SELF-DUAL CLONES COLLAPSED

Albert Vucaj
joint work with M. Bodirsky and D. Zhuk

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   - Primitive positive (pp) constructions
   - Minor conditions

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4. Open problems

Primitive positive constructions

We can consider the following posets on finite structures:

- pp definability: $\mathbb{A} \leq_{\mathrm{Def}} \mathbb{B}$ if $A = B$ and every relation in $\mathbb{B}$ has a pp definition in $\mathbb{A}$.

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- $\leq_{\mathrm{Def}}$, $\leq_{\mathrm{Int}}$, $\leq_{\mathrm{Con}}$ preserve the complexity of CSPs.
- CSP($\mathbb{A}$) is in P if $\mathbb{A} \not\leq_{\mathrm{Con}} \mathbb{K}_3$, and is NP-hard otherwise. (Bulatov '17; Zhuk '17)

The algebraic approach

Pol($\mathbb{A}$): the clone of polymorphisms of $\mathbb{A}$ (A finite).

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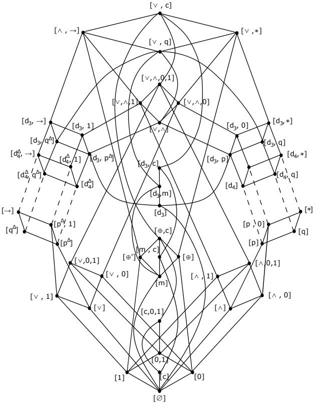
Note: If $\mathrm{Pol}(\mathbb{A}) \models \Sigma$ and $\mathbb{A} \leq_{\mathrm{Con}} \mathbb{B}$, then $\mathrm{Pol}(\mathbb{B}) \models \Sigma$.

Σ: minor-condition (a.k.a. Height 1 condition, linear Mal'cev condition)

$$f(\ldots) \approx g(\ldots).$$

How powerful are pp constructions?

Chapter 1: "Two"



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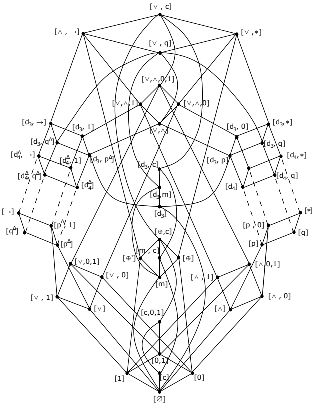

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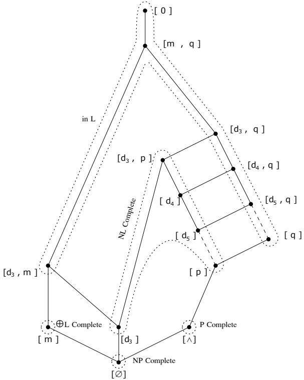

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Theorem (Yanov, Muchnik '59)

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Two clones $\mathcal{C}$ and $\mathcal{D}$ collapse if $\mathcal{C} \xrightarrow{\text{minor}} \mathcal{D}$ and $\mathcal{D} \xrightarrow{\text{minor}} \mathcal{C}$.

- If $\mathcal{C}$ has operation with image of size k, then $\mathcal{C}$ collapses with a clone on k elements.
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Non trivial collapses: it suffices to study idempotent clones!

$$f(x, \ldots, x) = x.$$

Self-dual clones

Definition

A function on $\{0, 1, 2\}$ *is self-dual if it preserves the relation*

$$C_3 = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \end{pmatrix}.$$

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  - continuum many self-dual clones;
  - they are all idempotent.

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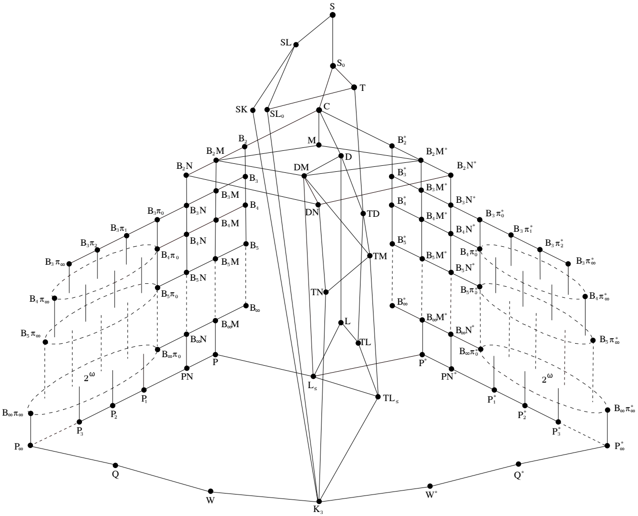

The lattice of self-dual clones

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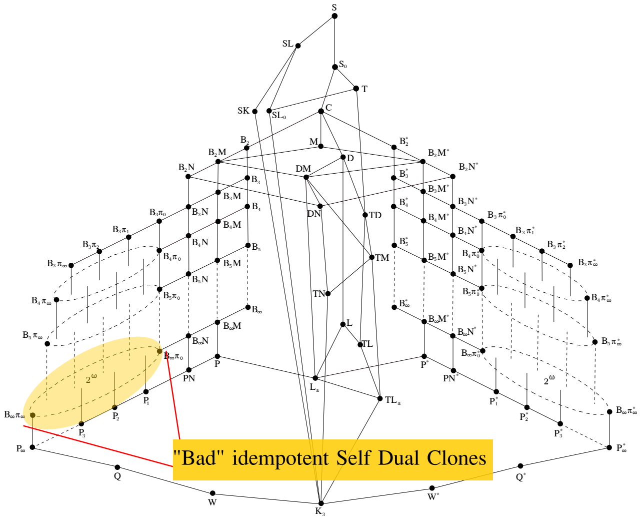

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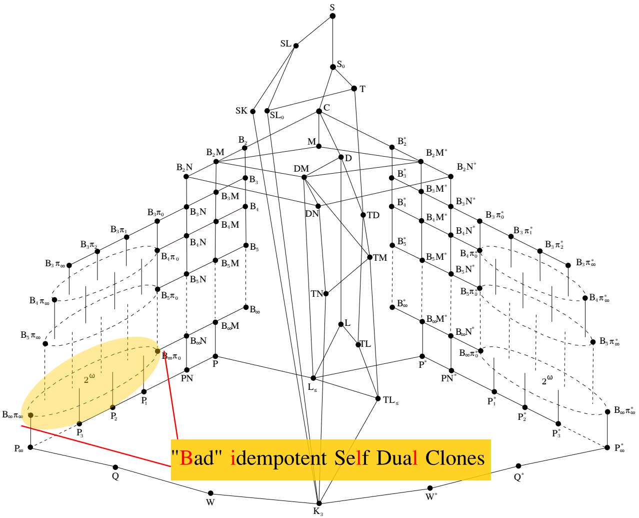

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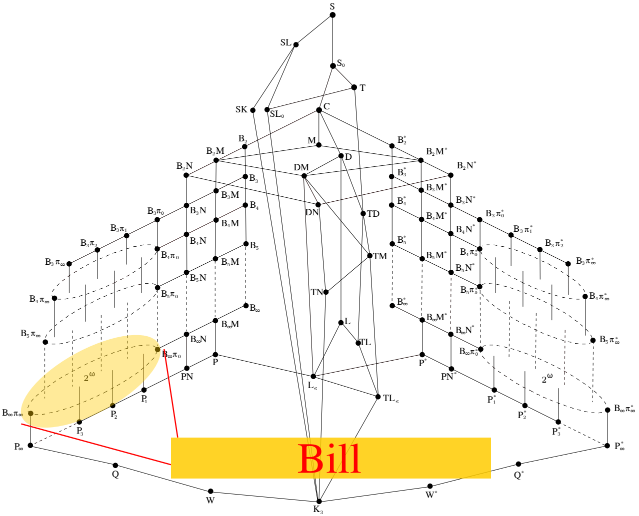
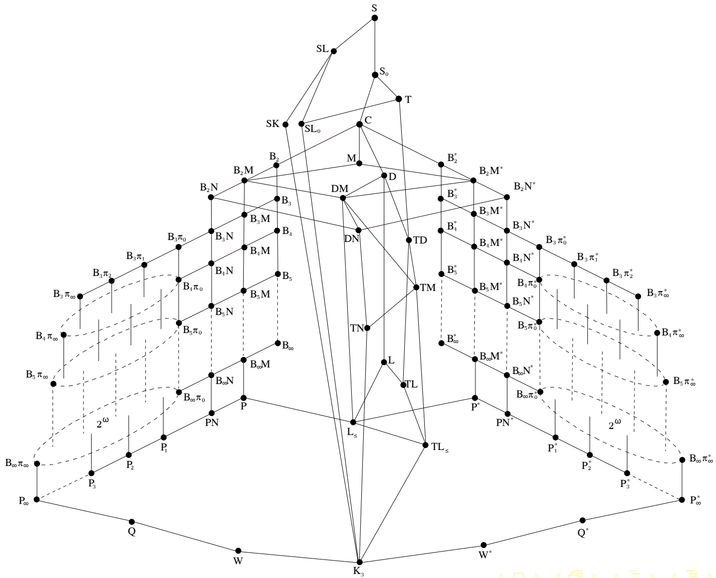


The lattice of self-dual clones

Alternative title

Kill Bill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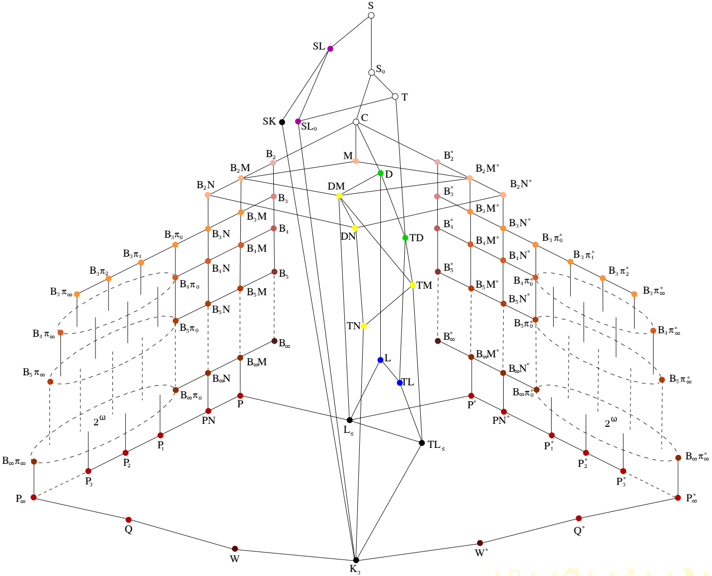

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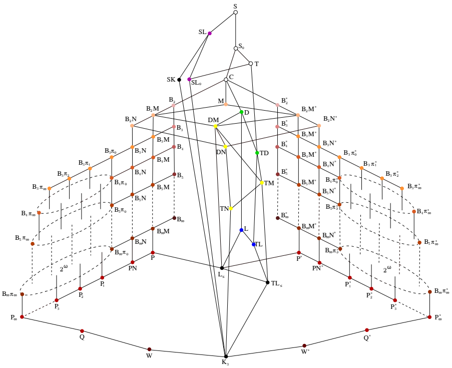

(Self-dual clones ; $\leq_{\mathrm{Def}}$)

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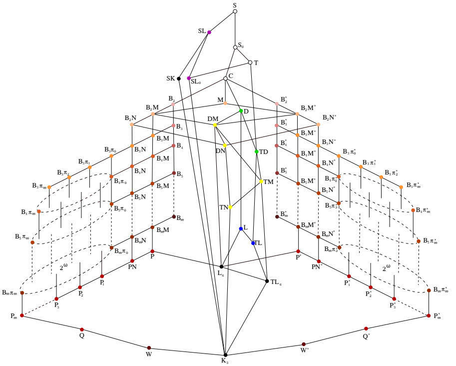

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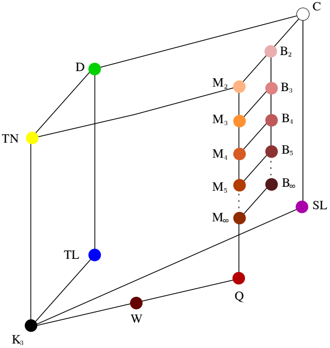

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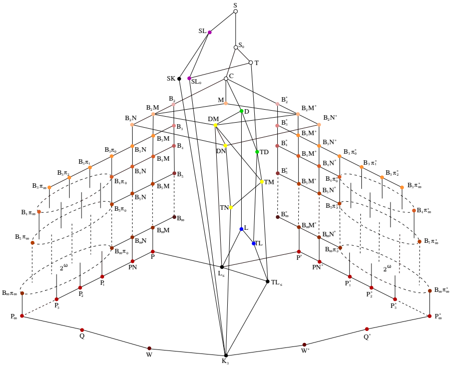

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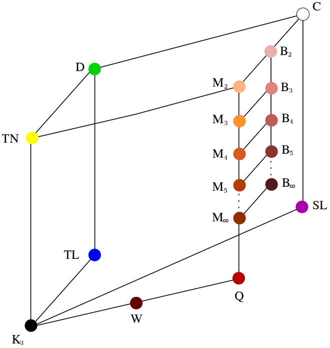

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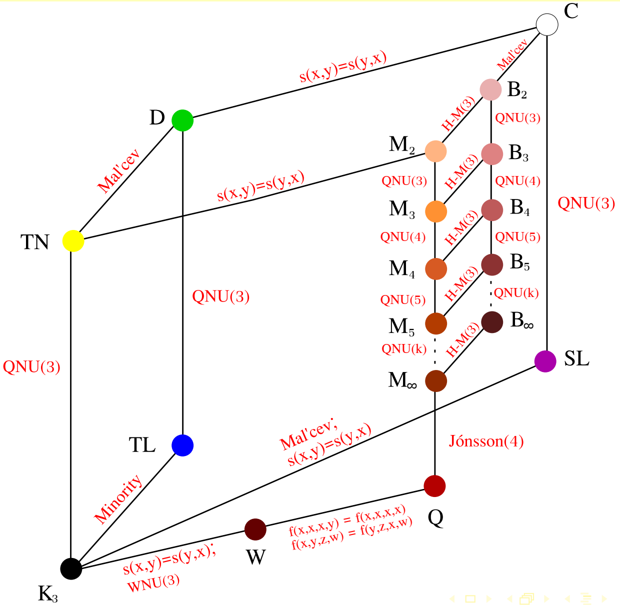
Theorem

There are only countably many self-dual clones on $\{0, 1, 2\}$ up to minor-equivalence.

Kill Bill



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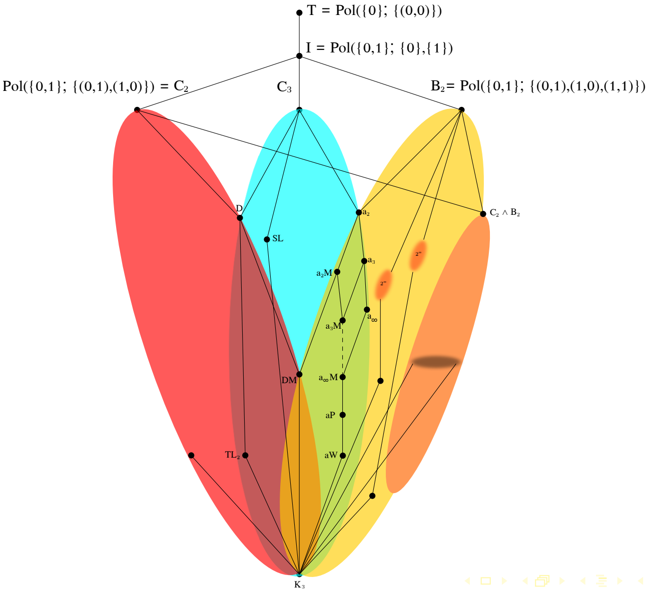
Kill Bill volume 2



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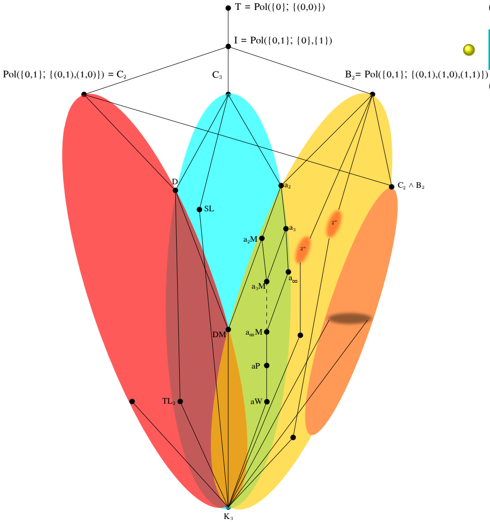

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- Region below $\mathcal{C}_3$: Completely described. (Bodirsky,V.,Zhuk)

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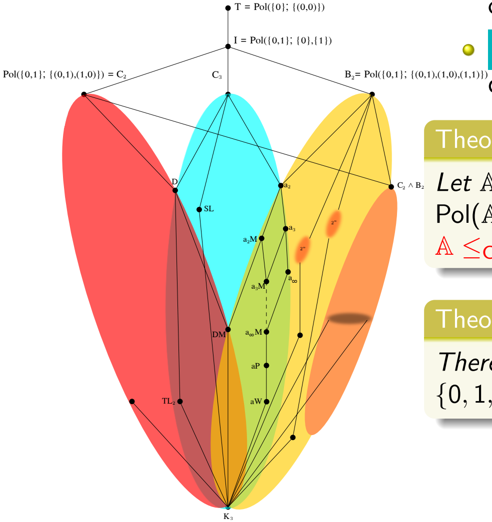

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Let $\mathbb{A}$ be a finite structure. Then either Pol($\mathbb{A}$) *has a Mal'cev operation or* $\mathbb{A} \leq_{\mathsf{Con}} \mathbb{B}_2$.

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There are only finitely many clones on $\{0, 1, 2\}$ *with a Mal'cev operation.*

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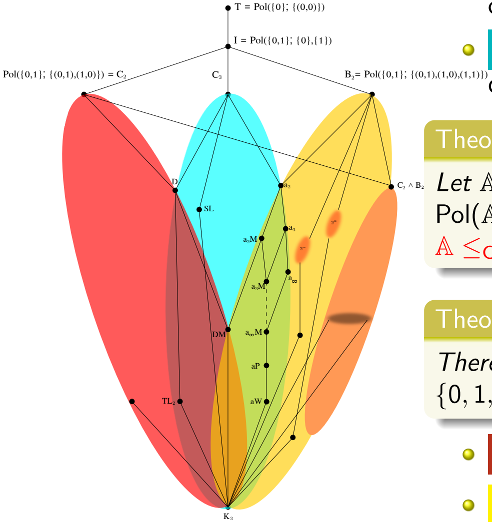

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- Region below $\mathcal{C}_2$: Tame!
- Region below $\mathcal{B}_2$: Wild!

Open problems

$E_n = \{0, \ldots, n-1\}$

$$\mathfrak{P}_n = (\text{Structures on } E_n ; \leq_{\mathsf{Con}})$$

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- Cardinality of $\mathfrak{P}_3$?

Open problems

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- Cardinality of $\mathfrak{P}_{\text{Fin}}$?

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- Cardinality of $\mathfrak{P}_{\text{Fin}}$?
- Is $\mathfrak{P}_{\text{Fin}}$ a lattice?

Thank you!

Zhuk's Katana

We define the following relations

$$C_3 = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \end{pmatrix}; \quad W = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 2 \end{pmatrix}; \quad B_n \quad = \{0,1\}^n \setminus \{(0,\ldots,0)\}.$$

The π-relations: $n, m \in \mathbb{N}$; $A_1 \cup \cdots \cup A_m = \{1, \ldots, n\}$,

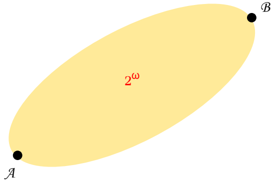
$$\pi_{A_1,\ldots,A_m}(x_1,\ldots,x_m,y_1,\ldots,y_n) = 1 \text{ if and only if}$$

1. $x_1, \ldots, x_m \in \{0,1\}$,
2. for every $i \in \{1,\ldots,m\}$, if $x_i = 0$ then $y_j \in \{0,1\}$ for every $j \in A_i$, and
3. not $x_1 = \cdots = x_m = y_1 = \cdots = y_n = 0$.

By Π_n^m we denote the set of all $(m+n)$-ary predicates $\pi_{A_1,\ldots,A_m}$. Finally:

$$\Pi^l = \bigcup_{3 \le m+n \le l} \Pi_n^m; \;\; \Pi = \bigcup_l \Pi^l.$$

Zhuk's Katana (4th power)



$$\textcolor{red}{\mathcal{A}} = \mathrm{Pol}(C_3, W, \Pi); \quad \textcolor{red}{\mathcal{B}} = \mathrm{Pol}\left(C_3, W, \bigcup_{i \geq 3} B_i\right).$$

$$\begin{aligned}
C_3^{\mathbb{F}} &:= \{(x,y) \in F^2 \mid C_3(x_0, y_0) \wedge x_1 = y_2 \wedge x_2 = y_3 \wedge x_3 = y_1\} \\
W^{\mathbb{F}} &:= \{(x,y) \in F^2 \mid W(x_0, y_0) \wedge x_3 = 0 \wedge y_3 \leq x_0\} \\
B_n^{\mathbb{F}} &:= \{(x^1, \dots, x^n) \in F^n \mid B_n(x_0^1, \dots, x_0^n) \wedge x_3^1 = \dots = x_3^n = 0\} \\
\pi_{A_1,\dots,A_m}^{\mathbb{F}} &:= \{(x^1, \dots, x^m, y^1, \dots, y^n) \in F^{n+m} \mid B_{m+n}(x_2^1, \dots, x_2^m, y_2^1, \dots, y_2^n) \\
&\wedge \bigwedge_{i \in \{1,\dots,m\}} (W(x_1^i, x_0^i + 2) \wedge x_3^i = 0) \\
&\wedge \bigwedge_{i \in \{1,\dots,m\}, j \in A_i} (W(x_0^i, y_0^j) \wedge y_3^j \leq x_0^i)\}
\end{aligned}$$