

## New polynomial wavelets and their use for the analysis of wall-bounded flows

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### Abstract :

The first part of the paper presents a construction of orthonormal polynomial wavelets on the interval by means of a Malvar-type decomposition in polynomial coefficient space. With this approach the localization of the wavelets could be improved substantially compared to an earlier construction by the authors. The new basis is then applied to the analysis of turbulence in the presence of walls. In particular, it allows to define local spectra in wall-normal direction. As an example, DNS data of the flow in a plane channel are considered. The extension to higher dimensions is performed as well.

### Key words :

Turbulence ; Wavelets ; near-wall flow

### 1 Introduction

The investigation of coherent structures in near-wall turbulent flows is a very active area of current research. It is motivated by their importance for many technical and environmental applications such as the resistance experienced by moving bodies, the exchange of heat and other quantities, etc., and hence also for active and passive control of turbulence in this respect. Due to their simultaneous localization in space and frequency, wavelets provide an attractive formalism for the analysis of turbulent flows Farge *et al.* (1996). Compared to the proper orthogonal decomposition, a wavelet decomposition typically applies many more basis functions, but on the other hand these carry a length scale which then allows to analyze coherent structures in this respect. Although wavelet analysis has frequently been applied to turbulent flows, the classical constructions are generally not suitable for dealing with geometrically confined situations. For this reason most investigations up to now were restricted to temporal analysis or periodic data in wall-parallel planes Farge *et al.* (1990); Dunn and Morrison (2000).

Fröhlich and Uhlmann (2001) devised discrete orthogonal wavelets based on Legendre polynomials for a wavelet transform on the interval. In order to be suitable for data analysis wavelets should fulfill the following properties :

1. Existence of a discrete orthogonal transform to avoid increase of data size.
2. Orthogonality with respect to unit weight to establish a relation to the physical energy
3. Symmetry of the absolute value
4. Good localization in physical space

When considering data on a bounded interval this immediately introduces the distance to the closest wall as an additional length scale. Exact shift invariance and symmetry are necessarily

lost in one or the other way if orthogonality is required, and exact scale invariance can not be maintained as a consequence. Hence, these properties can, right from the start, only be obtained with some relaxation. Suitable choices can however be made in order to provide a decomposition following at best the demands of the analyst. A detailed discussion is provided in Fröhlich and Uhlmann (2001).

In this reference, wavelets compliant with the above requirements were constructed by lumping normalized Legendre polynomials  $P_k$  such that the wavelet space for scale  $j$  is

$$W_j^{old} = \text{span}\{P_k, k = 2^j + 1, \dots, 2^{j+1}\} \quad j = 0, \dots \quad (1)$$

using the result of earlier work by Fischer and Prestin (1997). In Fröhlich and Uhlmann (2001) and Uhlmann and Fröhlich (2002), this basis was used for the analysis of turbulent flows, e.g. in terms of local wall-normal spectra. Unfortunately, the decay of these wavelets in space is only linear. This is remedied with the new construction presented in the next section.

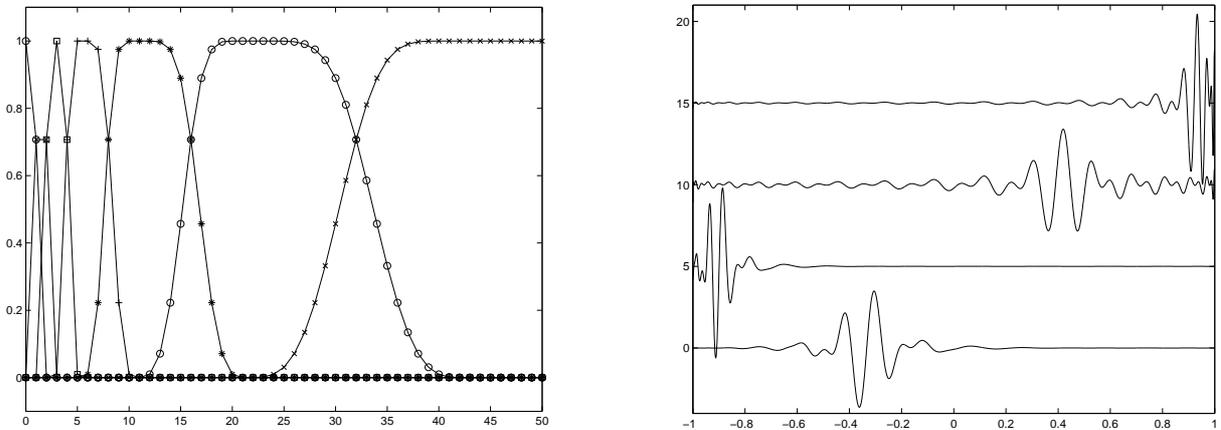


Figure 1: Left: Window functions  $w_j(k)$  with respect to the polynomial degree  $k$  generating the partitioning into wavelet spaces for  $j = -1, 0, 1, \dots$  (zoom around the origin). Right: wavelet functions for scale index  $j = 5$  (shifted vertically by multiples of 5). The lower two correspond to the new basis with rapid decay, the upper two to the old basis with linear decay.

## 2 Polynomial wavelets with rapid decay

After some research we found that the present goal can be achieved through a continuation and suitable combination of earlier work by Malvar (1990), Coifman and Meyer (1991), Auscher *et al.* (1992), and Kilgore *et al.* (1997). The basic idea is to relax the index bounds for  $k$  in the definition of the spaces  $W_j$  while still conserving  $\dim\{W_j\} = 2^j$  and orthogonality. This can be accomplished by a Malvar-type decomposition in “frequency”, here represented by the integer degrees of the basic polynomials. For the latter we select

$$P_k(x) = \sqrt{k + \frac{1}{2}} L_k(-x) \quad (2)$$

where  $L_k$  is the classical Legendre polynomial. The wavelets of scale  $j$  on the interval  $x \in [-1, 1]$  are then defined by

$$\psi_{ji}(x) = \sum_{k=0}^{\infty} w_j(k) b_{ji}(k) P_k(x) \quad j = 0, 1, \dots \quad i = 0, \dots, 2^j - 1 \quad (3)$$

with

$$w_j(x) = \begin{cases} g\left(\frac{k-\kappa_j}{\Delta\kappa_j}\right) & , k \in [\kappa_j - \Delta\kappa_j, \kappa_j + \Delta\kappa_j] \\ 1 & , k \in [\kappa_j + \Delta\kappa_j, \kappa_{j+1} - \Delta\kappa_{j+1}] \\ g\left(\frac{\kappa_{j+1}-k}{\Delta\kappa_{j+1}}\right) & , k \in [\kappa_{j+1} - \Delta\kappa_{j+1}, \kappa_{j+1} + \Delta\kappa_{j+1}] \\ 0 & , \text{else} \end{cases} \quad (4)$$

and

$$b_{ji}(k) = (-1)^{i+1} \sqrt{\frac{2}{\kappa_{j+1} - \kappa_j}} \cos\left(\pi \frac{k - \kappa_j}{\kappa_{j+1} - \kappa_j} \left(i + \frac{1}{2}\right)\right) \quad . \quad (5)$$

Here, we choose

Eq. (5) corrected w.r. to earlier versions !!!

$$\kappa_j = 2^j, \quad \Delta\kappa_j = \kappa_j/3 \quad j = 0, 1, \dots \quad . \quad (6)$$

With  $\kappa_{-1} = 0$  and  $w_{-1}(0) = 1$  one gets  $V_0 = W_{-1} = \text{span}\{P_0\}$ . For the mollifying function  $g : [-1, 1] \rightarrow [0, 1]$  we use the suggestion of Daubechies (1992) developed for Meyer wavelets. The resulting windows are represented in left part of Fig. 1. Examples of the new wavelet functions are depicted in the right part of this figure. They have a decay rate of  $\psi \sim x^{-5}$ , in the sense that this behavior is observed over some distance for large scale index  $j$  and remote from the interval boundaries. Two corresponding functions from the old basis are added for comparison. A detailed discussion of the construction and its properties will be provided elsewhere,  $\dim\{V_j\} = \dim\{W_j\}$ , e.g. unavoidably yields odd wavelets. The wavelets according to (3) are unevenly distributed over the interval and their characteristic “frequency” of oscillation changes with position as visible in Fig. 1. This is accounted for by an appropriate way of distorting the usual scalogram, a technique developed in Fröhlich and Uhlmann (2001). A local spectral-like analysis at a given point in space is unaffected by this issue.

### 3 Higher dimensions and local wavelet spectra

The extension to higher dimensions can be accomplished in two ways. The first is based on a multi-dimensional multiresolution analysis combining wavelets and scaling functions. The second employs tensor products of wavelets only. In our earlier work we found that the second approach is preferable here.

In both approaches wavelets of different type in different directions can be coupled. The data analyzed below result from DNS of plane channel flow performed with periodic conditions in streamwise and spanwise direction. When considering a scalar signal  $u(x, y)$  in two-dimensional slices with  $x$  the periodic streamwise or spanwise coordinate and  $y$  the bounded wall-normal coordinate it is appropriate to employ periodic wavelets in  $x$ , denoted  $\tilde{\psi}$ . The tensor product decomposition then reads

$$u(x, y) = \text{scaling functions} + \sum_{j_x=0}^J \sum_{j_y=0}^J \sum_{i_x=0}^{2^{j_x}-1} \sum_{i_y=0}^{2^{j_y}-1} u_{i_x, i_y}^{j_x, j_y} \tilde{\psi}_{j_x, i_x}(x) \psi_{j_y, i_y}(y) \quad . \quad (7)$$

The coefficients  $u_{i_x, i_y}^{j_x, j_y}$  of the development are obtained by computing the scalar product of the signal with the corresponding wavelet functions, which is possible due to the orthogonality of the bases. For  $\tilde{\psi}$  we take cubic spline wavelets.

Local spectra can then, roughly speaking, be obtained by specifying a position  $(x, y)$ , selecting the shift indices  $i_x, i_y$  such that the corresponding wavelets are located at or close to

this point, and plotting the coefficients  $u_{i_x, i_y}^{j_x, j_y}$  with respect to  $j_x, j_y$ . A statistical average  $\langle \cdot \rangle$ , e.g. over different planes from one flow, yields a physically significant quantity. With two independent indices,  $j_x$  and  $j_y$ , however, the interpretation is complex due to the large amount of information. Here, we therefore consider the one-dimensional case where an average is performed over all  $i_x, j_x$ , in other words over all points with the same wall distance  $y$ . this yields a one-dimensional spectrum which can be obtained as

$$E(y; j_y) = \left\langle 2^{j_y} \sum_{j_x, i_x} \frac{1}{\Delta k_y} \left( (u_{i_x, i_y(y)}^{j_x, j_y})^2 \right) \right\rangle . \quad (8)$$

This is equivalent to directly performing one-dimensional transforms, a procedure which in fact was employed to generate the figures below. For ease of interpretation and comparison between different locations  $y$ , the independent variable  $j_y$  is replaced by the scale number  $s_y$ . It is a measure of the period of oscillation in space of the wavelet  $\psi_{j_y, i_y}(y)$  as defined in Fröhlich and Uhlmann (2001), in other words it is the spatial scale characterizing this function. The factor  $2^{j_y}$  allows to compare a lokal spectrum with a global spectrum where the  $2^{j_y}$  coefficients on each scale are taken into account. In analogy to Fourier analysis the pseudo-wavenumber  $k_y = 1/s_y$  is defined and used to obtain the pre-multiplied spectra used in the literature with Fourier spectra.

#### 4 Results

The data considered here result from a DNS of plane channel flow at  $Re_\tau = 590$  performed by the second author. In this DNS, the domain of  $2\pi \times 2 \times \pi$  was discretized using  $600 \times 385 \times 600$  discrete Fourier-Chebyshev-Fourier modes. Statistics from 150 streamwise-wall-normal planes gathered over one flow-through time were used.

The analysis has been performed for all three velocity components, so that  $u_\alpha$  ( $\alpha = 1, 2, 3$ ) is one component of the velocity fluctuation and the corresponding pre-multiplied spectra are denoted  $k_y E_{11}$ ,  $k_y E_{22}$ ,  $k_y E_{33}$ , respectively. They are displayed in Figure 2. The large-scale limit of the wall-normal coordinate in wall units is imposed by the channel height of  $2Re_\tau$  in these units. For a given Reynolds number, increasing the wall distance yields a pre-multiplied spectrum which is shifted towards larger scales. The exited scales are roughly proportional to the wall distance. As observed in Liu *et al.* (2001), this is consistent with the attached-eddy hypothesis of Townsend (1976) stating that eddies centered at a distance  $y$  from the wall extend down to the wall and hence live on scales of order  $y$ . The behaviour of the three velocity components is quite similar when analyzed in the present basis. It seems, however, that at  $y^+ = 10$  the relative fine-scale activity for  $s_y^+$  around 20 is somewhat larger for  $u$  than for the other two components while for large  $y^+$  the energy in  $u$  lives at slightly larger scales compared to  $v$  and  $w$ . Before definitive conclusions in terms of physical properties can be drawn additional statistics should be compiled, though.

In Figure 2 the spectra obtained with the present localized basis are also compared to the ones obtained when using the linearly decaying Legendre wavelets of Fröhlich and Uhlmann (2001). The different values of the maxima result from the normalization and the logarithmic horizontal axis for  $s_y^+$ . All peaks are now much more pronounced. This is a substantial improvement with respect to the earlier results. Further investigations can now be performed with respect to two-dimensional local wavelet spectra and other geometries.

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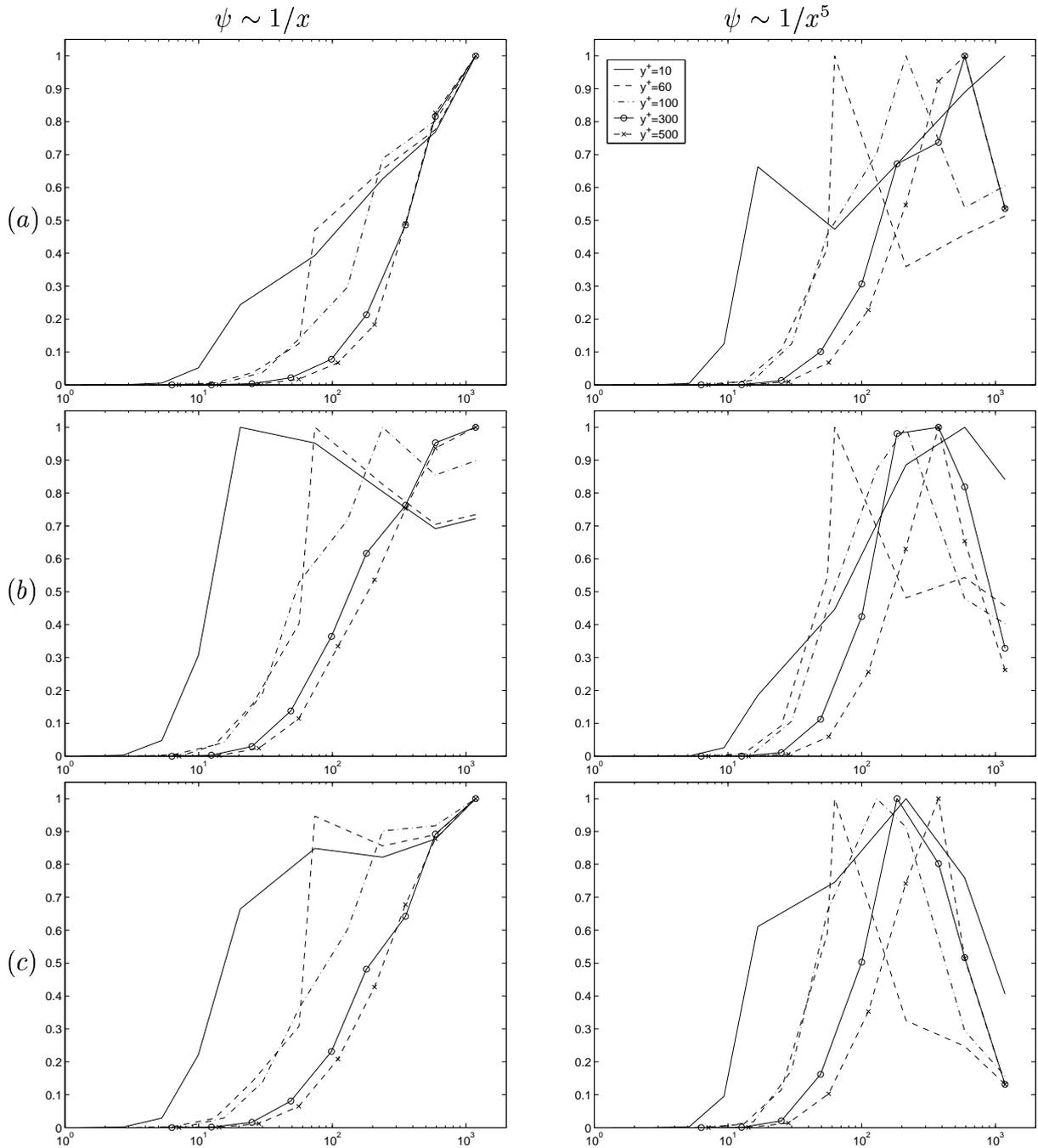


Figure 2: Pre-multiplied wall-normal power spectra  $k_y E_{\alpha\alpha}(y^+; k_y)$  as a function of scale  $s_y^+$  in plane channel flow at  $Re_\tau = 590$  with  $y^+ = \{10, 60, 100, 300, 500\}$ . Left: old wavelet basis with linear decay, right: new wavelet basis with asymptotic decay  $\sim 1/x^5$ . Line styles are given in the upper right picture. From top to bottom the graphs show the spectra related to (a) streamwise, (b) wall-normal and (c) spanwise velocity components. All curves are normalized with respect to their maximum.

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