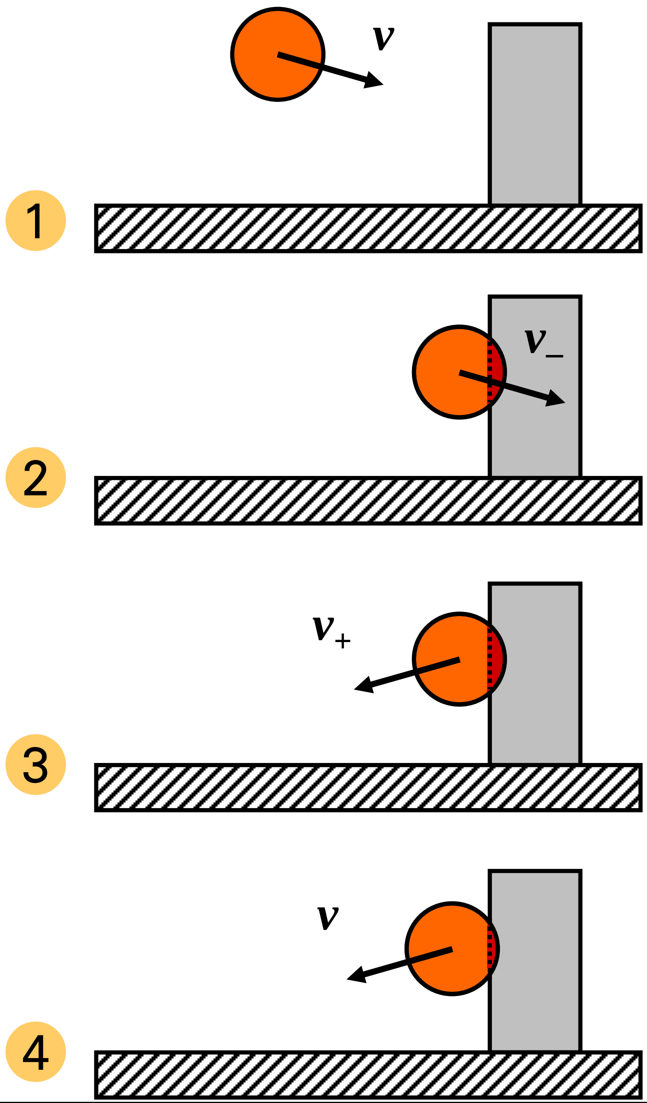
Terms – exact Simulation

- free motion
- collision
  - two (moving) objects are in a point contact **and**
  - have a relative velocity towards each other
- collision handling
  - Determine instantly the new object velocities / momenta without a change of positions
- free motion

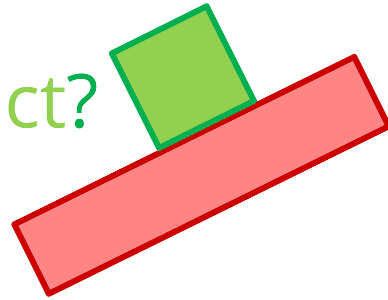


Terms – discrete time

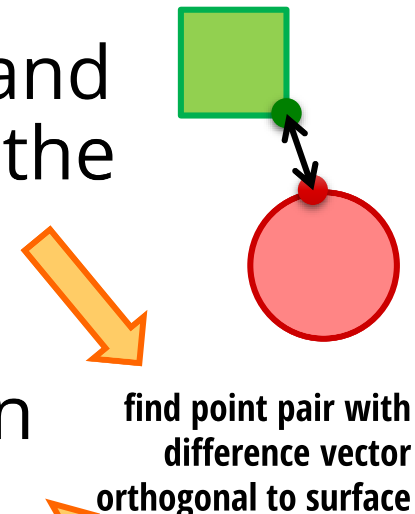
- free motion
- Simulation misses time point of contact
- collision
  - two (moving) objects touch in extended contact with relative velocity pointing towards each other
- collision handling
- contact without collision
- free motion



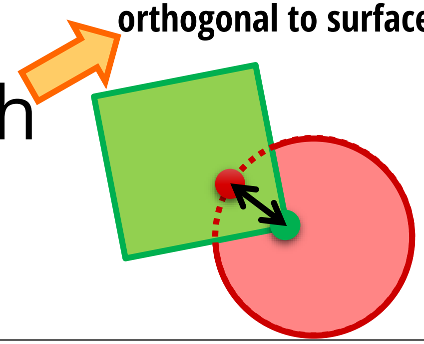
Different queries in collision handling:



- Which objects collide which are in contact?
- Do objects in contact stay in contact? (check relative velocity)
- How far apart are the nearest objects and in which direction is the connection of the shortest distance? (control size of time step)



- When within time step did first collision occur?
- How deeply did objects penetrate each other? (penetration depth)



- How to treat collisions and contacts?

Simulation Loop

- Contact search
  - <u>broad phase:</u> quickly find all collision pairs and possibly more
  - <u>narrow phase:</u> check collision pairs and analyze found intersections
- Filtering of contact points
  - Contacts where contact pairs move away from each other can be ignored
- Collision Resolution
  - Use energy and momentum conservation for a consistent recalculation of momenta only
- Integration
  - An integration step of the time evolution function time

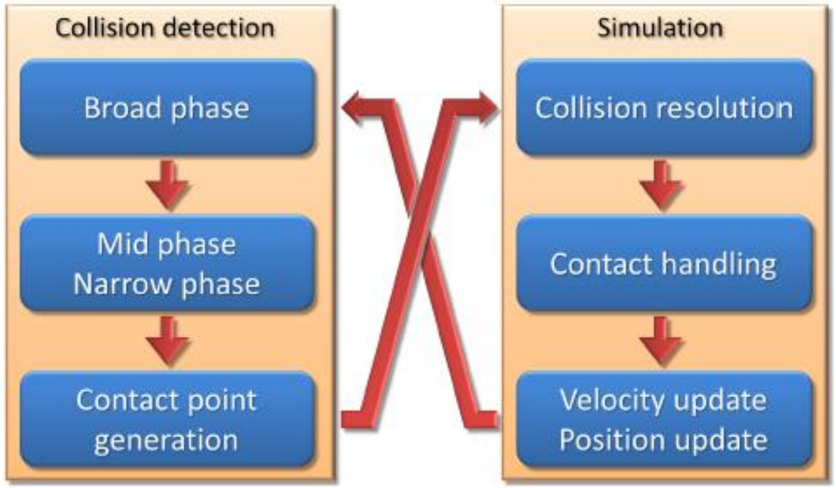


Figure 3: *A modular phase description of the sub tasks of a rigid body simulator helps decomposing a large complex system into simpler components.*

Impact & Friction

Impact types

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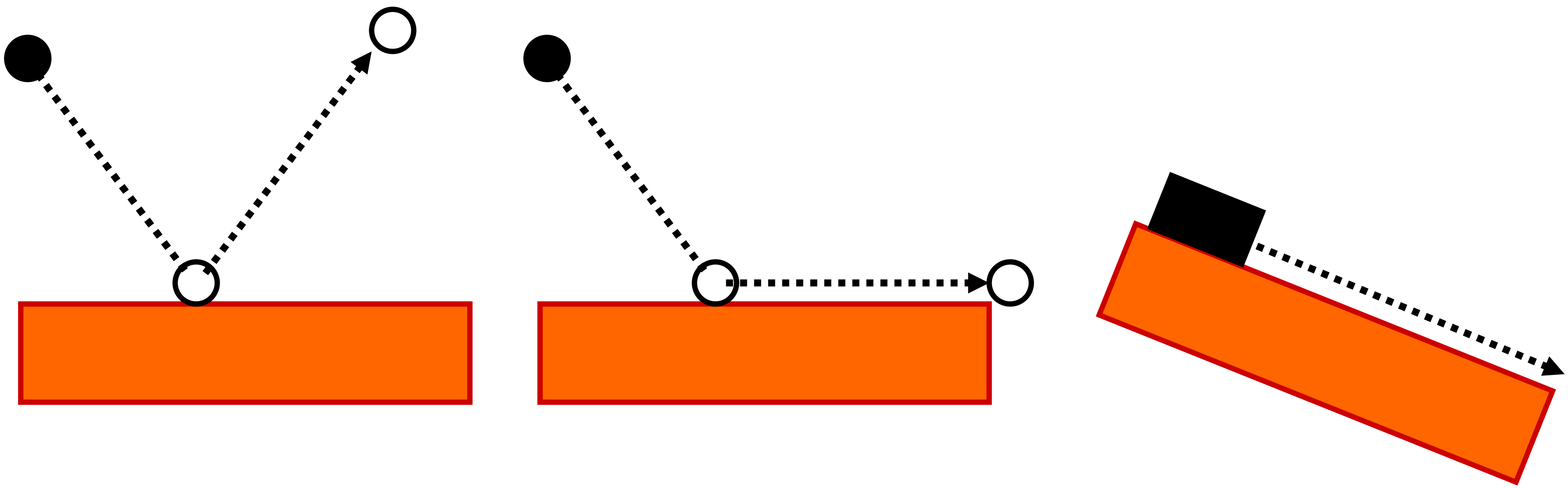
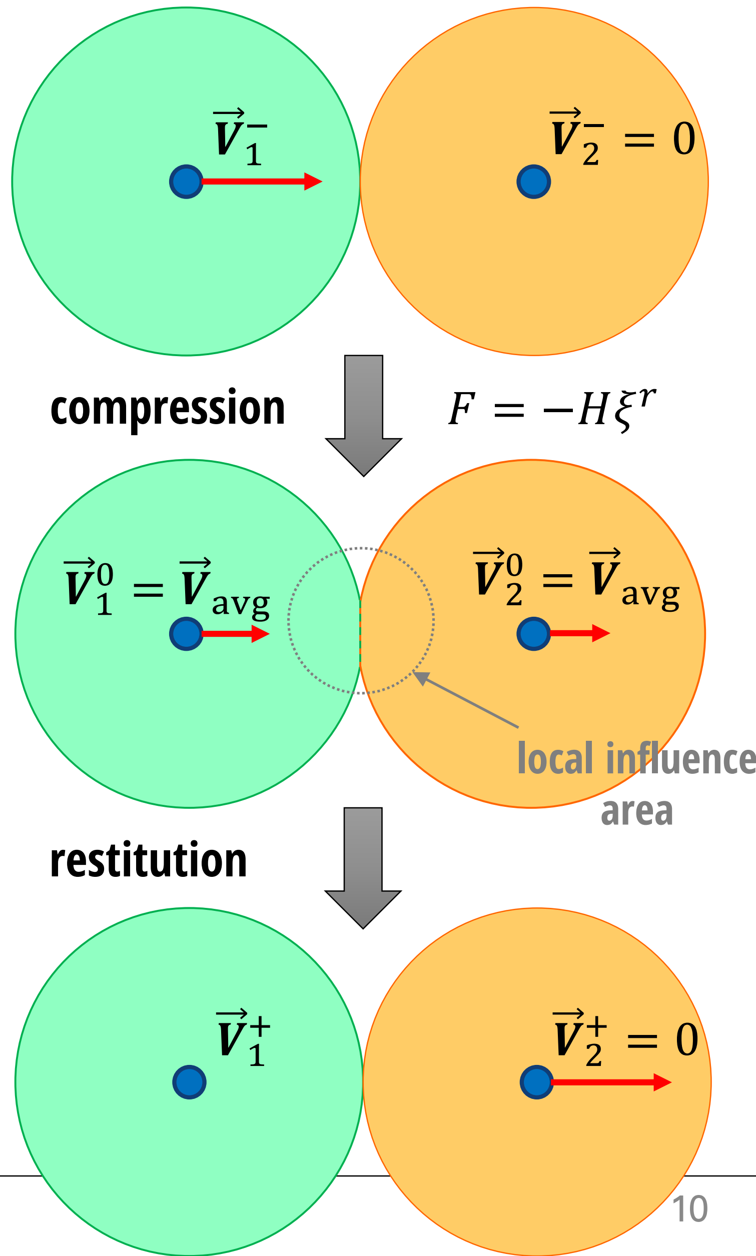
Example: Elastic vs. Inelastic impacts:



- In both elastic and inelastic impacts, we assume that speeds change instantaneously.
- This means that the collision time must be determined precisely in order to simulate the collision correctly.
- After an inelastic collision or after a special initial condition, persistent contacts with slipping under friction can occur.

Elastic Impact Details

- during contact local surroundings of elastic bodies become springs following Hertz's force law
$$F = -H \cdot \xi^r$$
with displacement ξ, exponent $r = 1.5$ and Hertz constant H.
- During compression phase kinetic energy is converted into deformation energy with small maximum displacement ξ_{max} until relative velocity vanishes
- In [partially] elastic impacts the restitution phase transfers deformation energy back to kinetic energy and transfers velocity to second body.



Hertz Constant

- The constant H depends on local
  1. surface curvature $\rho = \frac{1}{R}$
  2. shear modulus $G = \frac{F \cdot l}{A \cdot \Delta x}$
  3. Poisson's ratio $\nu = \frac{\Delta L'}{\Delta L}$
- In case of spheres of equal material and radius one gets

$$H = \frac{4}{3} \cdot \frac{G}{1-\nu} \cdot \sqrt{\frac{R}{2}}$$

- otherwise see original paper of Hertz from 1881 (German only):

  Hertz, H.: Über die Berührung fester elastischer Körper, Journal für die reine und angewandte Mathematik, Bd. 92 (1882), S. 156-171
- for values of G and ν see (pdf)

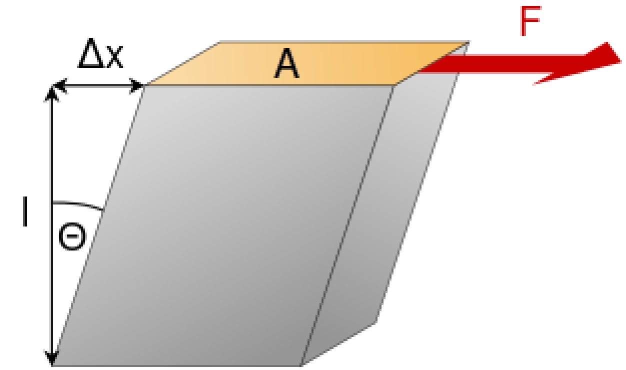


illustration for shear modulus © Wikipedia

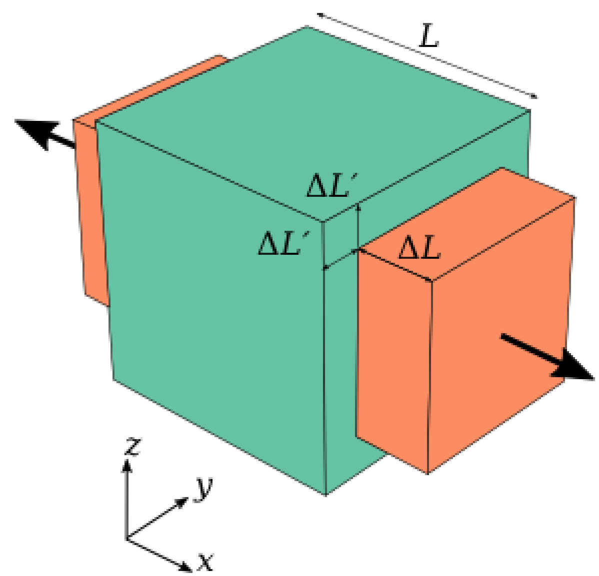
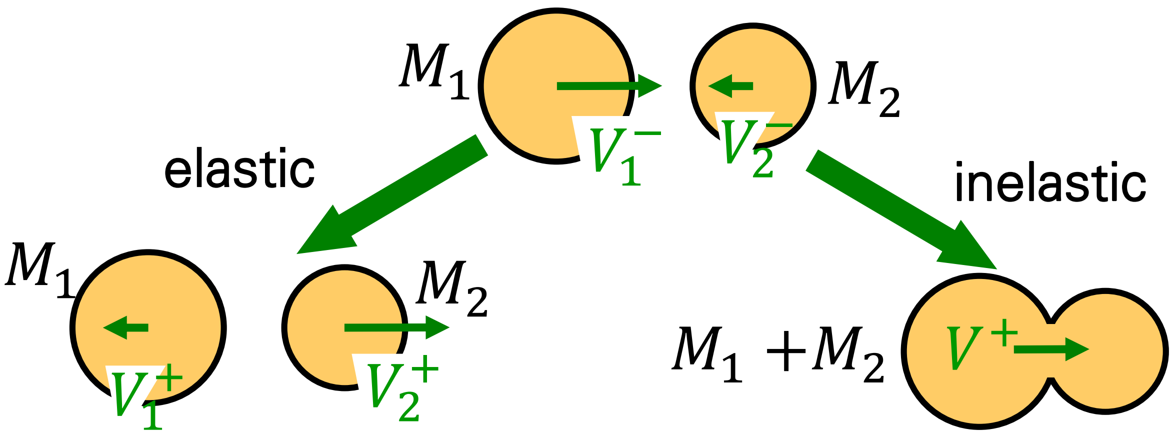


illustration for Poission's ratio © Wikipedia

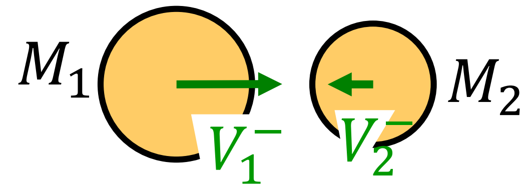
Rigid Body Impact – Central Case

- Basic consideration: two bodies of mass M_1 and M_2 meet centrally
- In an elastic collision the bodies touch each other briefly and bounce off each other without loss of energy
- After an inelastic impact both bodies coalesce and unite into one large body



Rigid Body Impact – Central Case

- In both cases, momentum is conserved
- in inelastic impact, energy is lost in deformation



Inelastic

- conserv. of momentum

$$M_1 V_1^- + M_2 V_2^- = (M_1 + M_2) V^+$$

$$V^+ = V_{\text{avg}} = \frac{M_1 V_1^- + M_2 V_2^-}{M_1 + M_2}$$

- deformation work

$$W = \frac{M_1 (V_1^-)^2 + M_2 (V_2^-)^2 - (M_1 + M_2)(V_1^+)^2}{2}$$

$$W = \frac{M_1 M_2}{M_1 + M_2} (V_2^- - V_1^-)^2$$

Elastic

- conserv. of momentum

$$M_1 V_1^- + M_2 V_2^- = M_1 V_1^+ + M_2 V_2^+$$

$$M_1 (V_1^- - V_1^+) = M_2 (V_2^+ - V_2^-) \quad ①$$

- conserv. of energy

$$M_1 \left((V_1^-)^2 - (V_1^+)^2 \right) = M_2 \left((V_2^+)^2 - (V_2^-)^2 \right) \quad ②$$

$$V_1^- + V_1^+ = V_2^+ + V_2^- \qquad ②/① = ③$$

- yielding

$$V_{1|2}^+ = V_{\text{avg}} \pm \frac{M_{2|1}}{M_1 + M_2} (V_2^- - V_1^-)$$

Coefficient of restitution

- nearly all materials are partially elastic
- This is quantified with the coefficient of restitution (COR) $\varepsilon = \dfrac{\text{relative velocity after impact}}{\text{relative velocity before impact}}$
  - $\varepsilon = 0$ ... inelastic
  - $0 < \varepsilon < 1$ ... partially elastic
  - $\varepsilon = 1$ ... elastic
- impact inverts direction of relative velocity:

$$V_2^+ - V_1^+ = V_{\mathrm{rel}}^+ = -\varepsilon V_{\mathrm{rel}}^- = -\varepsilon(V_2^- - V_1^-)$$

- The individual velocities after impact and the deformation energy are computed according to:

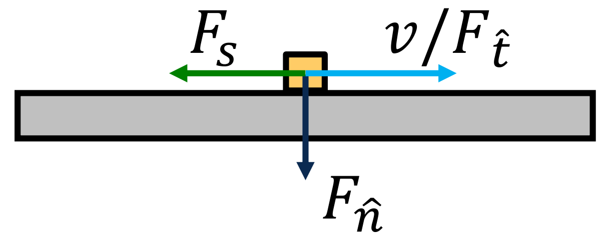
$$V_{1|2}^+ = V_{\mathrm{avg}} \pm \varepsilon \frac{M_{2|1}}{M_1 + M_2} V_{\mathrm{rel}}^-$$

$$W = \frac{M_1 M_2}{M_1 + M_2} (V_{\mathrm{rel}}^-)^2 (1 - \varepsilon^2), \varepsilon \in [0,1]$$

Friction Forces

Dry Friction

- friction forces act between rough bodies
- direction is in contact plane opposite to relative velocity v or external force $F_{\hat{t}}$
- absolute value proportional to normal force $F_{\hat{n}}$
- state of relative movement defines coefficient of friction μ_* $\mu_r \ll \mu_k < \mu_s$
  - static friction μ_s
  - kinectic or sliding friction $\mu_k < \mu_s$
  - rolling friction $\mu_r \ll \mu_k$



$$F_s = \mu_s F_{\hat{n}}$$

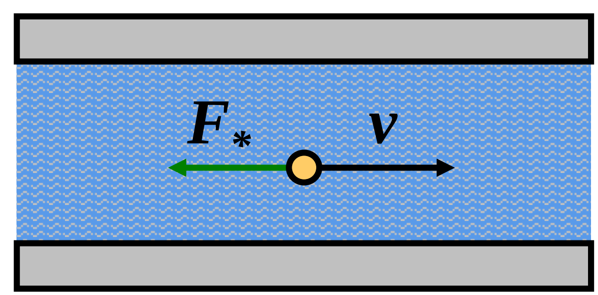
$$F_k = \mu_k F_{\hat{n}}$$

$$F_r = \mu_r F_{\hat{n}}$$

Viscous Friction

- Fluid friction in Laminar flows F_L
  - sphere radius r
  - dynamic viscosity η
- Fluid friction in Turbulent flows F_T
  - cross-sectional area A
  - numerical drag coefficient c_W
  - air density ρ

$$F_L = 6\pi\eta r v$$



$$F_T = c_W A \frac{\rho}{2} v^2$$

Measurements

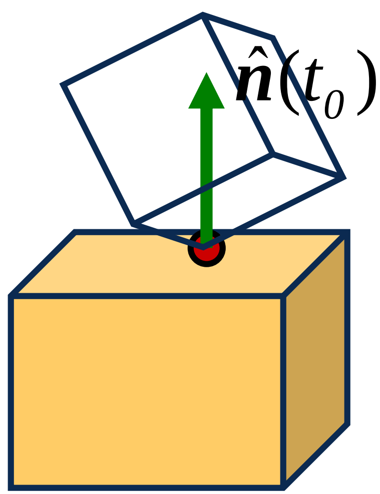
- COR and coefficient of friction are **not** properties of an object or of a material.
- both are properties of a contact of two objects
- Typically used values can be found here:

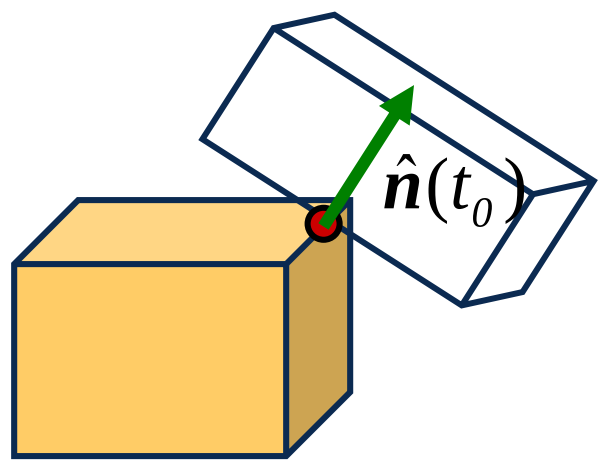
Material 1:	Material 2:	Mu static:	Mu dynamic:	Restitution coefficient:
Dry steel	Dry steel	0.70	0.57	0.80
Greasy steel	Dry steel	0.23	0.16	0.90
Greasy steel	Greasy steel	0.23	0.16	0.90
Dry aluminium	Dry steel	0.70	0.50	0.85
Dry aluminium	Greasy steel	0.23	0.16	0.85
Dry aluminium	Dry aluminium	0.70	0.50	0.85
Greasy aluminium	Dry steel	0.30	0.20	0.85
Greasy aluminium	Greasy steel	0.23	0.16	0.85
Greasy aluminium	Dry aluminium	0.30	0.20	0.85
Greasy aluminium	Greasy aluminium	0.30	0.20	0.85

Contacts

Contact Types (non-degenerate)

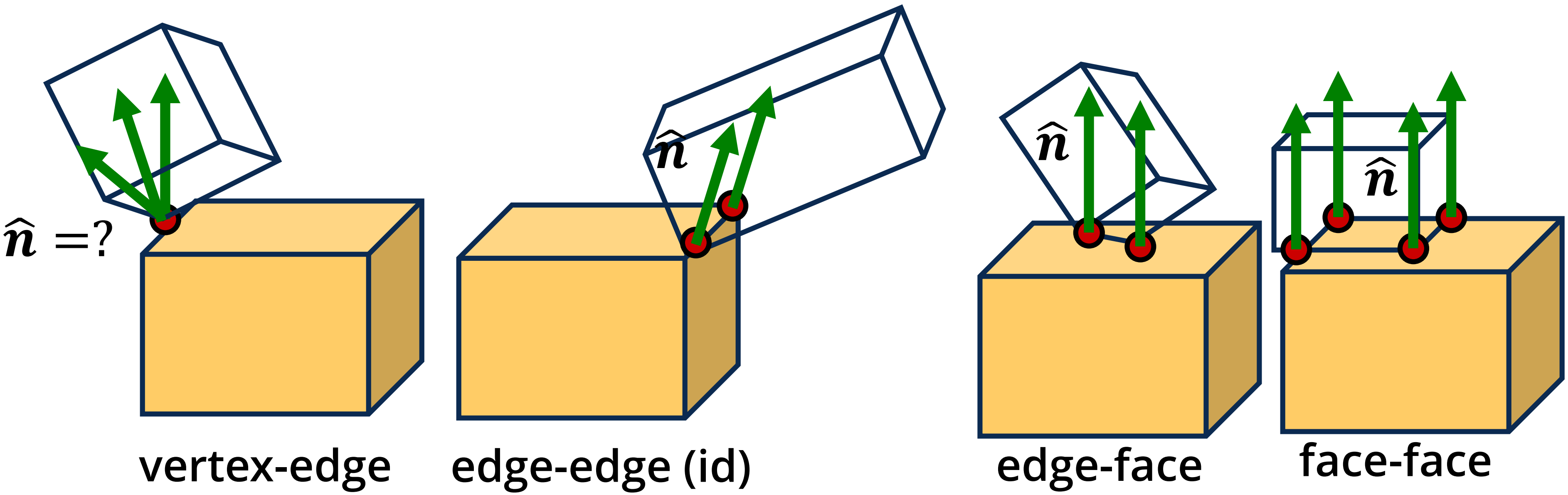
- The contact normal, which indicates the direction perpendicular to the contact, is important for the treatment
- for polyhedral 2 non-degenerated contact types arise
  - vertex-face: contact normal is identical with face normal
  - edge-edge: Contact normal results from the cross product of the edge directions





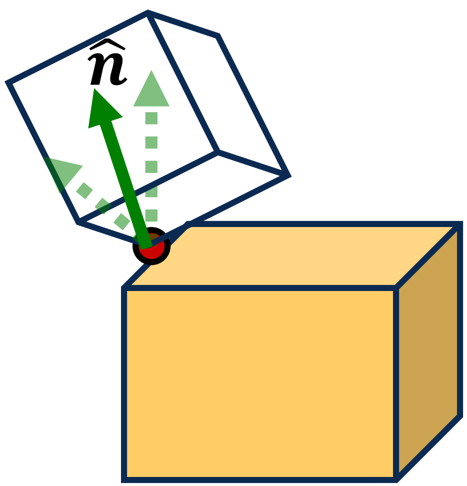
Contact Types (degenerate)

- Vertex-vertex, vertex-edge and edge-edge (id) contacts happens with vanishing probability but allow for multiple contact normal directions. This contact is therefore called indeterminate contact.
- Edge-face and face-face are extended contacts between edge/face and face. Both can be handled through multiple point contacts

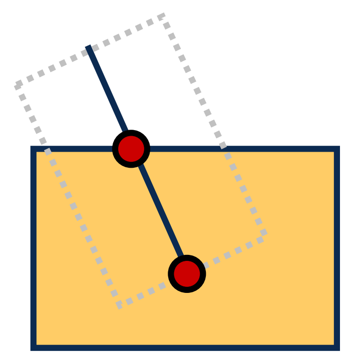


Handling Degenerate Contacts

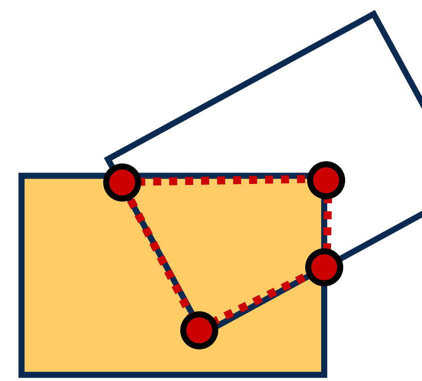
- Indeterminate contacts are problematic as they lead to NP hard contact resolution problems.
  This can be circumvented by choosing an arbitrary contact normal and imagining that one body is virtually extended orthogonal to this direction.
- In case of edge-face and face-face contacts the overlap region is found by clipping and replaced by multiple point contacts one for each corner



vertex-edge



edge-face



face-face

Broad Phase – example 2D

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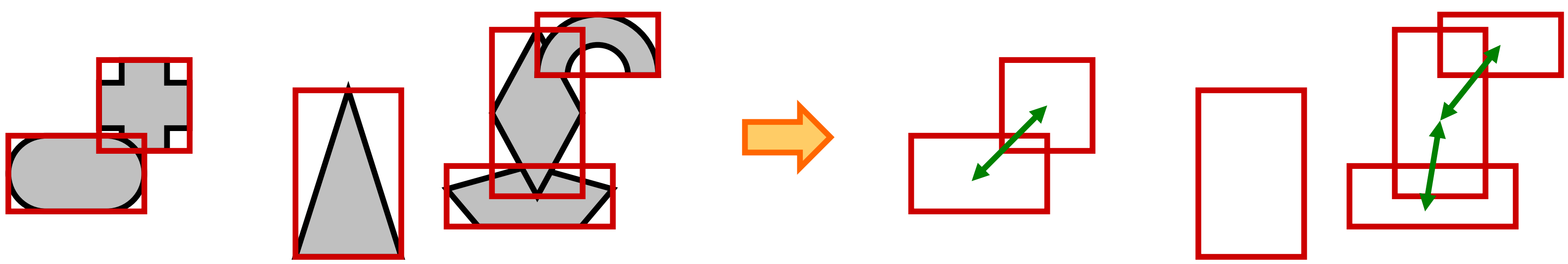
Rectangle intersection problem:

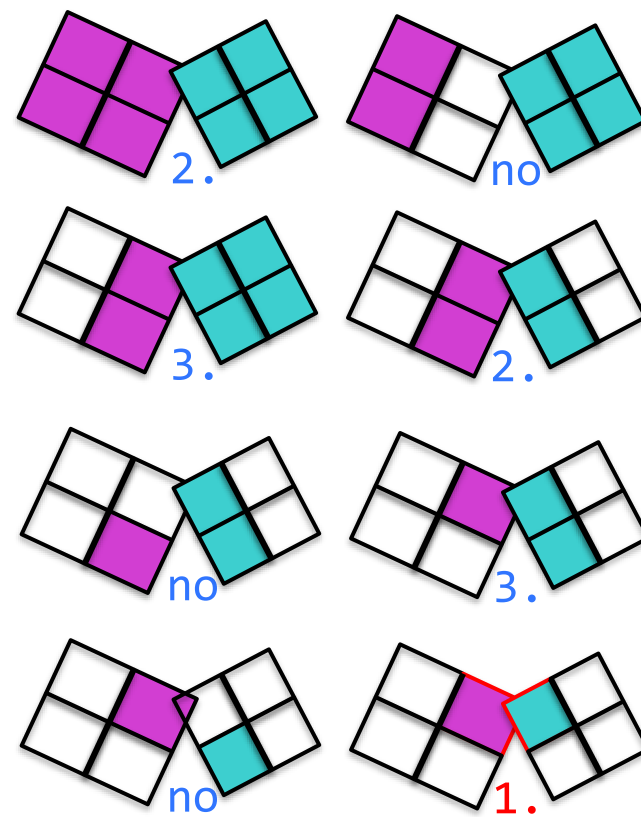
- given: n rectangles in 2D
- wanted: all k intersecting pairs of rectangles
- Approach … $O(n \log n + k)$
  - Sort rectangle edges along x-axis orthogonal to Sweep-Line in y-direction … $O(n \log n)$
  - For each rectangle in enter event sort y-intervall of rectangle into interval tree data structure and report all intersections … accumulates to $O(n \log n + k)$
  - For each rectangle exit event erase y-intervall of rectangle from interval tree data structure … accumulates to … $O(n \log n)$



Broad Phase – BVH collision test

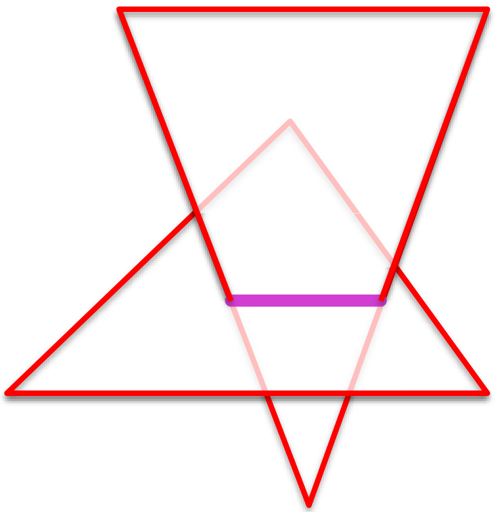
- For two objects with BVHs the collision test only descents down the hierarchy if the BVs of the root nodes intersect
- Heuristically it first descents into node of larger volume

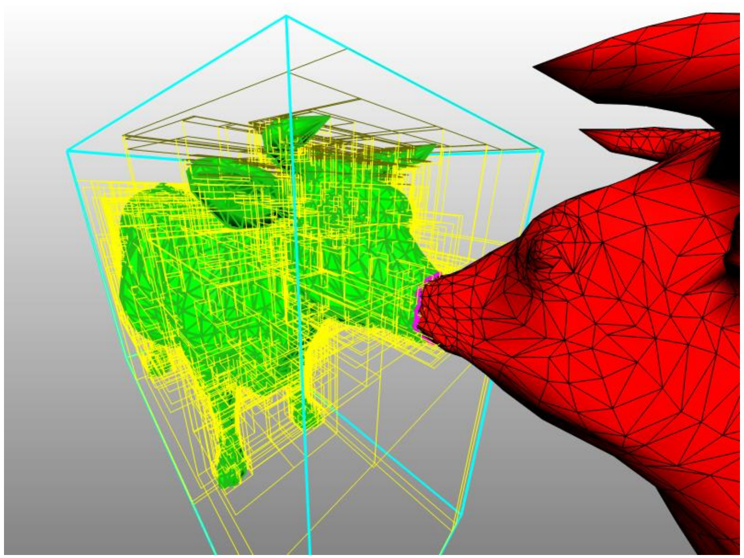
```
bool isColliding(N A, N B)
   if not BV_intersects(A, B) then false;
   if A.isLeaf() and B.isLeaf() then
1.      return primitiveIntersections(A, B)
   if (!A.isLeaf() and A.volume()>B.volume())
         or (B is leaf) then
2.      ∀a∈A.children:
         if isColliding(a,B) return true;
      return false;
   else
3.      ∀b∈B.children:
         if isColliding(A,b) return true;
      return false;
```

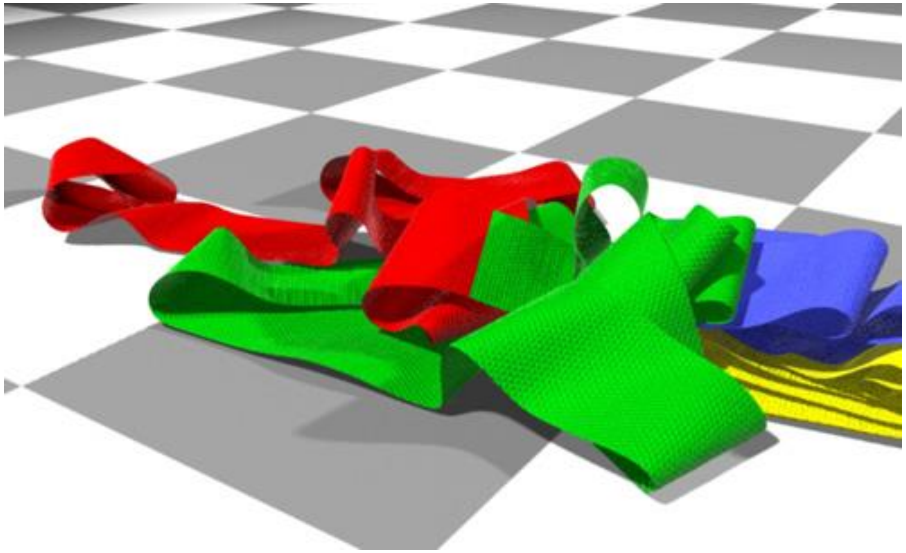


Near Phase – Analyzing Contacts

- The contact between two triangle meshes is composed of loops of line segments resulting from triangle-triangle intersection tests
- Bounding volume hierarchies can be used to filter the necessary triangle-triangle intersection tests.



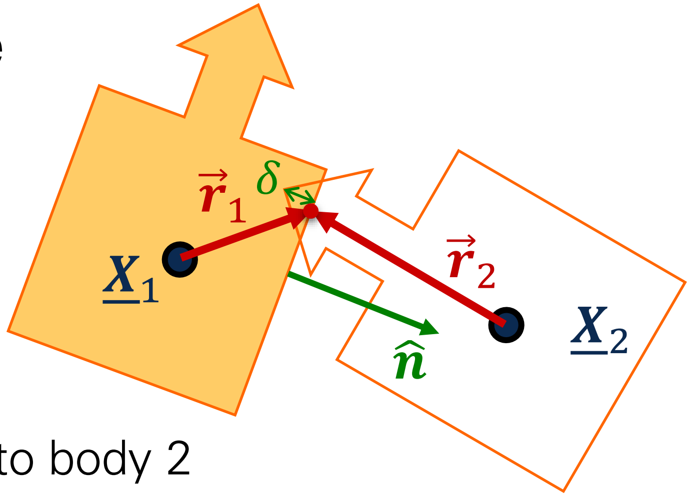




Contact Extraction Summary

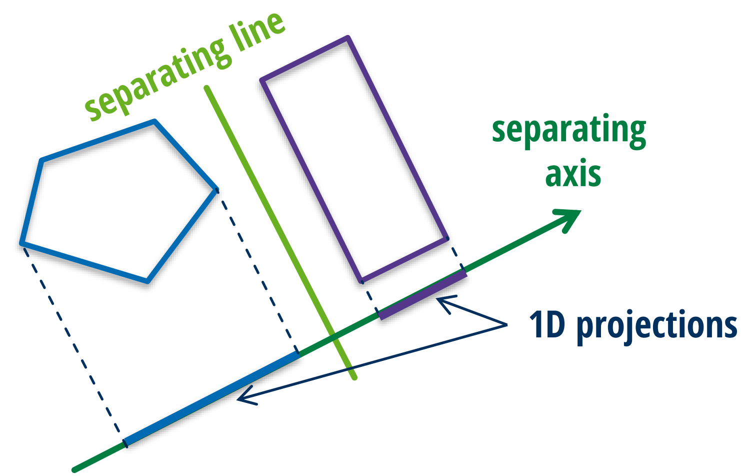
- After bodies have been moved, new contacts can arise
- Broad phase enumerates potential contact pairs
- Near phase filters out actual contact pairs and computes for each contact pair a list of point contacts
- Per point contact compute relative velocity. If this is positive, we have vanishing point contact that can be discarded.

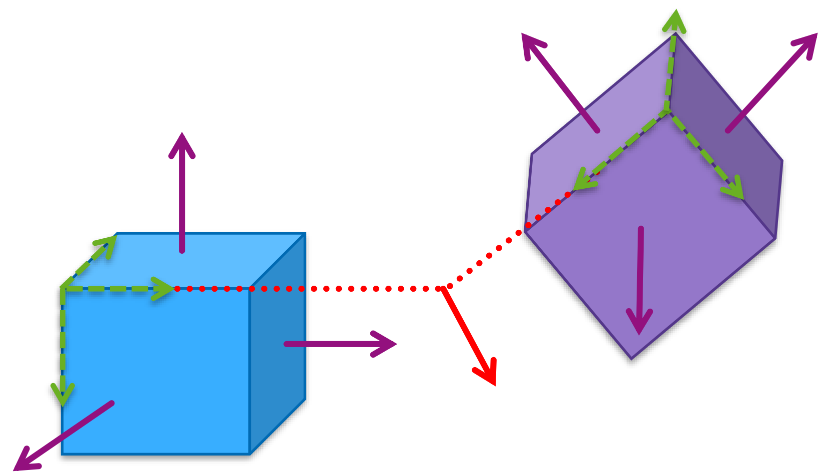
➔ list of contact pairs with at least one non-vanishing contact point, where per pair we store:
- indices of colliding bodies
- contact type with feature indices
- list of contact points with
  - positional vertex $\vec{r}_{i\in\{1,2\}}$
  - contact normal $\hat{n}$ pointing from body 1 to body 2
  - penetration depth δ



Near Phase with SAT for OOBB

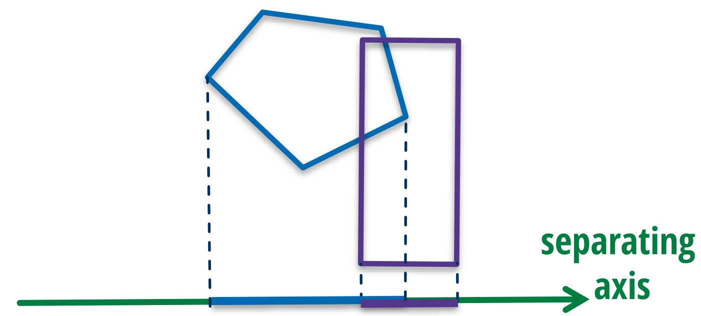
- According to separating axis theorem there is no intersection between two convex bodies if one separating axis exist where 1D projections do not intersect
- For OOBB one computes one axis per body face normal $(3 + 3)$ and edge-edge (3×3) pair
- Observation: separating axis corresponds to contact normal

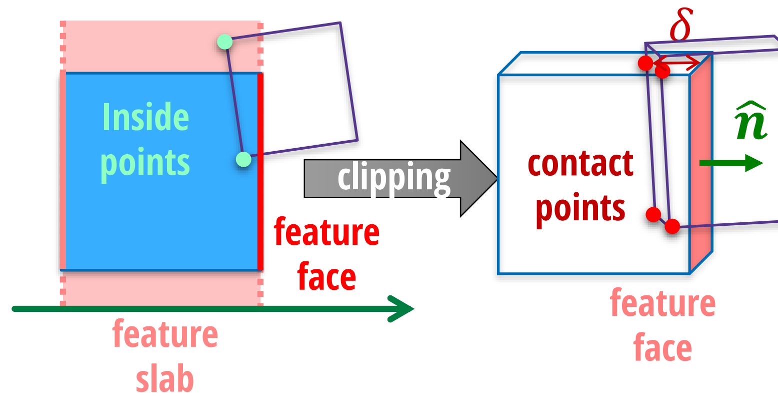


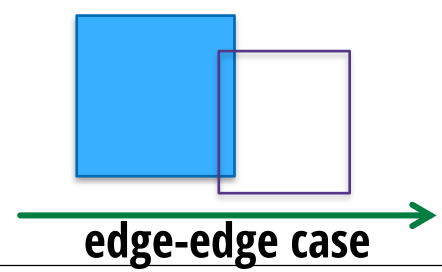


Near Phase with SAT for OOBB

- In case of intersection, all 1D projections produce penetration
- For contact analysis select 1D projection with minimal depth
- If axis is face normal of body 1
    - find vertices of body 2 inside of feature slab and select feature face with smaller max penetration depth
    - classify contact type based on inside count (vertex-face if #=1; edge-face if #=2; face-face otherwise)
    - for edge/face-face contacts perform clipping on full box of body 1
    - determine contact points, contact normal orientation (from 1 to 2) and penetration depth
- similarly, for edge-edge case



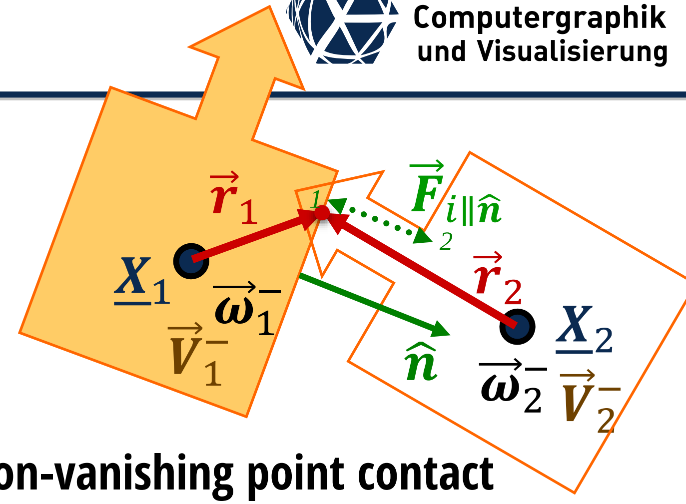




Collision Handling
Single Contact

Friction-Less Impact

- In case of collision, the contact normal direction of the relative velocity $v^-_{\mathrm{rel}\|\hat{\boldsymbol{n}}}$ is ≤ 0.
- We want to update $\vec{\boldsymbol{V}}_i$ and $\vec{\boldsymbol{\omega}}_i$ such that $\vec{\boldsymbol{v}}^+_{\mathrm{rel}\|\hat{\boldsymbol{n}}}$ becomes ≥ 0 by updating linear and angular momenta of the rigid bodies:
  $\vec{\boldsymbol{P}}^+_i = \vec{\boldsymbol{P}}^-_i + \Delta\vec{\boldsymbol{P}}_i \qquad \vec{\boldsymbol{L}}^+_i = \vec{\boldsymbol{L}}^-_i + \Delta\vec{\boldsymbol{L}}_i$
- To determine the impulses $\Delta\vec{\boldsymbol{P}}_i$ and $\Delta\vec{\boldsymbol{L}}_i$ we imagine per body a contact force $\vec{\boldsymbol{F}}_i$ at contact point along the contact normal over short timespan Δt.
- 3rd Newton's law ensures momentum preservation and single impulse $\Delta\vec{\boldsymbol{p}} = \vec{\boldsymbol{F}}_2 \cdot \Delta t$
- final normal impulse $\Delta p_{\|\hat{\boldsymbol{n}}}$ comes from energy conservat.



non-vanishing point contact

contact velocities: $\vec{\boldsymbol{v}}^-_i = \vec{\boldsymbol{V}}^-_i + \vec{\boldsymbol{\omega}}^-_i \times \vec{\boldsymbol{r}}_i$

relative velocities: $\vec{\boldsymbol{v}}^-_{\mathrm{rel}} = \vec{\boldsymbol{v}}^-_2 - \vec{\boldsymbol{v}}^-_1$

$$\Delta\vec{\boldsymbol{P}}_i = \vec{\boldsymbol{F}}_i \cdot \Delta t$$

force impact

$$\Delta\vec{\boldsymbol{L}}_i = \vec{\boldsymbol{r}}_i \times \vec{\boldsymbol{F}}_i \cdot \Delta t$$

$$\vec{\boldsymbol{F}}_1 = -\vec{\boldsymbol{F}}_2$$

3rd Newton's law

$$\Delta\vec{\boldsymbol{P}}_i = \mp\Delta\vec{\boldsymbol{p}}$$

$$\Delta\vec{\boldsymbol{L}}_i = \mp\vec{\boldsymbol{r}}_i \times \Delta\vec{\boldsymbol{p}}$$

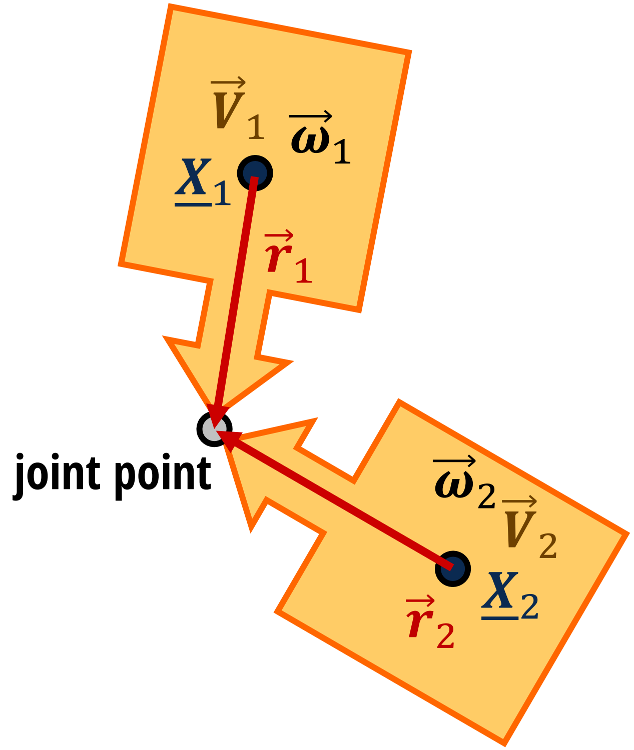
$$\Delta E_{\mathrm{tot}}(\Delta p_{\|\hat{\boldsymbol{n}}} \cdot \hat{\boldsymbol{n}}) = 0$$

The symbol Δ denotes the change of a quantity as its value after the impact minus the value before the impact and is a linear operator

Joints

- Two bodies can be attached to each other with hinge joints at a joint location
- The joint location defines per body a local positional vector $\vec{\boldsymbol{r}}_i$
- Again we can compute the relative velocity $\vec{\boldsymbol{v}}_{\mathrm{rel}}^{-}$ at the joint point from the linear and angular velocities
- To ensure the joint constraint we have to ensure

$$\vec{\boldsymbol{v}}_{\mathrm{rel}}^{+} = 0$$



Inverse Mass Matrix of Contact

- For each body i in the contact we introduce the inverse mass matrix

$$\boldsymbol{K}_i = \frac{\mathbf{1}}{M_i} + \boldsymbol{r}_i^{*\,t} \boldsymbol{I}_i^{-1} \boldsymbol{r}_i^*$$

where $\boldsymbol{r}_i^*$ is the cross product matrix of $\vec{\boldsymbol{r}}_i$ and $\mathbf{1}$ is the identity matrix

- This allows to relate the change in contact point velocity to the force impact

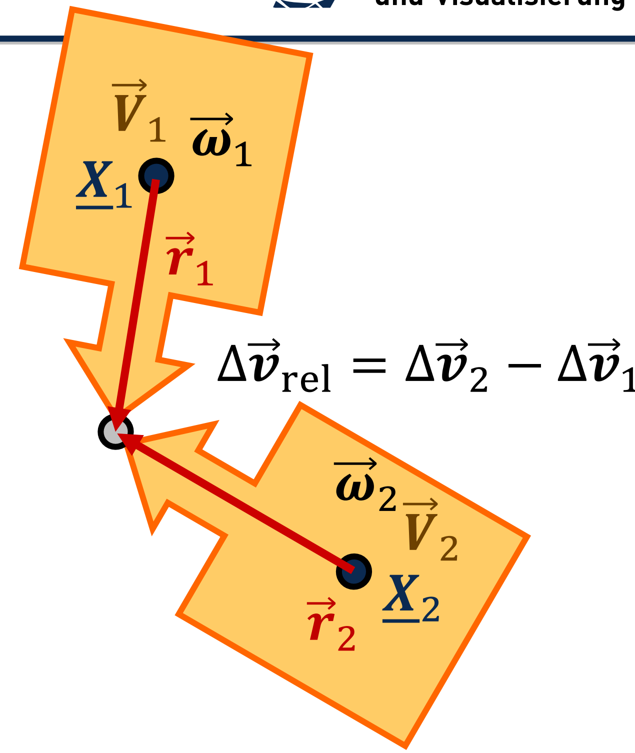
$$\Delta\vec{\boldsymbol{v}}_i = \mp \boldsymbol{K}_i \cdot \Delta\vec{\boldsymbol{p}}$$

- The K-matrix sum

$$\boldsymbol{K} = \boldsymbol{K}_1 + \boldsymbol{K}_2$$

relates the change in relative contact point velocities to the impulse:

$$\Delta\vec{\boldsymbol{v}}_{\mathrm{rel}} = \boldsymbol{K} \cdot \Delta\vec{\boldsymbol{p}}$$



$$\Delta\vec{\boldsymbol{v}}_i = \Delta\vec{\boldsymbol{V}}_i + \Delta\vec{\boldsymbol{\omega}}_i \times \vec{\boldsymbol{r}}_i$$

$$\vec{\boldsymbol{V}}_i = \frac{1}{M_i}\vec{\boldsymbol{P}}_i \qquad \vec{\boldsymbol{\omega}}_i = \boldsymbol{I}_i^{-1}\vec{\boldsymbol{L}}_i$$

$$\Delta\vec{\boldsymbol{P}}_i = \mp\Delta\vec{\boldsymbol{p}} \qquad \Delta\vec{\boldsymbol{L}}_i = \mp\vec{\boldsymbol{r}}_i \times \Delta\vec{\boldsymbol{p}}$$

Derivation of K-Matrix

$$\Delta\vec{v}_i = \Delta\vec{V}_i + \Delta\vec{\omega}_i \times \vec{r}_i$$

$$\Delta\vec{v}_i = \frac{1}{M_i}\Delta\vec{P}_i$$

$$\Delta\vec{\omega}_i = I_i^{-1}\Delta\vec{L}_i$$

$$\Delta\vec{v}_i = \frac{1}{M_i}\Delta\vec{P}_i + \left(I_i^{-1}\Delta\vec{L}_i\right) \times \vec{r}_i$$

$$\Delta\vec{P}_i = \mp\Delta\vec{p}$$

$$\Delta\vec{L}_i = \mp\vec{r}_i \times \Delta\vec{p}$$

$$\Delta\vec{v}_i = \mp\frac{1}{M_i}\Delta\vec{p} \mp \left(I_i^{-1}(\vec{r}_i \times \Delta\vec{p})\right) \times \vec{r}_i$$

$$\Delta\vec{v}_i = \mp\frac{1}{M_i}\Delta\vec{p} \pm \vec{r}_i \times \left(I_i^{-1}(\vec{r}_i \times \Delta\vec{p})\right)$$

$$\Delta\vec{v}_i = \mp\frac{1}{M_i}\Delta\vec{p} \pm r_i^* \cdot \left(I_i^{-1}(r_i^* \cdot \Delta\vec{p})\right)$$

$$\Delta\vec{v}_i = \left(\mp\frac{1}{M_i}\mathbf{1} \pm r_i^* \cdot I_i^{-1} \cdot r_i^*\right)\Delta\vec{p}$$

$$\Delta\vec{v}_i = \mp\left(\frac{1}{M_i}\mathbf{1} + (r_i^*)^t \cdot I_i^{-1} \cdot r_i^*\right)\Delta\vec{p}$$

$$\Delta\vec{v}_i = \mp K_i\Delta\vec{p}$$

$$\Delta\vec{v}_{\mathrm{rel}} = \Delta\vec{v}_2 - \Delta\vec{v}_1$$

$$\Delta\vec{v}_{\mathrm{rel}} = K_2\Delta\vec{p} - (-K_1\Delta\vec{p})$$

$$\Delta\vec{v}_{\mathrm{rel}} = (K_1 + K_2)\Delta\vec{p}$$

$$\Delta\vec{v}_{\mathrm{rel}} = K \cdot \Delta\vec{p}$$

Change in Total Energy

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- Total kinetic energy E_{tot} sums from per body kinetic energy $E_{\text{kin},i}$, which splits into linear $E_{\text{lin},i}$ and angular energy $E_{\text{rot},i}$
- mass and inertia tensor can be combined with one velocity to momentum.
- The change in the different energies can be expressed for a vectorial per body impulse $\Delta\vec{\boldsymbol{P}}_i$ in surprisingly compact form
- When summing up to per body kinetic energies contact point velocities compactify even more
- Plugging in normal impulses the total energy results from relative contact point velocity and its change.

$$E_{\text{tot}} = E_{\text{kin},1} + E_{\text{kin},2}$$
$$E_{\text{kin},i} = E_{\text{lin},i} + E_{\text{rot},i}$$

$$E_{\text{lin},i} = \frac{1}{2} M_i \vec{\boldsymbol{V}}_i^2 = \frac{1}{2} \vec{\boldsymbol{V}}_i^T \vec{\boldsymbol{P}}_i$$
$$E_{\text{rot},i} = \frac{1}{2} \vec{\boldsymbol{\omega}}_i^T \boldsymbol{I} \vec{\boldsymbol{\omega}}_i = \frac{1}{2} \vec{\boldsymbol{\omega}}_i^T \vec{\boldsymbol{L}}_i$$

$$\Delta E_{\text{lin},i} = \left(\vec{\boldsymbol{V}}_i^- + \tfrac{1}{2}\Delta\vec{\boldsymbol{V}}_i\right)^T \Delta\vec{\boldsymbol{P}}_i$$

$$\Delta E_{\text{rot},i} = \left(\left(\vec{\boldsymbol{\omega}}_i + \tfrac{1}{2}\Delta\vec{\boldsymbol{\omega}}_i\right) \times \vec{\boldsymbol{r}}_i\right)^T \Delta\vec{\boldsymbol{P}}_i$$

$$\Delta\vec{\boldsymbol{v}}_i = \Delta\vec{\boldsymbol{V}}_i + \Delta\vec{\boldsymbol{\omega}}_i \times \vec{\boldsymbol{r}}_i$$

$$\Delta E_{\text{kin},i} = \left(\vec{\boldsymbol{v}}_i^- + \tfrac{1}{2}\Delta\vec{\boldsymbol{v}}_i\right)^T \Delta\vec{\boldsymbol{P}}_i$$

$$\Delta\vec{\boldsymbol{P}}_i = \mp\Delta p_{\parallel\hat{\boldsymbol{n}}}\hat{\boldsymbol{n}}$$

$$\Delta E_{\text{tot}} = \Delta p_{\parallel\hat{\boldsymbol{n}}}\left(v_{\text{rel}\parallel\hat{\boldsymbol{n}}}^- + \tfrac{1}{2}\Delta v_{\text{rel}\parallel\hat{\boldsymbol{n}}}\right)$$

Derivation of Change in Total Energy

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$$\Delta E_{\text{lin},i} = \tfrac{1}{2}\left(\vec{\boldsymbol{V}}_i^{+T}\vec{\boldsymbol{P}}_i^{+} - \vec{\boldsymbol{V}}_i^{-T}\vec{\boldsymbol{P}}_i^{-}\right)$$

$$2\Delta E_{\text{lin},i} = \left(\vec{\boldsymbol{V}}_i^{-} + \Delta\vec{\boldsymbol{V}}_i\right)^T\left(\vec{\boldsymbol{P}}_i^{-} + \Delta\vec{\boldsymbol{P}}_i\right) - \vec{\boldsymbol{V}}_i^{-T}\vec{\boldsymbol{P}}_i^{-}$$

$$= \vec{\boldsymbol{V}}_i^{-T}\Delta\vec{\boldsymbol{P}}_i + \Delta\vec{\boldsymbol{V}}_i^T\vec{\boldsymbol{P}}_i^{-} + \Delta\vec{\boldsymbol{V}}_i^T\Delta\vec{\boldsymbol{P}}_i$$

$$\vec{\boldsymbol{V}}_i^{-T}\Delta\vec{\boldsymbol{P}}_i = M_i\Delta\vec{\boldsymbol{V}}_i^T\vec{\boldsymbol{V}}_i^{-} = \Delta\vec{\boldsymbol{V}}_i^T\vec{\boldsymbol{P}}_i^{-}$$

$$\Delta E_{\text{lin},i} = \left(\vec{\boldsymbol{V}}_i^{-} + \tfrac{1}{2}\Delta\vec{\boldsymbol{V}}_i\right)^T\Delta\vec{\boldsymbol{P}}_i$$

$$\Delta E_{\text{rot},i} = \left(\vec{\boldsymbol{\omega}}_i + \tfrac{1}{2}\Delta\vec{\boldsymbol{\omega}}_i\right)^T\Delta\vec{\boldsymbol{L}}_i$$

$$= \left(\vec{\boldsymbol{\omega}}_i + \tfrac{1}{2}\Delta\vec{\boldsymbol{\omega}}_i\right)^T\left(\vec{\boldsymbol{r}}_i \times \Delta\vec{\boldsymbol{P}}_i\right)$$

$$= \left(\vec{\boldsymbol{\omega}}_i + \tfrac{1}{2}\Delta\vec{\boldsymbol{\omega}}_i\right)^T\boldsymbol{r}_i^{*}\Delta\vec{\boldsymbol{P}}_i$$

$$= \left(\boldsymbol{r}_i^{*T}\left(\vec{\boldsymbol{\omega}}_i + \tfrac{1}{2}\Delta\vec{\boldsymbol{\omega}}_i\right)\right)^T\Delta\vec{\boldsymbol{P}}_i$$

$$\Delta E_{\text{rot},i} = \left(\left(\vec{\boldsymbol{\omega}}_i + \tfrac{1}{2}\Delta\vec{\boldsymbol{\omega}}_i\right) \times \vec{\boldsymbol{r}}_i\right)^T\Delta\vec{\boldsymbol{P}}_i$$

$$\Delta E_{\text{kin},i} = \Delta E_{\text{lin},i} + \Delta E_{\text{rot},i}$$

$$= \left(\vec{\boldsymbol{V}}_i^{-} + \tfrac{1}{2}\Delta\vec{\boldsymbol{V}}_i + \left(\vec{\boldsymbol{\omega}}_i + \tfrac{1}{2}\Delta\vec{\boldsymbol{\omega}}_i\right) \times \vec{\boldsymbol{r}}_i\right)^T\Delta\vec{\boldsymbol{P}}_i$$

$$= \left(\vec{\boldsymbol{V}}_i^{-} + \vec{\boldsymbol{\omega}}_i \times \vec{\boldsymbol{r}}_i + \tfrac{1}{2}\left(\Delta\vec{\boldsymbol{V}}_i + \Delta\vec{\boldsymbol{\omega}}_i \times \vec{\boldsymbol{r}}_i\right)\right)^T\Delta\vec{\boldsymbol{P}}_i$$

$$\Delta\vec{\boldsymbol{v}}_i = \Delta\vec{\boldsymbol{V}}_i + \Delta\vec{\boldsymbol{\omega}}_i \times \vec{\boldsymbol{r}}_i$$

$$\Delta E_{\text{kin},i} = \left(\vec{\boldsymbol{v}}_i^{-} + \tfrac{1}{2}\Delta\vec{\boldsymbol{v}}_i\right)^T\Delta\vec{\boldsymbol{P}}_i$$

$$\Delta\vec{\boldsymbol{P}}_i = \mp\Delta p_{\parallel\hat{\boldsymbol{n}}}\hat{\boldsymbol{n}}$$

$$\Delta E_{\text{tot}} = \Delta E_{\text{kin},1} + \Delta E_{\text{kin},2} = \Delta p_{\parallel\hat{\boldsymbol{n}}}\left[-\left(\vec{\boldsymbol{v}}_1^{-} + \tfrac{1}{2}\Delta\vec{\boldsymbol{v}}_1\right)^T + \left(\vec{\boldsymbol{v}}_2^{-} + \tfrac{1}{2}\Delta\vec{\boldsymbol{v}}_2\right)^T\right]\hat{\boldsymbol{n}}$$

$$= \Delta p_{\parallel\hat{\boldsymbol{n}}}\left[\vec{\boldsymbol{v}}_2^{-} - \vec{\boldsymbol{v}}_1^{-} + \tfrac{1}{2}\left(\Delta\vec{\boldsymbol{v}}_2 - \Delta\vec{\boldsymbol{v}}_1\right)\right]^T\hat{\boldsymbol{n}}$$

$$\Delta E_{\text{tot}} = \Delta p_{\parallel\hat{\boldsymbol{n}}}\left(v_{\text{rel}\parallel\hat{\boldsymbol{n}}}^{-} + \tfrac{1}{2}\Delta v_{\text{rel}\parallel\hat{\boldsymbol{n}}}\right)$$

Impulse Transfer

Frictionless Impact

- for elastic impact:
  $\Delta E_{\text{tot}} = 0 = v^{-}_{\text{rel}\|\hat{\boldsymbol{n}}} + \frac{1}{2}\Delta v_{\text{rel}\|\hat{\boldsymbol{n}}}$
  we get
  $$\Delta v_{\text{rel}\|\hat{\boldsymbol{n}}} = -2v^{-}_{\text{rel}\|\hat{\boldsymbol{n}}}$$
- for inelastic impact:
  $$v^{+}_{\text{rel}\|\hat{\boldsymbol{n}}} = 0$$
  we get
  $$\Delta v_{\text{rel}\|\hat{\boldsymbol{n}}} = -v^{-}_{\text{rel}\|\hat{\boldsymbol{n}}}$$
- the partially elastic case uses COR ε:
  $$\Delta v_{\text{rel}\|\hat{\boldsymbol{n}}} = -(1+\varepsilon)v^{-}_{\text{rel}\|\hat{\boldsymbol{n}}}$$
- from normal component $K_{\|\hat{\boldsymbol{n}}} = \hat{\boldsymbol{n}}^T\boldsymbol{K}\hat{\boldsymbol{n}}$ of $\boldsymbol{K}$ matrix:

$$\Delta p_{\|\hat{\boldsymbol{n}}} = -(1+\varepsilon)\cdot v^{-}_{\text{rel}\|\hat{\boldsymbol{n}}}/K_{\|\hat{\boldsymbol{n}}}$$

$$\Delta\vec{\boldsymbol{p}} = \Delta p_{\|\hat{\boldsymbol{n}}}\cdot\hat{\boldsymbol{n}}$$

Joints

- We just need to ensure
  $$\vec{\boldsymbol{v}}^{+}_{\text{rel}} = \vec{\boldsymbol{0}}$$
- This yields
  $$\Delta\vec{\boldsymbol{v}}_{\text{rel}} = -\vec{\boldsymbol{v}}^{-}_{\text{rel}}$$
- Using the $\boldsymbol{K}$ matrix finally gives the necessary vectorial impulse

$$\Delta\vec{\boldsymbol{p}} = -\boldsymbol{K}^{-1}\cdot\vec{\boldsymbol{v}}^{-}_{\text{rel}}$$

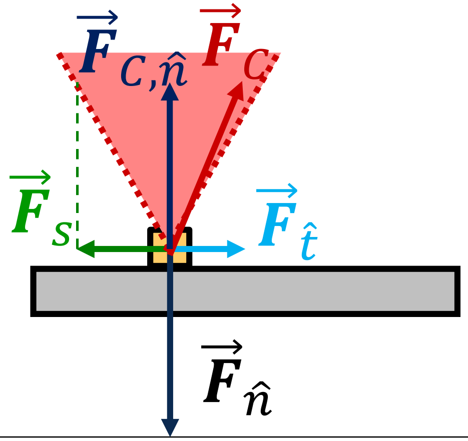
$$\Delta\vec{\boldsymbol{P}}_i = \mp\Delta\vec{\boldsymbol{p}}$$

impulse transfer

$$\Delta\vec{\boldsymbol{L}}_i = \mp\vec{\boldsymbol{r}}_i\times\Delta\vec{\boldsymbol{p}}$$

Coulomb Friction and Friction Cone

- In simulation we need to know when a contact changes from static to sliding
- This happens when an external force surpasses the static friction force $F_{\hat{t}} > F_s = \mu_s F_{\hat{n}}$
- At contact points the contact force is composed of the force acting against the normal force and the tangential external force: $\vec{\boldsymbol{F}}_C = -\vec{\boldsymbol{F}}_{\hat{n}} + \vec{\boldsymbol{F}}_{\hat{t}}$
- As long as $\vec{\boldsymbol{F}}_C$ is inside the friction cone $\left\{\mu_S^2 F_{C,\hat{n}}^2 \geq F_{C,\hat{t}}^2, F_{C,\hat{n}} \geq 0\right\}$ the contact stays static



Impact with Friction

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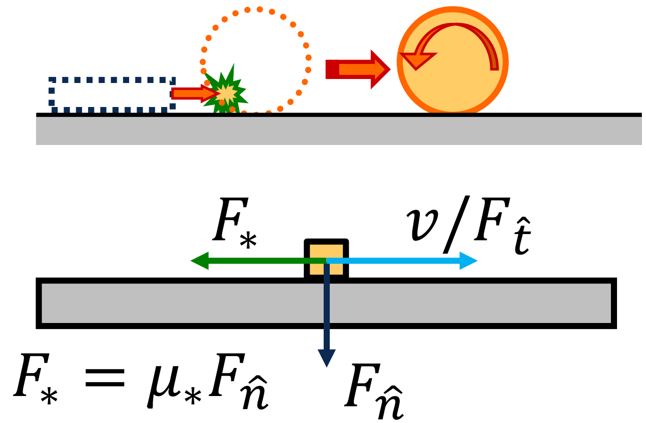
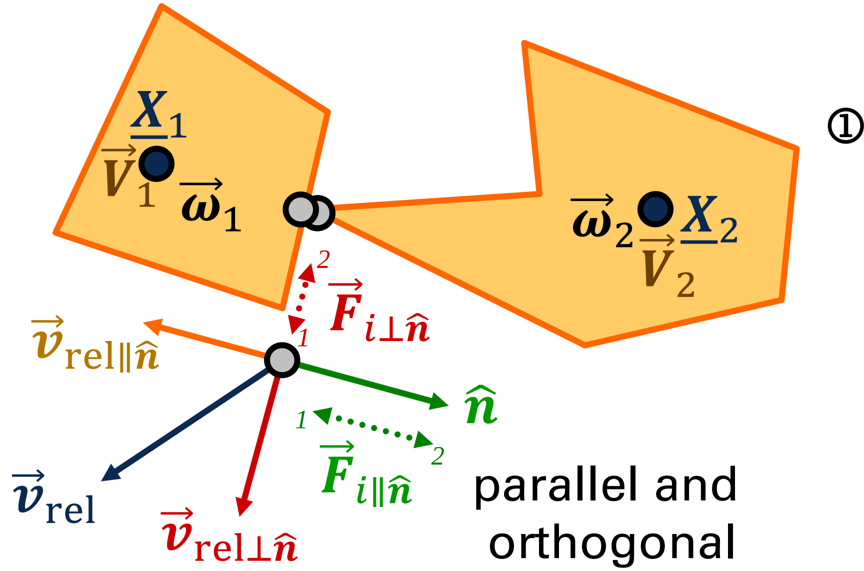
- In order to be able to generate a spin of the rigid bodies during the impact, frictional forces must be considered
- We again assume a homogeneous force impact over time Δt and translate the friction force laws into impulse laws:

① $$\Delta p_{*\|\hat{t}} = -\mu_* \Delta p_{\|\hat{n}}$$

- Tangential direction is computed from
$$\hat{t} = \text{normalize}\left(\vec{v}^-_{\text{rel}} - \vec{v}^-_{\text{rel}\|\hat{n}}\right)$$
- To decide whether to apply static or sliding friction we compute impulse for static friction from $v^+_{\text{rel}\|\hat{t}} = 0$ and check whether this is smaller than the one computed from ① with μ_s
- Otherwise static friction does not apply and one computes sliding friction from ① with μ_k and use this for friction



friction models: $\mu_r \ll \mu_k < \mu_s$



parallel and orthogonal force impact over Δt

Impact with Friction – Calculations

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- Preliminary computations: K-matrix, relative velocity at contact point and force impact normal direction $\Delta p_{\|\hat{\boldsymbol{n}}} = -v^-_{\mathrm{rel}\|\hat{\boldsymbol{n}}} / K_{\|\hat{\boldsymbol{n}}}$
- compute tangential direction & tangential part of K matrix $K_{\|\hat{\boldsymbol{t}}} = \hat{\boldsymbol{t}}^T \boldsymbol{K} \hat{\boldsymbol{t}}$
- compute force impact for static friction $\Delta p_{s\|\hat{t}}$ from constraint

$$v^+_{\mathrm{rel}\|\hat{\boldsymbol{t}}} = 0 \Longrightarrow \Delta v_{\mathrm{rel}\|\hat{t}} = -v^-_{\mathrm{rel}\|\hat{t}}$$

$$\Delta p_{s\|\hat{t}} = -v^-_{\mathrm{rel}\|\hat{t}} / K_{\|\hat{\boldsymbol{t}}}$$

- compute maximum force impact for static friction from

$$\Delta p_{s\|\hat{\boldsymbol{t}},\max} = -\mu_s \Delta p_{\|\hat{\boldsymbol{n}}}$$

- if $\Delta p_{s\|\hat{t}} \leq \Delta p_{s\|\hat{\boldsymbol{t}},\max}$ use $\Delta p_{s\|\hat{t}}$ as tangential force impact
- Otherwise compute force impact $\Delta p_{k\|\hat{\boldsymbol{t}}}$ for sliding friction

$$\Delta p_{k\|\hat{\boldsymbol{t}}} = -\mu_k \Delta p_{\|\hat{\boldsymbol{n}}}$$

  and use it as tangential force impact
- combine normal and tangential parts of impulse to $\Delta\vec{\boldsymbol{p}}$ before updating the momenta of the bodies

$$\Delta\vec{\boldsymbol{p}} = \Delta p_{\|\hat{\boldsymbol{n}}} \hat{\boldsymbol{n}} + \Delta p_{\|\hat{\boldsymbol{t}}} \hat{\boldsymbol{t}}$$

$$\Delta\vec{\boldsymbol{P}}_i = \mp \Delta\vec{\boldsymbol{p}}$$

impulse transfer

$$\Delta\vec{\boldsymbol{L}}_i = \mp \vec{\boldsymbol{r}}_i \times \Delta\vec{\boldsymbol{p}}$$

Collision Handling
Multiple Contacts

Collection of Formula

- Equations of Motion for each body i

$$\underline{\dot{\boldsymbol{X}}}_i = \boldsymbol{M}_i^{-1}\vec{\boldsymbol{P}}_i$$

$$\dot{\boldsymbol{R}}_i = \boldsymbol{I}_i^{-1}\boldsymbol{L}_i^*\boldsymbol{R}_i \qquad \boldsymbol{I}_i^{-1} = \boldsymbol{R}_i\boldsymbol{I}_{O,i}^{-1}\boldsymbol{R}_i^T$$

$$\dot{\vec{\boldsymbol{P}}}_i = \vec{\boldsymbol{F}}_{\mathrm{tot},i}$$

$$\dot{\vec{\boldsymbol{L}}}_i = \vec{\boldsymbol{T}}_{\mathrm{tot},i}$$

- Contact Impulses for each contact j:

$$\Delta\vec{\boldsymbol{P}}_i = \mp\Delta\vec{\boldsymbol{p}}_j$$

$$\Delta\vec{\boldsymbol{L}}_i = \mp\vec{\boldsymbol{r}}_i \times \Delta\vec{\boldsymbol{p}}_j$$

$$\Delta\vec{\boldsymbol{p}}_j = \Delta p_{\|\hat{\boldsymbol{n}}_j}\hat{\boldsymbol{n}}_j + \Delta p_{\|\hat{\boldsymbol{t}}_j}\hat{\boldsymbol{t}}_j \qquad \hat{\boldsymbol{t}}_j = \frac{\vec{\boldsymbol{v}}_{\mathrm{rel},j} - \vec{\boldsymbol{v}}_{\mathrm{rel}\|\hat{\boldsymbol{n}}_j}}{\left\|\vec{\boldsymbol{v}}_{\mathrm{rel},j} - \vec{\boldsymbol{v}}_{\mathrm{rel}\|\hat{\boldsymbol{n}}_j}\right\|}$$

$$\Delta p_{\|\hat{\boldsymbol{n}}_j} = -(1+\varepsilon)\cdot v^-_{\mathrm{rel}\|\hat{\boldsymbol{n}}_j}/K_{\|\hat{\boldsymbol{n}}_j}$$

$$\Delta p_{\|\hat{\boldsymbol{t}}_j} = \begin{cases} -v^-_{\mathrm{rel}\|\hat{\boldsymbol{t}}_j}/K_{\|\hat{\boldsymbol{t}}_j} & \text{if } \left|v^-_{\mathrm{rel}\|\hat{\boldsymbol{t}}_j}/K_{\|\hat{\boldsymbol{t}}_j}\right| \le \left|\mu_s \Delta p_{\|\hat{\boldsymbol{n}}_j}\right| \\ -\mu_k \Delta p_{\|\hat{\boldsymbol{n}}_j} & \text{otherwise} \end{cases}$$

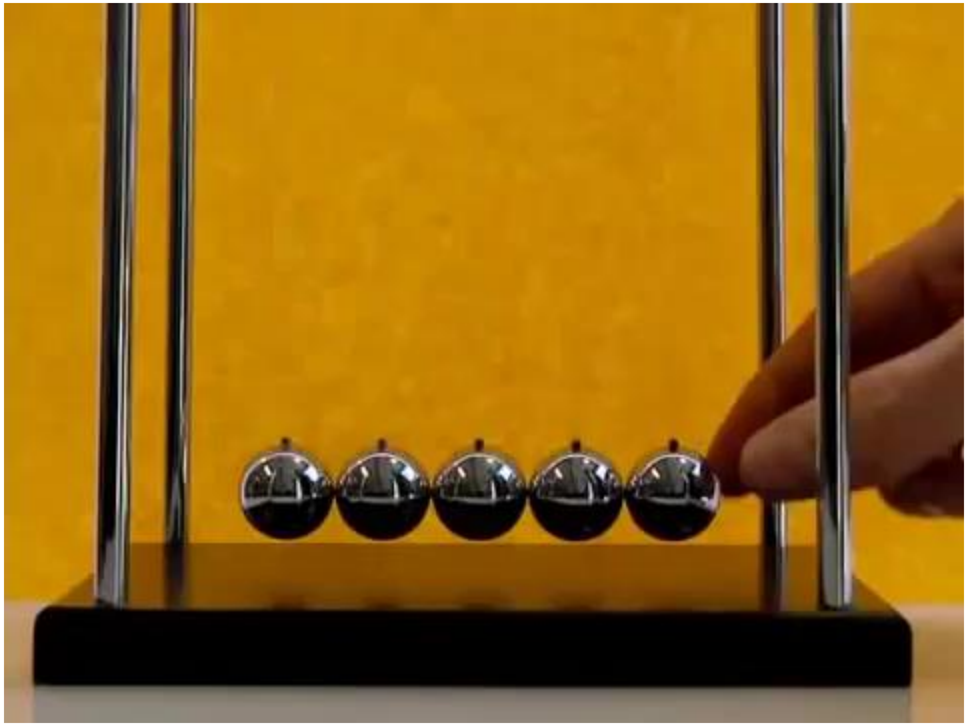
- Joint Impulse:

$$\Delta\vec{\boldsymbol{p}} = -\boldsymbol{K}^{-1}\cdot\vec{\boldsymbol{v}}^-_{\mathrm{rel}}$$

Newton’s Cradle

Computergraphik und Visualisierung

- For collisions with two and more contacts, energy and momentum conservation are not sufficient to uniquely describe motion:
- In Newton’s cradle a single ball could activate one or several balls. In case of several balls imagine that they move as if they were connected into one body. Then there is another solution for energy and momentum preservation and an infinite number of solutions between the ones were balls move asynchronously
- **Why do n balls transfer their momentum equally to again n balls on the other side??**



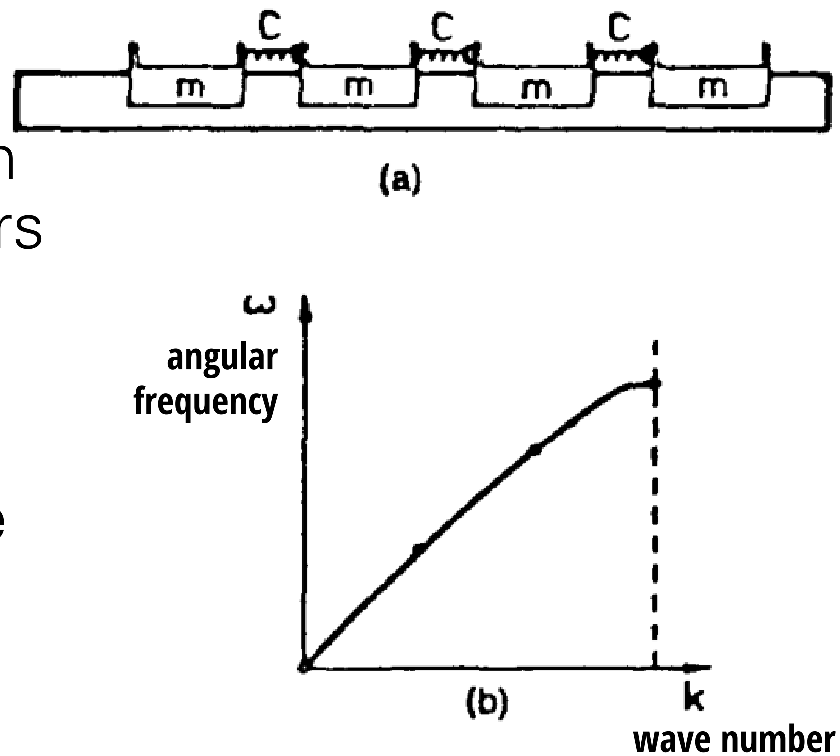
Video Collection

- https://www.youtube.com/watch?v=1W1Y64mzNio
- https://www.youtube.com/watch?v=uXI09nkg7mA
- https://www.youtube.com/watch?v=7qPvmYbfM6I

Analyzing Newton's Cradle

Computergraphik und Visualisierung

- Herrmann and Schmälzle performed an experiment where they replaced elastic balls with gliders of mass m on an air-track with spring bumpers with stiffness C.
- colliding one glider onto the remaining chain did not reproduce Newton's cradle like behavior but kind of random movement of all gliders
- an Eigen-frequency analysis of the mass-spring system showed dispersion: waves of different frequencies travel at different speeds in the system



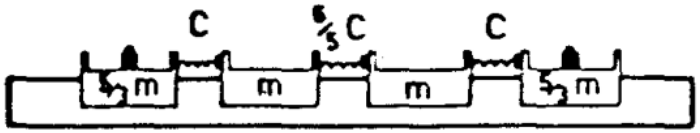
plane wave of angular frequency ω:

$$\Psi(x,t) = A \cdot \sin(kx + \omega t + \phi_0)$$

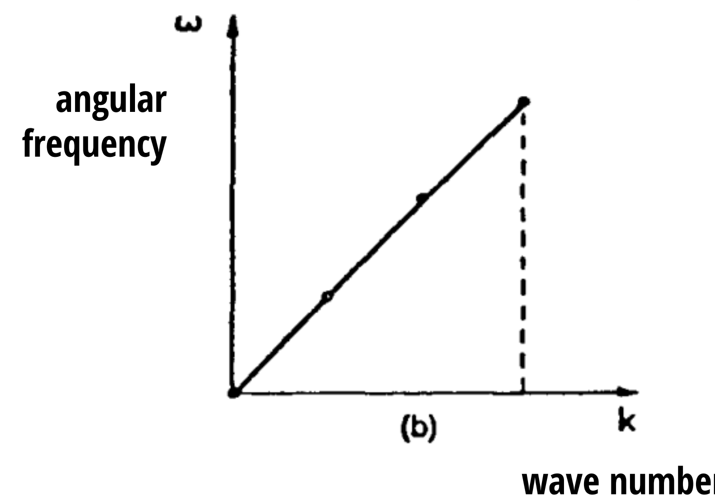
speed of plane wave: $v = \dfrac{\omega}{k}$

Analyzing Newton's Cradle

- The impact impulse generates waves of all frequencies. With dispersion they travel at different speeds and reach the right side at different times. First arrival activates first ball, second arrival second ball, etc.
- Herrmann and Schmälzle reconfigured the mass-spring system to a dispersion-free one such that all Eigen-frequencies have same speed.
- The dispersion-free system reproduced the behavior of Newton's Cradle.



(a)



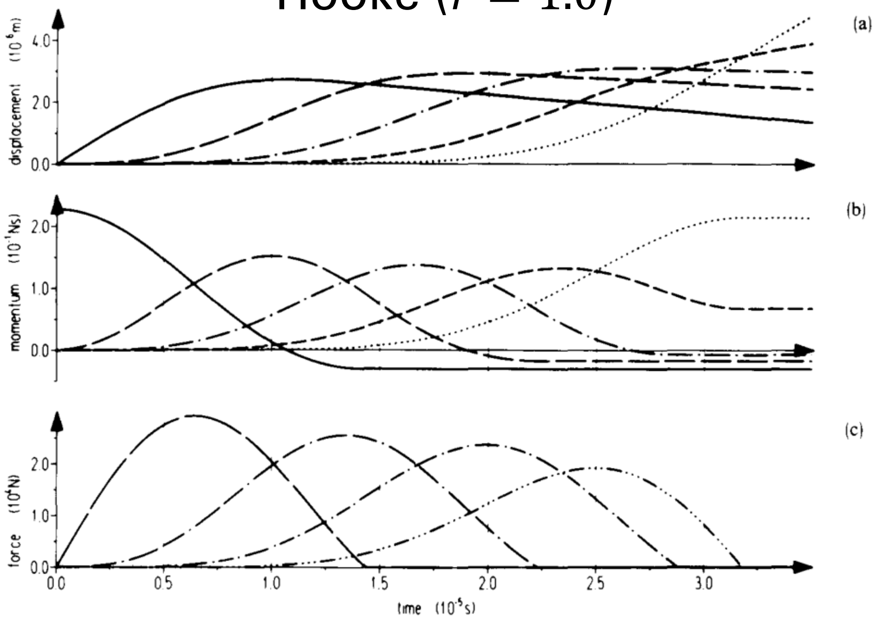
(b)

speed of plane wave: $v = \dfrac{\omega}{k}$

Analyzing Newton's Cradle

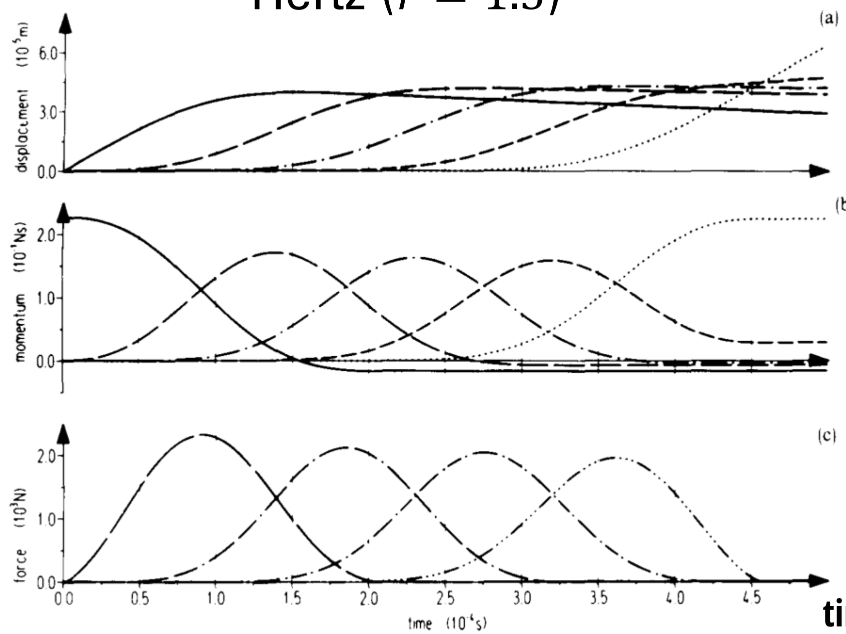
- Herrmann and Seitz did a computer simulation of a Newton's Cradel modelling the balls with elastic deformation
- They varied the exponent in the force law from Hooke ($r = 1.0$) over Hertz ($r = 1.5$) to a virtual value of $r = 4.0$.

Hooke ($r = 1.0$)



propagation time: $3.2 \cdot 10^{-5}$**sec**

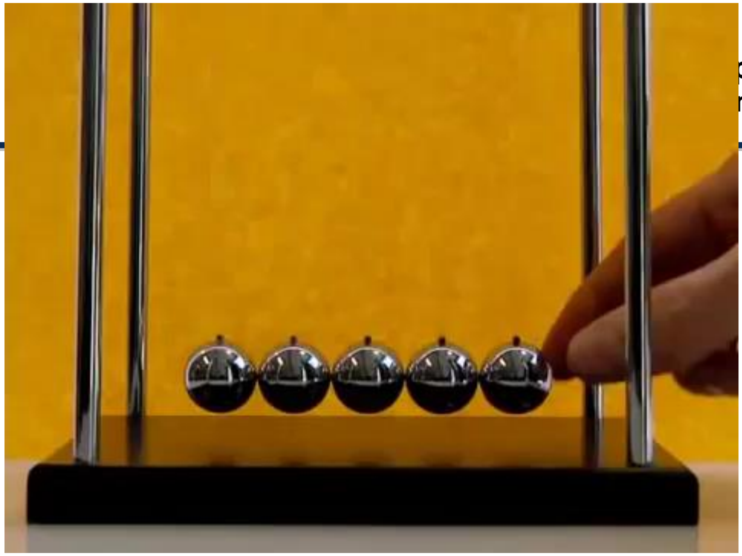
Hertz ($r = 1.5$)

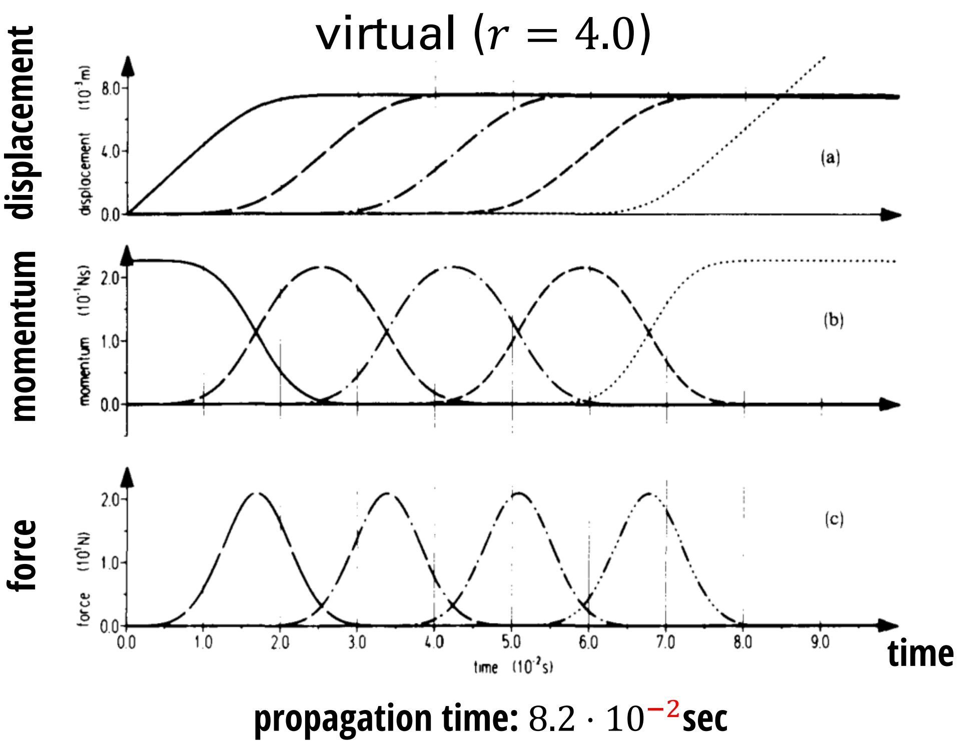


propagation time: $4.5 \cdot 10^{-4}$**sec**

Analyzing Newton's Cradle

- dispersion decreases with increasing exponent
- impact propagation time increases with increasing exponent
- Newton's Cradle has dispersion, such that the balls are not in contact anymore starting with second collision
- without contact there is no dispersion
- **Assumption in CG:** collisions are dispersion-free and propagate fast





Sequential Impulses

Sequential Impulses Loop

Overview

- only inelastic contacts
- allow and deal with penetration
- propagate remaining impulses to next time step

Simulation Loop

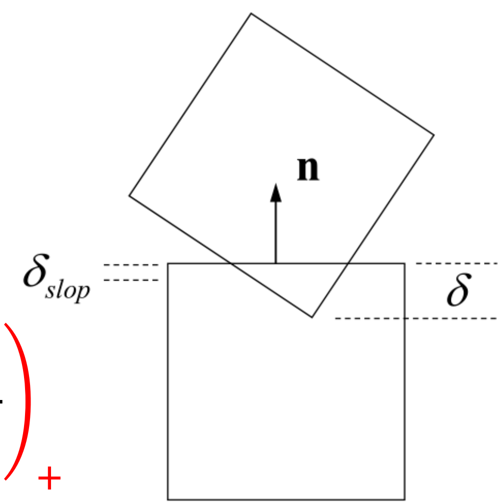
- compute list of contact points with penetration depths
  (ignore vanishing contact pts)
- compute external forces
- perform time integration step for momenta only
- apply impulses for inelastic impacts with friction and joint constraints with de-penetration bias in multiple iterations
  (remember remaining impacts and apply in beginning of next iteration)
- perform time integration step for position & orientation

Impulse Computation for Inelastic Impacts

- inelastic normal impulse $\Delta p_{\|\hat{\boldsymbol{n}}_j} = -v^-_{\mathrm{rel}\|\hat{\boldsymbol{n}}_j}/K_{\|\hat{\boldsymbol{n}}_j} \geq 0$
- friction impulse $\Delta p_{\|\hat{\boldsymbol{t}}_j} = \mathrm{clamp}\left(-v^-_{\mathrm{rel}\|\hat{\boldsymbol{t}}_j}/K_{\|\hat{\boldsymbol{t}}_j}, -\mu_s \Delta p_{\|\hat{\boldsymbol{n}}_j}, \mu_s \Delta p_{\|\hat{\boldsymbol{n}}_j}\right)$

(only static friction is considered here)

- de-penetration bias velocity
  - Only add bias if penetration δ is deeper than "slop": δ_{slop} (invisible offset)
  - Bias Factor: $\beta \in [0.1, 0.3]$ $(x)_+ := \max\{0, x\}$
  - Bias velocity: $v_{\mathrm{bias}} = \frac{\beta}{\Delta t}(\delta - \delta_{\mathrm{slop}})_+$
  - Biased normal impulse: $\Delta \tilde{p}_{(\|\hat{\boldsymbol{n}}_j} = \left(\frac{-v^-_{\mathrm{rel}\|\hat{\boldsymbol{n}}_j} + v_{\mathrm{bias}}}{K_{\|\hat{\boldsymbol{n}}_j}}\right)_+$



- joint impulses
  - Bias velocity from contact point difference: $\vec{\boldsymbol{v}}_{\mathrm{bias}} = \frac{\beta}{\Delta t}(\underline{\boldsymbol{x}}_1 - \underline{\boldsymbol{x}}_2)$
  - Biased impulse: $\Delta\vec{\tilde{\boldsymbol{p}}} = \boldsymbol{K}^{-1}\left(-\vec{\boldsymbol{v}}^-_{\mathrm{rel}} + \vec{\boldsymbol{v}}_{\mathrm{bias}}\right)$

Impulse Update with Accumulation

- per contact define accumulated impulses $\Delta P^{(0)}_{\|\hat{\boldsymbol{n}}_j} = 0$ & $\Delta P^{(0)}_{\|\hat{\boldsymbol{t}}_j} = 0$
- iterate $k = 1 \dots K$ times ($K = 10 - 50$) iterations
  - iterate contacts j in random order
    - compute relative velocity $v^{(k-1)}_{\text{rel}}$ based on current momenta
    - accumulate impulses with clamping:

$$\Delta\tilde{p}^{(k)}_{\|\hat{\boldsymbol{n}}_j} = \frac{\left(v_{\text{bias}} - v^{(k-1)}_{\text{rel}\|\hat{\boldsymbol{n}}_j}\right)}{K_{\|\hat{\boldsymbol{n}}_j}}, \qquad \Delta\tilde{p}^{(k)}_{\|\hat{\boldsymbol{t}}_j} = -\frac{v^{(k-1)}_{\text{rel}\|\hat{\boldsymbol{t}}_j}}{K_{\|\hat{\boldsymbol{t}}_j}}$$

$$\Delta P^{(k)}_{\|\hat{\boldsymbol{n}}_j} = \Delta P^{(k-1)}_{\|\hat{\boldsymbol{n}}_j} + \Delta\tilde{p}^{(k)}_{\|\hat{\boldsymbol{n}}_j}, \qquad \Delta P^{(k)}_{\|\hat{\boldsymbol{t}}_j} = \Delta P^{(k-1)}_{\|\hat{\boldsymbol{t}}_j} + \Delta\tilde{p}^{(k)}_{\|\hat{\boldsymbol{t}}_j}$$

$$\Delta p^{(k)}_{\|\hat{\boldsymbol{n}}_j} = \left(\Delta P^{(k)}_{\|\hat{\boldsymbol{n}}_j}\right)_+ - \Delta P^{(k-1)}_{\|\hat{\boldsymbol{n}}_j}, \qquad \Delta p^{(k)}_{\|\hat{\boldsymbol{t}}_j} = \text{clamp}\left(\Delta P^{(k)}_{\|\hat{\boldsymbol{t}}_j}, -\mu_s \Delta P^{(k)}_{\|\hat{\boldsymbol{n}}_j}, \mu_s \Delta P^{(k)}_{\|\hat{\boldsymbol{n}}_j}\right) - \Delta P^{(k-1)}_{\|\hat{\boldsymbol{t}}_j}$$

    - update momenta with $\Delta\vec{\boldsymbol{p}}^{(k)}_j = \Delta p^{(k)}_{\|\hat{\boldsymbol{n}}_j}\hat{\boldsymbol{n}}_j + \Delta p^{(k)}_{\|\hat{\boldsymbol{t}}_j}\hat{\boldsymbol{t}}_j$

Inter-timestep Persistence:

- Track contacts over time t and initialize impulses to final accumulated impulses $\Delta P^{(0)(t)}_{\|\hat{\boldsymbol{n}}|\hat{\boldsymbol{t}}_j} = \Delta P^{(K)(t-1)}_{\|\hat{\boldsymbol{n}}|\hat{\boldsymbol{t}}_j}$ of previous time step $t - 1$

Contact Force Approach

Simulation Loop

- compute graph of contact points (ignore vanishing contact points due to $v_{\mathrm{rel}}^{-} > 0$)
- propagate collisions by pair-wise partially elastic impacts with friction over graph
- compute external forces
- compute contact forces according to contact and joint constraints and add to external forces
- compute friction forces
- perform time integration step

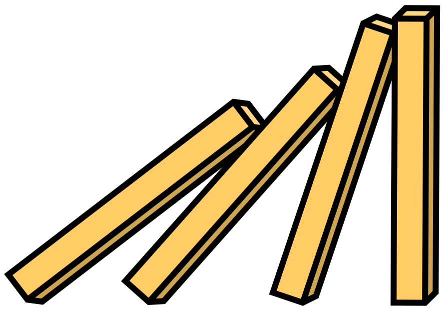
Contact Constraints

- given: n contact points with contact normals $\hat{\boldsymbol{n}}_j$
- compute: n contact forces f_j organizeable in a vector $\vec{\boldsymbol{f}}$

Conditions for calculating dynamically correct contact forces

1. The contact forces do not allow the bodies to inter-penetrate.
2. The contact forces can "push" but not "pull".
3. The contact forces occur only at contact points; once two bodies have separated at a contact point, there is no force between them at that contact point.
4. Viewed as a function of time, contact forces are continuous.

- in general there is more than one possible $\vec{\boldsymbol{f}}$ fulfilling all conditions, but all possibilities lead to the same motion



Contact Constraints – LCP

Distinguish contact points based on $v_{\mathrm{rel}\|\hat{\boldsymbol{n}},j}$ and $\dot{v}_{\mathrm{rel}\|\hat{\boldsymbol{n}},j}$ into

- non-vanishing if $v_{\mathrm{rel}\|\hat{\boldsymbol{n}},j} = 0$ and $\dot{v}_{\mathrm{rel}\|\hat{\boldsymbol{n}},j} = 0$
- vanishing if $v_{\mathrm{rel}\|\hat{\boldsymbol{n}},j} > 0$ or $\dot{v}_{\mathrm{rel}\|\hat{\boldsymbol{n}},j} > 0$

Conditions for calculating dynamically correct contact forces

1. no inter-penetration
   - $v_{\mathrm{rel}\|\hat{\boldsymbol{n}},j} > 0 \ldots$ can be ignored (safely vanishing)
   - $v_{\mathrm{rel}\|\hat{\boldsymbol{n}},j} = 0 \ldots$ then $\dot{v}_{\mathrm{rel}\|\hat{\boldsymbol{n}},j} \geq 0$, writeable as $\vec{\dot{\boldsymbol{v}}}_{\mathrm{rel}\|\hat{\boldsymbol{n}}} = \boldsymbol{A} \cdot \vec{\boldsymbol{f}} + \vec{\boldsymbol{b}} \geq \vec{\boldsymbol{0}}$
2. The contact forces can "push" but not "pull“ $f_j \geq 0$
3. contact forces vanish on separation: $f_j \cdot \dot{v}_{\mathrm{rel}\|\hat{\boldsymbol{n}},j} = 0$

- All three conditions can be combined into the linear complementary problem (LCP):

$$\vec{\boldsymbol{0}} \leq \left(\boldsymbol{A} \cdot \vec{\boldsymbol{f}} + \vec{\boldsymbol{b}}\right) \perp \vec{\boldsymbol{f}} \geq \vec{\boldsymbol{0}}$$

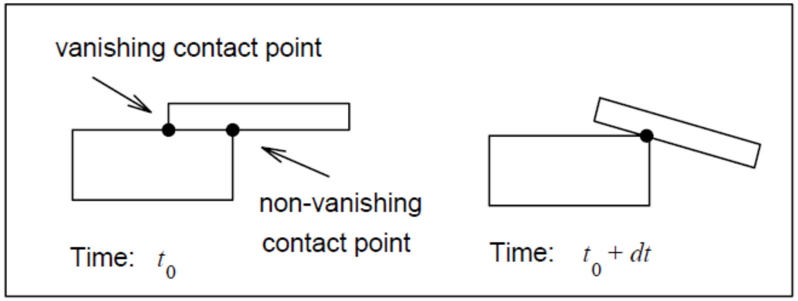


Figure 9. A vanishing contact point at time t_0. The bodies separate at the point immediately after time t_0.

Example of Derivation of Linear Constr.

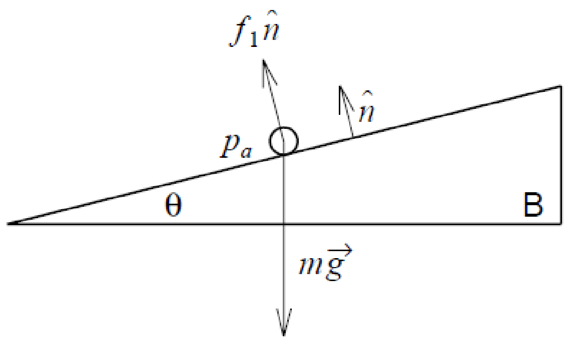
Computergraphik und Visualisierung

Variables

p_a	contact point	f_1	force magnitude
m	mass of A	$\vec{F}$	total force
$\vec{g}$	gravity vector	$\hat{n}$	unit normal

Relations

$$\vec{F} = m\vec{g} + f_1\hat{n} \qquad \ddot{p}_a = \frac{\vec{F}}{m} \qquad \hat{n}\cdot\vec{g} = -|\vec{g}|\cos\theta$$



Constraints

$$\ddot{\chi}_1 = \hat{n}\cdot\ddot{p}_a = \hat{n}\cdot\left(\frac{m\vec{g}+f_1\hat{n}}{m}\right) = \frac{m(\hat{n}\cdot\vec{g})+f_1}{m} \geq 0 \quad \text{or}$$

$$f_1 \geq -m(\hat{n}\cdot\vec{g}) = m|\vec{g}|\cos\theta$$

Figure 7. Constraint equations for a point mass A (with position p_a) resting on a fixed inclined plane B.

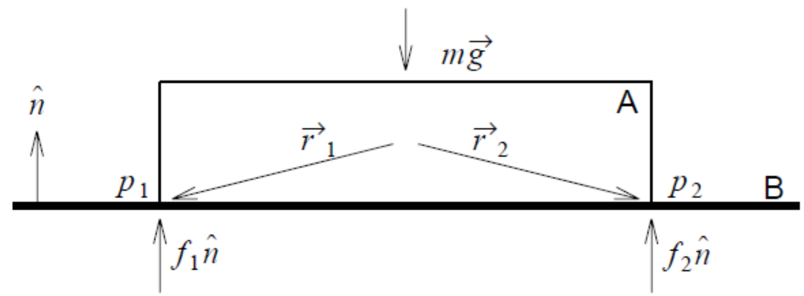
Variables

$p_{1,2}$	contact points	$\vec{r}_{1,2}$	body coordinates
$\vec{a}$	linear acceleration	$\vec{\alpha}$	angular acceleration
$f_{1,2}$	force magnitudes	$\vec{g}$	gravity vector
$\vec{F}$	total force	$\vec{\tau}$	total torque
m	mass	I	moment of inertia

Relations

$$\vec{F} = m\vec{g} + f_1\hat{n} + f_2\hat{n} \qquad \vec{\tau} = \vec{r}_1\times f_1\hat{n} + \vec{r}_2\times f_2\hat{n}$$

$$a = \frac{\vec{F}}{m} \qquad \vec{\alpha} = \frac{\vec{\tau}}{I} \qquad \ddot{p}_{1,2} = \vec{a} + \vec{\alpha}\times\vec{r}_{1,2} = \frac{\vec{F}}{m} + \frac{\vec{\tau}}{I}\times\vec{r}_{1,2}$$



Constraints

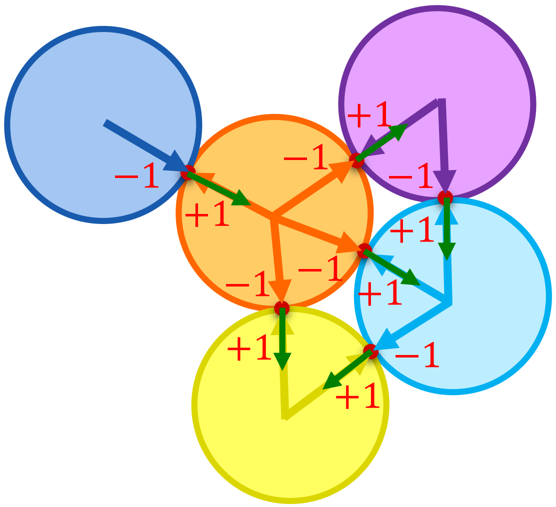
$$\ddot{\chi}_1 = \hat{n}\cdot\ddot{p}_1 = \hat{n}\cdot\left(\frac{m\vec{g}+f_1\hat{n}+f_2\hat{n}}{m} + \frac{\vec{r}_1\times f_1\hat{n}+\vec{r}_2\times f_2\hat{n}}{I}\times\vec{r}_1\right) \geq 0$$

$$\ddot{\chi}_2 = \hat{n}\cdot\ddot{p}_2 = \hat{n}\cdot\left(\frac{m\vec{g}+f_1\hat{n}+f_2\hat{n}}{m} + \frac{\vec{r}_1\times f_1\hat{n}+\vec{r}_2\times f_2\hat{n}}{I}\times\vec{r}_2\right) \geq 0$$

Figure 8. Constraint equations on the (unknown) contact force magnitudes $f_{1,2}$, for a block (A) supported by a fixed floor (B). The block is at rest.

contact graph:

- nodes are bodies; index: i
- edges are contacts; index: j
- contact normals: $\hat{\boldsymbol{n}}_j$
- positional vectors: $\vec{\boldsymbol{r}}_{ij}$
- contact normal signs: $s_{ij} = \pm 1$
- query nodes: $i \in \mathrm{bo}(j)$
- query edges: $j \in \mathrm{co}(i)$



$$\forall j: 0 \leq \dot{v}_{\mathrm{rel},j\|\hat{\boldsymbol{n}}_j} = \frac{\partial}{\partial t}\left\langle \hat{\boldsymbol{n}}_j, \sum_{i \in \mathrm{bo}(j)} s_{ij} \cdot \vec{\boldsymbol{v}}_{ij} \right\rangle$$

$$\dot{\vec{\boldsymbol{P}}}_i = \vec{\boldsymbol{F}}_{\mathrm{ext},i} + \sum_{\tilde{j} \in \mathrm{co}(i)} s_{i\tilde{j}}\, f_{\tilde{j}} \hat{\boldsymbol{n}}_{\tilde{j}}$$

$$\dot{\vec{\boldsymbol{L}}}_i = \vec{\boldsymbol{T}}_{\mathrm{ext},i} + \sum_{\tilde{j} \in \mathrm{co}(i)} s_{i\tilde{j}}\, f_{\tilde{j}} \vec{\boldsymbol{r}}_{i\tilde{j}} \times \hat{\boldsymbol{n}}_{\tilde{j}}$$

$$\boldsymbol{A} \cdot \vec{\boldsymbol{f}} + \vec{\boldsymbol{b}} \geq \vec{\boldsymbol{0}}$$

Constructing the LCP – Matrix A

$$\dot{v}_{\mathrm{rel},j\|\hat{\boldsymbol{n}}_j} = \frac{\partial}{\partial t}\left\langle \hat{\boldsymbol{n}}_j, \sum_{i\in \mathrm{bo}(j)} s_{ij}\cdot \vec{\boldsymbol{v}}_{ij}\right\rangle$$

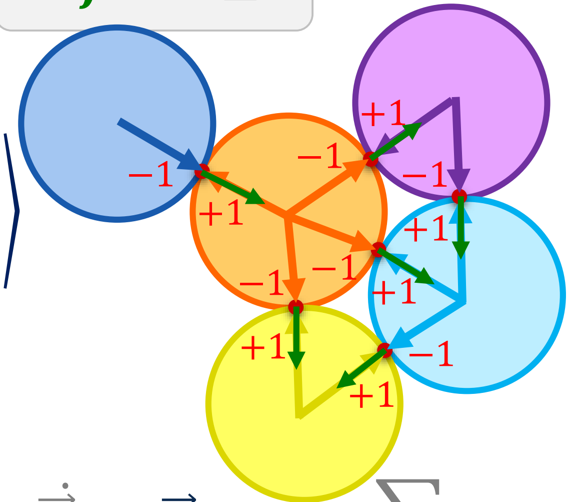
$$= \left\langle \hat{\boldsymbol{n}}_j, \sum_{i\in \mathrm{bo}(j)} s_{ij}\cdot \dot{\vec{\boldsymbol{v}}}_{ij}\right\rangle + \left\langle \dot{\hat{\boldsymbol{n}}}_j, \sum_{i\in \mathrm{bo}(j)} s_{ij}\cdot \vec{\boldsymbol{v}}_{ij}\right\rangle$$

depends on contact type and $\vec{\omega}_i$

depends on $\vec{V}_i$, $\vec{\omega}_i$ and $\vec{r}_{ij}$

both can be computed from $\vec{P}_i, \vec{L}_i, I_i^{-1}$ and contributes to b_j

$$\boldsymbol{A}\cdot\vec{f} + \vec{b} \geq \vec{\boldsymbol{0}}$$



$$\dot{\vec{P}}_i = \vec{\boldsymbol{F}}_{\mathrm{ext},i} + \sum_{\tilde{\jmath}\in \mathrm{co}(i)} s_{i\tilde{\jmath}}\, f_{\tilde{\jmath}} \hat{\boldsymbol{n}}_{\tilde{\jmath}}$$

$$\dot{\vec{L}}_i = \vec{\boldsymbol{T}}_{\mathrm{ext},i} + \sum_{\tilde{\jmath}\in \mathrm{co}(i)} s_{i\tilde{\jmath}}\, f_{\tilde{\jmath}} \vec{\boldsymbol{r}}_{i\tilde{\jmath}} \times \hat{\boldsymbol{n}}_{\tilde{\jmath}}$$

$$\dot{\vec{\boldsymbol{v}}}_{ij} = \dot{\vec{\boldsymbol{V}}}_i + \dot{\vec{\boldsymbol{\omega}}}_i \times \vec{\boldsymbol{r}}_{ij} + \vec{\boldsymbol{\omega}}_i \times \left(\vec{\boldsymbol{\omega}}_i \times \vec{\boldsymbol{r}}_{ij}\right)$$

$$\dot{\vec{\boldsymbol{V}}}_i = \frac{1}{M_i}\dot{\vec{\boldsymbol{P}}}_i \qquad \dot{\vec{\boldsymbol{\omega}}}_i = \boldsymbol{I}_i^{-1}\dot{\vec{\boldsymbol{L}}}_i + \dot{\boldsymbol{I}}_i^{-1}\vec{\boldsymbol{L}}_i$$

$$A_{jj} = \sum_{i\in \mathrm{bo}(j)} K_{ij\|\hat{\boldsymbol{n}}_j} \qquad A_{j\tilde{\jmath}} = \sum_{i\in \mathrm{bo}(j)} s_{ij}\cdot s_{i\tilde{\jmath}}\cdot\left\langle \hat{\boldsymbol{n}}_j, \frac{1}{M_i}\hat{\boldsymbol{n}}_{\tilde{\jmath}} + \boldsymbol{I}_i^{-1}\left(\vec{\boldsymbol{r}}_{i\tilde{\jmath}} \times \hat{\boldsymbol{n}}_{\tilde{\jmath}}\right) \times \vec{\boldsymbol{r}}_{ij}\right\rangle$$

Constructing the LCP – Vector $\vec{b}$

$$\boldsymbol{A}\cdot\vec{f}+\vec{b}\geq\vec{0}$$

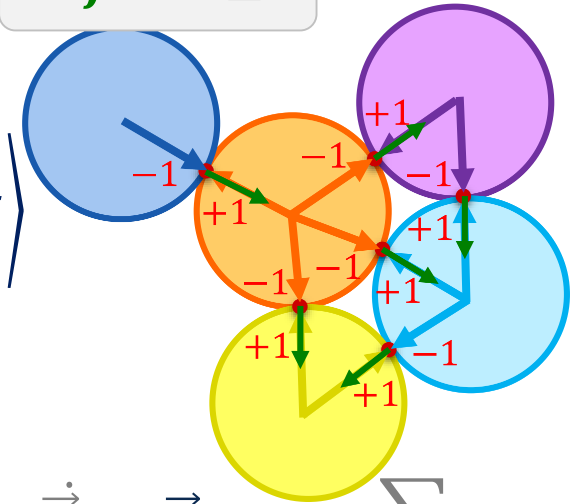
$$\dot{v}_{\mathrm{rel},j\|\hat{n}_j}=\frac{\partial}{\partial t}\left\langle\hat{\boldsymbol{n}}_j,\sum_{i\in\mathrm{bo}(j)}s_{ij}\cdot\vec{\boldsymbol{v}}_{ij}\right\rangle$$

$$=\left\langle\hat{\boldsymbol{n}}_j,\sum_{i\in\mathrm{bo}(j)}s_{ij}\cdot\dot{\vec{\boldsymbol{v}}}_{ij}\right\rangle+\left\langle\dot{\hat{\boldsymbol{n}}}_j,\sum_{i\in\mathrm{bo}(j)}s_{ij}\cdot\vec{\boldsymbol{v}}_{ij}\right\rangle$$

depends on contact type and $\vec{\omega}_i$

depends on $\vec{V}_i$, $\vec{\omega}_i$ and $\vec{r}_{ij}$

both can be computed from $\vec{P}_i, \vec{L}_i, I_i^{-1}$ and contributes to b_j



$$\dot{\vec{P}}_i=\vec{\boldsymbol{F}}_{\mathrm{ext},i}+\sum_{\tilde{j}\in\mathrm{co}(i)}s_{i\tilde{j}}\,f_{\tilde{j}}\hat{\boldsymbol{n}}_{\tilde{j}}$$

$$\dot{\vec{L}}_i=\vec{\boldsymbol{T}}_{\mathrm{ext},i}+\sum_{\tilde{j}\in\mathrm{co}(i)}s_{i\tilde{j}}\,f_{\tilde{j}}\vec{\boldsymbol{r}}_{i\tilde{j}}\times\hat{\boldsymbol{n}}_{\tilde{j}}$$

$$\dot{\vec{\boldsymbol{v}}}_{ij}=\dot{\vec{\boldsymbol{V}}}_i+\dot{\vec{\boldsymbol{\omega}}}_i\times\vec{\boldsymbol{r}}_{ij}+\vec{\boldsymbol{\omega}}_i\times\left(\vec{\boldsymbol{\omega}}_i\times\vec{\boldsymbol{r}}_{ij}\right)$$

$$\dot{\vec{\boldsymbol{V}}}_i=\frac{1}{M_i}\dot{\vec{\boldsymbol{P}}}_i\qquad\dot{\vec{\boldsymbol{\omega}}}_i=\boldsymbol{I}_i^{-1}\dot{\vec{\boldsymbol{L}}}_i+\dot{\boldsymbol{I}}_i^{-1}\vec{\boldsymbol{L}}_i=\boldsymbol{I}_i^{-1}\dot{\vec{\boldsymbol{L}}}_i+\boldsymbol{I}_i^{-1}\vec{\boldsymbol{L}}_i\times\vec{\boldsymbol{\omega}}_i$$

$$\vec{\boldsymbol{B}}_j=\sum_{i\in\mathrm{bo}(j)}s_{ij}\cdot\left(\frac{1}{M_i}\vec{\boldsymbol{F}}_{\mathrm{ext},i}+\left(\boldsymbol{I}_i^{-1}\left(\vec{\boldsymbol{T}}_{\mathrm{ext},i}+\vec{\boldsymbol{L}}_i\times\vec{\boldsymbol{\omega}}_i\right)\right)\times\vec{\boldsymbol{r}}_{ij}+\vec{\boldsymbol{\omega}}_i\times\left(\vec{\boldsymbol{\omega}}_i\times\vec{\boldsymbol{r}}_{ij}\right)\right)$$

$$b_j=\left\langle\hat{\boldsymbol{n}}_j,\vec{\boldsymbol{B}}_j\right\rangle+\left\langle\dot{\hat{\boldsymbol{n}}}_j,\sum_{i\in\mathrm{bo}(j)}s_{ij}\cdot\vec{\boldsymbol{v}}_i\right\rangle$$

Summary LCP-based Contact Forces

$$\vec{\boldsymbol{B}}_j = \sum_{i\in \mathrm{bo}(j)} s_{ij}\cdot\left(\frac{1}{M_i}\vec{\boldsymbol{F}}_{\mathrm{ext},i} + \left(\boldsymbol{I}_i^{-1}\left(\vec{\boldsymbol{T}}_{\mathrm{ext},i} + \vec{\boldsymbol{L}}_i \times \vec{\boldsymbol{\omega}}_i\right)\right)\times \vec{\boldsymbol{r}}_{ij} + \vec{\boldsymbol{\omega}}_i \times \left(\vec{\boldsymbol{\omega}}_i \times \vec{\boldsymbol{r}}_{ij}\right)\right)$$

$$A_{jj} = \sum_{i\in \mathrm{bo}(j)} K_{ij\|\hat{\boldsymbol{n}}_j}$$

$$b_j = \langle \hat{\boldsymbol{n}}_j, \vec{\boldsymbol{B}}_j\rangle + \left\langle \dot{\hat{\boldsymbol{n}}}_j, \sum_{i\in \mathrm{bo}(j)} s_{ij}\cdot \vec{\boldsymbol{v}}_i\right\rangle$$

$$A_{j\tilde{j}} = \sum_{i\in \mathrm{bo}(j)} s_{ij}\cdot s_{i\tilde{j}}\cdot\left\langle \hat{\boldsymbol{n}}_j, \frac{1}{M_i}\hat{\boldsymbol{n}}_{\tilde{j}} + \boldsymbol{I}_i^{-1}\left(\vec{\boldsymbol{r}}_{i\tilde{j}}\times\hat{\boldsymbol{n}}_{\tilde{j}}\right)\times\vec{\boldsymbol{r}}_{ij}\right\rangle$$

construct LCP

$$\vec{\mathbf{0}} \le \vec{\dot{\boldsymbol{v}}}_{\mathrm{rel}\|\hat{\boldsymbol{n}}} = \boldsymbol{A}\cdot\vec{\boldsymbol{f}} + \vec{\boldsymbol{b}} \perp \vec{\boldsymbol{f}} \ge \vec{\mathbf{0}}$$

solve LCP

momentum update:

$$\dot{\vec{\boldsymbol{P}}}_i = \vec{\boldsymbol{F}}_{\mathrm{ext},i} + \sum_{\tilde{j}\in \mathrm{co}(i)} s_{i\tilde{j}}\, f_{\tilde{j}}\hat{\boldsymbol{n}}_{\tilde{j}}$$

$$\dot{\vec{\boldsymbol{L}}}_i = \vec{\boldsymbol{T}}_{\mathrm{ext},i} + \sum_{\tilde{j}\in \mathrm{co}(i)} s_{i\tilde{j}}\, f_{\tilde{j}}\vec{\boldsymbol{r}}_{i\tilde{j}}\times\hat{\boldsymbol{n}}_{\tilde{j}}$$

Contact Constraints – LCP Solution

- Use spatial LCP solver from numerical recipes

- If contact points can be classified in vanishing and non-vanishing, the LCP becomes a linear programming problem
- Try classification on this example:

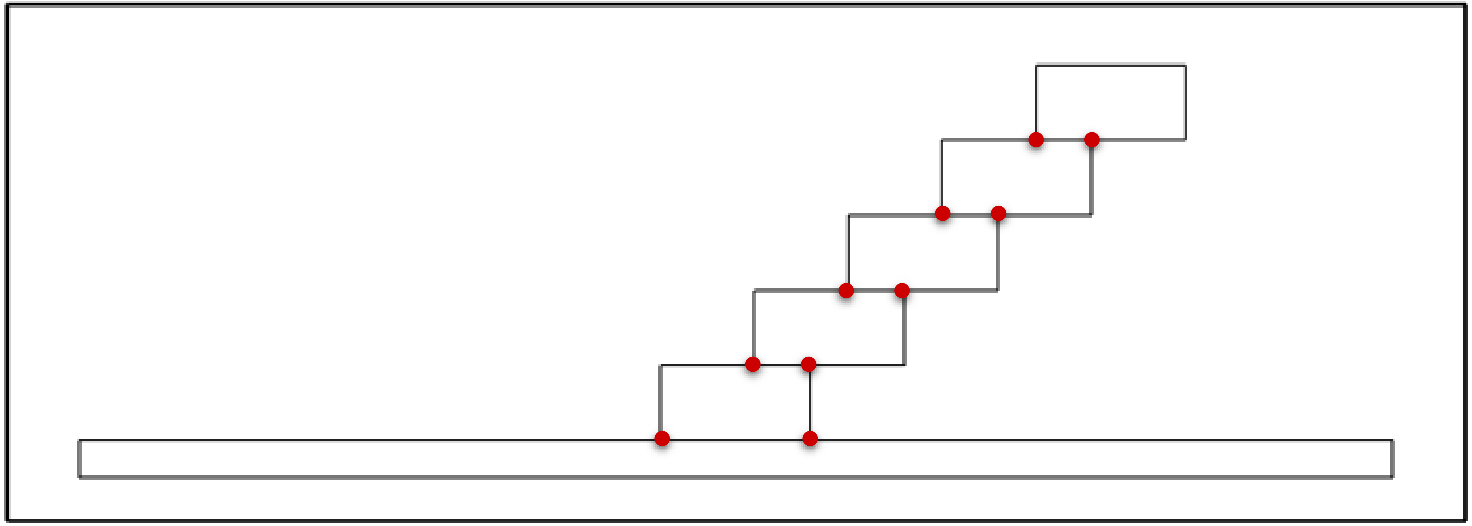
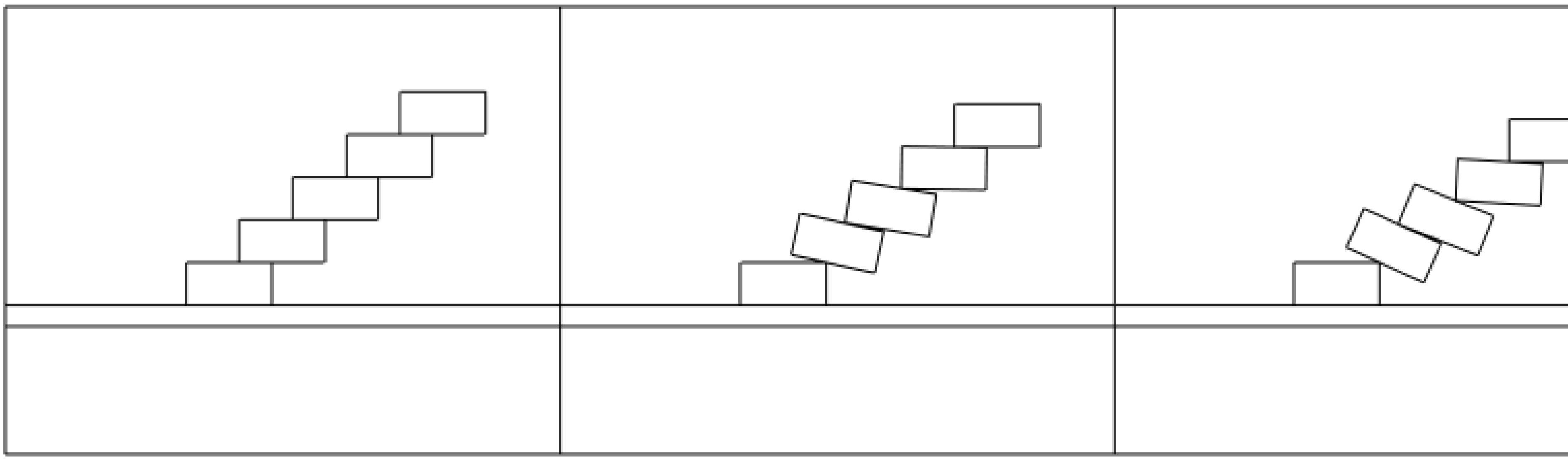


Figure 1. Overbalanced stack of bricks.

Contact Constraints – LCP Solution

(a) Overbalanced stack of bricks.



Summary

Summary

- One does not consider indeterminacy of contact normal in degenerate contacts to avoid NP hardness of collision handling
- Impacts with single contact can be solved with energy and momentum treatment through impulse transfer
- Assuming dispersion free rigid body systems we can treat multiple contact collisions through a sequence of single contact impacts
- Interpenetration from discrete time stepping can be cured by the introduction of bias velocities
- contact and joint constraints allow to compute contact forces through the solution of a LCP.

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