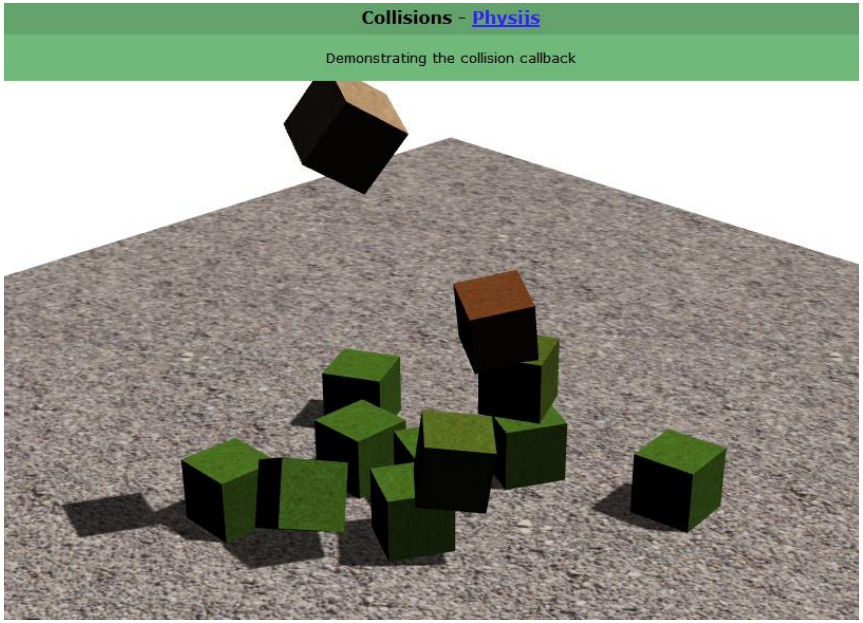
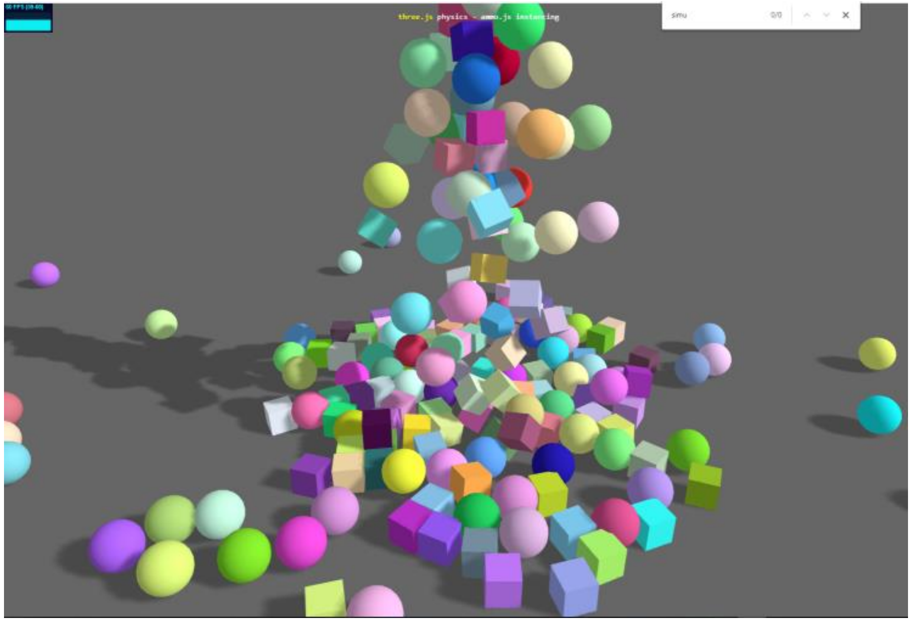
CG3 – Rigid Body Simulation

Rigid Body Simulation



https://github.com/chandlerprall/Physijs



https://threejs.org/examples/#physics_ammo_instancing

Box Demo

- Throw box into rotational motions along main axes
- One axis results in unstable rotations, why?

- Check out:
  - Intermediate axis theorem
  - Tennis Racket Theorem
  - Dzhanibekov Effect, or

https://www.camein.com/rotating-bodies-dzhanibekov-effect/

Content

- Motivation
- Placement $\underline{X}; R$
- Kinematics $\vec{V}; \vec{\omega}$
- Dynamics $M, \vec{P}, \vec{F}; I, \vec{L}, \vec{T}$
- Equations of Motion

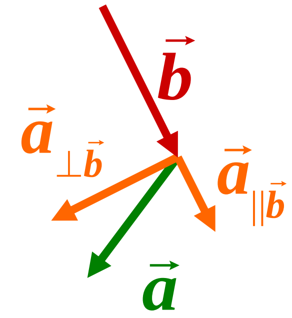
$$\vec{y} = \begin{pmatrix} \underline{X} \\ R \\ \vec{P} \\ \vec{L} \end{pmatrix} \ldots \dot{\vec{y}} = \begin{pmatrix} M^{-1}\vec{P} \\ I^{-1}\vec{L} \\ \vec{F} \\ \vec{T} \end{pmatrix}$$

Literatur

- Baraff, David. Rigid Body Simulation I + II (Siggraph course, Physically Based Modeling 1997) link
- Nolting, Wolfgang, Grundkurs Theoretische Physik 1, Klassische Mechanik, Band 1 Springer, 2. Auflage 2003

Vectors – Notation

- In the following, force vectors are usually decomposed into components that are parallel or perpendicular to another vector
- For this we introduce the two vectorial short notation that project perpendicularly or onto the vector.
- The same notation is used for the lengths of the respective components, except that the symbol is not written bold and without a vector.



$$\vec{\boldsymbol{a}}_{\|\vec{b}} = \left(\vec{\boldsymbol{a}}^T \vec{\boldsymbol{b}}\right)\vec{\boldsymbol{b}} / \vec{\boldsymbol{b}}^2$$

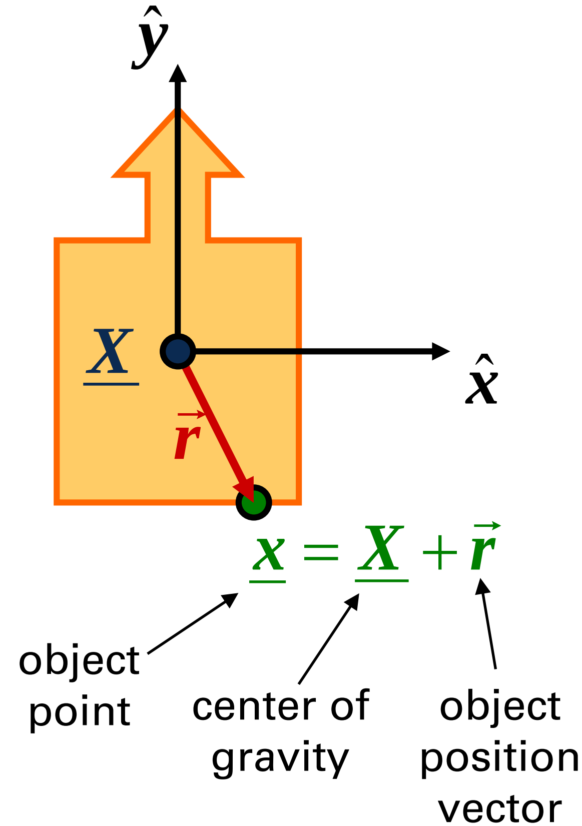
$$\vec{\boldsymbol{a}}_{\perp\vec{b}} = \vec{\boldsymbol{a}} - \vec{\boldsymbol{a}}_{\|\vec{b}}$$

$$a_{\|\vec{b}} = \left\|\vec{\boldsymbol{a}}_{\|\vec{b}}\right\|$$

$$a_{\perp\vec{b}} = \left\|\vec{\boldsymbol{a}}_{\perp\vec{b}}\right\|$$

Placement – Natural Coord. System I

- A rigid body can be positioned in space by means of an Euclidean transformation, i.e. a rotation and a translation.
- For this one defines a local object coordinate system $\mathcal{O}$ per rigid body.
- The natural origin is the center of mass $\underline{X}$ of the rigid body.
- A natural orientation results from the inertia tensor (see slides 12ff).
- In 2D the orientation is defined by an angle α. In general, a rotation matrix $\boldsymbol{R}$ can be used.



Placement – Center of Gravity

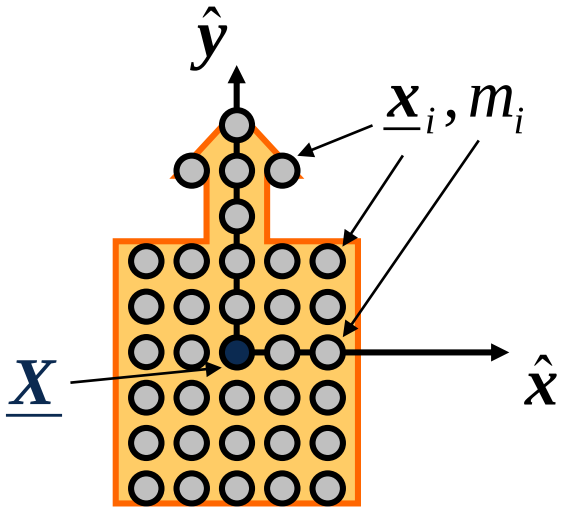
Discrete Case

- Rigid body is decomposed into discrete point masses $\underline{\boldsymbol{x}}_i, m_i$.
- The center of mass $\underline{\boldsymbol{X}}$ is the average position weighted with the point mass, which is derived from the affine combination

$$\underline{\boldsymbol{X}} = \frac{1}{M}\sum_{i} m_i \underline{\boldsymbol{x}}_i$$

- with the total mass M:

$$M = \sum_{i} m_i$$



Continuous Case

- here one defines the mass density

$$\rho = dM/dV$$

- summations become integrals:

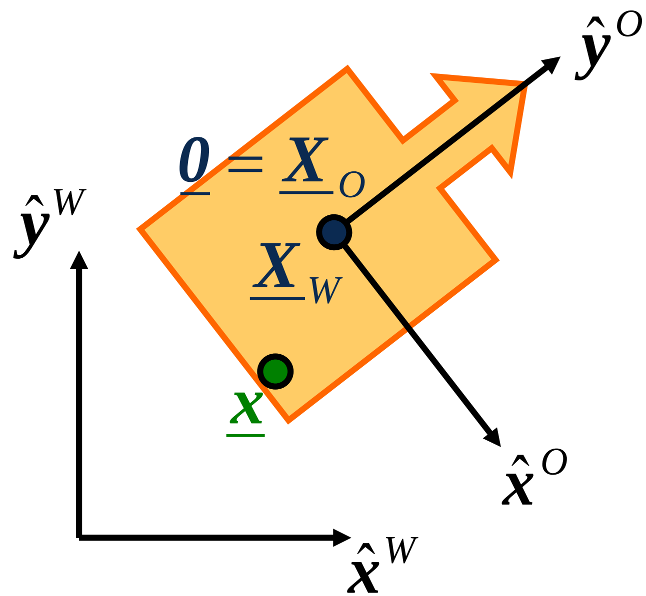
$$M = \int_V \rho(\underline{\boldsymbol{x}})dV$$

$$\underline{\boldsymbol{X}} = \frac{1}{M}\int_V \rho(\underline{\boldsymbol{x}})\,\underline{\boldsymbol{x}}dV$$

Placement

Notation

- W and O stand for world and object coordinate system.
- A subscript defines the coordinate system in which the vector components are given.
- For base vectors, the additional superscripts indicate which coordinate system is spanned by the base.
- For positioning, an Euclidean transformation from the natural object coordinates into world coordinates is given.



$$\underline{x}_W = R\left(\underline{x}_O - \underline{X}_O\right) + \underline{X}_W$$

$$\vec{r}_W = R\vec{r}_O$$

$$R = \begin{pmatrix} \hat{x}^O_W & \hat{y}^O_W \end{pmatrix}$$

$$\underline{x}_O = R^T\left(\underline{x}_W - \underline{X}_W\right) + \underline{X}_O$$

$$\vec{r}_O = R^T\vec{r}_W$$

Kinematics

- By derivation with respect to time one receives two contributions to the velocity (note that $\vec{r}_O$ does not change over time)

$$\underline{x}_W = R\vec{r}_O + \underline{X}_W$$

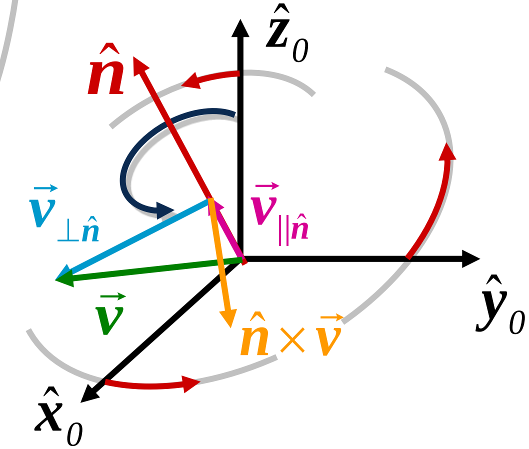
$$\dot{\underline{x}}_W = \dot{R}\vec{r}_O + \dot{\underline{X}}_W$$

$$\underbrace{\begin{pmatrix}\omega_x\\ \omega_y\\ \omega_z\end{pmatrix}}_{\vec{\omega}} \to \underbrace{\begin{pmatrix}0 & -\omega_z & \omega_y\\ \omega_z & 0 & -\omega_x\\ -\omega_y & \omega_x & 0\end{pmatrix}}_{\vec{\omega}^*}$$

Linear velocity

- Describes the uniform motion of the rigid body (which can also rotate around the center of mass).

$$\underline{V}_W = \dot{\underline{X}}_W$$



Angular velocity

- Describes the change of orientation
- For infinitesimal small dt rotation can be assumed to be constant around fixed axis $\hat{n}$
- with angular velocity defined as

$$\vec{\omega} = \frac{d\alpha}{dt}\hat{n}$$

$$\dot{R} = (\underbrace{\Re(\hat{n}, d\alpha)}R - R)/dt$$

$$\hat{n}\hat{n}^T + (1 - \hat{n}\hat{n}^T)\underbrace{\cos d\alpha}_{1} + \hat{n}^*\underbrace{\sin d\alpha}_{d\alpha}$$

$$\xrightarrow{d\alpha \to 0} = 1 + \hat{n}^* d\alpha$$

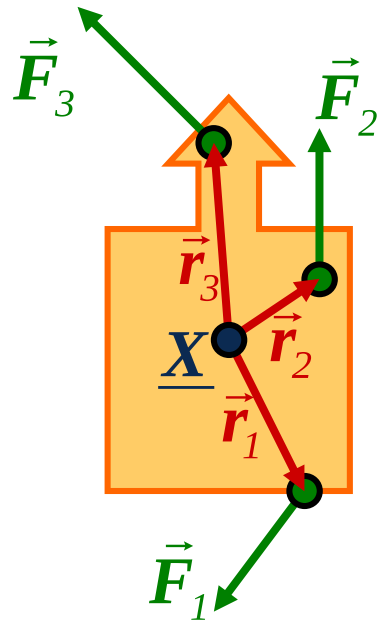
$$\dot{R} = \vec{\omega}^* R = \hat{n}^* \frac{d\alpha}{dt} R \quad \text{mit } \vec{\omega} = \frac{d\alpha}{dt}\hat{n}$$

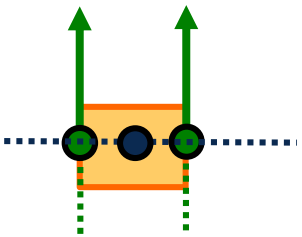
Dynamics – Forces and Accelerations

- In the following we assume the world coordinate system as default even if the subscript W is not given.
- As with kinematics, dynamics can be split into linear and angular motion.

Linear Dynamics

- All forces acting on the body are applied to the center of mass and added to the total force which, according to Newton, changes the linear velocity and the linear momentum.
- The procedure is equivalent to the one for point masses





simple example that shows, why force needs to be transported to centroid also orthogonal to force action lines

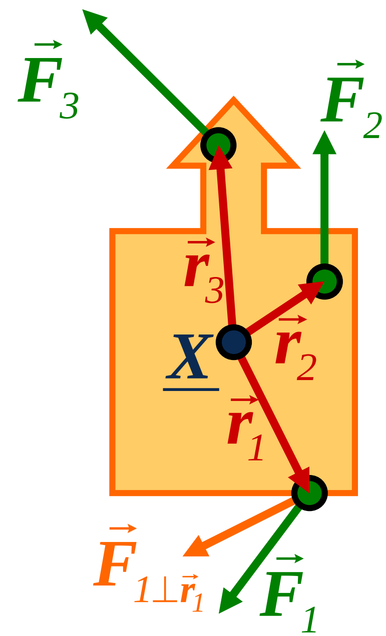
$$\vec{F}_{\text{tot}} = \sum_i \vec{F}_i = M\dot{\vec{V}} = \dot{\vec{P}}$$

with linear momentum $\vec{P} = M\vec{V}$

Dynamics – Torque I

Angular Dynamics

- Torque $\vec{T}$ is the rotational equivalent to force
- Torque measures lever action with that a force $\vec{F}_i$ acts on center of mass $\underline{X}$ when force is applied to an object position vector $\vec{r}_i$.
- Torque points along the rotation axis and in 2D orthogonal to 2D plane (up or down)
- The absolute value T can be computed from the length of $\vec{r}_i$, which is the length of the lever, and the component of the force orthogonal to $\vec{r}_i$.
- Any force $\vec{F}_i$ therefore acts twice – once for linear and once for the angular dynamics



only 2D

$$T_i = \pm r_i F_{i\perp\vec{r}_i}$$

$$T_{\text{tot}} = \sum_i T_i$$

3D case

$$\vec{T}_i = \vec{r}_i \times \vec{F}_i$$

$$\vec{T}_{\text{tot}} = \sum_i \vec{T}_i$$

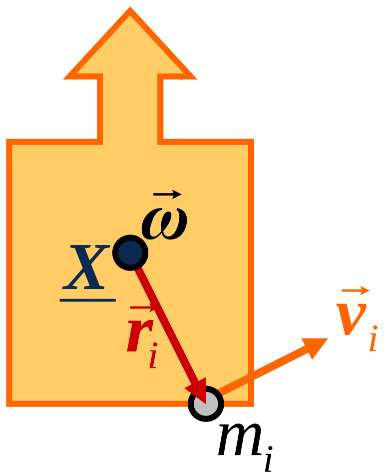
Dynamics – Torque II

Rotation & Newton's Laws

- For now, we will look at only one point mass, which circles around the center of mass with the angular velocity.
- For an acceleration on the orbit a force is needed which accelerates the point mass according to Newton's 2nd law.
- This force can be directly converted into the corresponding torque.
- If one uses the vector identity

$$\vec{a}\times\left(\vec{b}\times\vec{c}\right)=(\vec{a}\cdot\vec{c})\vec{b}-\left(\vec{a}\cdot\vec{b}\right)\vec{c}$$

the angular acceleration can be factored out also in the vectorial case.



only 2D

$$v_i = r_i\omega$$
$$F_i = m_i\dot{v}_i = m_i r_i\dot{\omega}$$
$$T_i = m_i r_i^2\dot{\omega}$$

3D case

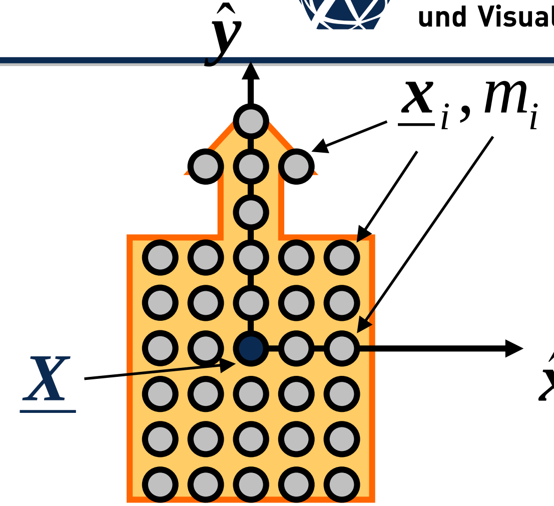
$$\vec{v}_i = \vec{\omega}\times\vec{r}_i$$
$$\vec{F}_i = m_i\dot{\vec{v}}_i = m_i\dot{\vec{\omega}}\times\vec{r}_i$$
$$\vec{T}_i = m_i\vec{r}_i\times\left(\dot{\vec{\omega}}\times\vec{r}_i\right) = m_i\left(\vec{r}_i^{\,2}\mathbf{1}-\vec{r}_i\vec{r}_i^{\,T}\right)\dot{\vec{\omega}}$$

Dynamics – Inertia Tensor I

Rotation & Newton's Laws

- Summing the contributions of all point masses yields the total torque
- This is proportional to the angular acceleration in 2D.
- The proportionality constant is called the moment of inertia and is the equivalent to the mass in linear dynamics.
- It grows quadratically in the distance to the axis of rotation.
- In 3D we obtain a symmetric 3x3-matrix
- In the continuous case, the inertia tensor results from

$$\boldsymbol{I} = \int \rho(\vec{\boldsymbol{r}})\left(\vec{\boldsymbol{r}}^{\,2} - \vec{\boldsymbol{r}}\vec{\boldsymbol{r}}^{T}\right)d\vec{\boldsymbol{r}}$$



2D case $T_{ges} = \left(\sum_i m_i r_i^2\right)\dot{\omega} = I\dot{\omega}$

moment of inertia: $I = \sum_i m_i r_i^2$

3D case $\vec{\boldsymbol{T}}_{ges} = \underbrace{\left(\sum_i m_i\left(\vec{\boldsymbol{r}}_i^{\,2} - \vec{\boldsymbol{r}}_i\vec{\boldsymbol{r}}_i^{T}\right)\right)}_{\boldsymbol{I}}\dot{\vec{\boldsymbol{\omega}}}$

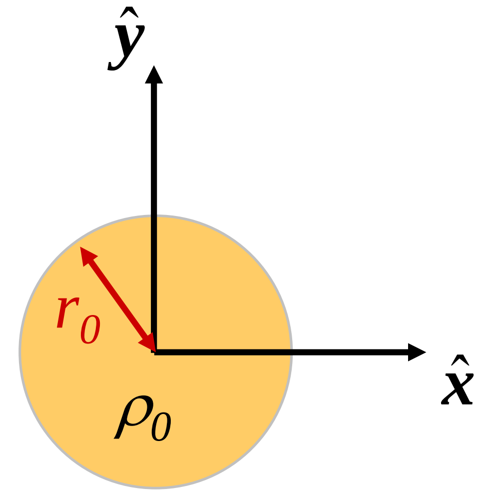
sym. inertia tensor: $\boldsymbol{I} = \sum_i m_i\left(\vec{\boldsymbol{r}}_i^2 - \vec{\boldsymbol{r}}_i\vec{\boldsymbol{r}}_i^T\right) \in \boldsymbol{R}^{3\times 3}$

$$= \sum_i m_i \begin{pmatrix} y_i^2 + z_i^2 & -x_i y_i & -x_i z_i \\ -x_i y_i & x_i^2 + z_i^2 & -y_i z_i \\ -x_i z_i & -y_i z_i & x_i^2 + y_i^2 \end{pmatrix}$$

Dynamics – Inertia Tensor II

Example of circular disk of radius r_0

- This is the 2D case and therefore a moment of inertia is calculated.
- Radius is r_0 and density constant equals ρ_0 over total disk
- The best way is to transform the integral into cylinder coordinates and integrate them by angle (yields a factor of 2π) and radius.
- The result can be interpreted in such a way that the circular disk has the same inertia with regard to rotation as a ring or point mass at radius $1/\sqrt{2}\, r_0$



$$I = \int \rho(\vec{\mathbf{r}}) r^2 \, dy dx$$

$$I = \int_0^{r_0} \int_0^{2\pi} \rho(\vec{\mathbf{r}}) r^2 [r \cdot d\phi \cdot dr]$$

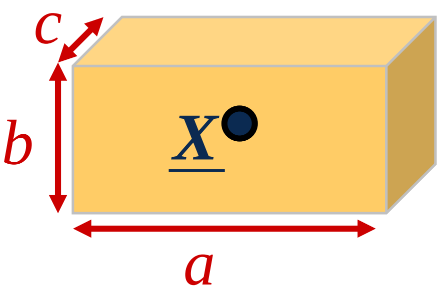
$$= 2\pi \int_0^{r_0} \rho_0 r^3 dr = \tfrac{1}{2}\pi \rho_0 r_0^4$$

$$= \tfrac{1}{2} M_0 r_0^2$$

Dynamics – Inertia Tensor III

Example Cuboid

- It is important to place the origin in the center of mass.
- By alignment to the main axes, a diagonal matrix with three main moments of inertia is obtained I_x, I_y, I_z:



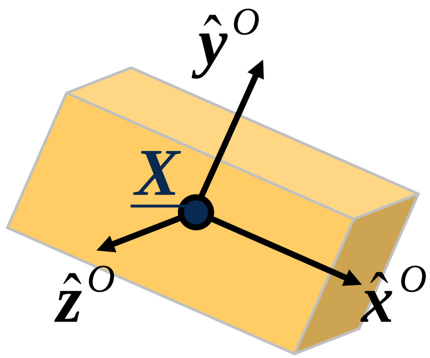
$$I = \int \rho(\vec{r})\left(\vec{r}^2 - \vec{r}\vec{r}^T\right)d\vec{r} = \int_{-c/2}^{c/2}\int_{-b/2}^{b/2}\int_{-a/2}^{a/2} \rho_0 \begin{pmatrix} y^2+z^2 & -xy & -xz \\ -xy & x^2+z^2 & -yz \\ -xz & -yz & x^2+y^2 \end{pmatrix} dxdydz$$

$$= \rho_0 \begin{pmatrix} abc\frac{b^2+c^2}{12} & 0 & 0 \\ 0 & abc\frac{a^2+c^2}{12} & 0 \\ 0 & 0 & abc\frac{a^2+b^2}{12} \end{pmatrix} = \frac{M_0}{12}\begin{pmatrix} b^2+c^2 & 0 & 0 \\ 0 & a^2+c^2 & 0 \\ 0 & 0 & a^2+b^2 \end{pmatrix} = \begin{pmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{pmatrix}$$

check out: http://www.cs.berkeley.edu/~jfc/mirtich/massProps.html

Dynamics – Natural Coord. System II

- Since the tensor of inertia is symmetric and positive definite, one can always find an orthonormal coordinate system in which the tensor is diagonal and completely defined by I_x, I_y, I_z.
- This is typically used as a natural coordinate system.
- Caution in contrast to mass, the inertia tensor must be transformed from the object to the world coordinate system. The rotation matrix of the orientation will be multiplied from the left and transposed from the right.



$$\mathbf{I}_O = \begin{pmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{pmatrix}$$

$$\mathbf{I}_W = \mathbf{R}\mathbf{I}_O\mathbf{R}^T$$

Dynamics – Angular Momentum

- To the relation between force and linear momentum corresonds a similar relation between torque and angular momentum
- Angular momentum $\vec{L}$ is a vectorial conserved quantity
- Inertia tensor changes over time with rotation matrix

$$I(t) = R(t) I_o R^T(t) \qquad \dot{R} = \omega^* R$$

$$\dot{I} = \dot{R} I_o R^T + R I_o \dot{R}^T$$

$$\dot{I} = \omega^* R I_o R^T - R I_o R^T \omega^* = \omega^* I - I \omega^*$$

- $\vec{L}$ and $\vec{\omega}$ are parallel if $\vec{\omega}$ points along a main axis of the inertia tensor, and it holds $\vec{T} = \dot{\vec{L}} = I\dot{\vec{\omega}}$
- In case of equal momentums of inertia all their linear combinations yield such main axes.

$$\vec{P} = M\vec{V} \Rightarrow \vec{F} = \dot{\vec{P}}$$

$$\Rightarrow \boxed{\vec{F} = M\dot{\vec{V}}}$$

$$\vec{L} = I\vec{\omega} \Rightarrow \vec{T} = \dot{\vec{L}}$$

$$\Rightarrow \vec{T} = I\dot{\vec{\omega}} + \dot{I}\vec{\omega}$$

$$\Rightarrow \vec{T} = I\dot{\vec{\omega}} + \omega^* I\vec{\omega} - I\,\underbrace{\omega^*\vec{\omega}}_{\vec{0}}$$

$$\Rightarrow \vec{T} = I\dot{\vec{\omega}} + \omega^* I\vec{\omega}$$

$$\Rightarrow \boxed{\vec{T} = I\dot{\vec{\omega}} + \omega^* \vec{L}}$$

Example: nutation of spinning top

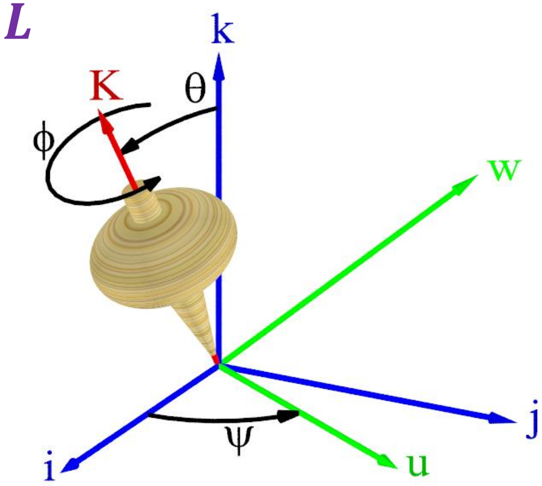
- If top spinning around main axis $\vec{K}$ that is tilted by angle θ from the vertical axis, gravity generates a torque $\vec{T}$ pointing along $\hat{u}$ axis in figure on right side
- This torque rotates main axis $\vec{K}$ around vertical axis
- Angular velocity of precession fulfills

$$\vec{T} = \vec{\omega}_{\mathrm{p}} \times \vec{L}$$

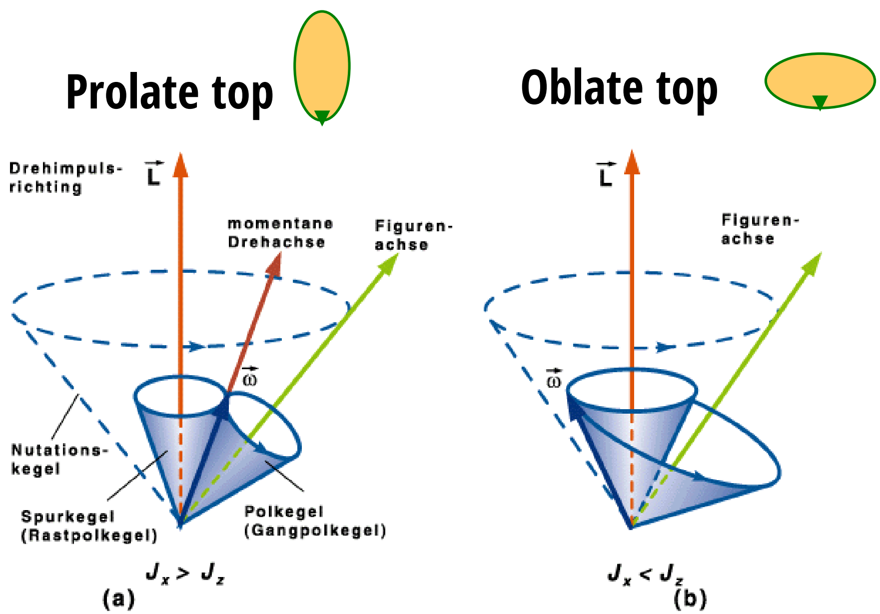
such that $\omega_{\mathrm{p}} = \dfrac{T}{\sin\theta \cdot L}$

- Precession speeds up with decrease in angular momentum
- If θ changes over time the motion is called nutation which can be described by cones

$$\vec{T} = I\dot{\vec{\omega}} + \omega^{*}\vec{L}$$



See also: https://www.youtube.com/watch?v=DG3TuMy0UAM



Taken from https://elearning.physik.uni-frankfurt.de/data

Equations of Motion

$\dot{R} = \vec{\omega}^* R \qquad \vec{L} = I\vec{\omega}$

- The state of the rigid body is uniquely defined by position, orientation, linear and angular momentum.
- The time evolution function is computed from previous observations in world space
- Here, only reciprocal values of mass and inertia tensor are needed.
- transformation of the inverse inertia tensor to world space:

$$I_W^{-1} = R I_O^{-1} R^T$$

- if a 0 is stored in reciprocal mass or tensor of inertia, this corresponds to an infinite mass

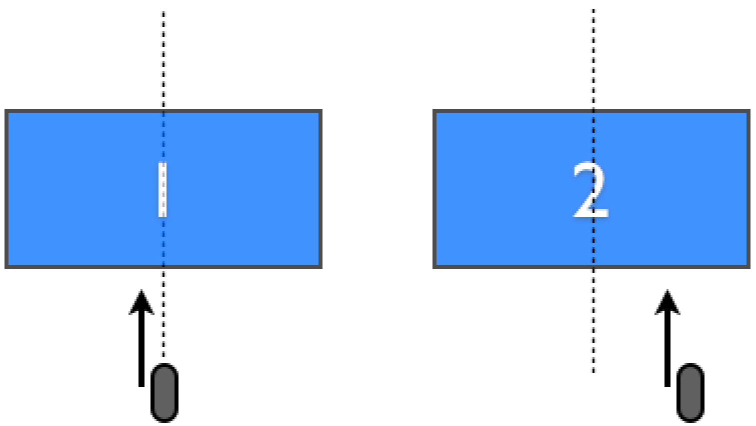
$$\vec{y} = \begin{pmatrix} \underline{X} \\ R \\ \vec{P} \\ \vec{L} \end{pmatrix}$$

$$\vec{f}(t, \vec{y}) = \begin{pmatrix} \dot{\underline{X}} \\ \dot{R} \\ \dot{\vec{P}} \\ \dot{\vec{L}} \end{pmatrix} = \begin{pmatrix} \frac{1}{M}\vec{P} \\ I^{-1}\vec{L}^* R \\ \vec{F}_{ges}(t, \vec{y}) \\ \vec{T}_{ges}(t, \vec{y}) \end{pmatrix}$$

- **Caution:** R must be orthogonalized after each integration step. This can be done, for example, with polar decomposition.

Bullet Block Experiment

- https://www.youtube.com/watch?v=vWVZ6APXM4w



- Which block flies higher?