Two-Element Structures modulo PP-Constructability

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1. The pp-constructability poset
   - ⋆ Partial order on the class of relational structures
   - ⋆ $\mathbb{A} \leq \mathbb{B}$ iff $\mathbb{A}$ "pp-constructs" $\mathbb{B}$
2. Two-Element Structures
   - Collapses
   - Separations
3. Applications for Complexity of Boolean CSPs

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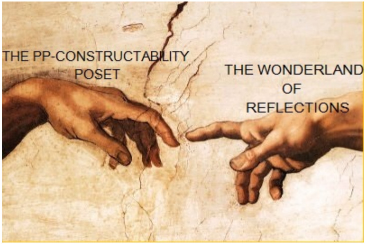
Genesis:
Barto, Opršal, Pinsker



Basics

Definition

- $\mathbb{A}$: *τ-structure*
- $\phi(x_1, \ldots, x_n)$: *τ-formula*

then the relation defined by ϕ *is the relation:*

$$\{(a_1, \ldots, a_n) \mid \mathbb{A} \models \phi(a_1, \ldots, a_n)\}$$

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$$\{(a_1, \dots, a_n) \mid \mathbb{A} \vDash \phi(a_1, \dots, a_n)\}$$

$\star$ If the involved formula ϕ is *primitive positive*, then this relation is said to be *pp*-definable in $\mathbb{A}$.

$\star$ A primitive positive formula is a formula of the form:

$$\exists \dots \exists\ (\dots \ \wedge \ \dots \ \wedge \ \dots)$$

PP-Construction

Definition

- $\mathbb{A}$, $\mathbb{B}$: *relational structures*

We say that $\mathbb{B}$ *is a pp-power of* $\mathbb{A}$ *if it is isomorphic to a structure with domain* A^n $(n \geq 1)$ *whose relations are pp-definable from* $\mathbb{A}$*.*

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We consider the following quasi-order:

$$\mathbb{A} \leq \mathbb{B} \text{ if and only if } \mathbb{A} \text{ pp-constructs } \mathbb{B}.$$

Every element in our poset is a $\equiv$-class where the equivalence relation is

$$A \equiv \mathbb{B} \text{ if and only if } \mathbb{B} \leq \mathbb{A} \leq \mathbb{B}.$$

Motivation

- Universal Algebra

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Theorem (Barto, Opršal, Pinsker, (2015))

- $\mathbb{A}$, $\mathbb{B}$: *finite relational structures*
- $\mathcal{A} := \mathrm{Pol}(\mathbb{A})$, $\mathcal{B} := \mathrm{Pol}(\mathbb{B})$: *their polymorphism clones.*

Then the following are equivalent:

1. $\mathbb{A}$ *pp-constructs* $\mathbb{B}$.
2. $\mathcal{B} \in \mathrm{ERP}^{\mathrm{fin}}\, \mathcal{A}$. *(remark:* $\mathrm{HSP}^{\mathrm{fin}}\, \mathcal{A} \subseteq \mathrm{ERP}^{\mathrm{fin}}\, \mathcal{A}$*)*
3. *There exists a minor-preserving map from* $\mathcal{A}$ *into* $\mathcal{B}$.
   (i.e., a mapping $h \colon \mathcal{A} \to \mathcal{B}$ *preserving height* 1*-identities)*

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- CSP

FACT: If $\mathbb{A} \leq \mathbb{B}$ then $\mathrm{CSP}(\mathbb{B})$ is log-space reducible to $\mathrm{CSP}(\mathbb{A})$.

Two-Element Structures

The previous theorem provides tools to prove **collapses** and **separations**.

Remark

If $\mathbb{A} \not\leq \mathbb{B}$ then there is a finite set of h1-identities which is satisfied by some operations in $\mathcal{A}$ but is not satisfied by any operation in $\mathcal{B}$.

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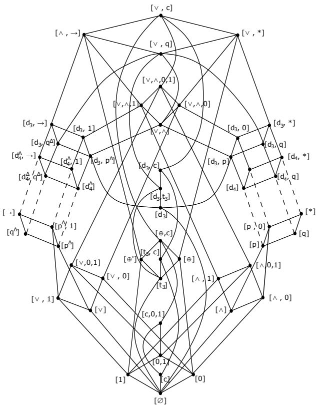

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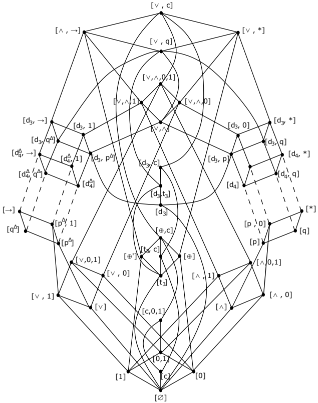

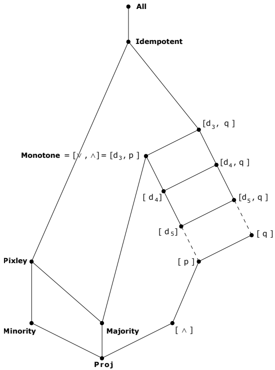

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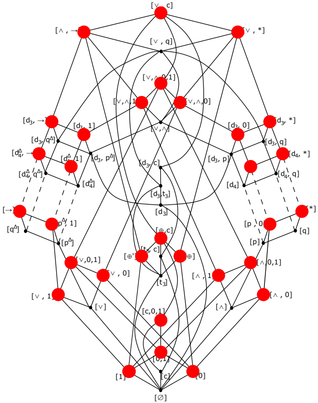
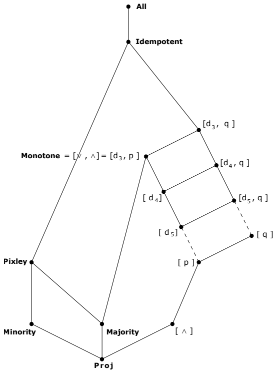
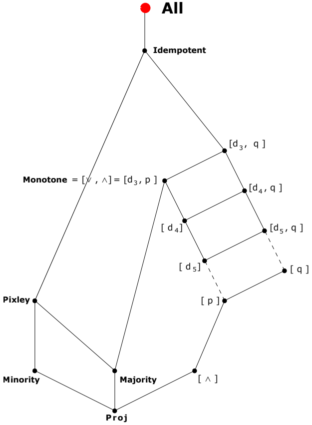




Two Examples of Collapse



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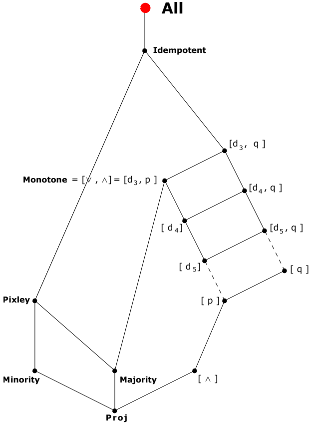

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Proposition

- $\mathcal{C}$ *clone.*
- $\mathcal{D}$ *be clones with a constant operation.*

Then $\mathcal{C} \leq \mathcal{D}$.

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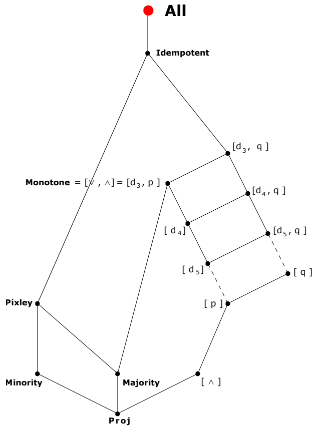

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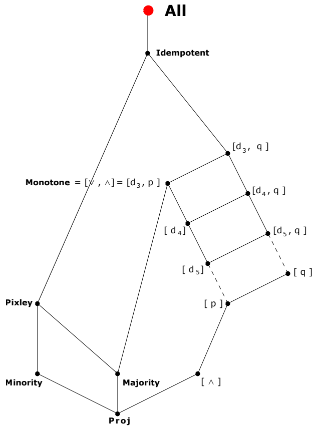

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⋆ Let **All** be the class of all clones with a constant operation .

All: Top-Element in $\mathfrak{L}$

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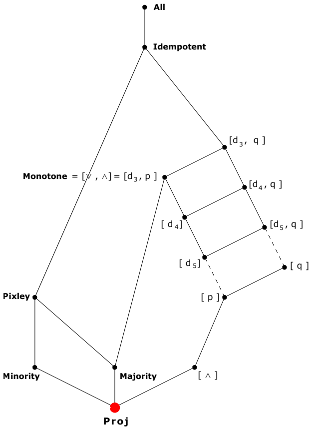

- Via PP-Construction
  - ⋆ NAE $:= \{0,1\}^3 \setminus \{(0,0,0),(1,1,1)\}$
  - ⋆ 1IN3 $:= \{(0,0,1),(0,1,0),(1,0,0)\}$.

Theorem (Folklore)

Let $\mathbb{A}$ be any structure. The following are equivalent:

1. $\mathbb{A}$ *pp-constructs* $\mathbb{NAE}$.
2. $\mathbb{A}$ *pp-constructs* $1\mathbb{IN}3$.

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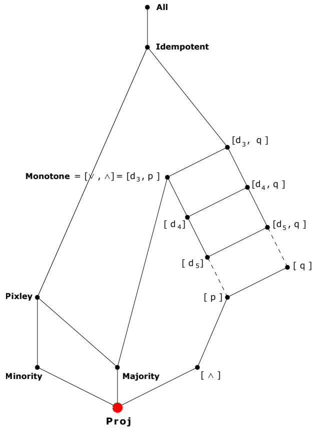

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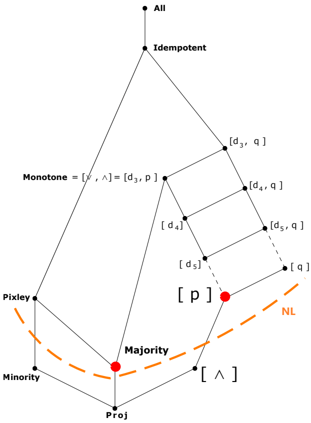
⋆ $\mathrm{Pol}(\mathbb{NAE}) = [c]$
⋆ $\mathrm{Pol}(1\mathbb{IN}3) = [\emptyset]$

$\Rightarrow$ $[c]$ and $[\emptyset]$ collapse.

Proj: Bottom-Element in $\mathfrak{L}$

An Example of Separation



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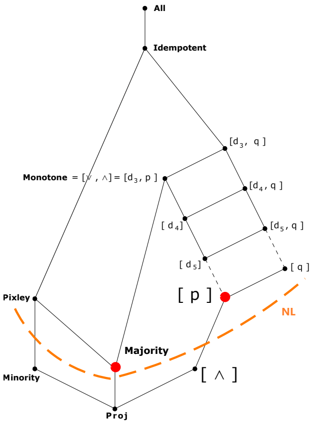

- To separate $[p]$ from $[\wedge]$: ($p(x, y, z) = x \wedge (y \vee z)$)

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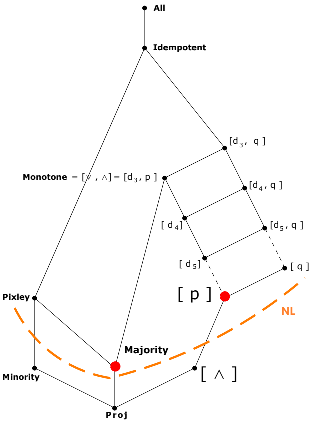

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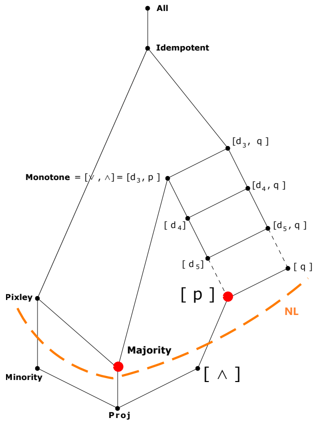

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Universal Algebra.

⋆ *Quasi-Jonssón terms* ['67]:

$$d_0(x, y, z) \approx d_0(x, x, x)$$
$$d_n(x, y, z) \approx d_n(z, z, z)$$
$$d_i(x, y, x) \approx d_i(x, x, x) \quad \text{for all } i$$
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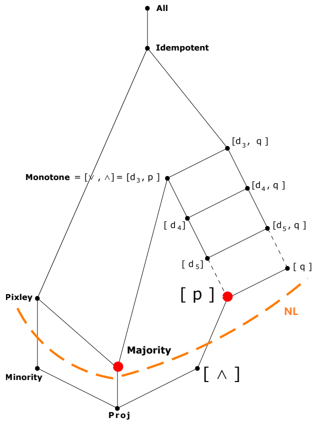

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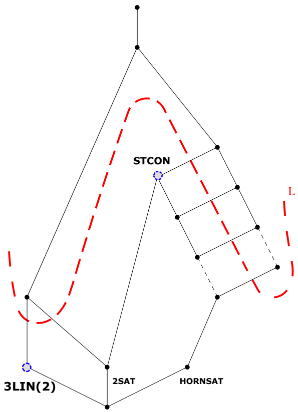
Theorem (Barto, Kozik, (2009))

"Jonssón" ⇒ bounded linear width.

On Complexity of Boolean CSPs



[1] Allender, Bauland, Immerman, Schnoor, Vollmer: *The complexity of satisfiability problems: Refining Schaefer's theorem*. [2009]

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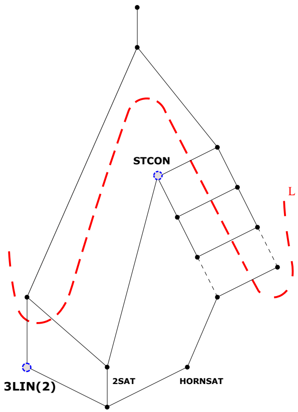

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Classification in [1] and Collapses — Membership to L

- Forbidding the pp-construction of structures:

⋆ $\mathbb{D}_{\text{STCON}} := \langle \{0,1\}, 0, 1, \leq \rangle$

⋆ $\mathbb{D}_{\text{3LIN(2)}}$: problem of solving systems of linear equations over Z_2

Conjecture (Larose, Tesson, (2009))

If a relational structure $\mathbb{A}$ pp-constructs neither $\mathbb{D}_{\text{3LIN(p)}}$, for any prime p, nor $\mathbb{D}_{\text{STCON}}$ then CSP($\mathbb{A}$) *is in L.*

Final Picture