Smooth Digraphs modulo PP-Constructability

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joint work with M. Bodirsky and F. Starke

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SSAOS, Karolinka, 3 September 2019

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The PP-Constructability Poset

- Height 1 identity: $f(x_1, \ldots, x_n) \approx g(y_1, \ldots, y_m)$ (x_i, y_j not necessarily distinct)

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$f(x) \approx f(y)$ ✔ $f(x, y) \approx f(y, x)$ ✔
$f(x, \ldots, x) \approx x$ ✖ $f(f(x, y), z) \approx f(x, f(y, z))$ ✖

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➢ F: Set of functions.

➢ Σ: Height 1 condition.

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➢ F: Set of functions.

➢ Σ: Height 1 condition.

Definition

We say that F **satisfies** Σ (F ⊨ Σ) *if there is a map* $\tilde{\cdot}$ *assigning to each function symbol occurring in* Σ *a function in* F, *such that for all* $f_\sigma \approx g_\tau \in \Sigma$ *we have* $\tilde{f}_\sigma = \tilde{g}_\tau$.

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Let $\mathbb{A}$, $\mathbb{B}$ be finite relational structures.

$$\begin{array}{ccc} \mathbb{B} & & \mathrm{Pol}(\mathbb{B}) \models \Sigma \\ \Big| & \textbf{iff} & \Uparrow \\ \mathbb{A} & & \mathrm{Pol}(\mathbb{A}) \models \Sigma \end{array}$$

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Main definition of this Talk

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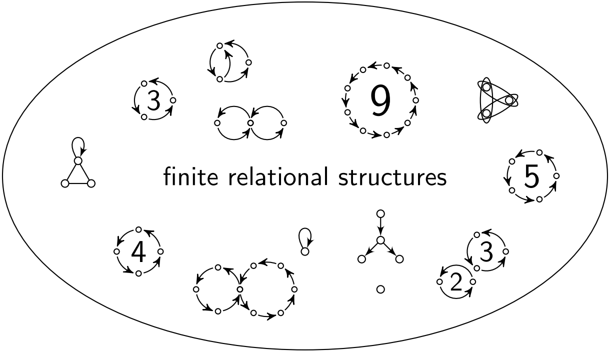

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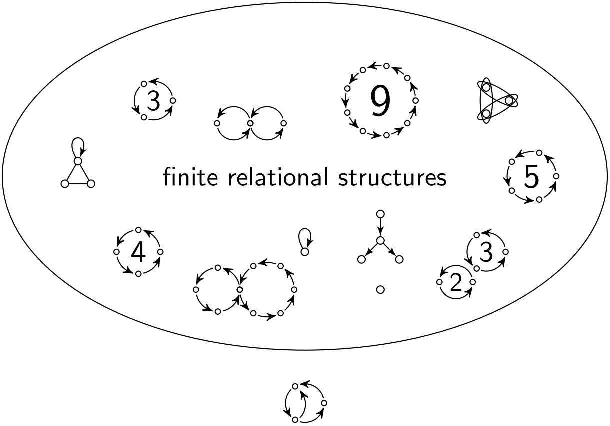

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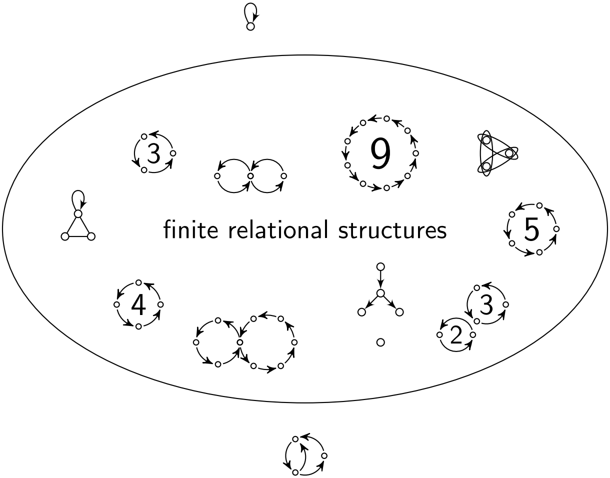

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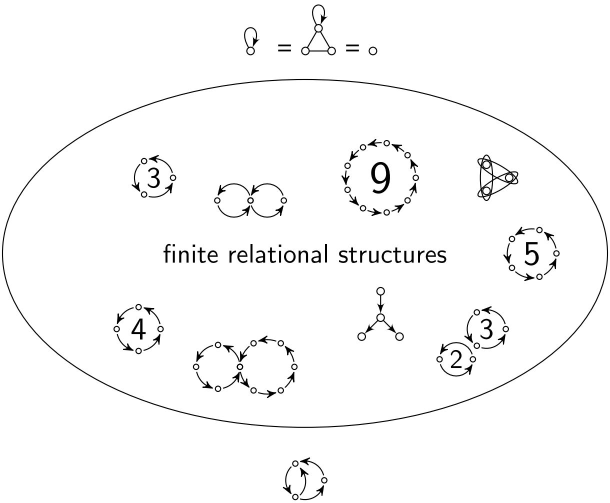

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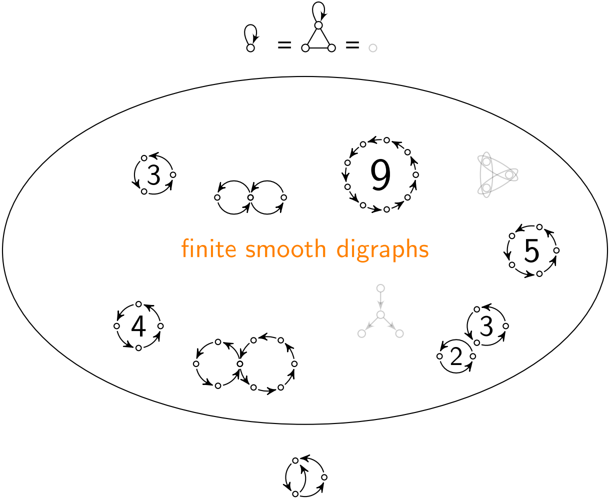

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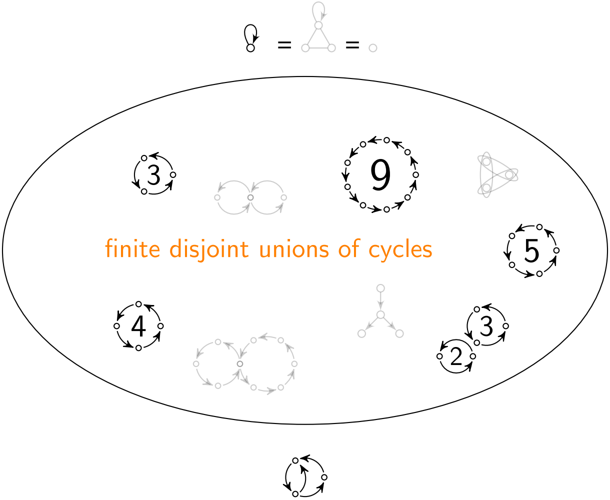

Why...

- PP-Constructability?

  ➢ $\mathbb{A}$ pp-constructs $\mathbb{B}$ iff there is a simple gadget reduction from CSP($\mathbb{B}$) to CSP($\mathbb{A}$). (Barto, Opršal, Pinsker '18)

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  ✔ Boolean Structures (Bodirsky, V. '19)

- Finite disjoint unions of directed cycles?

  ➢ Any fixed domain CSP is, up to log-space reductions, equivalent to a CSP whose template is a directed graph.
  (Bulín, Delic, Jackson, Niven '15)

  ➢ (Prime cyclic) Loop conditions.
  (Olšak; Gillibert, Jonušas, Pinsker; ...)

Loop conditions

Definition

Let $\sigma, \tau \colon [m] \to [n]$ be maps. A **loop condition** *is a height 1 identity of the form*

$$f(x_{\sigma(1)}, \ldots, x_{\sigma(m)}) \approx f(x_{\tau(1)}, \ldots, x_{\tau(m)}).$$

To any loop condition Σ we can assign a directed graph $\mathbb{G}_\Sigma$ in a natural way:

$$\mathbb{G}_\Sigma := ([n], \{(\sigma(i), \tau(i)) \mid i \in [m]\}).$$

Example

1. $\Sigma := f(x,y,z,x) \approx f(y,z,x,z)$. *Then* $\mathbb{G}_\Sigma \cong$ 
2. $\Sigma_2 := f(x,y) \approx f(y,x)$. *Then* $\mathbb{G}_{\Sigma_2} \cong$ 

Cyclic loop conditions

- If $\mathbb{G}_\Sigma$ is (isomorphic to) a disjoint union of directed cycles, then we say that Σ is a **cyclic loop condition**.
- Let $A = \{a_1, \ldots, a_n\} \subset \mathbb{N}^+$. We denote by Σ_A the cyclic loop condition associated to $\mathbb{C}_{a_1} \uplus \cdots \uplus \mathbb{C}_{a_n}$.

  (where $\mathbb{C}_a$ denotes the directed cycle of length a).

Definition

We say that Σ_P is a **prime cyclic loop condition** *if P is a set of prime numbers.*

From "h1 identities" to prime cyclic loop conditions

$$\begin{array}{c}\mathrm{Pol}(\mathbb{C}_A) \models \Sigma \\ \Uparrow \\ \mathrm{Pol}(\mathbb{B}) \models \Sigma\end{array}$$

$$\begin{array}{c}\mathbb{C}_6 \uplus \mathbb{C}_{20} \uplus \mathbb{C}_{15} \\ | \\ \mathbb{B}\end{array}$$

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$$\begin{array}{c} \mathbb{C}_6 \cup \mathbb{C}_{20} \cup \mathbb{C}_{15} \\ | \\ \mathbb{B} \end{array} \quad \text{iff} \quad \begin{array}{c} \mathrm{Pol}(\mathbb{B}) \not\models \Sigma_{6,20,15}, \\ \Sigma_{2,20,5},\ \Sigma_{3,10,15}, \\ \Sigma_{6,4,3},\ \Sigma_{3,5,15}, \\ \Sigma_{3,2,3} \end{array}$$

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Smooth Digraphs modulo PP-Constructability

$\mathbb{C}_1$

$$\begin{tikzcd}[arrows=dash]
 & \Sigma_{2,3,5} \arrow[dl] \arrow[d] \arrow[dr] & \\
\Sigma_{2,3} \arrow[d] \arrow[dr] & \Sigma_{2,5} \arrow[dl] \arrow[dr] & \Sigma_{3,5} \arrow[dl] \arrow[d] \\
\Sigma_{2} & \Sigma_{3} & \Sigma_{5}
\end{tikzcd}$$

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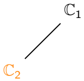

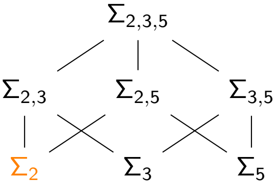

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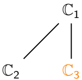

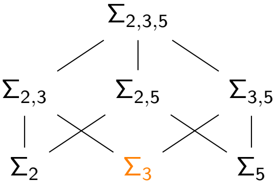

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$$
\begin{tikzcd}[arrows=-]
 & \mathbb{C}_1 \arrow[dl] \arrow[d] \\
\mathbb{C}_2 \arrow[d] & \mathbb{C}_3 \arrow[dl] \\
\mathbb{C}_6 &
\end{tikzcd}
$$

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\begin{tikzcd}[arrows=-]
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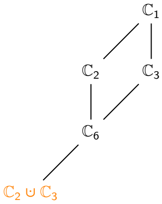

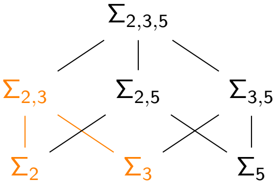

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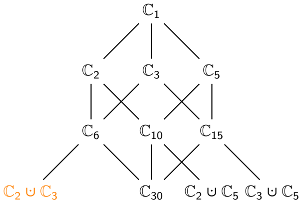

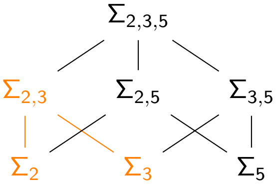

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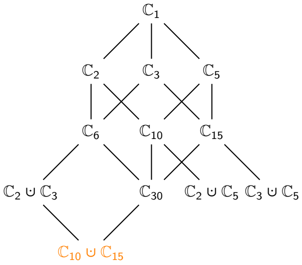

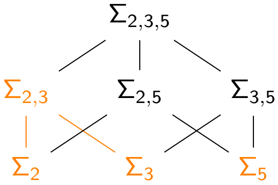

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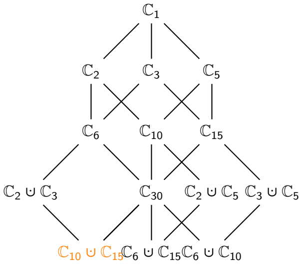

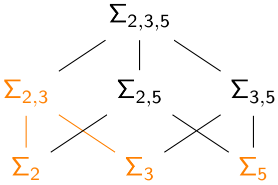

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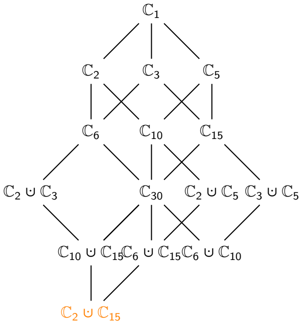

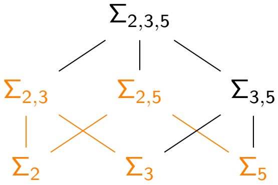

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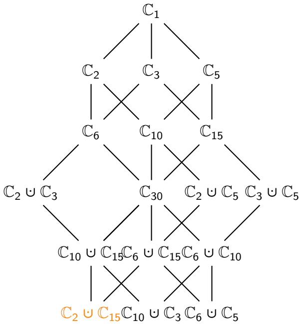

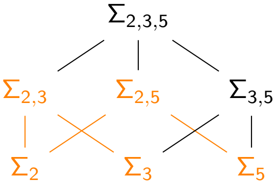

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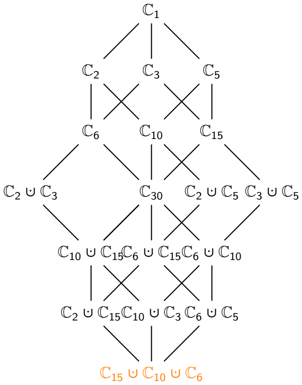

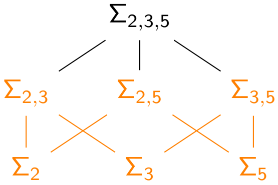

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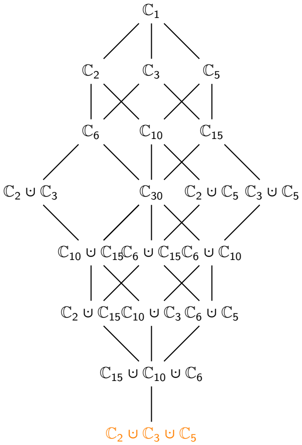

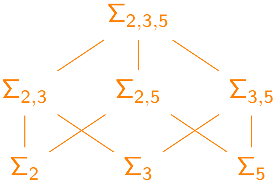

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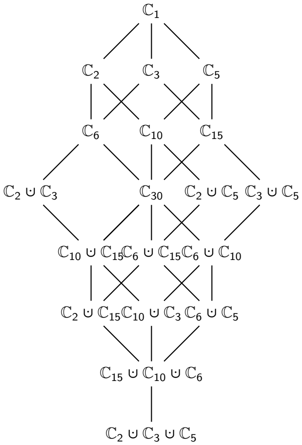

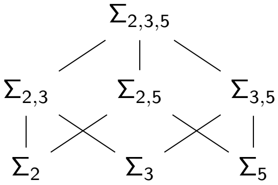

Main result

- $\mathrm{PL}(\mathbb{A}) := \{P \mid P$ a finite nonempty set of primes and $\mathrm{Pol}(\mathbb{A}) \not\models \Sigma_P\}$
- Let FP be the set of all finite nonempty sets of primes.

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- Let FP be the set of all finite nonempty sets of primes.

Theorem

The map

$$\begin{aligned} \mathrm{P}_{SD} &\to {\downarrow}(\mathrm{FP}) \cup \{\mathrm{FP}\} \\ [\mathbb{A}] &\mapsto \mathrm{PL}(\mathbb{A}) \end{aligned}$$

is an isomorphism where the order on the image set is reverse inclusion.

$$\begin{tikzcd}[column sep=small]
 & \mathbb{C}_1 \arrow[dl, dash] \arrow[dr, dash] & & & & \varnothing \arrow[dl, dash] \arrow[dr, dash] & \\
\mathbb{C}_2 \arrow[dr, dash] & & \mathbb{C}_3 \arrow[dl, dash] & & \{\Sigma_2\} \arrow[dr, dash] & & \{\Sigma_3\} \arrow[dl, dash] \\
 & \mathbb{C}_6 \arrow[d, dash] & & \mapsto & & \{\Sigma_3, \Sigma_2\} \arrow[d, dash] & \\
 & \mathbb{C}_2 \dot\cup \mathbb{C}_3 & & & & \{\Sigma_{2,3}, \Sigma_3, \Sigma_2\} &
\end{tikzcd}$$

Open problems

- Is P_{fin} a lattice?

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- Is every prime cyclic loop condition a prime Mal'cev condition?
  (Σ is **prime** if is not implied by the conjunction of two strictly weaker conditions)

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Open problems

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