Institut Camille Jordan
UMR 5208 du CNRS

Some progress on the unique ergodicity problem

Colin Jahel

Thèse de doctorat

Université Claude Bernard Lyon 1

n°. d'ordre: 2021LYSE1101

THESE de DOCTORAT DE L'UNIVERSITE DE LYON

opérée au sein de

Université Claude Bernard Lyon 1

École doctorale InfoMaths (ED 512)
Spécialité: Mathématiques

Thèse soutenue publiquement le 21 juin 2021 par
Colin Jahel

Des progrès sur le problème d'unique ergodicité

devant le jury composé de :

Dana Bartošová	Assistant Professor	(University of Florida)	Examinatrice
David Evans	Professor	(Imperial College London)	Examinateur
Aleksandra Kwiatkowska	Juniorprofessur & Assistant Professor	(Universität Münster & Uniwersytet Wrocławski)	Examinatrice
Damien Gaboriau	Directeur de recherche	(ENS de Lyon)	Président du jury
Julien Melleray	Maître de conférence	(UCBL)	Examinateur
Lionel Nguyen Van Thé	Maître de conférence	(Université Aix-Marseille)	Co-directeur de thèse
Katrin Tent	Professeure	(Universität Münster)	Rapporteuse
Todor Tsankov	Professeur des universités	(UCBL)	Co-directeur de thèse

et rapporté par :

Alexander Kechris	Professeur	(Caltech)

Contents

Remerciements

Je me dois d'exprimer en premier la profonde gratitude que j'ai pour mes directeurs de thèse, Lionel Nguyen Van Thé et Todor Tsankov. Vos approches pédagogiques complémentaires m'ont permis de me former en tant que mathématicien et je vous dois d'avoir découvert un univers immensément beau. Merci de m'avoir accompagné et soutenu, d'avoir su être présents à chaque fois que j'en ai eu besoin, de m'avoir laissé exprimer mes idées bonnes comme mauvaises et de m'apprendre à les exprimer. Merci pour votre patience et vos encouragements.

I am very thankful to Alekos Kechris and Katrin Tent who accepted to review this thesis. I hope it was pleasant to read. I am also thankful to Dana Bartošová, David Evans, Damien Gaboriau, Alessandra Kwiatkowska and Julien Melleray who accepted to be in my committee.

J'ai passé deux fois deux ans dans deux laboratoires, et j'ai donc eu la chance de rencontrer deux équipes différentes, qui m'ont énormément apporté. Je suis reconnaissant à tous les membres du groupes de travail Actions, mais aussi à ceux du groupe de travail de théorie ergodique de l'Université de Paris. Je tiens à remercier tout spécialement Damien Gaboriau, Tomás Ibarlucia, François Le Maître et Julien Melleray dont les portes étaient toujours ouvertes et qui m'ont entouré et supporté ces quatres années durant.

Je suis aussi très heureux des nombreuses rencontres que j'ai pu faire en conférence (lorsqu'on se déplaçait encore en conférence), I want to thank Honza and Matej who welcomed me to Praha. Thanks also to all the people that helped me in my research for a postdoc, especially Manuel Bordisky, Marcin Sabok and Clinton Conley. Je remercie aussi Claude Laflamme pour le sympathique accueil qu'il m'a fait au Canada.

I obviously need to thank the people who are on the cover of this thesis: Andy, Gianluca and Matthieu. The way we work together is the heart of my mathematical practice. Surprisingly, none of the work we did all the four of us together appears in this thesis, but working with you has been essential to my construction as a mathematician. Andy played the role of a third advisor in the first two years of my thesis and I am incredible thankful for all the discussions that we have had. As I write this, Gianluca is in my appartment and we are thinking on how to expand the most recent results present in this thesis. All the time you have been here has been a bliss. Thank you also for welcoming me to Lausanne when I was still a shy graduate student. Je suis aussi très chanceux d'avoir rencontré Matthieu et que nos intérêts aient été si compatibles. J'ai pu grâce à toi faire de nouvelles mathématiques dans lesquelles j'aurai probablement eu du mal à me plonger sans toi. Merci aussi pour les jeux, les rigolades et les balades, en espérant qu'il y en aura beaucoup d'autres.

Merci aussi aux doctorants de l'ICJ qui m'ont accueilli et avec lesquels j'ai pu discuter et boire bières et cafés. Je pense notamment à Lola, Marina, Vincent, Mélanie, Quentin, Rémy, Marion, Martin et Kyriaki.

Je tiens à spécialement remercier mes cobureaux, dont le sens de l'humour et le goût pour la bonne cuisine ont fait de mes années en présentiel un vrai plaisir. Merci donc à Reda le tout premier qui m'a énormément soutenu lorsque je doutais de moi, Octave qui a réparé Thunderbird en appuyant sur F8 (et qui a fait pas mal d'autres choses, il faut bien le dire), Simon avec qui c'est toujours un bonheur de discuter, Simon qui est plein de ressources, Gauthier qui nous régale toujours d'humour et de gateaux, Gary que je n'ai pas vu autant que j'aimerais, Clément dont j'admire les convictions, Jules que j'espere avoir réussi à accueillir, Pattéo qui a accepté de partager son espace avec moi et Thomas avec qui je peux parler de tout ce que j'aime et me comprend si bien.

Il y a aussi tout mes compagnons de voyage qui ne sont pas dans les mêmes laboratoires que moi. Vous m'avez soutenus et porter pendant tout ce temps, je vous dois énormément. Merci donc à Isao toujours fidèle au poste, Felix qui fait secretement les mêmes maths que moi, Alexandre qui est d'aussi bonne humeur qu'il est inventif, Lucas qui nous régale d'anecdotes et de cuisine, Pierre qui est un guide et un modèle pour moi, Figaro qui s'enthousiasme pour tout ce que je fais, Inès petite soeur de mon coeur, Anne-Elisabeth dont j'admire les talents et la patience et Béné à qui je peux toujours parler et qui m'a tant porté.

Les paragraphes suivants ont été écrits lorsque je suis parti de Paris, c'est donc écrit par le Colin de juillet 2019.

Je dois commencer par ceux qui m'ont accueilli en premier : Antoine et Sacha. Merci à vous qui m'avez ouvert les portes de votre bel univers et m'avez fait sentir à ma place.

Je continuerai avec celles et ceux avec qui j'ai travaillé : Maud, merci pour ton incroyable énergie et ton optimisme ; Charles, pour toutes les aventures, merci pour ton amitié ; Wille, dont l'avis est toujours pertinent, merci pour nos (trop rares) discussions ; Amandine, nous nous sommes rencontrés tard, mais ce fut un très grand plaisir, j'éspère sincèrement que nous resterons proches ; et Léa, la plus expérimentée qui nous a dit tant aidés, merci pour tous les bons moments, passés et à venir.

Merci à Alexandre de m'avoir montré qu'on pouvait jouer au solitaire toute la matinée et faire une thèse brillante. Merci à mes camarades de promotion, Alex et Mario, d'avoir partagé les difficultés et les joies avec moi, vous m'avez donné beaucoup de force. Merci à Elie, dont l'enthousiasme est si contagieux, qui nous a tant remonté le moral. Merci à Alice, petite soeur de thèse qui partage sa richesse intérieure. Merci à Kevin, qui améliore son entourage de sa simple présence, pour le modèle que tu nous donnes. Merci Omar qui me surprend sans arrêt. Merci à Nicolas pour ton soutien et la profondeur à laquelle tu partages ton univers. Merci à Noélie pour toutes nos discussions et le beau voyage au Canada. Merci à Reda, excellent cobureau, avec qui j'ai a tellement rigolé. Merci à Andreas pour nos discussion mathématiques et plus généralement pour ta sympathie. Merci à Théophile pour ta verve et tes attentions. Merci à Mathieu, qui veut toujours jouer au loup garou. Merci à Andrei, qu'on ne voit jamais assez. Merci à Fabien qui nous a apporté beaucoup de rigolade. Merci à Corentin, éternellement sympathique. Merci à Tancrède, qui rayonne de gentillesse. Merci aussi à Minsung, Esther, Kevin, Oussama, Daniele, Charazade, Alice, Yann, Juan Pablo, Pierre-Marie, Alexande, Rahman, Leonardo et Jerôme. Vous n'avez pas tous un petit mot mais je ne vous oublierai pas pour autant.

Merci à mes parents et mes soeurs, qui me soutiennent inconditionnellement.

Merci à Marie. Ton soutien et ton sourire sont les réalités les plus précieuses de mon existence. Tu m'impressionnes à chaque instant, tu améliores tout ce que tu touches. Cette thèse t'es dédiée.

Résumé en français de la thèse

Mon travail se concentre sur une notion de dynamique des groupes topologiques qui spécialise la notion de moyennabilité, appelée unique ergodicité, et qui se définit comme suit :

Définition 1. Un groupe G est dit uniquement ergodique si tout G-flot $G \curvearrowright X$ minimal est le support d'une unique mesure invariante.

Ci-dessus, un *flot* est une action continue d'un groupe topologique sur un espace compact. Un flot est dit minimal lorsque l'orbite de chaque point est dense. De manière équivalente, un flot est minimal lorsqu'il n'admet pas de sous-flot propre. On peut facilement prouver, à l'aide du Lemme de Zorn, que tout flot admet un sous-flot minimal.

Il est à noter que la notion d'unique ergodicité se réfère généralement à une action plutôt qu'à un groupe. Une action est uniquement ergodique lorsqu'elle n'admet qu'une mesure invariante.

Pour des exemples de groupes uniquement ergodiques, on peut tout d'abord se tourner vers les groupes compacts : en utilisant l'unicité la mesure de Haar, on peut facilement montrer que tous les groupes compacts sont uniquement ergodiques. En revanche, lorsqu'on s'intéresse aux groupes polonais localement compacts mais non-compacts, la situation change radicalement :

Théorème 1. [JZ2] Les groupes polonais localement compacts non-compacts ne sont jamais uniquement ergodiques.

Ce théorème est un des principaux résultats du Chapitre 6. Trouver des groupes uniquement ergodiques intéressants nécessite donc de se tourner vers des groupes plus gros, c'est-à-dire non-localement compacts. Le premier exemple d'un tel groupe est dû à Glasner et Weiss en 2002, il s'agit de S_∞, le groupe de permutations des entiers, muni de la topologie de convergence simple. C'est ensuite en 2012 qu'Angel, Kechris et Lyons montrent que plusieurs sous-groupes fermés de S_∞ possèdent également cette propriété. Leur méthode, qui est essentiellement de nature combinatoire, s'appuie largement sur le fait que les sous-groupes fermés de S_∞ peuvent être réalisés en tant que groupes d'automorphismes de certaines structures dénombrables dites homogènes, où tout isomorphisme entre sous-structures finies peut être étendu en un automorphisme global.

L'approche d'Angel, Kechris et Lyons repose également sur l'étude d'un flot particulier : le flot minimal universel d'un groupe. Ellis a montré en 1966 que tout groupe topologique G admet un unique flot minimal universel $\mathrm{M}(G)$, c'est-à-dire un flot minimal qui se surjecte de manière G-équivariante sur tout autre flot minimal.

Angel, Kechris et Lyons ont démontré une caractérisation utile de l'unique ergodicité:

Théorème 2. [AKL] Un groupe G est uniquement ergodique ssi $G \curvearrowright \mathrm{M}(G)$ n'a qu'une seule mesure invariante.

Ce théorème est aussi la base de mes travaux : à l'aide d'une description efficace du flot minimal universel, on peut obtenir des résultats d'unique ergodicité. Le premier exemple d'un tel résultat se trouve dans le chapitre 3 :

Théorème 3 ([JZ2])**.** Soit H, K, G des groupes polonais tels que

$$1 \to H \to G \xrightarrow{\pi} K \to 1$$

est une suite exacte.

Si $\mathrm{M}(H)$ et $\mathrm{M}(K)$ sont métrisables, alors $\mathrm{M}(G)$ l'est aussi. Si de plus H et K sont uniquement ergodiques, G l'est aussi.

Le chapitre 4 quant à lui s'intéresse aux mesures invariantes de certaines actions de groupes d'automorphismes de structures homogènes. On y montre en particulier le résultat suivant:

Théorème 4 ([JT])**.** Soit $\mathbb{F}$ une structure homogène $\aleph_0$-catégorique, éliminant faiblement les imaginaires, sans algébricité et transitive. L'une des deux conclusions suivantes est vraie:

1) $\mathbb{F}$ admet un ordre définissable.

2) $\mathrm{Aut}(\mathbb{F}) \curvearrowright \mathrm{LO}(\mathbb{F})$ est uniquement ergodique.

Ce théorème a plusieurs implications intéressantes, il permet entre autres de retrouver de nombreux résultats d'unique ergodicité de groupes (et en donne de nouveaux). Il permet aussi de montrer que certains groupes ne sont pas moyennables, ou encore de faire apparaître des propriétés combinatoires dans certains cas.

Enfin les chapitres 2 et 5 s'intéressent chacun à l'étude de l'unique ergodicité de groupes d'automorphismes de structures bien particulières.

Le chapitre 2 s'intéresse au 2-graphe, un hypergraphe aux propriétés combinatoires particulières, qui rendent le flot minimal universel de son groupe d'automorphismes singulier parmi les flots minimaux universels connus. Je présente dans ce chapitre la méthode combinatoire d'Angel, Kechris et Lyons pour prouver l'unique ergodicité. Le lecteur trouvera une autre preuve, celle-ci dynamique, à la fin du chapitre 4.

Enfin le chapitre 5 traite l'unique ergodicité du groupe d'automorphismes du graphe semigénérique. Il s'agit du premier de mes résultats de thèse et repose sur une étude dynamique et combinatoire du graphe semigénérique, pour laquelle la méthode d'Angel, Kechris et Lyons ne s'applique pas, et où il a donc fallu développer une stratégie nouvelle.

CHAPTER 1

Introduction - A love story of combinatorics and dynamics

1.1 Dynamics of groups

The work presented in this thesis is at the intersection of dynamics, probability and model theory. It focuses on a specialization of the notion of amenability for topological groups: unique ergodicity. This notion was introduced by Angel, Kechris and Lyons in [AKL], though the notion of a uniquely ergodic action has been around for much longer.

A topological group is a group together with a topology such that the product and inverse operations are continuous. If the topology is separable and completely metrizable, then the group is said to be Polish.

An action of a group G on a set X is a map $a\colon G \times X \to X$ such that

$$a(h, a(g,x)) = a(hg, x).$$

An action of a topological group G on a topological space X is continuous if the map $G \times X \to X$ is. Generally, we will not explicitly write a but instead $G \curvearrowright X$ to mean that there is an action. Moreover, we write $g \cdot x$ instead of $a(g,x)$.

A G-map is a map $\phi\colon X \to Y$ where $G \curvearrowright X$ and $G \curvearrowright Y$, and such that for all $x \in X$, $g \in G$,

$$\phi(g \cdot x) = g \cdot \phi(x).$$

A G-flow is a continuous action of a topological group G on a compact space. An invariant measure on a flow $G \curvearrowright X$ is a Borel measure μ on X such that for all $g \in G$ and $A \subset X$ measurable, $\mu(g \cdot A) = \mu(A)$. Given a group G, we talk about G-flows, G-invariance, etc... to refer to actions of G.

From this point on, unless specified otherwise, all the measures will be assumed to be Borel probability measures.

Let G be a topological group. A G-flow is said to be minimal if every orbit is dense. This is equivalent to saying that the flow admits no proper subflow, i.e. a closed subset invariant by the action. Using Zorn's Lemma, one can prove that any G-flow admits a minimal subflow.

In studying actions of groups, a natural class of groups that are well-behaved appears: amenable groups.

Definition 1.1.1. A topological group G is said to be amenable if every G-flow admits an invariant measure.

Examples of amenable groups include the group of permutations of $\mathbb{N}$ equiped with the pointwise convergence topology, that we denote by S_∞, but also $\mathbf{Z}^n$ for all $n \in \mathbb{N}$. All compact groups are also amenable. We will see more complicated examples later in this chapter. As an historical side note, this notion was introduced by Von Neumann, under a different name in [VN], with a very different characterization. His definition applies to discrete groups whereas the above definition has the benefit of being applicable to all topological groups.

A very important specialization of this notion is extreme amenability.

Definition 1.1.2. A topological group G is said to be extremely amenable if every G-flow admits a fixed point.

Another characterization would be that all minimal G-flows of an extremely amenable group G are trivial, that is, reduced to a singleton. Moreover, extremely amenable groups are indeed amenable, since the Dirac mass at a fixed point is an invariant measure. Later in this section we will see a very powerful way to prove the extreme amenability of some groups by connecting it to combinatorial properties.

Another strengthening of amenability and the key notion in this thesis is unique ergodicity.

Definition 1.1.3. A Polish group G is said to be uniquely ergodic if every minimal G-flow admits a unique G-invariant measure.

The expression uniquely ergodic usually refers to an action. An action is uniquely ergodic if it admits a unique invariant measure. This name is particularly well-chosen because of the following definition and theorem.

Definition 1.1.4. Let G be a Polish group acting continuously on a compact space X. A G-invariant measure ν is said to be *G-ergodic* if for all $A \subset X$ measurable such that

$$\forall g \in G,\ \nu(A \triangle g \cdot A) = 0,$$

we have $\nu(A) \in \{0, 1\}$.

We can now state the following (see [P3] Proposition 12.4):

Theorem 1.1.5. *Let G be a Polish group acting continuously on a compact space X. Let $P_G(X)$ denote the convex compact space of G-invariant measures on X. Then the extreme points of $P_G(X)$ are the G-ergodic invariant measures.*

This implies in particular that if an action has a unique invariant measure, it is necessarily ergodic.

Since the above defined notions require to work with *all* minimal flows in some sense, the following object can be very helpful when trying to understand them.

A famous theorem of Ellis [E2] states that any topological group G admits a unique *universal minimal flow* (UMF) that we denote by $\mathrm{M}(G)$. This means that for any minimal G-flow X there is a surjective G-map from $\mathrm{M}(G)$ to X. A proof of this theorem can be found in Chapter 3, Section 3.2.

It is easy to check that a characterization of amenability can now be stated as G is amenable iff $G \curvearrowright \mathrm{M}(G)$ admits an invariant measure. Similarly, G is extremely amenable iff $\mathrm{M}(G)$ is reduced to a singleton.

The characterization for unique ergodicity is not quite as easy to prove, however in [AKL] Theorem 8.1, the authors prove:

Proposition 1.1.6. *A Polish group G is uniquely ergodic iff $G \curvearrowright \mathrm{M}(G)$ admits a unique invariant measure.*

This characterization, however, leaves one big gap: can we describe the UMF of a group? In the next few paragraphs, we will try to answer this question as precisely as we can.

Let us first take a step back and observe that a compact group is its own UMF when considering the action of the group on itself via left-multiplication. Moreover, by existence of the Haar measure ([H1]), they are always amenable. By uniqueness of the Haar measure, they are always uniquely ergodic.

If we look at locally compact non-compact Polish groups, their UMFs are never metrizable (see [KPT] Appendix 2). Some are amenable ($\mathbb{Z}$ for instance) while some are not (free groups for instance). As for unique ergodicity, Andy Zucker and I proved:

Theorem 1.1.7. *Locally compact non-compact Polish group are never uniquely ergodic.*

This result is one of the main theorems of Chapter 6.

The main object of study in this thesis is therefore a class of non-locally compact groups coming from model theory, whose UMFs are metrizable and can be used to study amenability and unique ergodicity.

We are going to construct flows using closed subgroups of G. If G^* is a closed subgroup of G, then the space G/G^* has a natural uniformity and therefore admits a completion.

The uniformity of G/G^* is defined by entourages of the form U_V where $V \subset G$ is a neighbourhood of the identity and

$$U_V = \{(gG^*, vgG^*) : v \in V\}.$$

This is compatible with the topology given by the quotient metric:

$$d_{G/G^*}(hG^*, h'G^*) = \inf_{g^* \in G^*} d_G(hg^*, h'),$$

where d_G is a right-invariant metric on G, which exists because G is Polish. Therefore, we can complete G/G^* and we denote by $\widehat{G/G^*}$ its completion. Note that G/G^* is dense in $\widehat{G/G^*}$. For more details on completion of uniform spaces, see [AFP, Section 12]. If G^* a closed subgroup of G, it is called coprecompact if $\widehat{G/G^*}$ is compact.

1.2 Towards computing UMFs - The Kechris-Pestov-Todorcevic correspondence

In 2005, Kechris, Pestov and Todorcevic found a way to connect extreme amenability to a combinatorial property called the Ramsey Property. This was an outstanding development in the understanding of universal minimal flows. The aim of this section is to describe how this correspondence is used in dynamics. We will need some background in model theory.

A relational countable language $\mathcal{L}$ is a countable collection of symbols (relations), to each of which is associated a positive natural number, that we call its arity. A structure $\mathbf{M}$ in a language $\mathcal{L}$ is a domain, that we denote by $\mathrm{Dom}(\mathbf{M})$, and an interpretation of the symbols in $\mathcal{L}$, i.e. to each relation $R \in \mathcal{L}$ of arity r is associated a subset of $\mathrm{Dom}(\mathbf{M})^r$, that corresponds to the elements verifying the relation. For a structure $\mathbf{M}$ and R a symbol of arity r in its language, we write $R^{\mathbf{M}}(x_1, \ldots, x_r)$ to mean that $(x_1, \ldots, x_r)$ verifies R in $\mathbf{M}$. For a given structure, we call its *signature* the language it is expressed in. Usually, our countable structure will be assumed to have domain $\mathbb{N}$. Equality is always assumed to be in the language and to be interpreted as the usual identification. We will not write equality when we describe a language.

Remark that if $\mathcal{L} = (R_i)_{i \in I}$ for some I, a $\mathcal{L}$-structure $\mathbf{M}$ can be interpreted as an element m of $\prod_{i \in I} \{0, 1\}^{\mathbb{N}^{r_i}}$ where r_i is the arity of R_i. The identification goes as follows:

$$\forall x_1, \ldots, x_{r_i} \in \mathbb{N},\ R_i^{\mathbf{M}}(x_1, \ldots, x_{r_i}) \ \Leftrightarrow\ m_i(x_1, \ldots, x_{r_i}) = 1.$$

This identification gives us a natural topology for the space of $\mathcal{L}$-structures, as a subspace of a compact space.

A substructure of a given structure $\mathbf{A}$ is a structure whose domain is included in $\mathrm{Dom}(\mathbf{A})$ and the relations are the relations induced by restriction. An embedding from a structure $\mathbf{A}$ into a structure $\mathbf{B}$ in the same language $\mathcal{L}$ is a map f from $\mathrm{Dom}(\mathbf{A})$ to $\mathrm{Dom}(\mathbf{B})$ such that for any $R \in \mathcal{L}$ with arity r and $x_1, \ldots, x_r \in \mathrm{Dom}(\mathbf{A})$, we have $R^{\mathbf{A}}(x_1, \ldots, x_r) \Leftrightarrow R^{\mathbf{B}}(f(x_1), \ldots, f(x_r))$. If there is such an f, it needs to be injective. If there is such a map that is bijective, we say that $\mathbf{A}$ and $\mathbf{B}$ are isomorphic. If it is a bijection and $\mathbf{A} = \mathbf{B}$, we call it an automorphism of $\mathbf{A}$.

A class $\mathcal{F}$ of finite structures is a Fraïssé class if it contains structures of arbitrarily large (finite) cardinality and satisfies the following:

i) (Hereditary Property) If $A \in \mathcal{F}$ and B is a substructure of A, then $B \in \mathcal{F}$.

ii) (Joint Embedding Property) If $A, B \in \mathcal{F}$ then there exists $C \in \mathcal{F}$ such that A and B can be embedded in C.

iii) (Amalgamation Property) If $A, B, C \in \mathcal{F}$ and $f\colon A \to B$, $g\colon A \to C$ are embeddings, then there exists $D \in \mathcal{F}$ and $h\colon B \to D$, $l\colon C \to D$ embeddings such that $h \circ f = l \circ g$.

A Fraïssé class $\mathcal{F}$ admits a Fraïssé limit which is a countable structure whose age, i.e. the set of its finite substructures up to isomorphism, is $\mathcal{F}$. Fraïssé limits are *homogeneous*, i.e. any isomorphism between two finite substructures of the structure can be extended to an automorphism of the structure. The Fraïssé limit of a Fraïssé class is unique up to isomorphism. For more details on Fraïssé classes see [H2].

We now give a list of examples of Fraïssé classes and their limits that will be touched upon in the rest of the thesis.

Examples. *1) In the empty language, we have the class of finite sets. Its limit is just a countable set, we denote it by $\mathbb{N}$. Its automorphism group is denoted by S_∞.*

2) *In the language $\{E\}$ of arity 2, we have the class of finite graphs where E is interpreted as the edge relation. The limit, that we denote by R, is known as the random graph. This is because if we take a countable number of vertices and put an edge between two vertices with probability $1/2$ independently for each pair of vertices, we almost surely obtain a structure isomorphic to R.*

3) *In the language $\{\leq\}$, we have the class of finite linear orderings. Its limit is the only countable dense ordering without endpoints, an ordering isomorphic to $(\mathbb{Q}, \leq)$. Its automorphism group is usually written $\mathrm{Aut}(\mathbb{Q})$.*

4) *In the language $\{\leq\}$ we have the class of finite partial orderings, and its limit is known as the generic poset.*

5) *In the language $\{d_q\}_{q\in\mathbb{Q}}$ where all the relations are of arity 2, we have the class of finite metric spaces where $d_q(x, y)$ is interpreted as x and y are at distance q. The limit of this class is known as the rational Uryshon space.*

6) *In the language $\{\rightarrow\}$, we have the class of finite tournaments. Its limit is called the generic tournament.*

Two more examples of Fraïssé limits, the 2-graph and the semigeneric directed graph, are described in Chapter 2 and 5 respectively.

It is also important to note that automorphism groups of Fraïssé limits correspond exactly to closed subgroups of S_∞ (see [KPT] Section 2 and [H2] Theorem 4.1.4), the permutation group of $\mathbb{N}$. This is especially important because it means that any action of S_∞ on a space induces an action of $\mathrm{Aut}(\mathbb{F})$ for all Fraïssé limits $\mathbb{F}$. Similarly, an S_∞-invariant measure on a space is also a $\mathrm{Aut}(\mathbb{F})$-invariant measure on the same space. Note that neither ergodicity nor minimality of an action necessarily pass to the action of a subgroup.

The rest of this section is dedicated to the study of $\mathrm{Aut}(\mathbb{F})$-flows and how to compute the UMF of some groups.

1.2.1 Flows and expansions

Consider two languages $\mathcal{L} \subset \mathcal{L}^*$ and $\mathbf{A}$ a $\mathcal{L}^*$-structure. By $\mathbf{A}_{|\mathcal{L}}$, we mean the $\mathcal{L}$-structure that has domain $\mathrm{Dom}(\mathbf{A})$ and such that for all $R \in \mathcal{L}$ of arity r and $a_1, \ldots, a_r \in \mathrm{Dom}(\mathbf{A})^r$, we have

$$R^{\mathbf{A}_{|\mathcal{L}}}(a_1, \ldots, a_r) \Leftrightarrow R^{\mathbf{A}}(a_1, \ldots, a_r).$$

Let us take $\mathbb{F}$ and $\mathbb{F}^*$ two Fraïssé limits corresponding to the Fraïssé classes $\mathcal{F}$ and $\mathcal{F}^*$ respectively. We denote by $\mathcal{L}$ and $\mathcal{L}^*$ the signatures of $\mathbb{F}$ and $\mathbb{F}^*$ and assume that $\mathcal{L} \subset \mathcal{L}^*$. We say that $\mathbb{F}^*$ is an expansion of $\mathbb{F}$ if $\mathcal{F}^*_{|\mathcal{L}} := \{\mathbf{A}_{|\mathcal{L}} : \mathbf{A} \in \mathcal{F}^*\}$ is exactly $\mathcal{F}$. For a given stucture $\mathbf{A} \in \mathcal{F}$, we write $\mathcal{F}^*(\mathbf{A})$ for all $\mathbf{B} \in \mathcal{F}^*$ such that $\mathbf{B}_{|\mathcal{L}} = \mathbf{A}$.

If we write $\mathcal{L}^* = \mathcal{L} \cup \{R_i\}_{i\in I}$ for some set I and $G = \mathrm{Aut}(\mathbb{F})$, then we have $\mathbb{F}^* = (\mathbb{F}, (R_i^*))_{i\in I}$ and an action $G \curvearrowright \overline{G \cdot (R_i^*)_{i\in I}}$, where the closure is taken in $\prod_{i\in I}\{0,1\}^{\mathbb{N}^{r_i}}$ where r_i is the arity of R_i.

Remark that, if we write $G^* = \mathrm{Aut}(\mathbb{F}^*)$, which is the same as saying that $G^* \leq G$ and G^* stabilizes $(R_i^*)_{i\in I}$, then not only G^* is a closed subgroup of G, but we can also identify G/G^* with $G \cdot (R_i^*)_{i\in I}$. Indeed, we can consider the continuous map

$$\begin{aligned} G/G^* &\to G \cdot (R_i^*)_{i \in I} \\ gG^* &\mapsto g \cdot (R_i^*)_{i \in I}. \end{aligned}$$

If G^* is coprecompact, i.e. $\widehat{G/G^*}$ is compact, then the uniformities of the two spaces G/G^* and $G \cdot (R_i^*)_{i \in I}$ coincide and we can identify $\widehat{G/G^*}$ and $\overline{G \cdot (R_i^*)_{i \in I}}$. In this case we say that $\mathbb{F}^*$ is a precompact expansion of $\mathbb{F}$. There is a combinatorial characterization for precompactness: $\mathbb{F}^*$ is a precompact expansion of $\mathbb{F}$ if for all $\mathbf{A} \in \mathcal{F}$ there are finitely many $\mathbf{A}' \in \mathcal{F}^*$ such that $\mathbf{A}'_{|\mathcal{L}} = \mathbf{A}$. See [N] for more details on this identification.

Let us take $\mathbf{A} \subset \mathbb{F}$ finite and $\mathbf{A}^* \in \mathcal{F}^*(\mathbf{A})$, then we define

$$U_{\mathbf{A},\mathbf{A}^*} = \{E \in \overline{G \cdot (R_i^*)_{i \in I}} \,:\, E_{|\mathbf{A}} \simeq \mathbf{A}^*\}.$$

This family of sets, where $\mathbf{A}$ ranges over finite substructures of $\mathbb{F}$ and $\mathbf{A}^*$ ranges over $\mathcal{F}^*(\mathbf{A})$, is a clopen basis for the topology of the space $\overline{G \cdot (R_i^*)_{i \in I}}$.

This will allow us to work with a large class of flows that we further characterize in the rest of this section.

1.2.2 Universality of a flow

Consider $\mathcal{F}$ a Fraïssé class. For $A, B \in \mathcal{F}$ we call $\binom{B}{A}$ the space of embeddings of A into B. We say that $\mathcal{F}$ has the Ramsey property if for all $A, B \in \mathcal{F}$ and $k \in \mathbb{N}$, there exists $C \in \mathcal{F}$ such that $\binom{C}{A}$ and for any map $\gamma\colon \binom{C}{A} \to \{1, \dots, k\}$, there is B_0 a substructure of C isomorphic to B such that the restriction map of γ to $\binom{B_0}{A}$ is constant.

Theorem 1.2.1. [KPT] *Let $\mathbb{F}$ be a Fraïssé limit.* $\mathrm{Aut}(\mathbb{F})$ *is extremely amenable iff* $\mathrm{Age}(\mathbb{F})$ *has the Ramsey property.*

The following proposition allows us to connect extreme amenability and universality.

Proposition 1.2.2 ([N]). *Let G be a Polish group that admits an extremely amenable coprecompact subgroup G^*. The flow $\widehat{G/G^*}$ is universal, in the sense that there is a G-map $\hat{\phi}$ from $\widehat{G/G^*}$ to any G-flow. Moreover, if the flow is minimal, then $\hat{\phi}$ is also surjective.*

Proof. Let $G \curvearrowright X$ be a G-flow. Choose $x \in X$ a point fixed by G^* (which exists by extreme amenability) and consider the map

$$\begin{aligned} \phi\colon G/G^* &\to X \\ gG^* &\mapsto g \cdot x. \end{aligned}$$

This map is well-defined because x is fixed by G^* and $g \cdot x$ only depends on the class of g when taking the quotient. ϕ is also a uniformly continuous G-map because the action is continuous on a compact and G/G^* is precompact.

Since G/G^* is dense in $\widehat{G/G^*}$, we can continuously extend ϕ to $\widehat{G/G^*}$ and we call $\hat{\phi}$ the map thus obtained.

Finally, if $G \curvearrowright X$ is minimal, then the G-orbit of x is dense. Moreover, the G-orbit of x is equal to the image of ϕ. Therefore the image of ϕ is dense, therefore the image of $\hat{\phi}$ is X. $\square$

1.2.3 Minimality of a flow

Let us first look at an important G-flow when G is the automorphism group of a Fraïssé limit. Given a structure $\mathbf{A}$ we denote by $\mathrm{LO}(\mathbf{A})$ the space of linear orderings of the domain of $\mathbf{A}$. If we take $\mathbb{F}$ a Fraïssé limit and G its automorphism group, then $G \curvearrowright \mathrm{LO}(\mathbb{F})$. This action can be described as follows: for all $a, b \in \mathbb{F}$, $g \in G$ and $< \in \mathrm{LO}(\mathbb{F})$,

$$a(g\cdot <)b \Leftrightarrow g^{-1}a < g^{-1}b.$$

The minimality of $G \curvearrowright \mathrm{LO}(\mathbb{F})$ was first characterised by Kechris, Pestov and Todorcevic in [KPT] as being equivalent to $\mathcal{F}$ having the ordering property, i.e. for every $A \in \mathcal{F}$, there exists $B \in \mathcal{F}$ such that for any two linear orders $<$ and $<'$ on A and B respectively, there is an embedding of $(A,<)$ into $(B,<')$.

More generally, we want to talk about the *expansion property*. We say that $\mathcal{F}$ has the expansion property with respect to $\mathcal{F}^*$ if for all $\mathbf{A} \in \mathcal{F}$ there is a $\mathbf{B} \in \mathcal{F}$ such that for all $\mathbf{A}' \in \mathcal{F}^*$ and $\mathbf{B}' \in \mathcal{F}^*$ such that $\mathbf{A}'_{|\mathcal{L}} = \mathbf{A}$ and $\mathbf{B}'_{|\mathcal{L}} = \mathbf{B}$, $\mathbf{A}'$ embeds into $\mathbf{B}'$.

Proposition 1.2.3 ([P2]). *The flow $G \curvearrowright \overline{G \cdot (R_i^*)_{i\in I}}$ is minimal iff $\mathrm{Age}(\mathbb{F})$ has the expansion property with respect to $\mathrm{Age}(\mathbb{F}^*)$.*

For more details, see [N].

1.2.4 The Kechris-Pestov-Todorcevic correspondence

We can now combine all the above theorems to have a sufficient condition to compute the UMF of an automorphism group. This was actually proven to be the only possible setup where the universal minimal flow of G is metrizable.

Theorem 1.2.4 ([MNT],[BMT],[Z1]). *Let G be a Polish group. G has metrizable UMF iff there exists $G^* \leq G$ extremely amenable such that*

$$M(G) = \widehat{G/G^*}.$$

This theorem translates in terms of Fraïssé limits.

Theorem 1.2.5. *Let $\mathbb{F}$ be a Fraïssé limit. $\mathrm{Aut}(\mathbb{F})$ has metrizable UMF iff $\mathbb{F}$ admits a precompact expansion $\mathbb{F}^* = (\mathbb{F}, (R_i^*)_{i\in I})$ such that $\mathbb{F}^*$ has the Ramsey property and $\mathrm{Age}(\mathbb{F})$ has the expansion property with respect to $\mathrm{Age}(\mathbb{F}^*)$. In this case,*

$$\mathrm{M}(\mathrm{Aut}(\mathbb{F})) = \mathrm{Aut}(\mathbb{F}) \curvearrowright \overline{\mathrm{Aut}(\mathbb{F}) \cdot (R_i^*)_{i\in I}}.$$

To conclude this subsection, let us present the UMF associated to some automorphism groups.

Fraïssé class	UMF of Aut($\mathbb{F}$)
Finite sets (no relation)	Linear orderings
Finite graphs	Linear orderings
Finite metric spaces with distances 0, 1 and 3	Convex linear orderings
Finite partial orderings	Linear orderings extending the generic poset
Finite directed graphs	Linear orderings

We will see in chapters 2 and 5 two examples of more complex UMFs for the 2-graph and the semigeneric directed graph respectively.

1.3 Unique ergodicity

Now that we have set up our framework, we can start the study of some uniquely ergodic groups.

A famous unique ergodicity result is due to Angel, Kechris and Lyons and if one denotes by R the Rado graph (the Fraïssé limit of finite graphs) it states:

Theorem 1.3.1. [AKL] $\mathrm{Aut}(R)$ *is uniquely ergodic.*

In the same paper, they ask the following question which guided my work:

Question 1.3.2. *Let G be an amenable Polish group with metrizable universal minimal flow. Is G uniquely ergodic?*

Angel, Kechris and Lyons also provide a proof of unique ergodicity for the Fraïssé limit of graphs, K_n-free graphs for $n \in \mathbb{N}$, metric spaces and r-uniform hypergraphs.

In [PS], using methods from [AKL], Pawliuk and Sokić extended the catalogue of uniquely ergodic automorphism groups with the automorphism groups of homogeneous directed graphs, which were all classified by Cherlin (see [C2]), leaving as an open question only the case of the semigeneric directed graph. This case is dealt with in Chapter 5 of this thesis.

We quickly explain the general ideas behind their proof here. A similar proof can be found in Chapter 2.

Rather that proving directly unique ergodicity, they prove that some Fraïssé classes have the Quantitative Ordering Property and show that this is equivalent to unique ergodicity of the automorphism group of the Fraïssé limit.

Definition 1.3.3. Let $\mathcal{F}$ be a Fraïssé class. We say that $\mathcal{F}$ has the Quantitative Ordering Property if for all $\mathbf{A} \in \mathcal{F}$ and $\varepsilon > 0$ there is $\mathbf{B} \in \mathcal{F}$ in which $\mathbf{A}$ embeds and $\mathcal{E} \subset \mathrm{Emb}(\mathbf{A}, \mathbf{B})$ such that for any $<_{\mathbf{A}} \in \mathrm{LO}(\mathbf{A})$ and $<_{\mathbf{B}} \in \mathrm{LO}(\mathbf{B})$ we have

$$\left| \frac{|\mathrm{Emb}((\mathbf{A}, <_{\mathbf{A}}), (\mathbf{B}, <_{\mathbf{B}})) \cap \mathcal{E}|}{|\mathcal{E}|} - \frac{1}{|\mathrm{LO}(\mathbf{A})|} \right| \leq \varepsilon.$$

The proof of Quantitative Ordering Property for finite graphs by Angel, Kechris and Lyons relies on taking a random graph $\mathbf{G}$ similar to the random graph but on finitely many edges and using the McDiarmid inequality (see [M2]) to prove that with high probability it verifies the conditions of Quantitative Ordering Property for a given graph H.

1.3.1 Amenability of automorphism groups

We will need to define measures on various spaces. Fortunately, we have a very powerful tool for that: Carathéodory's extension Theorem. We state it as it is stated in [K1] where it is Theorem A1.1. A field $\mathcal{A}$ in a set S is a family of subsets of S stable by finite intersection and complementation and that contains S. A measure on $\mathcal{A}$ is a finitely additive function $\mu \colon \mathcal{A} \to [0, 1]$, such that $\mu(S) = 1$ and if $(A_i)_{i \in \mathbb{N}}$ is a decreasing family with empty intersection, then $\mu(A_n) \to_n 0$.

Theorem 1.3.4 (Carathéodory's extension)**.** *Any measure on a field $\mathcal{A}$ can be extended to a unique measure on the σ-field generated by $\mathcal{A}$.*

Let us see how we use this theorem in our context.

Let $\mathbb{F}$ and $\mathbb{F}^* = (\mathbb{F}, (R_i^*)_{i\in I})$ be two Fraïssé limits such that $\mathbb{F}^*$ is an expansion of $\mathbb{F}$ with the expansion property. We denote by G and G^* the respective automorphism groups of $\mathbb{F}$ and $\mathbb{F}^*$. We give a criterion to define a measure on $\overline{G \cdot (R_i^*)_{i\in I}}$. We denote by $\mathcal{F}$ and $\mathcal{F}^*$ the respective ages of $\mathbb{F}$ and $\mathbb{F}^*$. Recall that we have a clopen basis for $\overline{G \cdot (R_i^*)_{i\in I}}$ given by,

$$U_{\mathbf{A},\mathbf{A}^*} = \{E \in \overline{G \cdot (R_i^*)_{i\in I}} \ : \ E_{|\mathbf{A}} \simeq \mathbf{A}^*\},$$

for $\mathbf{A} \subset \mathbb{F}$ finite and $\mathbf{A}^* \in \mathcal{F}^*(\mathbf{A})$.

We denote by $\mathcal{U}$ the family $U_{\mathbf{A},\mathbf{A}^*}$ where $\mathbf{A}$ and $\mathbf{A}^*$ vary over the finite substructures of $\mathbb{F}$ and $\mathcal{F}^*(\mathbf{A})$. Remark that the σ-field generated by this family is the Borel σ-field on $\widehat{G/G^*}$. To use Theorem 1.3.4, we would also need to know that this family is stable under intersection, unfortunately this is not the case. However, the intersection of two sets in $\mathcal{U}$ is actually a disjoint union of sets in $\mathcal{U}$. Therefore if we consider $\mathcal{U}'$ the collection of finite intersections of elements of $\mathcal{U}$, by Theorem 1.3.4, a measure on $\mathcal{U}$ extends to a measure on the Borel sets of $\widehat{G/G^*}$. Therefore, if a measure on $\mathcal{U}$ extends to a measure on $\mathcal{U}'$, then it extends to the Borel σ-field on $\widehat{G/G^*}$. We use this to get:

Theorem 1.3.5. *The following conditions are sufficient for a measure $\mu \colon \mathcal{U} \to [0,1]$ to be extendable to a unique measure:*

1) *For all $\mathbf{A} \in \mathbf{F}$, $\sum_{\mathbf{A}^* \in \mathcal{F}^*(\mathbf{A})} \mu(U_{\mathbf{A},\mathbf{A}^*}) = 1$.*
2) *For all $\mathbf{A}, \mathbf{A}^*$ and $\mathbf{B}$ whose domain has one more point than $\mathbf{A}$, we have*

$$\mu(U_{\mathbf{A},\mathbf{A}^*}) = \sum_{\{\mathbf{B}^* :\ \mathbf{A}^* = \mathbf{B}^*_{|A}\}} \mu(U_{\mathbf{B},\mathbf{B}^*}).$$

Note that the condition for decreasing families with empty intersection is always satisfied because of Cantor's intersection theorem.

This theorem means in particular that a measure on $\widehat{G/G^*}$ is entirely determined by its values on the family of clopen sets generating the topology.

Corollary 1.3.6. *Using the same notations as above, if we now assume that the number of expansions in $\mathcal{F}^*$ of a given $\mathbf{A} \in \mathcal{F}$ only depends on the size of $\mathbf{A}$, then the action $G \curvearrowright \widehat{G/G^*}$ admits an invariant measure defined as:*

$$\mu(U_{\mathbf{A},\mathbf{A}^*}) = \frac{1}{|\mathcal{F}^*(\mathbf{A})|}.$$

This result is Theorem 4.1 in [PS].

1.4 Summary of the chapters

The chapters of this thesis are independent and can be read in any order. For the sake of clarity and independence, some notions may be defined several times in different chapters. Except for Chapter 2, all chapters follow closely submitted or accepted papers.

1.4.1 Getting started on an example: the 2-graph

This is a short chapter that aims at presenting in more details the ideas present in [AKL] and [PS]. The main idea is to present a Fraïssé class whose automorphism group has an interesting UMF. We present the work of [EHKN] who considered this example for the first time. We show that the automorphism group of this structure is uniquely ergodic, using the method from [AKL]

Note that this connects to Chapter 4, where the last subsection is devoted to proving unique ergodicity of the automorphism group of this structure using tools from this chapter.

1.4.2 Structure of M(G) and unique ergodicity for group extensions

Let G be a Polish group, and suppose $H \subseteq G$ is a closed, normal subgroup. Setting $K = G/H$, we have that K is also a Polish group, and the quotient map $\pi\colon G \to K$ is a continuous, open homomorphism. In this setting, we say that G is an extension of H by K. This is the same as saying that

$$1 \to H \to G \xrightarrow{\pi} K \to 1$$

is a short exact sequence. In this setting, there is a natural G-map from $\mathrm{M}(G)$ to $\mathrm{M}(K)$ because $\mathrm{M}(K)$ is a minimal G-flow. With Andy Zucker, we proved that:

Theorem 1.4.1. [JZ2] *If $\mathrm{M}(H)$ and $\mathrm{M}(K)$ are metrizable, then so is $\mathrm{M}(G)$. Furthermore, letting $\pi\colon \mathrm{M}(G) \to \mathrm{M}(K)$ be the natural map, we have that $\pi^{-1}(\{y\})$ is a minimal H-flow for every $y \in \mathrm{M}(K)$. Moreover, if both H and K are uniquely ergodic, then G is also uniquely ergodic.*

This result was already known for semidirect products due to Pawliuk and Sokic in [PS].

1.4.3 Unique ergodicity of the action on the space of linear orderings

The results in this chapter take a different approach to unique ergodicity. Rather than looking at the universal minimal flow of a given group, we look at specific actions and study their possible invariant probability measures.

For a homogeneous structure $\mathbb{F}$ there are two $\mathrm{Aut}(\mathbb{F})$-flows we study.

1) $\mathrm{Aut}(\mathbb{F}) \curvearrowright [0,1]^{\mathbb{F}}$ by permuting the coordinates.

   This flow always admits some invariant probability measures of the form $\nu^{\mathbb{F}}$ for some probability measure ν on $[0,1]$.

2) If we denote by $\mathrm{LO}(\mathbb{F})$ for the space of linear orderings of $\mathbb{F}$, there is an action of $\mathrm{Aut}(\mathbb{F})$ on $\mathrm{LO}(\mathbb{F})$, defined as

   $$a(g\cdot <)b \Leftrightarrow g^{-1}a < g^{-1}b.$$

   This flow always admits an invariant measure μ called the uniform measure that is such that for all pairwise different $a_1, \dots, a_n \in \mathbb{F}$, we have

   $$\mu(a_1 < \ldots < a_n) = \frac{1}{n!}.$$

The aim of this chapter is to present a class of structures for which the above invariant measures are the only ones for these flows.

We define here the model-theoretic assumptions that we will need to express the theorems. The definitions here are given from a permutation group perspective and may require some work to prove that they are equivalent their original formulation. Let $\mathbb{F}$ be a Fraïssé limit, we say that:

1) $\mathbb{F}$ has no algebraicity if for any tuple $\overline{a} \in \mathbb{F}$, for any $x \notin \overline{a}$, $G_{\overline{a}} \cdot x$ is infinite, where $G_{\overline{a}}$ denotes the stabilizer of $\overline{a}$ for the action $\mathrm{Aut}(\mathbb{F}) \curvearrowright \mathbb{F}$.

2) $\mathbb{F}$ is $\aleph_0$-categorical if for all $n \in \mathbb{N}$, $G \curvearrowright \mathbb{F}^n$ has finitely many orbits.

3) $\mathbb{F}$ has weak elimination of imaginaries if for every proper, open subgroup $V < \mathrm{Aut}(\mathbb{F})$, there exists k and a tuple $\bar{a} \in M^k$ such that $G_{\overline{a}} \leq V$ and $[V : G_{\bar{a}}] < \infty$.

4) $\mathbb{F}$ is said to be transitive if for any $a, b \in \mathbb{F}$, there is $g \in \mathrm{Aut}(\mathbb{F})$ such that $g(a) = b$.

Theorem 1.4.2, which I used in the proof of Theorem 1.4.3, first appeared in [T2].

Theorem 1.4.2. [JT] *Let $\mathbb{F}$ be an $\aleph_0$-categorical, transitive structure with no algebraicity that admits weak elimination of imaginaries. Let Z be a standard Borel space and consider the natural action* $\mathrm{Aut}(\mathbb{F}) \curvearrowright Z^{\mathbb{F}}$. *Then the only invariant, ergodic probability measures on $Z^{\mathbb{F}}$ are product measures of the form $\lambda^{\mathbb{F}}$, where λ is a probability measure on Z.*

Theorem 1.4.3. [JT] *Let $\mathbb{F}$ be a transitive, $\aleph_0$-categorical structure with no algebraicity that admits weak elimination of imaginaries. Consider the action* $\mathrm{Aut}(\mathbb{F}) \curvearrowright \mathrm{LO}(\mathbb{F})$. *Then exactly one of the following holds:*

1. *The action* $\mathrm{Aut}(\mathbb{F}) \curvearrowright \mathrm{LO}(\mathbb{F})$ *has a fixed point (i.e., there is a definable linear order on $\mathbb{F}$);*

2. *The action* $\mathrm{Aut}(\mathbb{F}) \curvearrowright \mathrm{LO}(\mathbb{F})$ *is uniquely ergodic.*

One motivation for this result is that in many cases, $\mathrm{LO}(\mathbb{F})$ is the universal minimal flow of the group. I hope that this will lead to a better understanding for the more general Question 1.3.2.

Many previously known examples of uniquely ergodic groups fall under the scope of this theorem. Moreover, we get some interesting consequences, for instance the following non-amenability result.

Corollary 1.4.4. *Suppose that $\mathbb{F}$ satisfies the assumptions of Theorem 1.4.3 and let $G =$* $\mathrm{Aut}(\mathbb{F})$. *If the action $G \curvearrowright$* $\mathrm{LO}(\mathbb{F})$ *is not minimal and has no fixed points, then G is not amenable.*

Finally, a new subsection is present in this thesis and not in the associated paper: the proof that the automorphism group of the 2-graph is uniquely ergodic using a dynamical proof. This connects to Chapter 2.

1.4.4 The case of the semigeneric directed graph

Using a classification of homogeneous directed graphs by Cherlin ([C2]) and methods from [AKL], Pawliuk and Sokić were able to prove in [PS] that the answer to Question 1.3.2 was positive for all automorphism groups of homogeneous directed graphs, except for one case where their method did not apply: the semigeneric directed graph. One of my first pieces of work consisted in filling that gap.

Theorem 1.4.5. [J] *The automorphism group of the semigeneric directed graph is uniquely ergodic.*

The proof relies on the ergodic decomposition theorem, allowing one to show that any invariant probability measure satisfies certain independence properties. It is interesting to remark that I was not able to describe precisely how generally this proof applies. However, it certainly applies to more than just the semigeneric graph and the hope would be to use this proof in combination with results from chapter 4 to prove more general unique ergodicity results.

1.4.5 A minimal model-universal flow for locally compact Polish groups

The universal minimal flow is a minimal flow which maps onto any other minimal flow; by understanding the properties of this one object, we can better understand the collection of all minimal flows. With Andy Zucker, we proved that when G is a locally compact Polish group, there exists another minimal flow which is universal in a different sense, in that it contains a copy of any probability measure-preserving free action. Similarly, this "universal minimal model" can help shed light on the dynamical properties of a given locally compact group.

By a *G-system*, we mean a Borel G-action on a standard Lebesgue space (X, μ) which preserves μ. We say that a G-system (X, μ) is *free* if the set

$$\mathrm{Free}(X) := \{x \in X : \forall g \in (G \setminus \{1_G\})\ gx \neq x\}$$

has measure 1 (remark that this set is Borel because G is locally compact). We say that a G-flow Y is *model-universal* if for every free G-system (X, μ), there is ν a G-invariant probability measure on Y with $(X, \mu) \cong (Y, \nu)$.

This work is a generalisation of a work of Weiss in [W], who proved that all countable discrete groups admit a minimal model-universal flow.

Theorem 1.4.6. [JZ1] *Let G be a locally compact, non-compact Polish group. Then there exists a minimal model-universal flow for G.*

As a corollary, we get:

Theorem 1.4.7. [JZ1] *Let G be a locally compact non-compact Polish group. Then there is a minimal G-flow with multiple invariant probability measures. In particular, G is not uniquely ergodic.*

This result was suggested in [AKL] (see p. 2063).

CHAPTER 2

Getting started on an example: the 2-graph

This is based on joint work with Gianluca Basso. I thank Jan Hubička for presenting this example to me.

2.1 Definitions

A k-uniform hypergraph $\mathbf{M}$ for some $k \in \mathbb{N}^*$ is a structure whose signature is a k-ary relation R so that for all $x_1, \ldots, x_k \in \mathbf{M}$ and $\sigma \in S_k$ we have:

$$R^{\mathbf{M}}(x_1, \ldots, x_n) \Leftrightarrow R^{\mathbf{M}}(x_{\sigma(1)}, \ldots, x_{\sigma(k)})$$

and

$$\neg R^{\mathbf{M}}(x_1, x_1, x_2, \ldots, x_{k-1}).$$

For example, graphs are 2-hypergraphs.

Consider the class $\mathcal{H}$ of *even hypergraphs*[1], i.e. the class of finite 3-hypergraphs such that the number of hyperedges on any 4 vertices is even. It is a Fraïssé class (see [EHKN]). We denote by $\mathbb{H}$ the Fraïssé limit of $\mathcal{H}$. This limit is called the 2-graph. We write $R^{\mathbb{H}}$ for the hyperedge relation in $\mathbb{H}$.

There is a map from graphs to even hypergraphs by the following operation: from a graph, one obtains an even hypergraph with the same domain by putting an hyperedge between 3 vertices iff there is an even number of edges between those vertices in the original graph. One can check that this always gives an even hypergraph. For a graph $\mathbf{A}$, we call reduct of $\mathbf{A}$ the even hypergraph thus obtained, and we denote it by $\mathrm{red}_{\mathbb{H}}(\mathbf{A})$.

For a given even hypergraph $\mathbf{H}$, a graph on the same vertex set as $\mathbf{H}$ whose reduct is isomorphic to $\mathbf{H}$ is called a *graphing* of $\mathbf{H}$.

An important remark is that a graphing of an even hypergraph $\mathbf{H}$ is entirely determined by the edge relations between one point $a \in \mathbf{H}$ and all the other points in $\mathbf{H}$. Indeed, if we want to know if there is an edge between two points x and x', we have the following possibilities:

[1] This is also called a 2-graph. In the spirit of clarity we chose this renaming.

1) There is an hyperedge (a, x, x') in $\mathbf{H}$. In this case, there is an edge between x and x' iff there is an odd number of edges between a and $\{x, x'\}$.

2) There is no hyperedge (a, x, x') in $\mathbf{H}$. In this case, there is an edge between x and x' iff there is an even number of edges between a and $\{x, x'\}$.

We remark in particular that there are 2^{n-1} graphings of a given even hypergraph on n vertices.

Take $\mathbf{G}$ and $\mathbf{G}'$ two finite graphs with the same vertex set D. We denote by E the edge relation for graphs. We say that $\mathbf{G}'$ is in the switching class of $\mathbf{G}$ if there is $A \subset D$ such that for all $x, y \in A$, $E^{\mathbf{G}}(x,y) \iff E^{\mathbf{G}'}(x,y)$; for all $x, y \notin A$, $E^{\mathbf{G}}(x,y) \iff E^{\mathbf{G}'}(x,y)$; and for all $x \in A$ and $y \notin A$, $E^{\mathbf{G}}(x,y) \iff \neg E^{\mathbf{G}'}(x,y)$. Remark that by symmetry of the edge relation, this is the same as saying or all $x \notin A$ and $y \in A$, $E^{\mathbf{G}}(x,y) \iff \neg E^{\mathbf{G}'}(x,y)$. In this context, we call $\mathbf{G}'$ the switching of $\mathbf{G}$ by A.

Remark that the reduct of a graph $\mathbf{G}$ is isomorphic to the reduct of another graph $\mathbf{G}'$ iff $\mathbf{G}'$ is isomorphic to a graph in the switching class of $\mathbf{G}$, indeed this operation will not change the parity of the number of edges in a triplet of vertices. In particular, the class of graphings of a given even hypergraph can be recovered by all the switchings of any of its graphings. One way to see this is simply to observe that there are exactly 2^{n-1} different switchings of a graph on n vertices, which corresponds to the number of graphings of the reduct of this graph.

The UMF of $\mathrm{Aut}(\mathbb{H})$ was computed by Evans, Hubička, Konečný and Nešetřil in [EHKN]. It is $G \curvearrowright \mathrm{LO}(\mathbb{H}) \times \mathrm{Gr}(\mathbb{H})$, where $\mathrm{LO}(\mathbb{H})$ is the space of linear orderings of $\mathbb{H}$ and $\mathrm{Gr}(\mathbb{H})$ the space of graphings of $\mathbb{H}$. Let us take $(<, E) \in \mathrm{LO}(\mathbb{H}) \times \mathrm{Gr}(\mathbb{H})$, $g \in G$, then for all $a, b \in \mathbb{H}$, we have

$$a(g\cdot <)b \Leftrightarrow g^{-1}(a) < g^{-1}(b)$$

and

$$g \cdot E(a,b) \Leftrightarrow E(g^{-1}(a), g^{-1}(b)).$$

It will be useful to consider $\mathcal{H}^*$ the Fraïssé class of graphed ordered even hypergraphs. We call $\mathbb{H}^*$ its Fraïssé limit. Recall that for $\mathbf{1} \in \mathcal{H}$, $\mathcal{H}^*(\mathbf{A})$ designates the class of structures in $\mathcal{H}^*$ such that the induced even hypergraph is isomorphic to $\mathbf{A}$.

Using Corollary 1.3.6, we get that $\mathrm{Aut}(\mathbb{H})$ is amenable, because for a given hypergraphs on n vertices, there are $n!2^{n-1}$ ordered graphings of it.

2.2 Unique ergodicity

In the rest of the chapter, $G = \mathrm{Aut}(\mathbb{H})$. We give two proofs of the fact that G is uniquely ergodic. The first one follows closely the original paper of Angel Kechris and Lyons, even though it requires some adjustements, using among other things ideas from [PS]. Another proof of this result can be found in the last subsection of Chapter 4

Definition 2.2.1. Let $\mathcal{F}$ and $\mathcal{F}^*$ be two Fraïssé classes such that $\mathcal{F}^*$ is an expansion of $\mathcal{F}$ with the expansion property. We say that $\mathcal{F}$ has the Quantitative

Expansion Property relative to $\mathcal{F}^*$ if for all $\mathbf{A} \in \mathcal{F}$, there is a number $\rho(\mathbf{A}) \geq 0$ such that for all $\varepsilon > 0$ there is $\mathbf{B} \in \mathcal{F}$ in which $\mathbf{A}$ embeds such that for any $\mathbf{A}^* \in \mathcal{F}^*(\mathbf{A})$ and $\mathbf{B}^* \in \mathcal{F}^*(\mathbf{B})$ we have

$$\left| \frac{|\mathrm{Emb}(\mathbf{A}^*, \mathbf{B}^*)|}{|\mathrm{Emb}(\mathbf{A}, \mathbf{B})|} - \rho(\mathbf{A}) \right| \leq \varepsilon.$$

The following Theorem is taken from [AKL]. We give the proof for convenience of the reader.

Theorem 2.2.2. *Let $\mathcal{F}$ and $\mathcal{F}^*$ be two Fraïssé classes such $\mathcal{F}^*$ an expansion of $\mathcal{F}$ with the expansion property. We also assume that $\mathcal{F}$ has the Quantitative Expansion Property with respect to $\mathcal{F}^*$. We write $\mathbb{F}$ and $\mathbb{F}^*$ for the limits of $\mathcal{F}$ and $\mathcal{F}^* = (\mathbb{F}, (R_i)_{i\in I})$. If we assume that $\mathrm{Aut}(\mathbb{F}) \curvearrowright \overline{\mathrm{Aut}(\mathbb{F}) \cdot (R_i)_{i\in I}}$ admits an invariant measure, then $\mathrm{Aut}(\mathbb{F}) \curvearrowright \overline{\mathrm{Aut}(\mathbb{F}) \cdot (R_i)_{i\in I}}$ is uniquely ergodic.*

In particular this theorem can be used to prove unique ergodicity of $\mathrm{Aut}(\mathbb{F})$ when $\mathbb{F}^*$ is a Ramsey precompact expansion of $\mathbb{F}$ with the Quantitative Expansion Property.

Proof. Take ν an $\mathrm{Aut}(\mathbb{F})$-invariant measure on $\overline{\mathrm{Aut}(\mathbb{F}) \cdot (R_i)_{i\in I}}$. We want to show that $\nu(U_{A,A^*}) = \rho(\mathbf{A})$ for all $\mathbf{A} \in \mathcal{F}$ and $\mathbf{A}^* \in \mathcal{F}^*(\mathbf{A})$. Let us fix $\mathbf{A}$ and $\varepsilon > 0$ and take $\mathbf{B}$ as in definition 2.2.1. We also fix $\mathbf{A}^* \in \mathcal{F}^*(\mathbf{A})$ and set $\mathbf{B}^* \in \mathcal{F}^*(\mathbf{B})$ a random variable $\beta_*\nu$ where β is the restriction map from $\overline{\mathrm{Aut}(\mathbb{F}) \cdot (R_i)_{i\in I}}$ to $\mathbf{B}$.

We take ϕ a uniform random embedding of $\mathbf{A}$ in $\mathbf{B}$. We look at the event C="ϕ is an embedding of $\mathbf{A}^*$ in $\mathbf{B}^*$". Let us fix a embedding ψ of $\mathbf{A}$ in $\mathbf{B}$. Since $\mathbf{B}^*$ has distribution $\beta_*\nu$, we have $\mathbb{P}(C|\phi = \psi) = \nu(U_{A,A^*})$, therefore $\mathbb{P}(C) = \nu(U_{A,A^*})$. However, by the definition of QEP, we have $|\mathbb{P}(C) - \rho(\mathbf{A}^*)| \leq \varepsilon$, therefore we have the result. ☐

Let us now prove that $\mathcal{H}$ has the Quantitative Expansion Property. We will use McDiarmid's inequality from [M3].

Theorem 2.2.3. *Let $n \in \mathbb{N}$, $Z = (Z_1, \ldots, Z_n)$ be a family of iid random variable on $\{0,1\}$ and $f\colon \{0,1\}^n \to \mathbf{R}$ such that there is a family $(a_i)_{i\leq n} \in \mathbf{R}^n$ that verifies $|f(z) - f(z')| \leq a_i$ whenever $z(k) = z'(k)$ for $k \neq i$ and $z(i) = 1 - z'(i)$. Then, for all $L > 0$, we have*

$$\mathbb{P}(|f(Z) - \mathbb{E}(f(Z))| \geq L) \leq 2\exp\left(-\frac{2L}{\sum_{i=1}^n a_i^2}\right).$$

Proposition 2.2.4. *$\mathcal{H}$ has the Quantitative Expansion Property with respect to $\mathcal{H}^*$.*

Proof. Let us fix $\mathbf{A} \in \mathcal{H}$ with k vertices. The aim of the proof is to construct a $\mathbf{B} \in \mathcal{H}$ on $n \geq k$ vertices such that for any $\mathbf{A}^* \in \mathcal{H}^*(\mathbf{A})$ and $\mathbf{B}^* \in \mathcal{H}^*(\mathbf{B})$, we have:

$$\left| \frac{|\mathrm{Emb}(\mathbf{A}^*, \mathbf{B}^*)|}{|\mathrm{Emb}(\mathbf{A}, \mathbf{B})|} - \frac{1}{k!2^{k-1}} \right| \leq C\sqrt{\frac{\log(n)}{n}}$$

where C is a constant depending only on k. This is enough to prove the Quantitative Expansion Property, because for n large enough, $C\sqrt{\frac{\log(n)}{n}}$ is arbitrarily small.

We take $\mathcal{G}$ a uniform random graph on a vertex set V of size n, i.e for any two vertices of V, we put an edge between them with probability $1/2$, independently for each pair of vertices. We will prove that with non-zero probability, the reduct of $\mathcal{G}$ is as required. In particular, we will prove the property using switchings of $\mathcal{G}$ to describe the graphings of the reduct of $\mathcal{G}$.

Let $N(\mathbf{A}, \mathcal{G})$ denote the number of embeddings of $\mathbf{A}$ in $\mathrm{red}_{\mathbb{H}}(\mathcal{G})$. We remark that $E := \mathbb{E}[N(\mathbf{A}, \mathcal{G})] = 2^{k-1}n(n-1)...(n-k+1)2^{-\frac{k(k-1)}{2}}$. Indeed, fix an embedding φ of the domain of $\mathbf{A}$ in V and $\mathbf{B}$ a graphing of $\mathbf{A}$. The probability that φ is an embedding of $\mathbf{B}$ in $\mathcal{G}$ is $2^{-\frac{k(k-1)}{2}}$. There are $n(n-1)...(n-k+1)$ possible φ and 2^{k-1} possible $\mathbf{B}$. By summing over φ and $\mathbf{B}$ we have the result.

We define

$$f(\mathbf{A}, \mathcal{G}) = \frac{N(\mathbf{A}, \mathcal{G})}{E}$$

which is a function of $\binom{n}{2}$ iid variables, each indicating the absence or presence of an edge in $\mathcal{G}$. Adding or removing an edge to $\mathcal{G}$ changes $N(\mathbf{A}, \mathcal{G})$ by at most $k(k-1)(n(n-1)...(n-k+3))$. Indeed, this counts every possible embedding using this specific pair of vertices. Therefore f satisfies the conditions of Theorem 2.2.3 with $a_1 = c_1 n^{-2}$, where c_1 (as well as all the c_j we will define in the rest of the proof) is a positive constant depending only on k. We therefore have, for any $D > 0$,

$$\mathbb{P}(|f(\mathbf{A}, \mathcal{G}) - 1| \geq D) \leq 2\exp\left(-\frac{2D^2}{\binom{n}{2}c_1^2 n^{-4}}\right) \leq \exp\left(-c_2 D^2 n^2\right).$$

Let us now set $\mathbf{A}^* = (\mathbf{A}, <^{\mathbf{A}}, \mathrm{E}^{\mathbf{A}}) \in \mathcal{H}^*(\mathbf{A})$. We fix $<_V$, $U \subset V$. We define $\mathcal{G}^*$ as the random structure in $\mathcal{H}^*$ where the even hypergraph structure is the reduct of $\mathcal{G}$, the ordering is $<_V$ and the graphing is the switching of $\mathcal{G}$ by U. We define $N^*(\mathbf{A}^*, \mathcal{G}^*)$ to be the number of embeddings of $\mathbf{A}^*$ in $\mathcal{G}^*$. Remark that $\mathbb{E}(N^*(\mathbf{A}^*, \mathcal{G}^*)) = \frac{E}{k!2^{k-1}}$. We define

$$f^*(\mathbf{A}^*, \mathcal{G}^*) = \frac{N^*(\mathbf{A}^*, \mathcal{G}^*)}{E}.$$

Here, adding or removing an edge to $\mathcal{G}$ changes $N^*(\mathbf{A}^*, \mathcal{G}^*)$ by at most $\binom{k}{2}\binom{n-2}{k-2} \leq \binom{k}{2}n^{k-2}$. So as before, we have

$$\mathbb{P}\left(|f^*(\mathbf{A}^*, \mathcal{G}^*) - \frac{1}{k!2^{k-1}}| \geq D\right) \leq 2\exp\left(-\frac{2D^2}{\binom{n}{2}c_3 n^{-4}}\right) \leq \exp\left(-c_4 D^2 n^2\right).$$

Summing over all possible $<^{\mathbf{A}}$, $\mathrm{E}^{\mathbf{A}}$, $<_V$ and $U \subset V$, we have that except with probability $c_5 n! 2^{n-1}\exp(-D^2 c_6 n^2)$, we have simultaneously

$$|f(\mathbf{A}, \mathcal{G}) - 1| < D$$

and

$$|f^*(\mathbf{A}^*, \mathcal{G}^*) - \frac{1}{k!2^{k-1}}| < D$$

for all expansions of $\mathbf{A}$ and $\mathcal{G}$. Remark that the we only count 2^{n-1} possibilities for $U \subset V$, because U and $V \backslash U$ give the same switching.

We choose $D = c_7\sqrt{\frac{\log(n)}{n}}$ with c_7 chosen so that $c_5 n! 2^{n-1}\exp(-c_6 D^2 n^2) < 1$ for all $n \geq 1$. This implies that there exists a (deterministic) graph $\widetilde{\mathbf{B}}$ satisfying all the above inequalities simutaneously. If we denote by $\mathbf{B} \in \mathcal{H}$ its reduct, then we have

$$\left|\frac{|\mathrm{Emb}(\mathbf{A},\mathbf{B})|}{E} - 1\right| < D$$

and

$$\left|\frac{|\mathrm{Emb}(\mathbf{A}^*,\mathbf{B}^*)|}{E} - \frac{1}{k!2^{k-1}}\right| < D,$$

for all $\mathbf{A}^* \in \mathcal{H}^*(\mathbf{A})$ and $\mathbf{B}^* \in \mathcal{H}^*(\mathbf{B})$.

The required inequality then follows with $\rho(\mathbf{A}) = \frac{1}{k!2^{k-1}}$. Indeed, for all $\mathbf{A}^* \in \mathcal{H}^*(\mathbf{A})$ and $\mathbf{B}^* \in \mathcal{H}^*(\mathbf{B})$:

$$\begin{aligned}
\left|\frac{|\mathrm{Emb}(\mathbf{A}^*,\mathbf{B}^*)|}{|\mathrm{Emb}(\mathbf{A},\mathbf{B})|} - \frac{1}{k!2^{k-1}}\right| \leq & \left|\frac{|\mathrm{Emb}(\mathbf{A}^*,\mathbf{B}^*)|}{|\mathrm{Emb}(\mathbf{A},\mathbf{B})|} - \frac{|\mathrm{Emb}(\mathbf{A}^*,\mathbf{B}^*)|}{E}\right| \\
& + \left|\frac{|\mathrm{Emb}(\mathbf{A}^*,\mathbf{B}^*)|}{E} - \frac{1}{k!2^{k-1}}\right| \\
\leq & \left|\frac{|\mathrm{Emb}(\mathbf{A}^*,\mathbf{B}^*)|}{|\mathrm{Emb}(\mathbf{A},\mathbf{B})|} - \frac{|\mathrm{Emb}(\mathbf{A}^*,\mathbf{B}^*)|}{E}\right| + D \\
\leq & \frac{|\mathrm{Emb}(\mathbf{A}^*,\mathbf{B}^*)|}{|\mathrm{Emb}(\mathbf{A},\mathbf{B})|} \cdot \left|\frac{|\mathrm{Emb}(\mathbf{A},\mathbf{B})|}{E} - 1\right| + D \\
\leq & 2D \\
= & 2c_7\sqrt{\frac{\log(n)}{n}}.
\end{aligned}$$

□

CHAPTER 3

Stability by extension

This is joint work with Andy Zucker, it follows closely [JZ2].
In this chapter, the actions are on the right.

3.1 Introduction

Let G be a Polish group, and suppose $H \subseteq G$ is a closed, normal subgroup. Setting $K = G/H$, we have that K is also a Polish group, and the quotient map $\pi\colon G \to K$ is a continuous, open homomorphism. In this setting, we say that G is an *extension of K by H*. This is the same as saying that

$$1 \to H \to G \xrightarrow{\pi} K \to 1$$

is a short exact sequence. Examples of group extensions include group products, semidirect products but also more complicated ones as we illustrate in section 5.

Our aim in this chapter is to describe $\mathrm{M}(G)$ using information about $\mathrm{M}(H)$ and $\mathrm{M}(K)$. In particular, knowing that $\mathrm{M}(H)$ and $\mathrm{M}(K)$ have nice properties, we would like to show that $\mathrm{M}(G)$ also shares these properties. The first theorem shows that metrizability of the universal minimal flow is preserved under group extension and also elaborates on the interaction between $\mathrm{M}(G)$ and $\mathrm{M}(K)$. Notice that $\mathrm{M}(K)$ is a minimal G-flow under the action $x \cdot g := x \cdot \pi(g)$, so there is a G-map from $\mathrm{M}(G)$ to $\mathrm{M}(K)$. We also denote this G-map by π for reasons to be explained in Section 3.2.

Theorem 3.1.1. *Let $1 \to H \to G \xrightarrow{\pi} K \to 1$ be a short exact sequence of Polish groups. If $\mathrm{M}(H)$ and $\mathrm{M}(K)$ are metrizable, then so is $\mathrm{M}(G)$. Furthermore, letting $\pi\colon \mathrm{M}(G) \to \mathrm{M}(K)$ be the canonical map, we have that $\pi^{-1}(\{y\})$ is a minimal H-flow for every $y \in \mathrm{M}(K)$.*

Using this description of $\mathrm{M}(G)$, we are also able to prove that when the universal minimal flows are metrizable, then unique ergodicity is stable under group extension.

Theorem 3.1.2. *With H, G, and K as in Theorem 3.1.1, then if both H and K are uniquely ergodic, then G is also uniquely ergodic.*

We briefly discuss the organization of the chapter. Section 3.2 gives background on the Samuel compactification of a topological group and the universal minimal flow. Section 3.3 proves the main technical lemma regarding the *almost periodic points* of a G-flow. Section 3.4 proves the two main theorems. Section 3.5 provides examples of Polish group extensions. Section 3.6 provides a more combinatorial proof of part of Theorem 3.1.1 by using the connections between topological dynamics and Ramsey theory. Finally, Section 3.7 collects some open questions inspired by our work.

Aknowledgements: The authors thank Lionel Nguyen Van Thé and Todor Tsankov for their advice and comments during the writing of this paper.

3.2 Background

For this section, let G be any topological group. A *G-ambit* is a pair (X, x_0) with X a G-flow and $x_0 \in X$ a distinguished point with dense orbit. If (Y, y_0) is another ambit, we say that $\phi\colon X \to Y$ is a *map of ambits* if ϕ is a G-map and $\phi(x_0) = y_0$. Notice that there is at most one map of ambits from (X, x_0) to (Y, y_0). One can in fact construct the *greatest ambit*, denoted $(\mathrm{S}(G), 1_G)$, which is an ambit admitting a map of ambits onto any other ambit and is unique up to isomorphism. The orbit $1_G \cdot G$ is homeomorphic to (and identified with) G, and $\mathrm{S}(G)$ is often called the *Samuel compactification* of G. As an example, when G is a discrete group, then $\mathrm{S}(G) \cong \beta G$, the space of ultrafilters on G. In general, $\mathrm{S}(G)$ has the following universal property: if X is a compact space and $f\colon G \to X$ is left-uniformly continuous, then f can be continuously extended to $\mathrm{S}(G)$. For two different constructions of $\mathrm{S}(G)$, see [KPT] or [Z3].

The universal property of $\mathrm{S}(G)$ allows us to give $\mathrm{S}(G)$ the structure of a *compact left-topological semigroup*, a compact space S endowed with a semigroup structure so that for each $s \in S$, the map $\lambda_s\colon S \to S$ given by $\lambda_s(t) = st$ is continuous. Fix $p \in S(G)$. Then $(\overline{p \cdot G}, p)$ is an ambit, so there is a unique map of ambits $\lambda_p\colon (\mathrm{S}(G), 1_G) \to (\overline{p \cdot G}, p)$. Now given $p, q \in \mathrm{S}(G)$, we declare that $p \cdot q = \lambda_p(q)$. Associativity follows because $\lambda_p \circ \lambda_q$ and λ_{pq} are both G-maps sending 1_G to pq, hence they must be equal.

More generally, let X be a G-flow. Given $x \in X$, then $(\overline{x \cdot G}, x)$ is an ambit, so there is a unique map of ambits $\lambda_x\colon \mathrm{S}(G) \to X$. Given $p \in \mathrm{S}(G)$, we often write $x \cdot p := \lambda_x(p)$. Notice that if $p, q \in \mathrm{S}(G)$, then $x(pq) = (xp)q$, so the semigroup $\mathrm{S}(G)$ acts on X in a manner which extends the G-action.

For a more detailed account of the theory of compact left-topological semigroups, Chapters 1 and 2 of [HS1] are a great reference (but note the left-right switch between that reference and the presentation here). We will need the following facts, all of which can be found there. Fix a compact left-topological semigroup S.

1. Every compact left-topological semigroup S contains an *idempotent*, an element $u \in S$ with $u \cdot u = u$.

2. A *right ideal* is any $I \subseteq S$ for which $I \cdot s \subseteq I$ for every $s \in S$. Notice that if $p \in I$, then $p \cdot S \subseteq I$ is a closed right ideal. It follows that every right ideal contains a minimal right ideal which must be closed.

3. If $I \subseteq S$ is a minimal right ideal, then I is a compact left-topological semigroup in its own right, so contains an idempotent. If $u \in I$ is an idempotent, then $uI = I$, so $up = p$ for every $p \in I$.

4. If $I \subseteq S$ is a minimal right ideal and $p \in I$, then $S \cdot p$ is a minimal *left* ideal, and $pS \cap Sp = Ip$, which is a group. If $u \in Ip$ is the identity of this group, then $Ip = Iu$. So for every $p \in I$, there is an idempotent $u \in I$ with $p \in Iu$.

We now apply this to $\mathrm{S}(G)$. First note that the minimal right ideals of $\mathrm{S}(G)$ are exactly the minimal subflows of $\mathrm{S}(G)$. Notice also that every minimal subflow of $\mathrm{S}(G)$ is universal, simply by the universal property of $\mathrm{S}(G)$. We argue that $\mathrm{M}(G)$ is unique up to isomorphism, a classical theorem of Ellis [E2]. Fix $M \subseteq \mathrm{S}(G)$ a minimal right ideal, and let $u \in M$ be an idempotent. Suppose $\phi\colon M \to M$ is a G-map. Then by Fact 3, we have $\phi(p) = \phi(up) = \phi(u)p$ for any $p \in M$, hence $\phi = \lambda_{\phi(u)}|_M$. By Fact 4, we have $\phi(u) \in Mv$ for some idempotent v. Then since Mv is a group with identity v, we can find $q \in M$ with $q\phi(u) = v$. Notice that $\lambda_v|_M$ is the identity on M (since $M = vM$). Since $\lambda_v|_M = \lambda_q \circ \lambda_{\phi(u)}|_M$, the map $\lambda_{\phi(u)}|_M = \phi$ must be a bijection, hence a G-flow isomorphism. If N is another minimal flow which is universal, then let $\psi\colon M \to N$ and $\theta\colon N \to M$ be G-maps. If $M \not\cong N$, then $\theta \circ \psi$ is not injective, contradicting that every G-map from M to itself is an isomorphism.

Furthermore, suppose X is a minimal G-flow, and suppose ϕ and ψ are two G-maps from M to X. Let $u \in M$ be an idempotent, and consider $\psi^{-1}(\{\phi(u)\}) \subseteq M$. If $p \in \psi^{-1}(\{\phi(u)\})$, then $\psi(pu) = \psi(p)u = \phi(u)u = \phi(uu) = \phi(u)$. It follows that $\psi \circ \lambda_p = \phi$, i.e. there is only one G-map from M to X up to isomorphism.

Now suppose K is another topological group and that $\pi\colon G \to K$ is a continuous surjective homomorphism. We note that every K-flow is also a G-flow, where if X is a K-flow, $x \in X$, and $g \in G$, we set $x \cdot g = x \cdot \pi(g)$. The map π continuously extends to a map from $\mathrm{S}(G)$ to $\mathrm{S}(K)$, which we also denote by π. If $M \subseteq \mathrm{S}(G)$ is a minimal subflow, then $\pi[M] \subseteq \mathrm{S}(K)$ is also minimal and is isomorphic to $\mathrm{M}(K)$.

3.3 Almost periodic points

This section proves the following key propositions which will be used in the proof of Theorems 3.1.1 and 3.1.2. Throughout this section, we consider a Polish group H, which will be the same H that appears in the main theorems. We fix on H a compatible left-invariant metric d of diameter one, and for $c > 0$, we set $U_c := \{g \in H : d(1_H, g) < c\}$.

Given an H-flow X, the *almost periodic* points of X, denoted $\mathrm{AP}(X)$, are those points in X belonging to minimal subflows.

Proposition 3.3.1. *Let H be a Polish group, and suppose that $\mathrm{M}(H)$ is metrizable. Then for any H-flow X, the set $\mathrm{AP}(X) \subseteq X$ is closed.*

The assumption that $\mathrm{M}(H)$ is metrizable is essential. Hindman and Strauss in [HS2] show that when $H = \mathbb{Z}$ and $X = \beta\mathbb{Z}$, then $\mathrm{AP}(X) \subseteq X$ is not even

Borel. In a work in preparation, Bartošová and Zucker generalize this, showing that for any Polish group H with $\mathrm{M}(H)$ non-metrizable and $X = S(H)$, then $\mathrm{AP}(X) \subseteq X$ is not Borel.

On $\mathrm{AP}(X)$, the relation given by $E(x, y)$ iff x and y belong to the same minimal subflow of X is an equivalence relation, and one can ask about the complexity of this equivalence relation. It turns out that in the setting of Theorem 3.3.1, this equivalence relation is as nice as possible.

Proposition 3.3.2. *In the setting of Proposition 3.3.1, the equivalence relation* $E \subseteq \mathrm{AP}(X) \times \mathrm{AP}(X)$ *is closed.*

Combining the key results from [MNT] and [BMT], we have the following.

Fact 3.3.3. *Whenever* $\mathrm{M}(H)$ *is metrizable, then there is an extremely amenable, co-precompact subgroup* $H^* \subseteq H$ *so that* $\mathrm{M}(H) \cong \widehat{H^*\backslash H}$*, the left completion of the right coset space.*

In particular, $\mathrm{M}(H)$ comes equipped with a canonical compatible metric ∂ inherited from the metric d on H. The key property we need about this metric is the following.

Lemma 3.3.4. *Let H be a Polish group, and assume* $\mathrm{M}(H) \cong \widehat{H^*\backslash H}$ *is metrizable with the compatible metric ∂ inherited from d. Then whenever $\partial(p, q) < c$ and $A \ni p$ is open, we have* $q \in \overline{AU_c}$.

Proof. Fix sequences $p_n, q_n \in H$ with $H^*p_n \to p$ and $H^*q_n \to q$. We may assume for every $n < \omega$ that $d(H^*p_n, H^*q_n) < c$. By modifying q_n if necessary, we may assume $p_n^{-1}q_n \in U_c$. Now if $A \ni p$ is open, then eventually $H^*p_n \in A$. Then $H^*q_n \in AU_c$, implying that $q \in \overline{AU_c}$ as desired. □

We now assume $\mathrm{M}(H)$ metrizable with a compatible metric ∂ as in Lemma 3.3.4, and we fix an H-flow X. Consider some collection $\{X_i : i \in I\}$ of minimal subflows of X; we will treat I as a directed partial order. For each $i \in I$, let $\phi_i \colon \mathrm{M}(H) \to X_i$ be an H-map. The key lemma regards the right action of $\mathrm{S}(H)$ on X. In general, this action is not continuous, but the lemma states that in this setting, we recover some fragments of continuity.

Lemma 3.3.5. *Suppose we have* $p, q \in \mathrm{M}(H)$ *with* $\phi_i(p) \to x$ *and* $\phi_i(q) \to y$*. Suppose* $r \in \mathrm{S}(H)$ *with* $pr = q$*. Then* $xr = y$.

Proof. Fix an open $B \ni y$, and fix a net $(g_j)_{j \in J}$ from H with $g_j \to r$. We want to show that eventually $xg_j \in B$. Find some open $C \ni y$ and $\epsilon > 0$ with $\overline{CU_\epsilon} \subseteq B$. Eventually $\partial(pg_j, q) < \epsilon$; fix such a g_j. Eventually $\phi_i(q) \in C$, so by Lemma 3.3.4 for such $i \in I$ we have $pg_j \in \overline{\phi_i^{-1}(C)U_\epsilon} \subseteq \phi_i^{-1}(\overline{CU_\epsilon})$. So $\phi_i(pg_j) = \phi_i(p)g_j \in \overline{CU_\epsilon}$. As this is true for all large enough $i \in I$, we have $xg_j \in \overline{CU_\epsilon} \subseteq B$ as desired. □

The other lemma we will need allows us to express points in $\overline{\mathrm{AP}(X)}$ as limits of certain nice nets. The proof is almost identical to that of Lemma 3.3.5.

Lemma 3.3.6. *Suppose ϕ_i are as above, and suppose we have $p_i \in \mathrm{M}(H)$ with $\phi_i(p_i) \to x \in X$. If $p_i \to p \in \mathrm{M}(H)$, then $\phi_i(p) \to x$.*

Proof. Fix an open $B \ni x$; we want to show that eventually $\phi_i(p) \in B$. Find some open $C \ni x$ and $\epsilon > 0$ so that $\overline{CU_\epsilon} \subseteq B$. Eventually we have $\phi_i(p_i) \in C$ and $\partial(p, p_i) < \epsilon$. For such $i \in I$, since $\phi_i^{-1}(C) \ni p_i$, we have by Lemma 3.3.4 that $p \in \overline{\phi_i^{-1}(C)U_\epsilon} = \overline{\phi_i^{-1}(CU_\epsilon)} \subseteq \phi^{-1}(\overline{CU_\epsilon})$. It follows that $\phi_i(p) \in B$ as desired. $\square$

We can now easily complete the proof of both key propositions. First suppose $x_i \in \mathrm{AP}(X)$ with $x_i \to x$. Each x_i belongs to some minimal flow X_i, so fix H-maps $\phi_i \colon \mathrm{M}(H) \to X_i$. Also fix $p_i \in \mathrm{M}(H)$ with $\phi_i(p_i) = x_i$. By passing to a subnet, we may assume $p_i \to p$, so by Lemma 3.3.6, we have $\phi_i(p) \to x$. Now fix a minimal subflow $M \subseteq \mathrm{S}(H)$, and consider the H-flow isomorphism $\lambda_p|_M \colon M \to \mathrm{M}(H)$. If $u \in M$ is such that $pu = p$, then by Lemma 3.3.5, we have $xu = x$, i.e. that $x \in \lambda_x[M]$, showing that $x \in \mathrm{AP}(X)$ as desired.

For the second proposition, suppose $(x_i, y_i) \in E$ with $x_i \to x$ and $y_i \to y$. Much as above, we may assume that there are $p, q \in \mathrm{M}(H)$ with $\phi_i(p) \to x$ and $\phi_i(q) \to y$. Now suppose $r \in \mathrm{S}(H)$ with $pr = q$. By Lemma 3.3.5, we have $xr = y$. It follows that $(x, y) \in E$ as desired.

3.4 Abstract proof of Theorems 3.1.1 and 3.1.2

This section applies the key propositions from Section 3.3 to prove the two main theorems from the introduction. Fix a short exact sequence $1 \to H \to G \to K \to 1$ of Polish groups, and let d be a compatible left-invariant metric on G with diameter 1. Then d induces compatible left-invariant metrics on H and K, which we also denote by d. Given $c > 0$, we set $U_c = \{g \in G : d(1_G, g) < c\}$. Then $U_c \cap H$ is the ball of radius c around $1_H = 1_G$ in H, and HU_c is the ball of radius c around $1_K = H$ in K.

We first tackle Theorem 3.1.1. The assumption that $\mathrm{M}(K)$ is metrizable is only used at the very end, but the assumption that $\mathrm{M}(H)$ is metrizable is used throughout the proof. Indeed, the proof proceeds by viewing $\mathrm{M}(G)$ as an H-flow. We write $\mathrm{AP}_H(\mathrm{M}(G))$ for those points in $\mathrm{M}(G)$ belonging to minimal H-subflows.

Lemma 3.4.1. *The set $\mathrm{AP}_H(\mathrm{M}(G)) \subseteq \mathrm{M}(G)$ is G-invariant, hence dense.*

Proof. Suppose $X \subseteq \mathrm{M}(G)$ is a minimal H-subflow. Fix $g \in G$. Then $XgH = XHg = Xg$, so Xg is an H-flow. Now suppose $y \in Xg$. Then $yHg^{-1} = yg^{-1}H \subseteq X$ is dense, so also $yH \subseteq Xg$ is dense, showing that Xg is also a minimal H-subflow. $\square$

Proof of Theorem 3.1.1. By Proposition 3.3.1 and Lemma 3.4.1, we must have $\mathrm{AP}_H(\mathrm{M}(G)) = \mathrm{M}(G)$, i.e. every point in $\mathrm{M}(G)$ belongs to a minimal H-subflow. Furthermore, by Proposition 3.3.2, the relation E defined by $E(x, y)$ iff $x \in \overline{y \cdot H}$ is a closed equivalence relation on $\mathrm{M}(G)$. Then $Y = \mathrm{M}(G)/E$ is a compact Hausdorff space and since the projection of the action of G is

H-invariant, Y is a K-flow. This flow is minimal by minimality of the action of G on $\mathrm{M}(G)$, hence it is metrizable and has cardinality $\mathfrak{c}$. Each equivalence class is a minimal H-flow, hence is metrizable and has cardinality $\mathfrak{c}$. This means that $\mathrm{M}(G)$ has cardinality at most $\mathfrak{c}$ and by [Z3] Proposition 2.7.5, if $\mathrm{M}(G)$ were non-metrizable, it would have cardinality $2^{\mathfrak{c}}$.

Furthermore, note that for every $y \in \mathrm{M}(K)$, the fiber $\pi^{-1}(\{y\})$ is an H-flow, giving us a map $\psi \colon Y \to \mathrm{M}(K)$. As Y is minimal and $\mathrm{M}(K)$ is the universal minimal flow, we must have ψ an isomorphism, i.e. each fiber $\pi^{-1}(\{y\})$ is a minimal H-flow. □

We now turn towards the proof of Theorem 3.1.2, so assume $\mathrm{M}(H)$ and $\mathrm{M}(K)$ are metrizable and that both H and K are uniquely ergodic. The main idea of the proof is to apply the following measure disintegration theorem (see [F2] Theorem 5.8 and Proposition 5.9).

Theorem 3.4.2. *Let X, Y be standard Borel spaces and $\phi \colon X \to Y$ a Borel map. Let $\mu \in \mathrm{P}(X)$ and $\nu = \phi_* \mu$, then there is a Borel map $y \mapsto \mu_y$ from Y to $P(X)$ such that:*

i) $\mu_y(\phi^{-1}(\{y\}) = 1$

ii) $\mu = \int \mu_y \mathrm{d}\nu(y)$.

Moreover, if there is another such map $y \mapsto \mu'_y$, then for ν-almost all y, $\mu_y = \mu'_y$.

We apply the theorem with $X = \mathrm{M}(G)$, $Y = \mathrm{M}(G)/E = \mathrm{M}(K)$, and $\phi = \pi$. First note that G is amenable (see the proof in Section 3.7), so let us take μ any G-invariant measure. Then $\nu = \phi_* \mu$ is K-invariant, hence it is the unique such measure. The following lemma gives us the uniqueness of the disintegration, hence the unique ergodicity of G and the proof of Theorem 3.1.2.

Lemma 3.4.3. *For ν-almost all $y \in Y$, μ_y is H-invariant.*

Proof. We remark that by the uniqueness of the decomposition, it is easy to establish that for any countable set $(h_n)_{n \in \mathbb{N}}$ of elements of H, ν-almost surely μ_y is $(h_n)_{n \in \mathbb{N}}$-invariant for all $n \in \mathbb{N}$. Since H is Polish, we can assume that $(h_n)_{n \in \mathbb{N}}$ is dense in H. Since the set of $h \in H$ such that $h \cdot \mu_y = \mu_y$ is closed, it follows that for any $y \in Y$ with μ_y $(h_n)_{n \in \mathbb{N}}$-invariant, we in fact have that μ_y is H-invariant. □

3.5 Examples

In this section, we give several examples of short exact sequences appearing in the realm of non-archimedian Polish groups. The main application of Theorems 3.1.1 and 3.1.2 occurs in Subsection 3.5.1, where we discuss wreath products. Subsection 3.5.2 describes instances of more general Polish group extensions, where the main theorems don't apply as clearly.

3.5.1 Wreath products

The simplest (non trivial) setting in which short exact sequences appear is the one where we have H a Polish group, K is a Polish group that acts on a countable set X and $G = H^X \rtimes K$. The product is defined as:

$$((h_a)_{a\in X}, \sigma) \cdot ((g_a)_{a\in X}, \tau) = ((h_a g_{\sigma(a)})_{a\in X}, \sigma\tau)$$

This is a short exact sequence where H^X is the normal subgroup and K is the quotient.

We apply the main theorems to prove the following.

Theorem 3.5.1. *Letting $G = H^X \rtimes K$, if $\mathrm{M}(H)$ and $\mathrm{M}(K)$ are metrizable, then so is $\mathrm{M}(G)$. Under those hypotheses, if H and K are uniquely ergodic, then so is G.*

The proof relies on the following lemma.

Lemma 3.5.2. *Let $(G_i)_{i\in\mathbb{N}}$ be a family of groups such that $\mathrm{M}(G_i)$ is metrizable for all $i \in \mathbb{N}$, and set $G = \prod_{i\in\mathbb{N}} G_i$, then $\mathrm{M}(G)$ is metrizable. Moreover, if G_i is uniquely ergodic for all $i \in \mathbb{N}$, then so is G.*

Proof. Using Fact 3.3.3, we know that there exists a sequence $(G_i^*)_{i\in\mathbb{N}}$ such that G_i^* is an extremely amenable, closed, co-precompact subgroup of G_i.

Let us consider $G^* = \prod_{i\in\mathbb{N}} G_i^*$ as a subgroup of G. It is a closed subgroup and is extremely amenable, as the property of being extremely amenable is closed under arbitrary (not just countable) products.

The observation $\widehat{G^*\backslash G} = \prod_{i\in\mathbb{N}} \widehat{G_i^*\backslash G_i}$ gives the co-precompactness of G^*. Hence $\mathrm{M}(G) = \widehat{G^*\backslash G}$ and is metrizable.

This also implies unique ergodicity of G, for let μ_i be the unique G_i-invariant measure on $\mathrm{M}(G_i)$ and μ any G-invariant measure on $\mathrm{M}(G)$. The pushfoward of μ on $\prod_{i<n} \mathrm{M}(G_i)$ has to be equal to $\mu_0 \otimes \cdots \otimes \mu_{n-1}$ for all $n \in \mathbb{N}$, hence μ is uniquely determined on the basic open set of the topology of $\mathrm{M}(G)$ and is therefore uniquely determined. □

Proof of Theorem 3.5.1. By Theorems 3.1.1 and 3.1.2, it is enough to show that if $\mathrm{M}(H)$ is metrizable (and uniquely ergodic), then so is $\mathrm{M}(H^X)$. Lemma 3.5.2 gives us exactly that. □

Note that this result was already proven by Pawliuk and Sokic ([PS] Theorem 2.1) and Sokic ([M1] Proposition 5.2) in the case where H and K are automorphism groups of Fraïssé limits.

3.5.2 Beyond semi-direct products

We now consider group extensions which are not semidirect products. This subsection does not contain any particular applications of the main theorem, but is included to give some more understanding of how diverse short exact sequences of Polish groups can be. The reason that applying the main theorems is difficult here is that often, the closed subgroup H is equally difficult to work with as G itself.

To show that certain group extensions are not semi-direct products, we briefly discuss some of the properties of those group extensions that are. Let H and K be topological groups, and suppose we are given a homomorphism $\phi\colon K \to \mathrm{Aut}(H)$. We will write $\phi(k)$ as ϕ_k to simplify notation. Then we can endow $H \times K$ with a group operation, where we define $(h_0, k_0) \cdot (h_1, k_1) = (h_0\phi_{k_0}(h_1), k_0k_1)$. Now suppose ϕ has the property that whenever $k_i \to k \in K$ and $h_i \to h \in H$, we have $\phi_{k_i}(h_i) \to \phi_k(h) \in H$. For example, when H is locally compact, this property says that ϕ is continuous when $\mathrm{Aut}(H)$ is given the compact-open topology. In this case, $H \times K$ endowed with the above operation is a topological group, and we denote this by $H \rtimes^{\phi} K$, or $H \rtimes K$ if ϕ is understood. Setting $G = H \rtimes K$, we identify H with the closed normal subgroup $\{(h, 1_K) : h \in H\}$, and the quotient G/H is isomorphic to K, showing that G is an extension of K by H.

If $1 \to H \to G \xrightarrow{\pi} K \to 1$ is a short exact sequence of topological groups, we say that the sequence *splits continuously* if there is a continuous homomorphism $\alpha\colon K \to G$ so that $\pi \circ \alpha = \mathrm{id}_K$, the identity map on K. Such an α will always have closed image. When $G = H \rtimes K$, one can define $\alpha(k) = (1_H, k)$. Conversely, if $\alpha\colon K \to G$ is a continuous homomorphism with closed image and $\pi \circ \alpha = \mathrm{id}_K$, then we obtain a homomorphism $\phi\colon K \to \mathrm{Aut}(H)$ given by $\phi_k(h) = \alpha(k)h\alpha(k^{-1})$. Then ϕ satisfies the required continuity property, and we have $G \cong H \rtimes^{\phi} K$.

Ordered homogeneous metric space with distances 1, 3 and 4.

We first consider the countable homogeneous metric space with distances 1, 3 and 4, which we denote by $\mathbb{F}$. This is the Fraïssé limit of those metric spaces with distances belonging to the set $\{1, 3, 4\}$. In $\mathbb{F}$, there are infinitely many infinite equivalence classes of points at distance 1 and such that the distance between two non-equivalent points is 3 or 4 at random.

We now consider an extension $\mathbb{F}^*$ of this structure where on each equivalence class we generically put a dense linear ordering, and we leave points between different classes unordered. We set $G = \mathrm{Aut}(\mathbb{F}^*)$.

Letting H be the subgroup that stabilizes every class set-wise, then $H \subseteq G$ is a closed normal subgroup. Moreover, it is easy to prove via a back and forth that the quotient $H\backslash G$ is isomorphic to S_∞.

This extension cannot be a semi-direct product. To prove this we consider the element σ of S_∞ that swaps 0 and 1, leaving all other points fixed. We have $\sigma^2 = \mathrm{id}_{\mathbb{N}}$. Suppose now that G is indeed a semidirect product; letting $\alpha\colon S_\infty \to G$ be a continuous homomorphism with $\pi \circ \alpha = \mathrm{id}_{S_\infty}$, then the element $g^* = \alpha(\sigma)$ has order 2 in G and permutes two equivalence classes, say A and B. If we look at the action of g^* on a class C which g^* fixes set-wise, then g^* defines an automorphism of $(\mathbb{Q}, <)$ of order 2, thus it acts trivially on C. Now given $x \in A$, let $y = g^*(x) \in B$. Then using the homogeneity of $\mathbb{F}^*$, we can find $z \in C$ with $d(x, z) = 3$ and $d(y, z) = 4$. Therefore we must have $g^*(z) \neq z$, a contradiction.

The switching group

We now consider the structure $\mathbb{F}$ formed by first considering the Rado graph $\mathbb{H} = (V, E)$. The structure $\mathbb{F}$ has domain V and the language only has one

4-ary relation symbol R with the following condition:

$$R^{\mathbb{F}}(x,y,w,z) \Leftrightarrow \Big((E^{\mathbb{H}}(x,y) \wedge E^{\mathbb{H}}(w,z)) \vee (\neg E^{\mathbb{H}}(x,y) \wedge \neg E^{\mathbb{H}}(w,z) \Big)$$

where x, y, w, z are vertices. We obviously have $\mathrm{Aut}(\mathbb{H}) \lhd \mathrm{Aut}(\mathbb{F})$. The quotient is $\mathbb{Z}/2\mathbb{Z}$.

Again, this is not a semi-direct product, otherwise we would have f an involution of the vertices such that $E(x,y) \Leftrightarrow \neg E(f(x), f(y))$, which is impossible because we would have $E(x, f(x))$, $\neg E((f(x), f^2(x))$, and $f^2(x) = x$.

Partitioned $(\mathbb{Q}, <)$

We partition $\mathbb{Q}$ in dense codense classes that we name $(E_i)_{i\in\mathbb{N}}$. We define an equivalence relation E on $\mathbb{Q}$:

$$E(x,y) \Leftrightarrow \exists i \in \mathbb{N} \ : \ (x,y) \in E_i.$$

We let G be the subgroup of $\mathrm{Aut}(\mathbb{Q}, <)$ fixing the equivalence relation E, and we let H be the subgroup of G that fixes each E_i setwise; it is normal in G.

Again, a torsion argument allows us to prove this is not a semi direct product.

3.6 Combinatorial proof of Theorem 3.1.1

This section provides a combinatorial proof of a weakening of Theorem 3.1.1; this proof does not show that each fiber is a minimal H-flow. The advantage of this proof is that it is "quantitative" in a sense that will be made precise. We will first reprove the theorem in the case that G is non-Archimedean, and then discuss the general case.

3.6.1 The non-Archimedean case

We first assume that G, hence also H and K, are non-Archimedean. So fix $\{U_n : n < \omega\}$ a base at 1_G of clopen subgroups. Then $\{H \cap U_n : n < \omega\}$ and $\{\pi[U_n] : n < \omega\}$ are bases of clopen subgroups at the identity in H and K, respectively. For instance, if $G = \mathrm{Aut}(\mathbf{K})$ for some Fraïssé structure $\mathbf{K} = \mathrm{Flim}(\mathcal{K})$, and we write $\mathbf{K} = \bigcup_n \mathbf{A}_n$ as an increasing union of finite substructures, then we can let U_n be the pointwise stabilizer of $\mathbf{A}_n$. We will need the following definition, which we translate from the Fraïssé setting to the group setting.

We consider the left coset space $G_n := G/U_n$, which is countable. When $U_n = \mathrm{Stab}(\mathbf{A}_n)$, then G_n can be identified with the set $\mathrm{Emb}(\mathbf{A}_n, \mathbf{K})$. The group G acts on G_n on the left in the natural way. If X is a compact space, then X^{G_n} becomes a right G-flow, where for $\gamma \in X^{G_n}$ and $g_0 \in G$, we define $\gamma \cdot g_0$ via $\gamma \cdot g_0(g_1 U_n) := \gamma(g_0 g_1 U_n)$.

Definition 3.6.1. *Fix $n, m < \omega$. We say that the clopen subgroup $U_n \subseteq G$ has* Ramsey degree *$m < \omega$ if the following both hold.*

1. *For any $r < \omega$ and any coloring $\gamma \colon G_n \to r$, there is some $\delta \in \overline{\gamma \cdot G}$ and some $F \subseteq r$ with $\delta \in F^{G_n}$ and $|F| \le m$. Equivalently, for any γ as above, there is $p \in \mathrm{S}(G)$ with $|\mathrm{Im}(\gamma \cdot p)| \le m$.*

2. *There is a surjective coloring $\gamma : G_n \to m$ so that $\overline{\gamma \cdot G}$ is a minimal G-flow. We often call such colorings* minimal, *or* G-minimal *to emphasize the group.*

When $U_n = \mathrm{Stab}(\mathbf{A}_n)$, then Definition 3.6.1 coincides with the Ramsey degree (for embeddings) of $\mathbf{A}_n \in \mathcal{K}$. We then have the following theorem.

Theorem 3.6.2 ([Z2]). $\mathrm{M}(G)$ *is metrizable iff for every $n < \omega$, the subgroup U_n has finite Ramsey degree.*

By Theorem 3.6.2 and our assumption that $\mathrm{M}(H)$ and $\mathrm{M}(K)$ are metrizable, we know that for every $n < \omega$, the subgroups $H \cap U_n \subseteq H$ and $\pi[U_n] \subseteq K$ have finite Ramsey degrees m_H and m_K, respectively. We will use these to bound the Ramsey degree of U_n. The following proposition will prove Theorem 3.1.1 in the non-Archimedean case.

Proposition 3.6.3. *With m_H and m_K as above, the Ramsey degree m_G of U_n satisfies $m_G \leq m_H \cdot m_K$.*

Proof. Write $H_n := H/(H \cap U_n)$ and $K_n := K/\pi[U_n]$. Then we have an inclusion map $H_n \hookrightarrow G_n$ as well as a projection map $\pi_n : G_n \to K_n$, both of which respect the various left actions. Furthermore, the equivalence relation E_n induced by π_n is exactly the orbit equivalence relation of the left H-action on G_n. From now on, we will view H_n as a subset of G_n.

Let $\gamma\colon G_n \to r$ be a coloring. Find $\{g_k : k < \omega\} \subseteq G$ so that $\{g_k \cdot H_n : k < \omega\}$ lists the E_n-classes in G_n. We now inductively define a sequence of colorings $\{\gamma_k : k < \omega\} \subseteq r^{G_n}$. Set $\gamma = \gamma_0$. If γ_k is defined for some $k < \omega$, consider $\gamma_k \cdot g_k|_{H_n}$. We can find $p_k \in \mathrm{S}(H) \subseteq \mathrm{S}(G)$ so that $|\gamma_k \cdot g_k \cdot p_k[H_n]| \leq m_H$. Then set $\gamma_{k+1} = \gamma_k \cdot g_k \cdot p_k \cdot g_k^{-1}$. Note that $|\gamma_{k+1}[g_k H_n]| \leq m_H$. Also notice that $g_k \cdot p_k \cdot g_k^{-1} \in S(H)$, implying that for any $i \leq k$, we have $\gamma_{k+1}[g_i H_n] \subseteq \gamma_k[g_i H_n]$. Let $\delta\colon G_n \to r$ be any cluster point of $\{\gamma_k : k < \omega\}$. Then for each $k < \omega$, $\delta[g_k H_n]$ is a subset of r of size at most m_H. This allows us to produce a finite coloring $\eta\colon K_n \to [m_H]^{\leq r}$, and we can find $q \in S(G)$ with $|\eta \cdot \pi(q)[K_n]| \leq m_K$. It follows that $\delta \cdot q[G_n] \leq m_H \cdot m_K$ as desired. $\square$

3.6.2 The general case

In the general case, we will need the following analogue of Theorem 3.6.2. If G is a Polish group equipped with a compatible left-invariant metric of diameter 1 and X is a compact metric space, then $\mathrm{Lip}_G(X)$ will denote the space of 1-Lipschitz functions from G to X. When endowed with the topology of pointwise convergence, $\mathrm{Lip}_G(X)$ becomes a compact space. We have a right action of G on $\mathrm{Lip}_G(X)$; if $f \in \mathrm{Lip}_G(X)$ and $g \in G$, then $f \cdot g$ is given by $f \cdot g(h) = f(gh)$. This action is continuous, turning $\mathrm{Lip}_G(X)$ into a G-flow.

Theorem 3.6.4. $\mathrm{M}(G)$ *is metrizable iff for every $c > 0$, there is $k < \omega$ so that if X is a compact metric space and $f \in \mathrm{Lip}_G(X)$, there is $\phi \in \overline{f \cdot G}$ so that $\overline{\phi[G]} \subseteq X$ can be covered by k-many balls of radius c.*

Remark. *One should think of $k < \omega$ above as the "Ramsey degree" of the constant $c > 0$, and we will use this terminology.*

Proof. Assume first that $M(G)$ is metrizable. Fact 3.3.3 gives $\mathrm{M}(G)$ a canonical compatible metric ∂ which satisfies the following property: for any G-map $\psi\colon \mathrm{M}(G) \to \mathrm{S}(G)$, for any compact metric space X, and for any 1-Lipschitz function $f\colon G \to X$, then upon extending f to $\mathrm{S}(G)$, the function $f \circ \psi$ is 1-Lipschitz for ∂.

For any function $\phi \in \overline{f \cdot G}$ with $\overline{\phi \cdot G}$ minimal, there is a minimal subflow $M \subseteq \mathrm{S}(G)$ and $p \in M$ with $\phi = f \cdot p$. It follows that $\overline{\phi[G]} = f[M]$. Given $c > 0$, pick $k < \omega$ so that $\mathrm{M}(G)$ can be covered by k-many ∂-balls of radius c. Then if $\psi\colon \mathrm{M}(G) \to M$ is an isomorphism, $f \circ \psi$ is 1-Lipschitz and the result follows.

For the other direction, suppose $\mathrm{M}(G)$ is not metrizable. Fixing minimal $M \subseteq \mathrm{S}(G)$ and mimicking the proof of Lemma 2.5 in [BMT], we can find a constant $c > 0$, $\{p_n : n < \omega\} \subseteq M$, and functions $\{f_n : n < \omega\} \subseteq \mathrm{Lip}_G([0,1])$. with $f_n(p_n) = c$ and $f_m(p_n) = 0$ whenever $m \neq n$. Then for any $N < \omega$, we can form $f\colon G \to [0,1]^N$ given by $f = (f_n)_{n<N}$. If $p \in M$, then $f \cdot p \in \mathrm{Lip}_G([0,1]^N)$ has minimal orbit closure, but covering $\overline{f \cdot p[G]}$ requires at least N-many balls of radius c, and $N < \omega$ is arbitrary. $\square$

We now return to the setting where $1 \to H \to G \to K \to 1$ is a short exact sequence of Polish groups, and we assume that $\mathrm{M}(H)$ and $\mathrm{M}(K)$ are metrizable. Therefore given $c > 0$, we let $m_H, m_K < \omega$ be the Ramsey degrees of c for H and K, respectively.

Proposition 3.6.5. *Suppose $c > 0$ and $m_H, m_K < \omega$ are as above. Then letting m_G denote the Ramsey degree of $2c$ in G, we have $m_G \leq m_H \cdot m_K$.*

Proof. Let X be a compact metric space, and let $\gamma \in \mathrm{Lip}_G(X)$. Find group elements $\{g_i : i < [G:H]\}$ so that $\{g_i H : i < [G:H]\}$ lists the elements of K. We proceed much as in the non-Archimedean case. Set $\gamma = \gamma_0$. If γ_α is defined for some ordinal $\alpha < [G:H]$, find $p_\alpha \in S(H)$ with the property that $\overline{\gamma_\alpha \cdot g_\alpha \cdot p_\alpha[H]} \subseteq X$ can be covered by at most m_H-many balls of radius c. Then set $\gamma_{\alpha+1} = \gamma_\alpha \cdot g_\alpha \cdot p_\alpha \cdot g_\alpha^{-1}$. If γ_β has been defined for each $\beta < \alpha$ and α is a limit ordinal, let γ_α be any cluster point of $\{\gamma_\beta : \beta < \alpha\}$.

Letting $\kappa = [G:H]$ (so $\kappa = \omega$ or $\mathfrak{c}$), notice that for any $\alpha < \kappa$, we have $\overline{\gamma_\kappa[g_\alpha H]} \subseteq \overline{\gamma_{\alpha+1}[g_\alpha H]}$. Now form $K(X)$, the space of compact subsets of X equipped with the Hausdorff metric. Notice that since γ_κ is 1-Lipschitz, the function $\eta\colon K \to K(X)$ given by $\eta(g_\alpha H) = \overline{\gamma_\kappa[g_\alpha H]}$ is 1-Lipschitz. Find $q \in S(G)$ so that $\overline{\eta \cdot q[K]} \subseteq K(X)$ can be covered by at most m_K-many balls of radius c. It follows that $\overline{\gamma_\kappa \cdot q[G]} \subseteq X$ can be covered by at most $m_H \cdot m_K$ balls of radius $2c$. $\square$

3.7 Questions

Analyzing the fibers

Our first question is a strengthening of Theorem 3.6.4, where we aim to describe the minimal H-flows of the form $\pi^{-1}(\{y\})$ for $y \in \mathrm{M}(K)$.

Question 3.7.1. *Let $1 \to H \to G \xrightarrow{\pi} K \to 1$ be a short exact sequence of Polish groups with $\mathrm{M}(H)$ and $\mathrm{M}(K)$ metrizable. Is it the case that $\pi^{-1}(\{y\}) \cong \mathrm{M}(H)$ for every $y \in \mathrm{M}(K)$?*

In the interest of understanding this question better, we focus on the case that G is non-Archimedean, and we continue to use notation from Section 3.6.1. For each $n < \omega$, let $m_H(n)$, $m_G(n)$, and $m_K(n)$ denote the Ramsey degrees of the clopen subgroups $H \cap U_n$, U_n, and $\pi[U_n]$ in H, G, and K, respectively. In Proposition 3.6.3, we showed that $m_G(n) \leq m_H(n) \cdot m_K(n)$ for every $n < \omega$.

Proposition 3.7.2. *Suppose for each $n < \omega$, we have $m_G(n) = m_H(n) \cdot m_K(n)$. Then for any $y \in \mathrm{M}(K)$, we have $\pi^{-1}(\{y\}) \cong \mathrm{M}(H)$.*

We will need the following lemma.

Lemma 3.7.3. *Suppose for each $n < \omega$ that $\gamma_n \colon G_n \to m_G(n)$ is a surjective coloring with $\overline{\gamma_n \cdot G}$ minimal. Form $\gamma = (\gamma_n)_{n<\omega} \in \prod m_G(n)^{G_n}$, and assume also that $\overline{\gamma \cdot G}$ is minimal. Then $\mathrm{M}(G) \cong \overline{\gamma \cdot G}$.*

Remark. *Notice that we may then think of $\mathrm{M}(G)$ as the space of all such $\gamma = (\gamma_n)_{n<\omega}$.*

Proof. We use some of the ideas from [Z2]. Recall that we have $\mathrm{S}(G) = \varprojlim \beta G_n$. We let π_n denote the projection onto coordinate n. Explicitly, given $p \in \mathrm{S}(G)$, we have $\pi_n(p) = \lim_{g \to p} gU_n \in \beta G_n$. Notice that if $gU_n = hU_n$, then $\pi_n(pg) = \pi_n(ph)$. If $x \in \beta G_n$, we set $p \cdot x = \lim_{gU_n \to x} \pi_n(pg) \in \beta G_n$. If $p, q \in \mathrm{S}(G)$, we note that $\pi_n(pq) = p \cdot \pi_n(q)$.

Let $M \subseteq \mathrm{S}(G)$ be a minimal subflow. It is shown in [Z2] that $|\pi_n[M]| = m_G(n)$. The map $\lambda_\gamma \colon M \to \overline{\gamma \cdot G}$ given by $\lambda_\gamma(p) = \gamma \cdot p$ is a G-map, and it suffices to show that it is injective. Notice first that $\gamma \cdot p = (\gamma_n \cdot p)_{n<\omega}$ for any $p \in \mathrm{S}(G)$. Suppose $u \in M$ is an idempotent such that $\gamma = \gamma \cdot u$. For each $n < \omega$, consider the coloring $\lambda_u^n \colon G_n \to \pi_n[M]$ given by $\lambda_u^n(gU_n) = \pi_n(ug)$. Then λ_u^n and $\gamma_n \cdot u = \gamma_n$ are both surjective, minimal colorings each taking $m_G(n)$ values. Because $m_G(n)$ is the Ramsey degree of U_n in G, we must have that λ_u^n and γ_n are equivalent, i.e. for $g_0U_n, g_1U_n \in G_n$, we have $\pi_n(ug_0) = \pi_n(ug_1)$ iff $\gamma_n(g_0U_n) = \gamma_n(g_1U_n)$.

So suppose $\gamma \cdot p = \gamma \cdot q$ for some $p, q \in M$. In particular, upon extending the coloring to βG_n, we have $\gamma_n(\pi_n(p)) = \gamma_n(\pi_n(q))$. So also we have $u \cdot \pi_n(p) = u \cdot \pi_n(q)$. But since u is a left identity for M, we have $\pi_n(p) = \pi_n(q)$. Since $n < \omega$ is arbitrary, we have $p = q$ as desired. $\square$

Proof of Proposition 3.7.2. Suppose $m_G(n) = m_H(n) \cdot m_K(n)$. Let $\gamma = (\gamma_n)_{n<\omega}$ be as in Lemma 3.7.3, so that $\mathrm{M}(G) \cong \overline{\gamma \cdot G}$. It is enough to show that $\overline{\gamma \cdot H} \cong \mathrm{M}(H)$. From the proof of Theorem 3.1.1, we know that $\overline{\gamma \cdot H}$ is minimal, so also each $\overline{\gamma_n \cdot H}$ is minimal. These in turn factor onto $\overline{\gamma_n|_{H_n} \cdot H}$, so also $\overline{(\gamma_n|_{H_n})_{n<\omega} \cdot H}$ is minimal. We then note that since γ_n was a surjective, minimal $m_G(n)$-coloring, we must have that $\gamma_n|_{H_n}$ is a surjective, H-minimal $m_H(n)$-coloring. Therefore we are done by Lemma 3.7.3. $\square$

Unique ergodicity and open subgroups

Theorem 3.1.2 tells us that metrizability of the universal minimal flow and unique ergodicity are stable under group extension. Therefore we can add entries to the following table, describing which dynamical properties are preserved under which group-theoretic operations.

	Amen.	Ext. amen.	Met. of UMF	+ unique ergo.
Group ext.	✓	✓	✓	✓
Count. prod.	✓	✓	✓	✓
Direct lim.	✓	✓	$\times$	$\times$
Open subgrp	✓	✓	✓	?

The arguments for open subgroups can be found in the proof of Lemma 13 in [BPT]. We already proved that metrizability of the universal minimal flow and unique ergodicity are stable under group extension and countable products, thus we only need to prove that amenability and extreme amenability are stable under group extension and direct limits. We will also produce counter examples for the failure of stability of metrizability of the universal minimal flow and unique ergodicity under direct limits.

We will use the following characterization of amenability (see [P1]):

Definition 3.7.4. *A Polish group G is* amenable *if every continuous affine action of G on a compact convex subspace of a locally convex topological vector space admits a fixed point.*

As this definition is very close to extreme amenability, we will only produce the proof of stability of amenability, the proof for extreme amenability following the same steps.

Stability under group extension: We consider the short exact sequence

$$1 \to H \to G \xrightarrow{\pi} K \to 1,$$

where H and K are amenable, and an affine continuous action $G \curvearrowright X$ on a convex compact space. H acts on X and therefore admits fixed points. Since the action is affine, the set of fixed points is convex, it is closed by continuity of the action. Morevover, K acts on this set of fixed point, and by amenability this action also admits a fixed point. This point will then be G invariant.

Stability under direct limits: A group G is a *direct limit* of the sequence $(G_n)_{n<\omega}$ if we have

$$G_0 \leq \cdots \leq G_n \leq \cdots \leq G$$

and $\overline{\bigcup G_n}$ is dense in G. We are interrested in the case where G_n is amenable for all $n \in \mathbb{N}$.

Again, consider an affine continuous action $G \curvearrowright X$ on a convex compact space. $G_n \curvearrowright X$ admits a fixed point x_n. Since X is compact, $(x_n)_{n\in\mathbb{N}}$ admits a cluster point $x \in X$. Then x is G_n-invariant for all $n \in \mathbb{N}$, hence it is G-invariant.

Counterexamples: In [KPT] appendix 2, it is proven that non compact locally compact group have non metrizable universal minimal flows. Moreover, Weiss proved in [W] that discrete group are never uniquely ergodic. Hence, any locally finite discrete group produce the counterexample we need, for instance the group of permutations of $\mathbb{N}$ with finite support.

The next question concerns the question mark appearing in the array.

Question 3.7.5. *Let G be a uniquely ergodic Polish group with metrizable universal minimal flow and U an open subgroup. Is U uniquely ergodic?*

Another question concerns the connection between $\mathrm{M}(G)$ metrizable and unique ergodicity. For all known examples of uniquely ergodic Polish groups G, we have that $\mathrm{M}(G)$ is metrizable.

Question 3.7.6. *Let G be a uniquely ergodic Polish group. Is $\mathrm{M}(G)$ metrizable?*

CHAPTER 4

Action on the space of linear orderings

This is joint work with Todor Tsankov, it follows closely [JT].

4.1 Introduction

In this chapter, we will be interested in the invariant probability measures on dynamical systems of the automorphism group $\mathrm{Aut}(M)$ of a homogeneous structure M. More precisely, we will consider two specific systems: products of the type Z^M, where Z is a standard Borel space, and the compact space $\mathrm{LO}(M)$ of all linear orders on M.

Our study of invariant measures on product spaces of the type Z^M is inspired by the classical de Finetti theorem. One formulation of this theorem is that the only ergodic measures on Z^M invariant under the full symmetric group $\mathrm{Sym}(M)$ are product measures of the type λ^M, where λ is some probability measure on Z. (Recall that a measure is *ergodic* if the only elements of the measure algebra fixed by the group are $\emptyset$ and the whole space.) In our first theorem, we obtain the same conclusion under a weaker hypothesis: that the measure is invariant under the much smaller group $\mathrm{Aut}(M)$, provided that the structure M satisfies certain model-theoretic conditions. We will say that a structure M is *transitive* if the action $\mathrm{Aut}(M) \curvearrowright M$ is transitive.

Theorem 4.1.1. *Let M be an $\aleph_0$-categorical, transitive structure with no algebraicity that admits weak elimination of imaginaries. Let Z be a standard Borel space and consider the natural action* $\mathrm{Aut}(M) \curvearrowright Z^M$*. Then the only invariant, ergodic probability measures on Z^M are product measures of the form λ^M, where λ is a probability measure on Z.*

We will discuss the model-theoretic hypotheses of the theorem in detail in the next section, where we give all relevant definitions. Here we only remark that they are all necessary (with the possible exception of $\aleph_0$-categoricity) and that they are satisfied, for example, by the random graph, the homogeneous triangle-free graph, the dense linear order $(\mathbf{Q}, <)$, the universal, homogeneous partial order, and many other structures.

The ergodicity assumption in the theorem is not essential: one can obtain a description of all invariant measures using the ergodic decomposition theorem.

A different formulation of the theorem that does not involve the group and that would perhaps be more appealing to model theorists is the following. Let M satisfy the hypothesis of the theorem and let $\{\xi_a : a \in M\}$ be a family of random variables. Suppose that for all tuples $\bar{a}, \bar{b} \in M^k$ that have the same type, we have that $(\xi_{a_0}, \dots, \xi_{a_{k-1}})$ and $(\xi_{b_0}, \dots, \xi_{b_{k-1}})$ have the same distribution. Then the family $\{\xi_a : a \in M\}$ is conditionally independent over the tail σ-field $\mathcal{T}$. The tail σ-field is defined by

$$\mathcal{T} = \bigcap \big\{ \langle \xi_a : a \notin F \rangle : F \subseteq M \text{ finite} \big\},$$

where $\langle \cdot \rangle$ denotes the generated σ-field. It turns out that under the hypothesis of Theorem 4.1.1, the invariant σ-field and the tail σ-field coincide.

In fact, Theorem 4.1.1 is a consequence of a rather more general independence result that applies to any measure-preserving action of $\mathrm{Aut}(M)$ for any $\aleph_0$-categorical structure M (cf. Theorem 4.3.4). The proof is based on representation theory and the results of [T1].

The model-theoretic formulation also permits to use Fraïssé's theorem and apply Theorem 4.1.1 even in situations where there is no homogeneity or an obvious group present. For example, we can recover a theorem of Ryll-Nardzewski [RN], which is another well-known strengthening of de Finetti's theorem; cf. Corollary 4.3.7.

Theorem 4.1.1 was announced in the habilitation memoir of the second author [T2]. Later, some independent related work has been done by Ackerman [A1] and Crane–Towsner [CT]. They consider a different class of homogeneous structures (with combinatorial assumptions on the amalgamation) and use completely different methods.

Next we consider $\mathrm{Aut}(M)$-invariant probability measures on the compact space $\mathrm{LO}(M)$ of linear orders on M. The systematic study of these measures was initiated by Angel, Kechris, and Lyons in [AKL]. In most known examples, the universal minimal flow of $\mathrm{Aut}(M)$ is a subflow of the flow $\mathrm{LO}(M)$ of all linear orders on M. In these situations, classifying the invariant measures on $\mathrm{LO}(M)$ gives information about *all* minimal flows of the group as well as other properties of G that can be expressed dynamically.

In the present chapter, we adopt a new approach to the unique ergodicity problem on the space of linear orders, based on the generalization of de Finetti's theorem that we discussed above. It has the advantage of working under rather general model-theoretic assumptions (which are mostly necessary) and can also give information about the invariant measures even in the absence of unique ergodicity. Our main theorem is the following.

Theorem 4.1.2. *Let M be a transitive, $\aleph_0$-categorical structure with no algebraicity that admits weak elimination of imaginaries. Consider the action $\mathrm{Aut}(M) \curvearrowright \mathrm{LO}(M)$. Then exactly one of the following holds:*

1. *The action $\mathrm{Aut}(M) \curvearrowright \mathrm{LO}(M)$ has a fixed point (i.e., there is a definable linear order on M);*

2. *The action $\mathrm{Aut}(M) \curvearrowright \mathrm{LO}(M)$ is uniquely ergodic.*

Theorem 4.1.2 recovers almost all known results about unique ergodicity of $\mathrm{LO}(M)$. More specifically, it applies to the following structures:

- the random graph, the K_n-free homogeneous graphs, various homogeneous hypergraphs, and the universal homogeneous tournament [AKL];
- the generic directed graphs obtained by omitting a (possibly infinite) set of tournaments or a fixed, finite, discrete graph [PS].

The class of structures satisfying the hypothesis of Theorem 4.1.2 is quite a bit richer than the examples above. We should mention, however, that it does not cover all cases where unique ergodicity of the space of linear orders is known. The exception is the rational Urysohn space $\mathbf{U}_0$: it was proved in [AKL] that the action $\mathrm{Iso}(\mathbf{U}_0) \curvearrowright \mathrm{LO}(\mathbf{U}_0)$ is uniquely ergodic but $\mathbf{U}_0$ is not $\aleph_0$-categorical (as it has infinitely many 2-types). It also does not apply directly to prove unique ergodicity for proper subflows of LO, for example for the automorphism group of the countable-dimensional vector space over a finite field.

The proof of Theorem 4.1.2 is the object of Section 4.4, where it is stated as Theorem 4.4.1.

We also have an interesting corollary of Theorem 4.1.2 concerning amenability.

Corollary 4.1.3. *Suppose that M satisfies the assumptions of Theorem 4.1.2 and let $G = \mathrm{Aut}(M)$. If the action $G \curvearrowright \mathrm{LO}(M)$ is not minimal and has no fixed points, then G is not amenable.*

Corollary 4.1.3 applies for example to the automorphism groups of the universal homogeneous partial order and the circular directed graphs $\mathbf{S}(n)$ for $n \geq 2$, recovering results of Kechris–Sokić [KS] and Zucker [Z1], respectively.

Corollary 4.1.3 also has an interesting purely combinatorial consequence of which we do not know a combinatorial proof. Recall that a Fraïssé class $\mathcal{F}$ (or its Fraïssé limit) has the *Hrushovski property* if partial automorphisms of elements of $\mathcal{F}$ extend to full automorphisms of superstructures in $\mathcal{F}$. It has the *ordering property* if for every $A \in \mathcal{F}$, there exists $B \in \mathcal{F}$ such that for any two linear orders $<$ and $<'$ on A and B respectively, there is an embedding of $(A, <)$ into $(B, <')$. The Hrushovski and the ordering properties are important in the theory of homogeneous structures and in structural Ramsey theory but are not a priori related. We refer the reader to [KR] and [NR] for more details about them.

Corollary 4.1.4. *Suppose that the homogeneous structure M satisfies the assumptions of Theorem 4.1.2. If M has the Hrushovski property, then it has the ordering property.*

The chapter is organized as follows. In Section 4.2, we recall some prerequisites from model theory, mostly about imaginaries and M^{eq}. While using standard model-theoretic terminology, we give all definitions and proofs in the language of permutation groups in the hope of making the chapter more accessible to non-logicians. In Section 4.3, we recall some facts from representation theory and prove Theorem 4.1.1. Section 4.4 is devoted to the proof of Theorem 4.1.2 and its corollaries. Finally, in Section 4.5, we briefly discuss some examples and possible extensions of Theorem 4.1.2.

Aknowledgements: We would like to thank Itaï Ben Yaacov and David Evans for helping us eliminate imaginaries in some examples and Lionel

Nguyen Van Thé and Andy Zucker for useful discussions. Research was partially supported by the ANR project AGRUME (ANR-17-CE40-0026) and the *Investissements d'Avenir* program of Université de Lyon (ANR-16-IDEX-0005).

4.2 Preliminaries from model theory

We start by recalling some basic definitions. A *signature* $\mathcal{L}$ is a collection of relation symbols $\{R_i\}$ and function symbols $\{F_j\}$, each equipped with a natural number called its *arity*. An *$\mathcal{L}$-structure* is a set M together with interpretations for the symbols in $\mathcal{L}$: each relation symbol R_i of arity n_i is interpreted as an n_i-ary relation on M, that is, a subset of M^{n_i}, and each function symbol F_j of arity n_j is interpreted as a function $M^{n_j} \to M$. Functions of arity 0 are called *constants*. A *substructure* of M is a subset of M closed under the functions, equipped with the induced structure. The *age of M* is the collection of isomorphism classes of all finitely generated substructures of M. If $\bar{a}$ is a tuple from M, we denote by $\langle \bar{a} \rangle$ the substructure of M generated by $\bar{a}$. If the signature contains only relation symbols (which will usually be the case for us), then a substructure of M is just a subset with the induced relations.

The *automorphism group* of M, $\mathrm{Aut}(M)$, is the set of all permutations of M that preserve all relations and functions. $\mathrm{Aut}(M)$ is naturally a topological group if equipped with the pointwise convergence topology (where M is taken to be discrete). If M is countable, then $\mathrm{Aut}(M)$ is a Polish group. If $G = \mathrm{Aut}(M)$ and $A \subseteq M$ is a finite subset, we will denote by G_A the pointwise stabilizer of A in G. A basis at the identity of G is given by the subgroups $\{G_A : A \subseteq M \text{ is finite}\}$. A topological group which admits a basis at the identity consisting of open subgroups is called *non-archimedean*.

The *type* of a tuple $\bar{a} \in M^k$, denoted by $\mathrm{tp}\, \bar{a}$, is the isomorphism type of the substructure $\langle a_i : i < k \rangle$ (with the a_i named). Thus two tuples $\bar{a}$ and $\bar{b}$ have the same type (notation: $\bar{a} \equiv \bar{b}$) if the map $a_i \mapsto b_i$ extends to an isomorphism $\langle \bar{a} \rangle \to \langle \bar{b} \rangle$. A k-type is simply the type of some tuple $\bar{a} \in M^k$. The structure M is called *homogeneous* if for every two tuples $\bar{a}$ and $\bar{b}$ with $\bar{a} \equiv \bar{b}$, there exists $g \in \mathrm{Aut}(M)$ such that $g \cdot \bar{a} = \bar{b}$. We will say that M is *transitive* if there is only one 1-type, i.e., G acts transitively on M.

What we call *type* is usually called *quantifier-free type* in the model-theoretic literature. However, for homogeneous structures, which is our main interest here, the two notions coincide.

An *age* is a countable family of (isomorphism types of) finitely generated $\mathcal{L}$-structures that is *hereditary* (i.e., closed under substructures) and *directed* (i.e., for any two structures in the class, there is another structure in the class in which they both embed). If M is a given countable structure, its age is the collection of finitely generated structures that embed into it. If M is homogeneous, then its age has another special property called *amalgamation*. An age with amalgamation is called a *Fraïssé class*. Fraïssé's theorem states that conversely, any Fraïssé class is the age of a unique countable, homogeneous structure, called its *Fraïssé limit*. Thus in order to define a homogeneous structure, one needs only to specify its age; and, as already mentioned, combinatorial properties of the age are reflected in the dynamics of the automorphism group of the limit.

The structures that will be especially important for us are the $\aleph_0$-categorical ones. A structure is *$\aleph_0$-categorical* if its first-order theory has a unique countable model up to isomorphism. Another characterization that will be crucial is given by the Ryll-Nardzewski theorem: M is $\aleph_0$-categorical iff the diagonal action $\mathrm{Aut}(M) \curvearrowright M^k$ has finitely many orbits for every k (a permutation group with this property is called *oligomorphic*). In particular, if $\mathcal{L}$ is a signature that contains only finitely many relational symbols of each arity and no functions, then every homogeneous $\mathcal{L}$-structure is $\aleph_0$-categorical. Conversely, if M is any $\aleph_0$-categorical structure, one can render it homogeneous by expanding the signature to include all first-order formulas (this is another facet of the Ryll-Nardzewski theorem). As we never make assumptions about the signature, in what follows, we will tacitly assume that every $\aleph_0$-categorical is rendered homogeneous by this procedure. If G is any closed subgroup of the full permutation group $\mathrm{Sym}(N)$ of some countable set N, one can convert N into a homogeneous structure with $\mathrm{Aut}(N) = G$ by naming, for every k, each G-orbit on N^k by a k-ary relation symbol. If the action $G \curvearrowright N$ is oligomorphic, then the resulting structure will be $\aleph_0$-categorical.

For the rest of the chapter, we will only consider $\aleph_0$-categorical structures. In this setting, all model-theoretic information about M is captured by the actions $\mathrm{Aut}(M) \curvearrowright M^k$. We refer the reader to Hodges [H3] for more details on Fraïssé theory, $\aleph_0$-categorical structures, and their automorphism groups.

Let M be $\aleph_0$-categorical, $G = \mathrm{Aut}(M)$, and let $A \subseteq M$ be finite. The *algebraic closure of* A (denoted $\mathrm{acl}(A)$) is the union of all finite orbits of G_A on M. We will say that M *has no algebraicity* if the algebraic closure is trivial, that is, $\mathrm{acl}(A) = A$ for all finite $A \subseteq M$. By Neumann's lemma [H3, Lemma 4.2.1], having no algebraicity is equivalent to the following: for all finite $A, B \subseteq M$, there exists $g \in G$ such that $g \cdot A \cap B = \emptyset$.

An *imaginary element* of M is the equivalence class of a tuple $\bar{a} \in M^k$ for some G-invariant equivalence relation on M^k. We denote by M^{eq} the collection of all imaginaries. In symbols,

$$M^{\mathrm{eq}} = \bigsqcup \{M^k/E : k \in \mathbb{N} \text{ and } E \text{ is a } G\text{-invariant equivalence relation on } M^k\}.$$

It is clear that G also acts on M^{eq} and, moreover, the action $G \curvearrowright M^{\mathrm{eq}}$ is *locally oligomorphic*, i.e., it is oligomorphic on any union of finitely many G-orbits (see, e.g., [T1, Theorem 2.4]).

We can define for a finite $A \subseteq M^{\mathrm{eq}}$,

$$\mathrm{acl}^{\mathrm{eq}} A = \{e \in M^{\mathrm{eq}} : G_A \cdot e \text{ is finite}\}.$$

Similarly, we can define the *definable closure* as

$$\mathrm{dcl}^{\mathrm{eq}} A = \{e \in M^{\mathrm{eq}} : G_A \cdot e = \{e\}\}.$$

For arbitrary $A \subseteq M^{\mathrm{eq}}$, we define $\mathrm{acl}^{\mathrm{eq}} A$ to be the union of $\mathrm{acl}^{\mathrm{eq}} A'$ for all finite $A' \subseteq A$. Similarly for $\mathrm{dcl}^{\mathrm{eq}}$. A subset $A \subseteq M^{\mathrm{eq}}$ is *algebraically closed* if $\mathrm{acl}^{\mathrm{eq}} A = A$. In other words, A is algebraically closed if for all finite $A' \subseteq A$, $G_{A'}$ has only infinite orbits outside of A. We have the following basic properties of the algebraic closure.

Lemma 4.2.1. *The following hold for an $\aleph_0$-categorical M:*

1. *For all $A \subseteq M^{\mathrm{eq}}$, $\mathrm{acl}^{\mathrm{eq}} A$ is algebraically closed;*
2. *If $A, B \subseteq M^{\mathrm{eq}}$ are algebraically closed, then so is $A \cap B$.*

Proof. 1 A permutation group theoretic proof of this fact can be found for example in [ET, Lemma 2.4].

2 Suppose that $e \in (\operatorname{acl}^{\mathrm{eq}} C) \setminus (A \cap B)$ for some finite $C \subseteq A \cap B$. Then there are finite $A' \subseteq A, B' \subseteq B$ such that $C = A' \cap B'$. Suppose for definiteness that $e \notin A$. As A is algebraically closed, $G_C \cdot e \supseteq G_{A'} \cdot e$ is infinite, contradiction. □

M admits *elimination of imaginaries* if all imaginary elements are *interdefinable* with real tuples, that is, for every $e \in M^{\mathrm{eq}}$, there exists $k \in \mathbb{N}$ and a tuple $\bar{a} \in M^k$ such that $e \in \operatorname{dcl}^{\mathrm{eq}} \bar{a}$ and $\bar{a} \in \operatorname{dcl}^{\mathrm{eq}} e$, or equivalently, $G_e = G_{\bar{a}}$. M admits *weak elimination of imaginaries* if for every imaginary element $e \in M^{\mathrm{eq}}$, there exists a real tuple $\bar{a} \in M^k$ such that $e \in \operatorname{dcl}^{\mathrm{eq}} \bar{a}$ and $\bar{a} \in \operatorname{acl}^{\mathrm{eq}} e$. Equivalently, for every proper, open subgroup $V < G$, there exists k and a tuple $\bar{a} \in M^k$ such that $G_{\bar{a}} \leq V$ and $[V : G_{\bar{a}}] < \infty$.

The two hypothesis of no algebraicity and weak elimination of imaginaries combined give us a complete understanding of the $\operatorname{acl}^{\mathrm{eq}}$ operator.

Lemma 4.2.2. *Suppose that M is $\aleph_0$-categorical and that it has no algebraicity and admits weak elimination of imaginaries. Then for all $A, B \subseteq M$, we have that*

$$\operatorname{acl}^{\mathrm{eq}} A \cap \operatorname{acl}^{\mathrm{eq}} B = \operatorname{dcl}^{\mathrm{eq}}(A \cap B).$$

Proof. The $\supseteq$ inclusion being clear, we only check the other. We may assume that A and B are finite. Suppose that $e = [\bar{c}]_E \in \operatorname{acl}^{\mathrm{eq}} A$, where $\bar{c} \in M^k$ and E is a G-invariant equivalence relation. We will show that if $e \notin \operatorname{dcl}^{\mathrm{eq}} \emptyset$, then the tuple $\bar{c}$ is contained in A. Consider the group $H = G_{A \cup \{e\}}$. As $G_A \cdot e$ is finite, H has finite index in G_A, so it is open. By weak elimination of imaginaries, there exists a tuple $\bar{a}$ such that $G_{\bar{a}} \leq H$ and $[H : G_{\bar{a}}] < \infty$. In particular, $[G_A : G_{\bar{a}}] < \infty$. By the no algebraicity assumption, $\bar{a}$ must be contained in A, so, in particular, $H = G_A$, i.e., G_A fixes e. If $\bar{c}$ is not contained in A, then the orbit $G_A \cdot \bar{c}$ is infinite and is contained in $[\bar{c}]_E$, which together with weak elimination of imaginaries implies that $e \in \operatorname{dcl}^{\mathrm{eq}} \emptyset$, contradiction. Thus we conclude that $\bar{c}$ is contained in A. An analogous argument shows that $\bar{c}$ is also contained in B and hence, $e \in \operatorname{dcl}^{\mathrm{eq}}(A \cap B)$. □

4.3 Unitary representations and a generalization of de Finetti's theorem

Recall that a *unitary representation* of a topological group G is a continuous action on a complex Hilbert space $\mathcal{H}$ by unitary operators, or, equivalently, a continuous homomorphism from G to the unitary group of $\mathcal{H}$. A representation $G \curvearrowright \mathcal{H}$ is *irreducible* if $\mathcal{H}$ contains no non-trivial, G-invariant, closed subspaces.

In the case where M is an $\aleph_0$-categorical structure and $G = \operatorname{Aut}(M)$, the action $G \curvearrowright M^{\mathrm{eq}}$ gives rise to a representation $G \curvearrowright^{\lambda} \ell^2(M^{\mathrm{eq}})$ given by

$$(\lambda(g) \cdot f)(e) = f(g^{-1}e), \quad \text{where } f \in \ell^2(M^{\mathrm{eq}}), g \in G, e \in M^{\mathrm{eq}}.$$

It turns out that this representation captures all of the representation theory of G. More precisely, it follows from the results of [T1] that the following holds.

Fact 4.3.1. *Let M be an $\aleph_0$-categorical structure and let $G = \mathrm{Aut}(M)$. Then every unitary representation of G is a sum of irreducible representations and every irreducible representation is isomorphic to a subrepresentation of λ. In particular, every representation of G is a subrepresentation of a direct sum of copies of λ.*

Proof. The first claim is part of the statement of [T1, Theorem 4.2]. For the second, it follows from [T1, Theorem 4.2] that every irreducible representation of G is an induced representation of the form $\mathrm{Ind}_H^G(\sigma)$, where H is an open subgroup of G and σ is an irreducible representation of H that factors through a finite quotient $K = H/V$ of H. (We refer the reader to [T1] for the definition of induced representation and more details.) As $V \leq G$ is open, there exists a tuple $\bar{a}$ from M such that $G_{\bar{a}} \leq V$. Define the G-invariant equivalence relation E on $G \cdot \bar{a}$ by

$$(g_1 \cdot \bar{a})\, E\, (g_2 \cdot \bar{a}) \iff g_1 V = g_2 V$$

and note that $V = G_{[\bar{a}]_E}$. In particular, the quasi-regular representation $\ell^2(G/V)$ is isomorphic to the subrepresentation $\ell^2(G \cdot [\bar{a}]_E)$ of $\ell^2(M^{\mathrm{eq}})$. On the other hand,

$$\ell^2(G/V) \cong \mathrm{Ind}_V^G(1_V) \cong \mathrm{Ind}_H^G(\mathrm{Ind}_V^H(1_V)) \cong \mathrm{Ind}_H^G(\lambda_K),$$

where λ_K denotes the left-regular representation of K. As σ (being an irreducible representation of the finite group K) is a subrepresentation of λ_K, the result follows. □

If $\mathcal{H}$ is a Hilbert space, $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$ are subspaces with $\mathcal{H}_2 \subseteq \mathcal{H}_1 \cap \mathcal{H}_3$, we write $\mathcal{H}_1 \perp_{\mathcal{H}_2} \mathcal{H}_3$ if $\mathcal{H}_1 \ominus \mathcal{H}_2 \perp \mathcal{H}_3 \ominus \mathcal{H}_2$. If we let p_1, p_2, p_3 denote the corresponding orthogonal projections, this is equivalent to $p_3 p_1 = p_2 p_1$.

If $G \curvearrowright \mathcal{H}$ is a unitary representation of G and $A \subseteq M^{\mathrm{eq}}$, let

$$\mathcal{H}_A = \overline{\{\xi \in \mathcal{H} : G_{A'} \cdot \xi = \xi \text{ for some finite } A' \subseteq A\}}. \tag{4.1}$$

It is clear that $\mathcal{H}_A$ is a closed subspace of $\mathcal{H}$.

Proposition 4.3.2. *Let M be $\aleph_0$-categorical and $G = \mathrm{Aut}(M)$. Let A and B be algebraically closed subsets of M^{eq}. Then $\mathcal{H}_A \perp_{\mathcal{H}_{A \cap B}} \mathcal{H}_B$.*

Proof. As for any subset $C \subseteq M^{\mathrm{eq}}$, the projection p_C onto $\mathcal{H}_C$ commutes with direct sums and subrepresentations, by Fact 4.3.1, we can reduce to the case where $\mathcal{H} = \ell^2(M^{\mathrm{eq}})$ and $\pi = \lambda$. If $\xi \in \mathcal{H}$, we view it as a function $M^{\mathrm{eq}} \to \mathbf{C}$ and we let $\mathrm{supp}\,\xi = \{e \in M^{\mathrm{eq}} : \xi(e) \neq 0\}$.

The main observation is the following: if $C \subseteq M^{\mathrm{eq}}$ is algebraically closed, then

$$\mathcal{H}_C = \{\xi \in \mathcal{H} : \mathrm{supp}\,\xi \subseteq C\}.$$

The $\supseteq$ inclusion follows from the fact that vectors with finite support are dense. For the other inclusion, as the subspace on the right-hand side is closed, it suffices to see that for all finite $C' \subseteq C$ and all ξ fixed by $G_{C'}$, $\mathrm{supp}\,\xi \subseteq C$. Let $e \in M^{\mathrm{eq}}$ be such that $\xi(e) \neq 0$. As ξ is fixed by $G_{C'}$, it must be constant on the orbit $G_{C'} \cdot e$. As ξ is in ℓ^2, this implies that $G_{C'} \cdot e$ is finite, i.e., $e \in \mathrm{acl}^{\mathrm{eq}}\, C' \subseteq C$.

Now it follows from the hypothesis and Lemma 4.2.1 that

$$\mathcal{H}_A \ominus \mathcal{H}_{A \cap B} = \{\xi \in \mathcal{H} : \mathrm{supp}\,\xi \subseteq A \setminus B\} \quad \text{and}$$
$$\mathcal{H}_B \ominus \mathcal{H}_{A \cap B} = \{\xi \in \mathcal{H} : \mathrm{supp}\,\xi \subseteq B \setminus A\},$$

whence the result. □

Remark 4.3.3. A more model-theoretic treatment of similar ideas, using the formalism of semigroups of projections, can be found in [BIT].

Now consider a measure-preserving action $G \curvearrowright (X, \mu)$, where (X, μ) is a probability space. As G is not locally compact, one has to take some care how this is defined. We denote by $\mathrm{MALG}(X, \mu)$ the Boolean algebra of all measurable subsets of X with two such sets identified if their symmetric difference has measure 0. $\mathrm{MALG}(X, \mu)$ is naturally a metric space with the distance between A and B given by $\mu(A \triangle B)$. We denote by $\mathrm{Aut}(X, \mu)$ the group of all isometric automorphisms of $\mathrm{MALG}(X, \mu)$, that is, the group of all automorphisms of the Boolean algebra that also preserve the measure. $\mathrm{Aut}(X, \mu)$ is naturally a topological group if equipped with the pointwise convergence topology coming from its action on $\mathrm{MALG}(X, \mu)$. If (X, μ) is *standard* (i.e., X is a standard Borel space and μ is a Borel probability measure), then $\mathrm{Aut}(X, \mu)$ is a Polish group. For us, a *measure-preserving action* $G \curvearrowright (X, \mu)$ will mean a continuous homomorphism $G \to \mathrm{Aut}(X, \mu)$, that is, G acts on measurable sets and measurable functions (up to measure 0) but not necessarily on points. It is easy to see that if X is standard and one has a jointly measurable action on points $G \curvearrowright X$ that preserves the measure μ, then this gives an action in our sense. The converse is also true for non-archimedean groups but this is less obvious (see [GW2, Theorem 2.3]) and we will not need it.

If $\mathcal{F}_1, \mathcal{F}_2, \mathcal{G}$ are σ-fields in a probability space, we will denote by $\mathcal{F}_1 \perp\!\!\!\perp_{\mathcal{G}} \mathcal{F}_2$ the fact that $\mathcal{F}_1$ and $\mathcal{F}_2$ are *conditionally independent over* $\mathcal{G}$, i.e., $\mathbf{E}(\xi \mid \mathcal{F}_2\mathcal{G}) = \mathbf{E}(\xi \mid \mathcal{G})$ for every $\mathcal{F}_1$-measurable random variable ξ. If $\mathcal{G}$ is trivial, we will write simply $\mathcal{F}_1 \perp\!\!\!\perp \mathcal{F}_2$ and will say that $\mathcal{F}_1$ and $\mathcal{F}_2$ are independent. We will freely use the standard facts about conditional independence, as described, for example, in [K1], and that go in model theory by the name of *forking calculus*.

If $G = \mathrm{Aut}(M)$ and a measure-preserving action $G \curvearrowright X$ is given, for $A \subseteq M^{\mathrm{eq}}$, we denote by $\mathcal{F}_A$ the σ-field of measurable subsets of X generated by the $G_{A'}$-fixed subsets for all finite $A' \subseteq A$. The following is the main result of this section.

Theorem 4.3.4. *Let M be an $\aleph_0$-categorical structure and let $G = \mathrm{Aut}(M)$. Let $G \curvearrowright (X, \mu)$ be any measure-preserving action on a probability space. Then the following hold:*

1. *For all algebraically closed $A, B \subseteq M^{\mathrm{eq}}$, we have that $\mathcal{F}_A \perp\!\!\!\perp_{\mathcal{F}_{A \cap B}} \mathcal{F}_B$.*
2. *If M has no algebraicity and admits weak elimination of imaginaries, then for all $A, B \subseteq M$, we have that $\mathcal{F}_A \perp\!\!\!\perp_{\mathcal{F}_{A \cap B}} \mathcal{F}_B$.*

Proof. 1 Consider the *Koopman representation* $G \curvearrowright^{\pi} L^2(X)$ given by

$$(\pi(g) \cdot f)(x) = f(g^{-1} \cdot x), \quad \text{where } f \in L^2(X), g \in G, x \in X.$$

For $C \subseteq M^{\mathrm{eq}}$, we denote by $L^2(\mathcal{F}_C)$ the subspace of $L^2(X)$ consisting of all $\mathcal{F}_C$-measurable functions. Observe that if we write $\mathcal{H} = L^2(X)$, then $L^2(\mathcal{F}_C) = \mathcal{H}_C$ (as defined in (4.1)). To show the required independence, it suffices to see that for all $\eta_A \in L^2(\mathcal{F}_A)$, we have that

$$\mathbf{E}(\eta_A \mid \mathcal{F}_B) = \mathbf{E}(\eta_A \mid \mathcal{F}_{A \cap B})$$

(see, e.g., [K1, Proposition 5.6]). Recalling that the conditional expectation $\mathbf{E}(\cdot \mid \mathcal{F}_C)$ for functions in L^2 is just the projection operator onto $\mathcal{H}_C$, this follows directly from Proposition 4.3.2.

2 Denote $A' = \operatorname{acl}^{\mathrm{eq}} A$, $B' = \operatorname{acl}^{\mathrm{eq}} B$, $C = \operatorname{dcl}^{\mathrm{eq}}(A \cap B)$. By Lemma 4.2.1, A' and B' are algebraically closed and by Lemma 4.2.2, we have that $C = A' \cap B'$. Now 1 applied to A' and B' yields that $\mathcal{F}_{A'} \perp\!\!\!\perp_{\mathcal{F}_C} \mathcal{F}_{B'}$. It only remains to observe that $\mathcal{F}_C = \mathcal{F}_{A \cap B}$ and that $\mathcal{F}_A \subseteq \mathcal{F}_{A'}$, $\mathcal{F}_B \subseteq \mathcal{F}_{B'}$. □

Theorem 4.3.4 has the following immediate corollary, which can be viewed as a generalization of the classical theorem of de Finetti.

Corollary 4.3.5. *Let M be an $\aleph_0$-categorical structure with no algebraicity that admits weak elimination of imaginaries and let $G = \operatorname{Aut}(M)$. Consider a family of random variables $\{\xi_a : a \in M\}$ whose joint distribution is invariant under G. Then these variables are conditionally independent over the G-invariant σ-field. If the G-invariant σ-field is trivial and M is transitive, then the ξ_a are* i.i.d. *(independent, identically distributed).*

Proof. Let $a_1, \ldots, a_n, b_1, \ldots, b_m \in M$ with $\{a_1, \ldots, a_n\} \cap \{b_1, \ldots, b_m\} = \emptyset$. Then it follows from Theorem 4.3.4 2 that

$$(\xi_{a_1}, \ldots, \xi_{a_n}) \underset{\mathcal{F}_\emptyset}{\perp\!\!\!\perp} (\xi_{b_1}, \ldots, \xi_{b_m})$$

and it remains to observe that $\mathcal{F}_\emptyset$ is precisely the G-invariant σ-field.

If M is moreover transitive, the variables ξ_a must have the same distribution by G-invariance. □

Remark 4.3.6. Using the properties of independence and an inductive argument, it is possible to replace the no algebraicity and weak elimination of imaginaries assumption above with a slightly weaker one. Namely, we only need that $\operatorname{acl}^{\mathrm{eq}} A \cap \operatorname{acl}^{\mathrm{eq}} B = \operatorname{dcl}^{\mathrm{eq}} \emptyset$ for any disjoint A and B where B is a singleton, rather than for arbitrary A and B.

An action $G \curvearrowright (X, \mu)$ is called *ergodic* if the G-invariant σ-field is trivial. Thus in the case of ergodic actions, one obtains genuine independence in Corollary 4.3.5.

Another interesting remark is that by virtue of Fraïssé's theorem, Corollary 4.3.5 can be applied even in situations in which there is no obvious group around. This is best illustrated by the following example, which is a well-known theorem of Ryll-Nardzewski [RN]. If $\bar{\xi} = (\xi_0, \ldots, \xi_{n-1})$ and $\bar{\eta} = (\eta_0, \ldots, \eta_{n-1})$ are tuples of random variables, we use the notation $\bar{\xi} \equiv \bar{\eta}$ to signify that $\bar{\xi}$ and $\bar{\eta}$ have the same distribution.

Corollary 4.3.7 (Ryll-Nardzewski)**.** *Let μ be a Borel probability measure on $\mathbf{R}^{\mathbb{N}}$ and denote by $\xi_i \colon \mathbf{R}^{\mathbb{N}} \to \mathbf{R}$ the projection on the i-th coordinate. Suppose that for all $i_0 < \cdots < i_{k-1}$, we have that $(\xi_{i_0}, \ldots, \xi_{i_{k-1}}) \equiv (\xi_0, \ldots, \xi_{k-1})$. Denote by $\phi \colon \mathbf{R}^{\mathbb{N}} \to \mathbf{R}^{\mathbb{N}}$ the one-sided shift defined by $\phi(x_0, x_1, \ldots) = (x_1, x_2, \ldots)$ and suppose moreover that μ is ϕ-ergodic. Then μ is a product measure.*

Proof. Here the relevant structure is $(\mathbb{N}, <)$ which has no automorphisms. Its age is the class of finite linear orders. This age amalgamates and its Fraïssé limit is the countable dense linear order without endpoints $(\mathbf{Q}, <)$, which satisfies the hypothesis of Corollary 4.3.5. Consider the random variables $(\xi_a : a \in \mathbf{Q})$ whose distribution is defined by

$$(\xi_{a_0}, \dots, \xi_{a_{k-1}}) \equiv (\xi_0, \dots, \xi_{k-1}) \quad \text{for all } a_0 < \cdots < a_{k-1}.$$

In order to apply Corollary 4.3.5 and conclude, we only need to check that the $\mathrm{Aut}(\mathbf{Q})$-invariant σ-field is trivial. Suppose that S is an $\mathrm{Aut}(\mathbf{Q})$-invariant event. Let $\mathcal{F}_n$ be the σ-field generated by $\xi_0, \dots, \xi_{n-1}$. Then, by invariance, for every ϵ, there exists n and an $\mathcal{F}_n$-measurable event S_ϵ with $\mathbf{P}(S \triangle S_\epsilon) < \epsilon$. This shows that S is measurable with respect to the original σ-field $\bigvee_n \mathcal{F}_n$. As ϕ extends to an automorphism of $\mathbf{Q}$, we obtain that $\phi^{-1}(S) = S$ and we are done. □

4.4 Invariant measures on the space of linear orderings

In this section, we fix a homogeneous, $\aleph_0$-categorical structure M with no algebraicity that admits weak elimination of imaginaries and we let $G = \mathrm{Aut}(M)$. We denote by $\mathrm{LO}(M)$ the space of all linear orders on M, that is

$$\mathrm{LO}(M) = \{x \in 2^{M \times M} : x \text{ is a linear order}\}.$$

$\mathrm{LO}(M)$ is a closed subset of $2^{M \times M}$ and thus a compact space. If $x \in \mathrm{LO}(M)$, we will use the more traditional infix notation $a <_x b$ instead of $(a, b) \in x$. $\mathrm{Sym}(M)$ (and, in particular, G) acts naturally on $\mathrm{LO}(M)$ as follows:

$$a <_{g \cdot x} b \iff g^{-1} \cdot a <_x g^{-1} \cdot b.$$

Our goal is to study the G-invariant measures on $\mathrm{LO}(M)$. There is always at least one such measure μ_{u} which is, in fact, invariant under all of $\mathrm{Sym}(M)$. It is defined by

$$\mu_{\mathrm{u}}(a_0 <_x \cdots <_x a_{k-1}) = 1/k! \quad \text{for all distinct } a_0, \dots, a_{k-1} \in M.$$

Here and below, we use the usual notation from probability theory and write $a_0 <_x \cdots <_x a_{k-1}$ for the event $\{x \in \mathrm{LO}(M) : a_0 <_x \cdots <_x a_{k-1}\}$. We will call μ_{u} the *uniform measure*. Glasner and Weiss [GW1] have shown that it is the only measure invariant under the whole symmetric group. (The proof is simple: the way the tuple $(a_0, \dots, a_{k-1})$ is ordered gives a partition of $\mathrm{LO}(M)$ into $k!$ pieces and for every two elements of this partition, there is an element of $\mathrm{Sym}(M)$ that sends one to the other, so they must all have the same measure.)

Our main theorem is the following.

Theorem 4.4.1. *Let M be a transitive, $\aleph_0$-categorical structure with no algebraicity that admits weak elimination of imaginaries. Consider the action $G \curvearrowright \mathrm{LO}(M)$. Then exactly one of the following holds:*

1. *The action $G \curvearrowright \mathrm{LO}(M)$ has a fixed point (i.e., there is a definable linear order on M);*

2. *μ_{u} is the unique G-invariant measure on $\mathrm{LO}(M)$.*

We describe a method to construct the uniform measure on $\mathrm{LO}(M)$ that will help illustrate our strategy for the proof. Let

$$\Omega = \{z \in [0,1]^M : z(a) \neq z(b) \text{ for all } a \neq b\}$$

and define the map $\pi \colon \Omega \to \mathrm{LO}(M)$ by

$$a <_{\pi(z)} b \iff z(a) < z(b) \quad \text{for } a, b \in M. \tag{4.2}$$

The group G acts naturally on $[0,1]^M$, Ω is a G-invariant set, and the map π is G-equivariant. Thus any G-invariant measure on Ω gives rise, via π, to a G-invariant measure on $\mathrm{LO}(M)$. In view of Corollary 4.3.5, the only G-invariant, ergodic measures on $[0,1]^M$ are of the form λ^M, where λ is a measure on $[0,1]$. It is clear that $(\lambda^M)(\Omega) = 1$ iff λ is non-atomic and in that case, $\pi_*(\lambda^M) = \mu_{\mathrm{u}}$ (this is true because λ^M is $\mathrm{Sym}(M)$-invariant and as we noted above, μ_{u} is the only $\mathrm{Sym}(M)$-invariant measure on $\mathrm{LO}(M)$). What we aim to show below is that if M does not admit a G-invariant linear order, then the map π is invertible almost everywhere for *any* G-invariant, ergodic measure on $\mathrm{LO}(M)$.

Let μ be an ergodic, G-invariant measure on $\mathrm{LO}(M)$. We will use probabilistic notation: we will denote by $<_x$ (or only by $<$ if there is no danger of confusion) a random element of $\mathrm{LO}(M)$ chosen according to μ, by $\mathbf{P}$ the probability of events and by $\mathbf{E}$ the expectation. If A is an event, we denote by $\mathbf{1}_A$ its characteristic function. For every $a \in M$, we denote by $\mathcal{F}_a$ the σ-field fixed by G_a.

For every 2-type τ and every $a \in M$, we define a random variable η_a^τ by

$$\eta_a^\tau = \mathbf{P}(c < a \mid \mathcal{F}_a), \quad \text{where } \mathrm{tp}(ac) = \tau.$$

The definition above does not depend on c but only on τ. Indeed, if $c' \in M$ is another element with $\mathrm{tp}\, ac' = \tau$, and $\zeta_a^\tau = \mathbf{P}(c' < a \mid \mathcal{F}_a)$, then for every $\xi \in L^2(\mathcal{F}_a)$, invariance implies that $\langle \xi, \mathbf{1}_{c<a} \rangle = \langle \xi, \mathbf{1}_{c'<a} \rangle$, so $\eta_a^\tau = \zeta_a^\tau$ a.s.

Lemma 4.4.2. *The random variables $(\eta_a^\tau)_{a \in M}$ are i.i.d.*

Proof. This is a direct consequence of Corollary 4.3.5. □

The following is a basic fact about conditional expectation that we will need.

Lemma 4.4.3. *Let $X \geq 0$ be an integrable random variable, A be an event and $\mathcal{F}$ be a σ-field. Suppose that $X > 0$ on A. Then $\mathbf{E}(X \mid \mathcal{F}) > 0$ on A.*

Proof. Let $Y = \mathbf{E}(X \mid \mathcal{F})$. Suppose, towards a contradiction, that for some measurable C, $\int_C X > 0$ but $\int_C Y = 0$. In particular, $C \subseteq \{Y = 0\}$. As the set $\{Y = 0\}$ is in $\mathcal{F}$, we have:

$$0 < \int_C X \leq \int_{Y=0} X = \int_{Y=0} Y = 0,$$

contradiction. □

If τ is a 2-type and $a \in M$, we define

$$D_\tau(a) = \{b \in M : \mathrm{tp}\, ab = \tau\}.$$

The next lemma is the main tool that allows us to recover the order from the random variables $(\eta_a^\tau)_{a \in M}$.

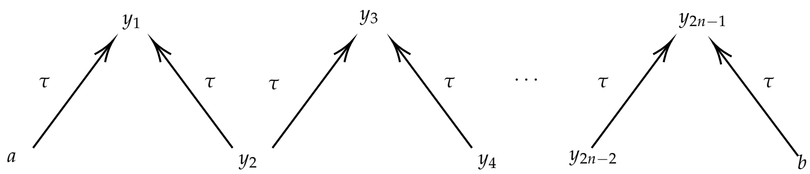


Figure 4.1: An alternating τ-path between a and b

Lemma 4.4.4. *Let the type τ and $a, b \in M$ be such that $D_\tau(a) \cap D_\tau(b) \neq \emptyset$. Then almost surely,*

$$a < b \implies \eta_a^\tau \leq \eta_b^\tau.$$

Moreover, for any $c \in D_\tau(a) \cap D_\tau(b)$, we have that almost surely,

$$a < c < b \implies \eta_a^\tau < \eta_b^\tau.$$

Proof. It follows from Theorem 4.3.4 that for distinct $a, b, c \in M$,

$$a < b, \mathcal{F}_b \underset{\mathcal{F}_a}{\perp\!\!\!\perp} c < a,$$

so

$$a < b \underset{\mathcal{F}_a\mathcal{F}_b}{\perp\!\!\!\perp} c < a. \tag{4.3}$$

Let $c \in D_\tau(a) \cap D_\tau(b)$. Using the fact that $\mathcal{F}_a \perp\!\!\!\perp_{\mathcal{F}_b} c < b$ (which follows from Theorem 4.3.4), (4.3), and their variants obtained by exchanging a and b, we have:

$$\begin{aligned}
\mathbf{E}(\mathbf{1}_{a<b} \mid \mathcal{F}_a\mathcal{F}_b)(\eta_b^\tau - \eta_a^\tau) &= \mathbf{E}(\mathbf{1}_{a<b} \mid \mathcal{F}_a\mathcal{F}_b)\big(\mathbf{P}(c < b \mid \mathcal{F}_b) - \mathbf{P}(c < a \mid \mathcal{F}_a)\big) \\
&= \mathbf{E}(\mathbf{1}_{a<b} \mid \mathcal{F}_a\mathcal{F}_b)\big(\,\mathbf{E}(\mathbf{1}_{c<b} \mid \mathcal{F}_b\mathcal{F}_a) - \mathbf{E}(\mathbf{1}_{c<a} \mid \mathcal{F}_a\mathcal{F}_b)\big) \\
&= \mathbf{E}(\mathbf{1}_{a<b}(\mathbf{1}_{c<b} - \mathbf{1}_{c<a}) \mid \mathcal{F}_a\mathcal{F}_b) \geq 0.
\end{aligned}$$

By Lemma 4.4.3, $\mathbf{E}(\mathbf{1}_{a<b} \mid \mathcal{F}_a\mathcal{F}_b)$ is a.s. strictly positive on the event $a < b$. This implies that on $a < b$, we have that $\eta_a^\tau \leq \eta_b^\tau$. The second assertion also follows from Lemma 4.4.3 because $\mathbf{1}_{a<b}(\mathbf{1}_{c<b} - \mathbf{1}_{c<a}) = 1$ on the event $a < c < b$ and hence, the last inequality is strict on that event. ☐

We will also need a combinatorial fact about 2-types. For a 2-type τ and $a, b \in M$, we say that $y_0, y_1, \ldots, y_{2n}$ is an *alternating τ-path* (or just a *τ-path* for brevity) between a and b if $y_0 = a$, $y_{2n} = b$ and $\operatorname{tp}(y_{2i}y_{2i+1}) = \operatorname{tp}(y_{2i+2}y_{2i+1}) = \tau$ for all $i = 0, \ldots, n-1$ and all of the nodes of the path are distinct. The *interior* of the path is the collection of all nodes except its endpoints. See Figure 4.1.

Lemma 4.4.5. *For all distinct $a, b \in M$ and any 2-type τ, there is $k \in \mathbb{N}$ such that for any finite $A \subseteq M$, there is an alternating τ-path of length k from a to b whose interior avoids A. In particular, there are infinitely many τ-paths between a and b of length k with pairwise disjoint interiors.*

Proof. Let us first prove that there is an alternating τ-path between a and b. Write $c \sim_\tau d$ if there is an alternating τ-path between c and d or $c = d$. We check that $\sim_\tau$ is an equivalence relation. Symmetry and reflexivity are clear from the definition. To check transitivity, consider a τ-path p_1 from c_0 to c_1

and a τ-path p_2 from c_1 to c_2 and suppose that $c_0 \neq c_2$. The concatenation $(y_0, \dots, y_{2n})$ of p_1 and p_2 satisfies all the conditions of a τ-path except possibly that vertices are distinct. Suppose for example that $y_i = y_j$ for some $i \neq j$. At least one of i and j is different from both 0 and $2n$; suppose for definiteness that this is true for i. By the no algebraicity assumption, there is an element $g \in G$ that fixes all points in $p_1 \cup p_2 \setminus \{y_i\}$ and such that $g \cdot y_i \notin p_1 \cup p_2$. Now replace y_i by $g \cdot y_i$. Thus we have reduced the number of coincidences in $p_1 \cup p_2$. We can repeat this procedure several times to finally conclude that there is a τ-path between c_0 and c_2 with distinct vertices.

By transitivity, there is $c \in M$ such that $\operatorname{tp} ac = \tau$. By the no algebraicity assumption, the orbit $G_c \cdot a$ is infinite, so the $\sim_\tau$-class of a is infinite. By transitivity and weak elimination of imaginaries, it follows that the $\sim_\tau$-class of a is all of M, so there is an alternating τ-path between a and b.

Now fix some alternating τ-path p between a and b and let k be the length of p. By the no algebraicity assumption, there is $g \in G_{ab}$ that moves p to a path whose interior is disjoint from A. $\square$

Denote by λ^τ the distribution of η_a^τ; this is a probability measure on $[0, 1]$ and by Lemma 4.4.2, it does not depend on a.

Lemma 4.4.6. *Suppose that the measure λ^τ is non-atomic. Then for all $a, b \in M$, we have that, almost surely,*

$$a < b \iff \eta_a^\tau < \eta_b^\tau.$$

Proof. First, we suppose that $D_\tau(a) \cap D_\tau(b) \neq \emptyset$. The contrapositive of the previous lemma gives us that in that case,

$$\eta_a^\tau < \eta_b^\tau \implies a < b. \tag{4.4}$$

Next we consider the general case. Suppose that there exists an alternating τ-path $y_0, \dots, y_{2n}$ such that

$$\eta_{y_0}^\tau < \eta_{y_2}^\tau < \cdots < \eta_{y_{2n}}^\tau. \tag{4.5}$$

Then for all i, $D_\tau(y_{2i}) \cap D_\tau(y_{2i+2}) \neq \emptyset$, so by the above observation, we obtain that $a = y_0 < y_2 < \cdots < y_{2n} = b$. Now condition on η_a^τ, η_b^τ and suppose that $\eta_a^\tau < \eta_b^\tau$. As the η_c^τ are i.i.d. with non-atomic distribution, for a fixed τ-path $(y_0, \dots, y_{2n})$ between a and b, (4.5) holds with positive probability that only depends on n. By Lemma 4.4.5, there exist infinitely many τ-paths of the same length between a and b with disjoint interiors and whether (4.5) holds for them are independent events with the same probability. Thus almost surely at least one of them happens and we conclude that (4.4) holds for all a, b. For the reverse implication, it suffices to notice that $\mathbf{P}(\eta_a^\tau = \eta_b^\tau) = 0$. $\square$

Lemma 4.4.6 allows us to conclude in the case where λ^τ is non-atomic.

Lemma 4.4.7. *Suppose that for some type τ, the distribution λ^τ is non-atomic. Then $\mu = \mu_{\mathrm{u}}$.*

Proof. Define $\rho\colon \mathrm{LO}(M) \to [0, 1]^M$ by $\rho(x)(a) = \eta_a^\tau(x)$ (ρ is defined μ-a.e.). By Lemma 4.4.6, $\pi \circ \rho = \mathrm{id}$ μ-a.e. (π is defined by (4.2)). By Lemma 4.4.2, $\rho_* \mu = (\lambda^\tau)^M$. Applying π to both sides, we obtain that

$$\mu = \pi_* \rho_* \mu = \pi_* (\lambda^\tau)^M = \mu_{\mathrm{u}}.$$ $\square$

Now we are left with the case where the distribution λ^τ has atoms for all 2-types τ and we will eventually conclude that there is a G-invariant linear order on M.

From here on, as we will deal with several measures simultaneously, we will incorporate the measure in our notation. If μ is an ergodic measure on $\mathrm{LO}(M)$, τ is a 2-type, and $p \in [0,1]$ is an atom for the distribution λ^τ, we define a new measure $\nu_{\mu,\tau,p}$ on basic clopen sets by

$$\mathbf{P}_{\nu_{\mu,\tau,p}}(a_0 < \cdots < a_{k-1}) = \mathbf{P}_\mu(a_0 < \cdots < a_{k-1} \mid \eta^\tau_{a_0} = \cdots = \eta^\tau_{a_{k-1}} = p), \quad (4.6)$$

where $a_0, \ldots, a_{k-1}$ are pairwise distinct elements of M. We note that as by Theorem 4.3.4,

$$a_0 < \cdots < a_{k-1}, \eta_{a_0}, \ldots, \eta_{a_{k-1}} \perp\!\!\!\perp \eta_{b_0}, \ldots, \eta_{b_{m-1}}$$

for any $\{b_0, \ldots, b_{m-1}\} \cap \{a_0, \ldots, a_{k-1}\}$, we can condition additionally on $\eta^\tau_{b_0} = \cdots = \eta^\tau_{b_{m-1}} = p$ on the right-hand side of (4.6) without changing the result.

For the next lemma, we will need the following well-known general ergodicity criterion.

Proposition 4.4.8. *Let G be a group and $G \curvearrowright (X, \mu)$ be a measure-preserving action. Suppose that the collection*

$$\{A \in \mathrm{MALG}(X, \mu) : \exists g \in G \; g \cdot A \perp\!\!\!\perp A\}$$

is dense in $\mathrm{MALG}(X, \mu)$. Then the action $G \curvearrowright X$ is ergodic.

Proof. Suppose that B is G-invariant. For every $\epsilon > 0$, there exist A and g such that $\mu(A \triangle B) < \epsilon$ and $g \cdot A \perp\!\!\!\perp A$. We have that

$$2(\mu(A) - \mu(A)^2) = \mu(A \triangle g \cdot A) \leq \mu(B \triangle g \cdot B) + 2\epsilon = 2\epsilon.$$

Taking a limit as $\epsilon \to 0$ yields that $\mu(B) - \mu(B)^2 = 0$, so that $\mu(B) = 0$ or 1. □

Lemma 4.4.9. *Let μ be a G-invariant, ergodic measure on $\mathrm{LO}(M)$, τ be a 2-type, and p be an atom for λ^τ. Then $\nu_{\mu,\tau,p}$ extends to a G-invariant, ergodic measure on $\mathrm{LO}(M)$.*

Proof. For brevity, write $\nu = \nu_{\mu,\tau,p}$. To define ν on a general clopen set U, we represent it as a disjoint union of basic clopen sets and use (4.6). It follows from the remark after (4.6) that this is well-defined and gives rise to a finitely additive measure on the Boolean algebra of clopen subsets of $\mathrm{LO}(M)$. Now it follows from the Carathéodory extension theorem that ν extends to a Borel measure on $\mathrm{LO}(M)$.

G-invariance of ν follows from (4.6) and the G-invariance of μ. Finally, ergodicity follows from Proposition 4.4.8, whose hypothesis is verified by virtue of Theorem 4.3.4 applied to μ. □

If τ is a 2-type, say that a measure μ on $\mathrm{LO}(M)$ *respects* τ if for all $a, b, c \in M$ such that $\operatorname{tp} ac = \operatorname{tp} bc = \tau$ and μ-a.e. $x \in \mathrm{LO}(M)$, c is not between a and b in the order $<_x$.

Lemma 4.4.10. *Let μ be a G-invariant, ergodic measure on $\mathrm{LO}(M)$, τ be a 2-type, and p be an atom for λ^τ. Let $\nu = \nu_{\mu,\tau,p}$. Then the following hold:*

1. *ν respects τ;*

2. *If τ' is a 2-type and μ respects τ', then ν respects τ'.*

Proof. 1 Let $a, b, c \in M$ be such that $\operatorname{tp} ac = \operatorname{tp} bc = \tau$. Using Lemma 4.4.4, we have that

$$\begin{aligned}\mathbf{P}_\nu(a < c < b) &= \mathbf{P}_\mu(a < c < b \mid \eta^\tau_a = \eta^\tau_b = \eta^\tau_c = p) \\ &\leq \frac{\mathbf{P}_\mu(a < c < b \text{ and } \eta^\tau_a = \eta^\tau_b)}{\mathbf{P}_\mu(\eta^\tau_a = \eta^\tau_b = \eta^\tau_c = p)} = 0.\end{aligned}$$

We obtain similarly that $\mathbf{P}_\nu(b < c < a) = 0$.

2 This is clear from the definition. □

Lemma 4.4.11. *Suppose that μ is a G-invariant, ergodic measure on* $\mathrm{LO}(M)$ *which respects all* 2*-types. Then μ is a Dirac measure.*

Proof. We will prove that the order between two elements $a, b \in M$ is almost surely determined by $\operatorname{tp} ab$. More formally, we will show that for all $a \neq b$, we have that a.s.,

$$\operatorname{tp} ab = \operatorname{tp} a'b' \implies \big(a < b \iff a' < b'\big).$$

What the hypothesis gives us is that for all $c, d, e \in M$, a.s.,

$$\operatorname{tp} ce = \operatorname{tp} de \implies \big(c < e \iff d < e\big). \tag{4.7}$$

Let $\tau = \operatorname{tp} ab = \operatorname{tp} a'b'$ and use Lemma 4.4.5 to construct a τ-path $a = y_0, \ldots, y_{2n} = a'$ from a to a' whose interior avoids b and b'. Applying (4.7) consecutively, we obtain that:

$$\begin{aligned}a < b \iff a < y_1 \iff y_2 < y_1 \iff y_2 < y_3 \iff \cdots \\ \iff y_{2n-1} < a' \iff a' < b',\end{aligned}$$

which concludes the proof. □

Now we can complete the proof of the theorem.

Proof of Theorem 4.4.1. Let μ be a G-invariant, ergodic measure on $\mathrm{LO}(M)$. Enumerate all 2-types as $\tau_0, \ldots, \tau_{n-1}$. If $\lambda^{\tau_0}_\mu$ is non-atomic, then we can apply Lemma 4.4.7 and conclude that $\mu = \mu_{\mathrm{u}}$. Otherwise, we construct a sequence of invariant, ergodic measures $\mu_0, \ldots, \mu_n$ such that for all $i < n$, μ_i respects $\tau_0, \ldots, \tau_{i-1}$ and $\lambda^{\tau_i}_{\mu_i}$ has an atom. Set $\mu_0 = \mu$ and suppose that μ_i is already constructed. Set $\mu_{i+1} = \nu_{\mu_i, \tau_i, p_i}$, where p_i is some atom for $\lambda^{\tau_i}_{\mu_i}$. By Lemma 4.4.10, μ_{i+1} respects $\tau_0, \ldots, \tau_i$. Moreover, $\lambda^{\tau_{i+1}}_{\mu_{i+1}}$ must have an atom: otherwise, by Lemma 4.4.7, $\mu_{i+1} = \mu_{\mathrm{u}}$, which is not possible because μ_{u} has full support and μ_{i+1} does not (as μ_{i+1} respects τ_i). Finally, apply Lemma 4.4.11 to conclude that μ_n is a Dirac measure, which, by invariance, implies that the action $G \curvearrowright \mathrm{LO}(M)$ has a fixed point.

Thus we have proved that either μ_{u} is the unique ergodic, invariant measure on $\mathrm{LO}(M)$ or there is a fixed point for the action. However, as convex combinations of ergodic measures are dense in the space of all invariant measures (see, e.g., [P3, Section 12]), this implies that in that case, μ_{u} is indeed the unique invariant measure. □

Proof of Corollary 4.1.3. Let $Z \subseteq \mathrm{LO}(M)$ be any minimal subsystem. By the hypothesis, Z is not a point and it is a proper subset of $\mathrm{LO}(M)$. If G is amenable, then there must be a G-invariant measure supported on Z, which contradicts Theorem 4.4.1. □

Proof of Corollary 4.1.4. By [KR, Proposition 6.4], the Hrushovski property implies that there are compact subgroups $K_0 \leq K_1 \leq \cdots$ of G with $\bigcup_n K_n$ dense in G. In particular, G is amenable.

If $K \leq G$ is any compact subgroup, then the orbits of K on M are finite. If M admits a G-invariant linear order, then the K-orbits must be trivial, so K is trivial. We conclude that there is no G-invariant linear order on M, so, by Corollary 4.1.3, the action $G \curvearrowright \mathrm{LO}(M)$ must be minimal. This implies that M has the ordering property (see [NVT, Theorem 4]). □

4.5 Examples and other invariant measures

4.5.1 Examples

We briefly discuss some examples that show that the assumptions of Theorem 4.1.1 and Theorem 4.1.2 are mostly necessary. This section requires more familiarity with Fraïssé theory than the rest of the paper.

Transitivity

Let $\mathcal{L}$ be a language with two unary predicates P and Q and consider the age consisting of all $\mathcal{L}$-structures for which $P \cap Q = \emptyset$ and every point satisfies either P or Q. Let M be its Fraïssé limit. Then one can randomly order M as follows. Let $(\xi_a : a \in M)$ be uniform, i.i.d. on $[0,1]$ and define an $\mathrm{Aut}(M)$-invariant random order $<$ on M by declaring all elements of P to be smaller than all elements of Q and $a < b \iff \xi_a < \xi_b$ if a and b both belong to P or to Q. This shows that the transitivity assumption in Theorem 4.1.2 is necessary.

No algebraicity

Let V be the countable-dimensional vector space over $\mathbf{F}_2$, the field with two elements. Let V^* be its dual: the space of linear maps from V to $\mathbf{F}_2$. V^* embeds as a subspace of $\mathbf{F}_2^V$ and, being a compact group, has a Haar measure which is invariant under the action of $\mathrm{Aut}(V)$. This gives an invariant, ergodic measure on $\mathbf{F}_2^V$ which is not a product measure and shows that one cannot omit the no algebraicity assumption in Theorem 4.1.1.

The same example also shows that this assumption cannot be omitted in Theorem 4.1.2. The universal minimal flow of $\mathrm{Aut}(V)$ is a proper subspace of $\mathrm{LO}(V)$ (see [KPT, Theorem 8.2]) and it carries a (unique) invariant measure [AKL, Section 10]. This measure can be obtained as a factor of the measure on $\mathbf{F}_2^V$ constructed above.

Weak elimination of imaginaries

In the presence of $\mathrm{Aut}(M)$-invariant equivalence relations on M, it is easy to construct distributions for $(\xi_a : a \in M)$ for which the random variables are not independent. One can, for example, toss a coin for each equivalence class and set $\xi_a = 0$ or 1 depending whether the coin toss for the class of a resulted in heads or tails.

In view of Remark 4.3.6, it is more interesting to ask whether the weak elimination of imaginaries assumption can be replaced just by requiring *primitivity* of the action $\mathrm{Aut}(M) \curvearrowright M$, that is, the absence of invariant equivalence relations on M. (This would also have the advantage of being much easier to check.) It turns out that the answer is negative, as the following example shows.

Let the signature $\mathcal{L}$ consist of two unary relations S_0 and S_1 and a binary relation R. We consider the class $\mathcal{A}$ of finite bipartite graphs viewed as $\mathcal{L}$-structures, where the two parts of the graphs are labeled by S_0 and S_1 and R is the edge relation. For a point a, we denote by $R(a)$ the set of R-neighbors of a. For elements of $\mathcal{A}$, we require furthermore that the degree of every point in S_0 is 2 and that $|R(a) \cap R(b)| \leq 1$ for all $a \neq b$ in S_0. It is easy to check that this is an amalgamation class; let N be the Fraïssé limit. Denote $M = \{a \in N : S_0(a)\}$ and $P = \{a \in N : S_1(a)\}$. The class $\mathcal{A}$ is not hereditary, so N is not fully homogeneous but we do have homogeneity for algebraically closed, finite substructures of N. A finite substructure $A \subseteq N$ is algebraically closed iff for every $a \in A \cap M$, the degree of a (calculated in A) is 2 (that is, $A \in \mathcal{A}$).

Now consider M as a structure on its own (in a different signature) with relations given by the traces of all definable relations on N. As N is $\aleph_0$-categorical, M is too. Using the homogeneity of N, it is easy to check that M has no algebraicity and that the action $\mathrm{Aut}(M) \curvearrowright M$ is primitive. Indeed, the action of $\mathrm{Aut}(M)$ on pairs of distinct elements of M has exactly two orbits: $\{(a, b) : |R(a) \cap R(b)| = 1\}$ and $\{(a, b) : R(a) \cap R(b) = \emptyset\}$ and none of them is an equivalence relation.

There is a homomorphism $\mathrm{Aut}(N) \to \mathrm{Aut}(M)$ given by the natural action of $\mathrm{Aut}(N)$ on M. As the elements of P can be recovered as imaginary elements of M, it turns out that this homomorphism is an isomorphism. With all of this in mind, it is easy to construct non-independent, $\mathrm{Aut}(M)$-invariant distributions of random variables $(\xi_a : a \in M)$. For example, we can start with i.i.d. $(\eta_b : b \in P)$ uniform in $[0, 1]$ and define

$$\xi_a = \min\{\eta_b : b \in P, a\, R\, b\}.$$

This also allows to construct non-uniform, invariant measures on $\mathrm{LO}(M)$: just define a random order $<$ on M as usual by $a < b \iff \xi_a < \xi_b$.

$\aleph_0$-categoricity.

We do not know whether $\aleph_0$-categoricity is necessary in either Theorem 4.1.1 or Theorem 4.1.2, although it is crucial for our proofs. In the absence of $\aleph_0$-categoricity, however, the other assumptions may need tweaking as the correspondence between model theory and permutation groups breaks down.

4.5.2 Other invariant measures on LO

One may ask, under the assumptions of Theorem 4.4.1, what other ergodic, invariant measures there are on $\mathrm{LO}(M)$ apart from the uniform measure and fixed points. A slight variation of the method we used to construct μ_{u} yields the following. Let λ be a probability measure on $[0,1]$ and let $S = \{z \in [0,1] : \lambda(\{z\}) > 0\}$ be the set of its atoms (it can be finite or countable). Let $F \subseteq \mathrm{LO}(M)$ be the set of G-fixed points (which, by Theorem 4.4.1, has to be non-empty if we want to construct anything interesting) and finally, let $f\colon S \to F$ be an arbitrary function. Note that $\aleph_0$-categoricity of M implies that F is finite. Let $\pi\colon [0,1]^M \to \mathrm{LO}(M)$ be defined (λ^M-a.e.) by

$$a <_{\pi(z)} b \iff z(a) < z(b) \text{ or } \big(z(a) = z(b) \text{ and } a <_{f(z(a))} b\big).$$

Then $\pi_*(\lambda^M)$ is an invariant, ergodic measure on $\mathrm{LO}(M)$.

For many structures M, the methods we developed for the proof of Theorem 4.4.1 can be used to show that all ergodic, invariant measures on $\mathrm{LO}(M)$ can be obtained as above; however, in the presence of definable cuts, more complicated constructions are possible. We just give one example.

Consider the language $\mathcal{L} = \{<, f\}$, where $<$ is a binary relation and f is a unary function. Let $\mathcal{A}$ be the age consisting of all finite $\mathcal{L}$-structures where $<$ is interpreted as a linear order and f is an involution without fixed points. It is easy to check that $\mathcal{A}$ amalgamates; let N be its Fraïssé limit. As for every n, the structure generated by n points is of size at most $2n$ and there are only finitely many structures of any given finite size, N is $\aleph_0$-categorical. Let $M = \{a \in N : f(a) < a\}$ and $M' = \{a \in N : f(a) > a\}$. It follows from homogeneity that M and M' are the two orbits of the action $\mathrm{Aut}(N) \curvearrowright N$. Now consider M as a structure on its own with relations defined as the traces of definable relations from N (in particular, the relations $a < b$, $f(a) < b$, $f(a) < f(b)$ for $a, b \in M$ are definable in the structure M). From a permutation group perspective, we can consider the homomorphism $\pi\colon \mathrm{Aut}(N) \to \mathrm{Sym}(M)$ given by the natural action $\mathrm{Aut}(N) \curvearrowright M$ and then $\mathrm{Aut}(M) = \overline{\pi(\mathrm{Aut}(N))}$ (this is because $\mathrm{Aut}(M)$ and $\mathrm{Aut}(N)$ have the same orbits on M^k for every k). It follows from the homogeneity of N that M is transitive, $\aleph_0$-categorical, and has no algebraicity. (The algebraic closure operator in N is given by $\mathrm{acl}(A) = A \cup f(A)$.) One can also verify weak elimination of imaginaries, for example using the criterion from [R, Proposition 10.1].

We can construct an invariant measure on $\mathrm{LO}(M)$ as follows. Let $(\eta_a)_{a \in M}$ be a collection of i.i.d., Bernoulli, $\{0,1\}$-valued random variables, where each of the two values is taken with probability $1/2$, and define a random order $\prec$ on M by

$$a \prec b \iff f^{\eta_a}(a) < f^{\eta_b}(b),$$

where $f^0 = \mathrm{id}$ and $f^1 = f$. This random order is different from the ones considered above.

4.5.3 Application: a dynamical proof of unique ergodicity of the automorphism group of the 2-graph

The notations in this subsection follow the ones from Chapter 2. For the sake of readability, we remind here all the basic properties of the 2-graph.

Recall that we call even hypergraph a finite 3-regular hypergraphs such that the number of hyperedges on any 4 vertices is even. The class of even hypergraphs is a Fraïssé class (see [EHKN]). We denote by $\mathbb{H}$ the Fraïssé limit of $\mathcal{H}$. This limit is called the 2-graph. We write $R^{\mathbb{H}}$ for the hyperedge relation in $\mathbb{H}$.

There is a map from graphs to even hypergraphs by the following operation: from a graph, one obtains an even hypergraph with the same domain by putting an hyperedge between 3 vertices iff there is an even number of edges between those vertices in the original graph. One can check that this always gives an even hypergraph. For a graph $\mathbf{A}$, we call reduct of $\mathbf{A}$ the even hypergraph thus obtained, and we denote it by $\mathrm{red}_{\mathbb{H}}(\mathbf{A})$.

For a given even hypergraph $\mathbf{H}$, a graph on the same vertex set as $\mathbf{H}$ whose reduct is isomorphic to $\mathbf{H}$ is called a *graphing* of $\mathbf{H}$.

An important remark is that a graphing of an even hypergraph $\mathbf{H}$ is entirely determined by the edges relation between one point $a \in \mathbf{H}$ and all the other points in $\mathbf{H}$. Indeed, if we want to know if there is an edge between to points x and x', we have the following possibilities:

1) There is an hyperedge (a, x, x') in $\mathbf{H}$. In this case, there is an edge between x and x' iff there is an odd number of edges between a and $\{x, x'\}$.

2) There is no hyperedge (a, x, x') in $\mathbf{H}$. In this case, there is an edge between x and x' iff there is an even number of edges between a and $\{x, x'\}$.

We remark in particular that there are 2^{n-1} graphings of a given even hypergraph on n vertices.

Take $\mathbf{G}$ and $\mathbf{G}'$ two finite graphs with the same vertex set D. We denote by E the edge relation for graphs. We say that $\mathbf{G}'$ is in the switching class of $\mathbf{G}$ if there is $A \subset D$ such that for all $x, y \in A$, $E^{\mathbf{G}}(x,y) \iff E^{\mathbf{G}'}(x,y)$; for all $x, y \notin A$, $E^{\mathbf{G}}(x,y) \iff E^{\mathbf{G}'}(x,y)$; and for all $x \in A$ and $y \notin A$, $E^{\mathbf{G}}(x,y) \iff \neg E^{\mathbf{G}'}(x,y)$. Remark that by symmetry of the edge relation, this is the same as saying or all $x \notin A$ and $y \in A$, $E^{\mathbf{G}}(x,y) \iff \neg E^{\mathbf{G}'}(x,y)$. In this context, we call $\mathbf{G}'$ the switching of $\mathbf{G}$ by A.

Remark that the reduct of a graph $\mathbf{G}$ is isomorphic to the reduct of another graph $\mathbf{G}'$ iff $\mathbf{G}'$ is isomorphic to a graph in the switching class of $\mathbf{G}$, indeed this operation will not change the parity of the number of edges in a triplet of vertices. In particular, the class of graphings of a given even hypergraph can be recovered by all the switchings of any of its graphing. One way to see this is simply to observe that there are exactly 2^{n-1} different switchings of a graph on n vertices, which corresponds to the number of graphings of the reduct of this graph.

The UMF of $\mathrm{Aut}(\mathbb{H})$ was computed by Evans, Hubicka, Konecny and Nesestril in [EHKN]. It is $G \curvearrowright \mathrm{LO}(\mathbb{H}) \times \mathrm{Gr}(\mathbb{H})$, where $\mathrm{LO}(\mathbb{H})$ is the space of linear orderings of $\mathbb{H}$ and $\mathrm{Gr}(\mathbb{H})$ the space of graphings of $\mathbb{H}$. Let us take $(<, E) \in \mathrm{LO}(\mathbb{H}) \times \mathrm{Gr}(\mathbb{H})$, $g \in G$, then for all $a, b \in \mathbb{H}$, we have

$$a(g \cdot <)b \Leftrightarrow g^{-1}(a) < g^{-1}(b)$$

and

$$g \cdot E(a,b) \Leftrightarrow E(g^{-1}(a), g^{-1}(b)).$$

It will be useful to consider $\mathcal{H}^*$ the Fraïssé class of graphed ordered even hypergraphs. We call $\mathbb{H}^*$ its Fraïssé limit.

We will use Theorem 4.3.4 to get the description of the ergodic invariant measures of some G-actions. Let us first show that $\mathbb{H}$ verifies the hypothesis of this Theorem.

First, we remark that $\aleph_0$-categoricity, the absence of algebraicity and transitivity are inherited from the random graph because $\mathbb{H}$ is one of its reducts. We spend some time on weak elimination of imaginaries. We use a criterion from Poizat, stated as Lemma 16.17 in [P4]:

Proposition 4.5.1. *Suppose $\mathbb{F}$ is $\aleph_0$-categorical with no algebraicity. Let $H = Aut(M)$. Then weak elimination of imaginaries is implied by $\langle H_B, H_C \rangle = H_{B\cap C}$ for all finite $B, C \subseteq M$, where H_B is the pointwise stabiliser of B.*

I am thankful to David Evans for pointing me towards this proposition.
We use it to prove:

Proposition 4.5.2. $\mathbb{H}$ *weakly eliminates imaginaries.*

Proof. Let B and C be finite substructures of $\mathbb{H}$. It is easy to see that $\langle G_B, G_C \rangle \leq G_{B\cap C}$. Moreover, $\langle G_B, G_C \rangle$ is closed in $G_{B\cap C}$, so we only have to prove that $\langle G_B, G_C \rangle$ is dense in $G_{B\cap C}$.

Let us take $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_n\}$ two finite substructures of $\mathbb{H}$ such that there is an element of $G_{B\cap C}$ that sends x_i to y_i for all $1 \leq i \leq n$. We want to prove that there is an element of $\langle G_B, G_C \rangle$ that sends x_i to y_i for all $1 \leq i \leq n$. Because of the absence of algebraicity of $\mathbb{H}$, up to using more elements of $\langle G_B, G_C \rangle$, we can assume that X and Y are disjoint and also disjoint from B and C.

Let us take a compatible graphing E of $B \cup C \cup X \cup Y$ such that for all $a \in B \cap C$ $E(a, x_i) \Leftrightarrow E(a, y_i)$ for all $1 \leq i \leq n$. This is always possible, because X and Y are in the same $G_{B\cap C}$-orbit. Indeed, one can take a graphing of $(B \cap C) \cup X$ and graph $(B \cap C) \cup Y$ as the image of the graphing of $(B \cap C) \cup X$ by an element of $G_{B\cap C}$ that sends X to Y. R Remark that this also forces the edges between X and Y, because of the parity condition. The rest of the edges are chosen so that E is compatible with the even hypergraph structure. Such a graphing exists because a graphing of a given even hypergraph can alway be extended to a graphing of a bigger even hypergraph. The proof of this fact is an easy induction.

Let us take $Z = \{z_1, \ldots, z_n\}$ with $z_i \notin B \cup C \cup X \cup Y$ and construct E' a graphing of $B \cup C \cup X \cup Y \cup Z$ such that for all $h, h' \in B \cup C \cup X \cup Y$, we have $E(h, h') \Leftrightarrow E'(h, h')$ and for all $1 \leq i \leq n$, and $b \in B, c \in C$, we have $E'(b, z_i) \Leftrightarrow E(b, x_i)$ and $E'(c, z_i) \Leftrightarrow E(c, y_i)$. There is no issue over the intersection of B and C, because of the way we constructed E.

Let us now consider L the reduct of $(B \cup C \cup X \cup Y \cup Z, E')$. It embeds in $\mathbb{H}$ in such a way that we can identify $B \cup C \cup X \cup Y$ to their original elements. We will also identify Z and its image.

By construction, there is $g_B \in G_B$ that sends X to Z and $g_C \in G_C$ that sends Z to Y, therefore $g_C \circ g_B$ is as wanted.

□

We can now state the result we will actually use:

Proposition 4.5.3. *Let $x_1, \ldots, x_n, y \in \mathbb{H}$. Let us denote by D_y the $G_{x_1,\ldots,x_n}$ orbit of y. Then the $G_{x_1,\ldots,x_n}$-ergodic and $G_{x_1,\ldots,x_n}$-invariant measures on 2^{D_y} are of the form* $\mathrm{Ber}(p)^{\otimes D_y}$ *where $p \in [0, 1]$ and* $\mathrm{Ber}(p)$ *stands for the Bernoulli distribution with parameter p.*

Proof. Let μ be a $G_{x_1,\ldots,x_n}$-ergodic invariant measure on 2^{D_y}. We prove that for any disjoints $A, B \subset D_y$, any two random variables η_A and η_B fixed by

$G_{x_1,\ldots,x_n} \cap G_A$ and $G_{x_1,\ldots,x_n} \cap G_B$ respectively, are independent. For this, we use Theorem 4.3.4 i).

We consider $\mathbb{H}_1$ the structure in the language $(R, c_1, \ldots, c_n)$, where R is a 3-ary relation (hyperedges) and $c_1, \ldots, c_n$ are constants. The domain of $\mathbb{H}_1$ is equal to the domain of $\mathbb{H}$, and the the interpretation of the constants are such that $x_i =^{\mathbb{H}_1} c_i$ for all $i \leq n$ and the interpretation of the relation R is such that $R^{\mathbb{H}}(z_1, z_2, z_3) \Leftrightarrow R^{\mathbb{H}_1}(z_1, z_2, z_3)$ for all z_1, z_2, z_3.

We first remark that $\mathrm{Aut}(\mathbb{H}_1) = G_{x_1,\ldots,x_n}$. Using the Ryll-Nardzewski characterisation of $\aleph_0$-categoricity, we can see that $\mathbb{H}_1$ is $\aleph_0$-categorical because $\mathrm{Aut}(\mathbb{H}_1) \curvearrowright \mathbb{H}_1^n$ has finitely many orbits for every $n \in \mathbb{N}$. Moreover, $\mathbb{H}_1$ weakly eliminates imaginaries. Indeed, let V be an open subgroup of $G_{x_1,\ldots,x_n}$. By weak elimination of imaginaries of G, there is a $\bar{z}$ such that $G_{\bar{z}} \leq V$ with finite index in V. Necessarily $G_{\bar{z}} \leq G_{x_1,\ldots,x_n}$, which by the no algebraicity assumption implies $\{x_1, \ldots, x_n\} \subset \bar{z}$, therefore we have weak elimination of imaginaries for $\mathbb{H}_1$.

Finally, by the no algebraicity of $\mathbb{H}$, $A \cup \{x_1, ..., x_n\}$ and $B \cup \{x_1, ..., x_n\}$ have to be algebraically closed. Therefore we can apply Theorem 4.3.4 i) to get $\eta_A \perp\!\!\!\perp_{\mathcal{F}_{x_1,\ldots,x_n}} \eta_B$. The ergodicity of μ tells us that $\mathcal{F}_{x_1,\ldots,x_n}$ is trivial, therefore the variables are independent.

$\square$

Proof of unique ergodicity

Let us first observe that $G \curvearrowright \mathrm{LO}$ is uniquely ergodic, by Theorem 4.4.1.

We can also prove that $G \curvearrowright \mathrm{Gr}$ is uniquely ergodic. We will denote by $\sim$ the relation of having an edge between two vertices. For all $a \in \mathbb{H}$, there is a continuous G-map π_a from Gr to $2^{\mathbb{H}\setminus\{a\}}$. The map associates to an ordered graphing x the function f_x such that $f_x(b) = 1$ iff $a \sim_x b$. This map is a bijection because of the parity condition on the edges with regard to the hyperedges of $\mathbb{H}$.

Lemma 4.5.4. *Let ν be a G-invariant measure on* Gr. *The pushforward $\nu_a = (\pi_a)_*\nu$ is* $\mathrm{Ber}(1/2)^{\otimes \mathbb{H}\setminus\{a\}}$.

Proof. By Proposition 4.5.3 the G_a-ergodic invariant measures of $2^{M\setminus\{a\}}$ are of the form $\mathrm{Ber}(p)^{\otimes M\setminus\{a\}}$.

We use the ergodicity decomposition theorem to get

$$\nu_a = \int_{[0,1]} \mathrm{Ber}(p)^{\otimes M\setminus\{a\}} \mathrm{d}\rho(p)$$

for some measure ρ on $[0,1]$.

Take x, x' such that $R^{\mathbb{H}}(a, x, x')$ and y, y' such that $\neg R^{\mathbb{H}}(a, y, y')$. Remark that $(x, x', R^{\mathbb{H}})$ is isomorphic to $(y, y', R^{\mathbb{H}})$, therefore there is an automorphism of $\mathbb{H}$ that sends (x, x') to (y,y′) and we have $\mathbb{P}_\nu(x \sim x') = \mathbb{P}_\nu(y \sim y')$. Let us compute this quantity in two different ways.

First, we see that $\mathbb{P}_\nu(x \sim x') = \int_{[0,1]} p^2 + (1-p)^2 \mathrm{d}\rho(p)$. Indeed $x \sim x'$ iff there is an even number of edges between a and (x, x'), therefore $\mathbf{1}_{x \sim x'} =$

$\mathbf{1}_{a\sim x, a\sim x'} + \mathbf{1}_{a\nsim x, a\nsim x'}$. We can therefore compute:

$$\begin{aligned}\mathbb{P}_\mu(x \sim x') &= \mathbb{E}_\nu[\mathbf{1}_{x\sim x'}] \\ &= \mathbb{E}_\nu[\mathbf{1}_{a\sim x, a\sim x'} + \mathbf{1}_{a\nsim x, a\nsim x'}] \\ &= \int_{[0,1]} \mathbb{E}_{\mathrm{Ber}(p)^{\otimes M\setminus\{a\}}}[\mathbf{1}_{a\sim x, a\sim x'} + \mathbf{1}_{a\nsim x, a\nsim x'}]\mathrm{d}\rho(p). \\ &= \int_{[0,1]} p^2 + (1-p)^2 \mathrm{d}\rho(p)\end{aligned}$$

Similarly, we have $\mathbb{P}_\mu(y \sim y') = \int_{[0,1]} 2p(1-p)\mathrm{d}\rho(p)$.

The equality $\mathbb{P}_\mu(x \sim x') = \mathbb{P}_\mu(y \sim y')$ implies that $\int_{[0,1]}(1-2p)^2\mathrm{d}\rho = 0$. Therefore $\rho = \delta_{1/2}$. □

This allows us to conclude that $G \curvearrowright \mathrm{Gr}$ is uniquely ergodic, because all invariant measures are sent to the same measure via the pushforward by π_a, and because π_a is a bijection, all invariant measures of $G \curvearrowright \mathrm{Gr}$ must be equal.

We can finally prove that $G \curvearrowright \mathrm{LO} \times \mathrm{Gr}$ is uniquely ergodic. Let μ be a G-ergodic invariant measure on $\mathrm{LO} \times \mathrm{Gr}$.

We will denote by $\widetilde{E}$ the random graph relation induced by μ on $\mathbb{H}$.

Recall that $R = (\mathbb{N}, E)$ is the random graph, of which $\mathbb{H}$ is a reduct. We will denote by $\simeq$ the graph isomorphism.

Proposition 4.5.5. *There exists a measure ν on $\mathrm{LO}(R)$ such that, for all $(x_1, \dots, x_n, E) \subset R$ and $(y_1, \dots, y_n, H)$ the reduct of $(x_1, \dots, x_n, E)$, we have that*

$$\nu(x_1 < \ldots < x_n) := \mu(y_1 < \ldots < y_n | (y_1, \dots, y_n, \widetilde{E}) \simeq (x_1, \dots, x_n, E))$$

Proof. Let us check, using Theorem 1.3.5 that ν can indeed be extended to a measure on $\mathrm{LO}(R)$. Conditions 1) is obviously satisfied, but we need prove 2).

Let us consider $(x_1, ..., x_n, z, E) \in R$ and $(y_1, ..., y_n, z', R^{\mathbb{H}})$ the associated reduct in $\mathbb{H}$. We want to prove that

$$\begin{aligned}\nu(x_1 < \cdots < x_n) = &\left(\sum_{1\le i\le n-1} \nu(x_1 \ldots < x_i < z < x_{i+1} < \ldots < x_n) \right) \\ &+ \nu(z < x_1 < \ldots < x_n) + \nu(x_1 < \ldots < x_n < z).\end{aligned}$$

We call $(*)$ this equality.

We define a family of events in $\mathrm{LO}(\mathbb{H})$:

$$U_i = \{y_1 \ldots < y_i < z' < y_{i+1} < \ldots < y_n\}$$

for i ranging from 1 to $n-1$.

We also set $U_0 = \{z < y_1 < \ldots < y_n\}$ and $U_n = \{y_1 < \ldots < y_n < z\}$.

In order not to have too heavy notations, $\overline{y}$ will denote $(y_1, \dots, y_n)$ and $\overline{x} = (x_1, \dots, x_n)$. If the edge set is not specified, $\overline{y}$ is assumed to have edge set $\widetilde{E}$ and $\overline{x}$ is assumed to have edge set E.

In those terms, $(*)$ becomes:

$$\mu(y_1 < \ldots < y_n | \overline{y} \simeq \overline{x}) = \sum_i \mu(U_i | (\overline{y}, z', \widetilde{E}) \simeq (\overline{x}, z, E))$$

This translates to

$$\frac{\mu(y_1 < \ldots < y_n \cap \overline{y} \simeq \overline{x})}{\mu(\overline{y} \simeq \overline{x})} = \sum_i \frac{\mu(U_i \cap (\overline{y}, z', \widetilde{E}) \simeq (\overline{x}, z, E))}{\mu((\overline{y}, z', \widetilde{E}) \simeq (\overline{x}, z, E))}$$

Remarking that since $G \curvearrowright \mathrm{Gr}$ is uniquely ergodic, we necessarily have:

$$\begin{aligned}\mu(\overline{y} \simeq \overline{x}) &= \frac{1}{2^{n-1}} \\ &= 2\mu((\overline{y}, z', \widetilde{E}) \simeq (\overline{x}, z, E)).\end{aligned}$$

Therefore, $(*)$ becomes

$$\mu(y_1 < \ldots < y_n \cap (\overline{y} \simeq \overline{x})) = 2\sum_i \mu(U_i \cap (\overline{y}, z', \widetilde{E}) \simeq (\overline{x}, z, E))$$

The following lemma is the final ingredient that will allow us to conclude.

Lemma 4.5.6. *Let $(x_1, \ldots, x_n, z, E) \in R$ and $(y_1, \ldots, y_n, z', R^{\mathbb{H}})$ be its reduct in $\mathbb{H}$. We have:*

$$\mu(U_i \cap \overline{y} \simeq \overline{x} \cap y_1 \sim_{\widetilde{E}} z') = \mu(U_i \cap \overline{y} \simeq \overline{x} \cap y_1 \nsim_{\widetilde{E}} z').$$

Proof. This is very similar to the proof of Lemma 4.5.4. We look at $D_{z'}$, the orbit of z' under $G_{y_1,\ldots,y_n}$. Let us denote by $\mathrm{Gr}_{\overline{y},\overline{x}}(D_{z'})$ is the space of graphings of $\{y_1, \ldots, y_n\} \cup D_y$ such that $(y_1, \ldots, y_n, \widetilde{E}) \simeq (x_1, \ldots, x_n, E)$. Consider π the map ψ from $\mathrm{Gr}_{\overline{y},\overline{x}}(D_{z'})$ to $2^{D_{z'}}$ that associates to a graphing x the function f_x such that $f_x(b) = 1$ iff $y_1 \sim_x b$ for all $b \in D_{z'}$. This map is a bijection because of the parity condition on the edges with regard to the hyperedges of $\mathbb{H}$.

We know by Proposition 4.5.3 that the $G_{y_1,\ldots,y_n}$ invariant ergodic measures on 2^{D_z} are of the form $\mathrm{Ber}(p)^{D_z}$.

Let us look at the measure $\mu_{y_1,\ldots,y_n}$ on $\mathrm{Gr}_{\overline{y},\overline{x}}(D_{z'})$ defined for all A measurable as:

$$\mu_{y_1,\ldots,y_n}(A) = \mu(A | y_1 < \ldots < y_n \text{ and } (y_1, \ldots, y_n, \widetilde{E}) \simeq (x_1, \ldots, x_n, E)).$$

The pushforward of $\mu_{y_1,\ldots,y_n}$ by π decomposes into $\int_{[0,1]} \mathrm{Ber}(p)^{D_z} \mathrm{d}\rho(p)$. Just as in the proof of Lemma 4.5.4, $\rho = \delta_{1/2}$, this gives us the result. □

By symmetry, we assume that $(x_1, z) \in E$ and we have

$$\begin{aligned}\mu(y_1 < \ldots < y_n \cap \overline{y} \simeq \overline{x}) &= \sum_i \mu(U_i \cap \overline{y} \simeq \overline{x}) \\ &= \sum_i \mu(U_i \cap \overline{y} \simeq \overline{x} \cap x_1 \sim y) + \mu(U_i \cap \overline{y} \simeq \overline{x} \cap x_1 \nsim y) \\ &= \sum_i 2\mu(U_i \cap \overline{y} \simeq \overline{x} \cap x_1 \sim y) \\ &= \sum_i 2\mu(U_i \cap (\overline{y}, z', \widetilde{E}) \simeq (\overline{x}, z, E)).\end{aligned}$$

Therefore $(*)$ is true and we have item 2) of Theorem 1.3.5, therefore ν is indeed well-defined. □

We define ν as in Proposition 4.5.5. It is easy to see that ν is an $\mathrm{Aut}(R)$-invariant measure on $\mathrm{LO}(R)$, hence it needs to be the uniform measure. Therefore μ is uniquely determined on a field generating the σ-algebra of $\mathrm{Gr} \times \mathrm{LO}$.

CHAPTER 5

The case of the semigeneric directed graph

This is a slightly modified version of [J].

5.1 Introduction

We denote $\mathbb{S}$, is the Fraïssé limit of the class $\mathcal{S}$ of simple, loopless, directed, finite graphs that verify the following conditions:

i) the relation $\perp$, defined by $x \perp y$ iff $\neg(x \to y \vee y \to x)$, is an equivalence relation,

ii) for any $x_1 \neq x_2, y_1 \neq y_2$ such that $x_1 \perp x_2$ and $y_1 \perp y_2$, the number of (directed) edges from $\{x_1, x_2\}$ to $\{y_1, y_2\}$ is even,

where $\to$ denotes the directed edge. We will refer to $\perp$-equivalence classes as columns and to the second condition as the parity condition. The $\perp$-class of an element $a \in \mathbb{S}$ will be referred to as $a^\perp$.

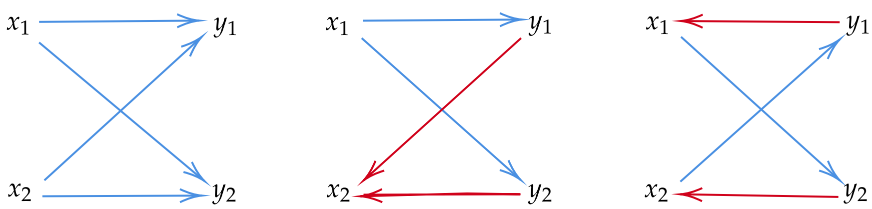


Figure 5.1: The three possible configurations (up to isomorphism) of two pairs of equivalent points respecting the parity condition.

More details on this structure will be given in the next section.

In this paper, we prove:

Theorem 1. The topological group $\mathrm{Aut}(\mathbb{S})$ is uniquely ergodic.

The method we use is different from the one found in [AKL] and [PS] since we do not work with the so-called "quantitative expansion property", but rather show that an ergodic measure can only take certain values on a generating part of the Borel sets. It is also different from the approach in [T2] (see Theorem 7.4) which only applies when the structure eliminates imaginaries. Our method relies on the idea that if there are equivalence classes in a structure and the universal minimal flow is essentially the convex orderings regarding the equivalence classes, then the ordering inside the equivalence classes and the ordering of the equivalence classes are independent, provided that the automorphism group behaves well enough.

Acknowledgements:

I am grateful to my PhD supervisors Lionel Nguyen Van Thé and Todor Tsankov for their helpful advice during my research on this paper. I also want to thank Miodrag Sokić for his comments on this paper. I thank the referee, whose comments helped me greatly improve the structure of the paper.

5.2 Preliminaries

The starting point of our proof is common with that of [AKL]: to prove that $\mathrm{Aut}(\mathbb{S})$ is uniquely ergodic, it suffices to show that one particular action is uniquely ergodic, namely, its universal minimal flow, $\mathrm{Aut}(\mathbb{S}) \curvearrowright \mathrm{M}(\mathrm{Aut}(\mathbb{S}))$. This is the unique minimal $\mathrm{Aut}(\mathbb{S})$-flow that maps onto any minimal $\mathrm{Aut}(\mathbb{S})$-flow (such a flow exists for any Hausdorff topological group by a classical result of Ellis, see [E2]); an explicit description was made by Jasiński, Laflamme, Nguyen Van Thé and Woodrow in [JLNVTW]. It is the space of expansions of $\mathbb{S}$ whose Age is a certain class $\mathcal{S}^*$.

Before describing this class, we give some more background on $\mathbb{S}$. Observe that the parity condition is equivalent to the fact for every $\mathbf{A} \in \mathcal{S}$ and two columns P, Q in $\mathbf{A}$, we have for all $x, x' \in P$,

$$(\forall y \in Q \ \left((x \to y) \Leftrightarrow (x' \to y)\right)) \text{ or } (\forall y \in Q \ \left((x \to y) \Leftrightarrow (y \to x')\right)).$$

This remark allows us to define the equivalence relation $\sim_Q$ on P as:

$$x \sim_Q x' \Leftrightarrow \ \forall y \in Q \ (x \to y \Leftrightarrow x' \to y).$$

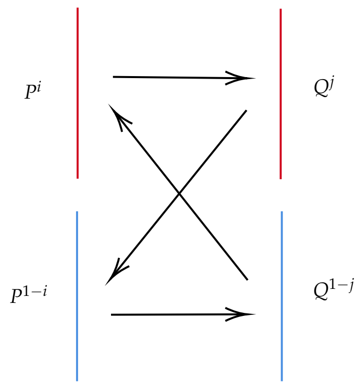
Note that as a consequence of the parity condition, we get that in $\mathbb{S}$,

$$\forall y \in Q \ (x \to y \Leftrightarrow x' \to y) \Leftrightarrow \ \exists y \in Q \quad (x \to y \text{ and } x' \to y).$$

We can now consider P^0 and P^1 the two $\sim_Q$ equivalence classes in P, and we have $P = P^0 \sqcup P^1$. Note that each of these class could be empty. Similarly, we have $Q = Q^0 \sqcup Q^1$, where Q^0 and Q^1 are $\sim_P$-equivalence classes. Note that at that stage, this labelling of these classes is arbitrary, which is crucial to the construction and understanding of $\mathcal{S}^*$ below. Indeed, the language of $\mathcal{S}^*$ has a binary relation R whose interpretation is mainly to give a proper labelling of those equivalence classes.

This description has an interesting consequence when we recall that there must be an edge between any two points of P and Q. Denote $P^i \to Q^j$ to mean for all $x \in P^i$ and $y \in Q^j$, we have $x \to y$. Then $P^i \to Q^j$, implies that $Q^j \to P^{1-i}$, $P^{1-i} \to Q^{1-j}$ and $Q^{1-j} \to P^i$. In particular, this means that for each $i \in \{0, 1\}$, there is a unique $j \in \{0, 1\}$ such that $P^i \to Q^j$.

The class $\mathcal{S}^*$ is the class of finite structures in the language $L = (\to, <, R)$, verifying :



1. $\mathcal{S}^*_{|\to} = \mathcal{S}$,

2. $<$ is interpreted as a linear ordering convex with respect to the columns, i.e. the columns are intervals for the ordering. For two columns P, Q, we will therefore write $P < Q$ to mean that for all $x \in P, y \in Q$ we have $x < y$.

3. For $\mathbf{A}^* \in \mathcal{S}^*$, the binary relation $R^{\mathbf{A}^*}$ verifies

   (a) For all $x, y \in \mathbf{A}^*$,
   $$R^{\mathbf{A}^*}(x,y) \Rightarrow \neg x \perp y.$$

   b) For all $x, y, y' \in \mathbf{A}^*$,
   $$(R^{\mathbf{A}^*}(x,y) \;\&\; y \perp y') \Rightarrow R^{\mathbf{A}^*}(x,y').$$

   c) For all $x, x', y \in \mathbf{A}^*$,
   $$(x \to y \;\&\; y \to x' \;\&\; x \perp x' \;\&\; x <^{\mathbf{A}^*} y) \Rightarrow (R^{\mathbf{A}^*}(x,y) \Leftrightarrow \neg R^{\mathbf{A}^*}(x',y)).$$

Observe that in a structure $\mathbf{A}^* \in \mathcal{S}^*$, $R^{\mathbf{A}^*}$ gives us a proper labelling of the $\sim_Q$-equivalence classes in P when $P < Q$. In particular, we can render the arbitrary decomposition $P = P^0 \sqcup P^1$, $Q = Q^0 \sqcup Q^1$ canonical by setting

$$x \in P^1 \Leftrightarrow (\forall y \in Q \quad R^{\mathbf{A}^*}(x,y))$$

and

$$y \in Q^1 \Leftrightarrow (\forall x \in P \quad (y \to x \Leftrightarrow R^{\mathbf{A}^*}(x,y))).$$

A remarkable property of this decomposition is that the edge relation is actually entirely defined by it. Indeed, take two columns P, Q in $\mathbf{A}^*$ that we decompose as above in $P = P^0 \sqcup P^1$, $Q = Q^0 \sqcup Q^1$. We know, by construction of R on Q, that $Q^1 \to P^1$. As we observed before, this means that $P^1 \to Q^0$, $P^0 \to Q^1$ and $Q^0 \to P^0$.

Another point of view on this expansion is given in [JLNVTW]. Take $\mathbf{A} \in \mathcal{S}$ with n columns $P_1, \ldots, P_n$ and an expansion $\mathbf{A}^* \in \mathcal{S}^*$. The expansion $\mathbf{A}^*$ is interdefinable with a structure $\mathbf{A}^{**}$ in the language $\{\to, <, L_{i,f}\}$ where $L_{i,f}$ is a unary predicate for all $i \in \{1, \ldots, n\} = [n]$ and $f \in 2^{[n]\setminus i}$. We have $\mathbf{A}^*_{|\to,<} = \mathbf{A}^{**}_{|\to,<}$. Assuming that $P_1 <^{\mathbf{A}^*} \ldots <^{\mathbf{A}^*} P_n$, then we define

$$L^{\mathbf{A}^{**}}_{i,f} = \{x \in P_i : \; \forall j \in [n]\setminus i, y \in P_j \quad (f(j) = 1 \Leftrightarrow R^{\mathbf{A}^*}(x,y)\}.$$

Denote $\mathcal{M} \subset \{0,1\}^{\mathbb{S}^2} \times \{0,1\}^{\mathbb{S}^2}$ the space of expansions of $\mathbb{S}$ whose Age is exactly $\mathcal{S}^*$. We will denote $E = (<^E, R^E)$ the elements of $\mathcal{M}$, by identification with the structure that can be inferred from the expansion. The result shown in [JLNVTW] is:

Theorem 5.2.1. *The universal minimal flow of* $\mathrm{Aut}(\mathbb{S})$ *is* $\mathrm{Aut}(\mathbb{S}) \curvearrowright \mathcal{M}$.

We are interested in showing that the $\mathrm{Aut}(\mathbb{S})$-invariant measures on $\mathcal{M}$ are all equal. A useful tool of measure theory is the following Lemma (see [G] Theorem 3.5)

Lemma 5.2.2. *Let μ and ν be two probability measures defined on a σ-field $\mathcal{E}$. If there is a family $(A_n)_{n\in\mathbb{N}} \in \mathcal{E}^{\mathbb{N}}$ stable under intersection that generates $\mathcal{E}$ and such that for all $n \in \mathbb{N}$, $\mu(A_n) = \nu(A_n)$, then $\mu = \nu$.*

The rest of this section is devoted to describing a family $\mathcal{P}$ of clopen sets that generate the Borel sets of $\mathcal{M}$. The sets of our family $\mathcal{P}$ are of the form

$$U_{(x_i)_{i=1}^n,(\varepsilon_i^j)_{1\le i<j\le n}} \cap V_{(a_1^1,\dots,a_{i_1}^1),\dots,(a_1^k,\dots,a_{i_k}^k)} \subset \mathcal{M}.$$

They are defined as follows.

Let $(x_i)_{i=1}^n$ be in different columns. Let $(\varepsilon_i^j)_{i<j\le n} \in \{0,1\}^{\binom{n}{2}}$. An element $E = (<^E, R^E) \in \mathcal{M}$ belongs to $U_{(x_i)_{i=1}^n,(\varepsilon_i^j)_{1\le i<j\le n}}$ iff the following conditions are satisfied :

1. $(x_1^\perp <^E \dots <^E x_n^\perp)$
2. for $k < l$,
$$R^E(x_k, x_l) \Leftrightarrow (x_k \to x_l)^{\varepsilon_k^l}.$$
where for all $x, y \in \mathbb{S}$ and $\varepsilon \in \{0,1\}$, $(x \to y)^\varepsilon$ means $(x \to y)$ if $\varepsilon = 1$ and $\neg(x \to y)$ otherwise.

The rest of R on those columns can be recovered from this by construction of $\mathcal{S}^*$. Indeed, observe that for all $x \in x_k^\perp$, $y \in x_l^\perp$, we have

$$R^E(x,y) \Leftrightarrow \left(\left(x \sim^{\mathbb{S}}_{x_l^\perp} x_k \text{ and } R^E(x_k,x_l)\right) \text{ or } \left(x \nsim^{\mathbb{S}}_{x_l^\perp} x_k \text{ and } \neg R^E(x_k,x_l)\right)\right).$$

An important remark is that if we have a different family $(x_1', \dots, x_n')$ such that $x_i \perp x_i'$, then there is a family $(\alpha_i^j)_{1\le i<j\le n}$ such that

$$U_{(x_i)_{i=1}^n,(\varepsilon_i^j)_{1\le i<j\le n}} = U_{(x_i')_{i=1}^n,(\alpha_i^j)_{1\le i<j\le n}}.$$

This can be achieved by taking $\alpha_i^j = \varepsilon_i^j$ if $x_i \sim_{x_j^\perp} x_i'$ and $\alpha_i^j = 1 - \varepsilon_i^j$ otherwise.

An additional remark that will be useful throughout the paper is that for a given family $(x_1, \dots, x_n)$ of elements taken in different columns,

$$\mathcal{M} = \bigsqcup_{\sigma\in S_n,(\varepsilon_i^j)_{1\le i<j\le n}} U_{(x_{\sigma(i)})_{i=1}^n,(\varepsilon_i^j)_{1\le i<j\le n}}.$$

We also define

$$V_{(a_1^1,\dots,a_{i_1}^1),\dots,(a_1^k,\dots,a_{i_k}^k)} = \{E \in \mathcal{M} : \\ (a_1^1 <^E \cdots <^E a_{i_1}^1) \wedge \cdots \wedge (a_1^k <^E \cdots <^E a_{i_k}^k)\}$$

where $(a_i^j \perp a_{i'}^{j'})$ iff $j = j'$.

This collection of sets is a generating family for the open sets of our space, so it is also a generating family for the Borel sets.

To use Lemma 5.2.2, we would also need to know that this family is stable under intersection, unfortunately this is not the case. However, the intersection of two sets in $\mathcal{P}$ is actually a disjoint union of sets in $\mathbb{P}$. Therefore if we consider $\mathcal{P}'$ the collection of finite intersection of elements of $\mathbb{P}$, the evaluation of a measure on an element of $\mathbb{P}'$ is determined by the evaluation of the measure on $\mathbb{P}$. By Lemma 5.2.2, any measure is entirely characterized by its evaluation on elements of $\mathbb{P}'$, so it is characterized by its evaluation on elements of $\mathbb{P}$.

5.3 Invariant measures

From this point on, we denote $G = \mathrm{Aut}(\mathbb{S})$. Let us first define μ_0 a G-invariant probability measure on $\mathcal{M}$. We define μ_0 by:

$$\mu_0\left(U_{(x_i)_{i=1}^n,(\varepsilon_i^j)_{1\leq i<j\leq n}} \cap V_{(a_1^1,\dots,a_{i_1}^1),\dots,(a_1^k,\dots,a_{i_k}^k)}\right) = \frac{1}{n!2^{\binom{n}{2}}} \frac{1}{\prod_{j=1}^{k} i_j!}.$$

We call μ_0 the uniform measure. It is proven in [PS] that this measure is well-defined on all Borel sets and that it is G-invariant. We want to show that it is actually the only invariant measure. By Lemma 5.2.2, we only have to check that the invariant measures coincide on $U_{(x_i)_{i=1}^n,(\varepsilon_i^j)_{1\leq i<j\leq n}} \cap V_{(a_1^1,\dots,a_{i_1}^1),\dots,(a_1^k,\dots,a_{i_k}^k)}$.

Before proving Theorem 1 we need to prove the following preliminary results:

Proposition 5.3.1. *For all $(x_i)_{i=1}^n$ such that $\neg(x_i \perp x_j)$ for $i \neq j$ and $(\varepsilon_i^j)_{i<j\leq n} \in 2^{\binom{n}{2}}$, we have:*

$$\mu\left(U_{(x_i)_{i=1}^n,(\varepsilon_i^j)_{1\leq i<j\leq n}}\right) = \frac{1}{n!2^{\binom{n}{2}}}.$$

Proposition 5.3.2. *For all $(a_1^1,\dots,a_{i_1}^1,\dots,a_1^k,\dots,a_{i_k}^k)$ such that $a_i^j \perp a_{i'}^{j'}$ iff $j = j'$, we have:*

$$\mu\left(V_{(a_1^1,\dots,a_{i_1}^1),\dots,(a_1^k,\dots,a_{i_k}^k)}\right) = \frac{1}{\prod_{j=1}^{k} i_j!}.$$

Similar results were proven in [PS]. We will prove those results using different methods. The proof of Proposition 5.3.2 is very similar to what we will do later on in order to conclude and contains the key argument of this paper.

For proofs of Proposition 5.3.2 and Theorem 1, we will need an ergodic decomposition theorem, thus we need to define the notion of ergodicity.

Definition 5.3.3. Let Γ be a Polish group acting continuously on a compact space X. A Γ-invariant measure ν is said to be Γ-*ergodic* if for all A measurable such that

$$\forall g \in \Gamma \ \nu(A \triangle g \cdot A) = 0,$$

we have $\nu(A) \in \{0, 1\}$.

We can now state the following (see [P3] Proposition 12.4):

Theorem 5.3.4. *Let Γ be a Polish group acting continuously on a compact space X. Let $P(X)$ denote the space of probability measures on X and $P_\Gamma(X) = \{\mu \in P(X) : \Gamma \cdot \mu = \mu\}$. Then, the extreme points of $P_\Gamma(X)$ are the Γ-ergodic invariant measures.*

We will also need to use Neumann's Lemma (see [C1], Theorem 6.2) :

Theorem 5.3.5. *Let H be a group acting on Ω with no finite orbit. Let Γ and Δ be finite subsets of Ω, then there is $h \in H$ such that $h \cdot \Gamma \cap \Delta = \varnothing$.*

The remaining of the section will be divided in three subsections. One for the proof of Proposition 5.3.1, one for the proof of Proposition 5.3.2 and finally one for the proof of Theorem 1.

5.3.1 Proof of Proposition 5.3.1

For this proof, we will need the following technical lemma.

Lemma 5.3.6. *Let $k < n$, let $P_1, \dots, P_n$ be different columns in $\mathbb{S}$ and let $y_1 \in P_1, \dots, y_k \in P_k$. Take a given family $\varepsilon_i^j \in \{0,1\}$ where $1 \leq i < j \leq n$ and $k < j$. Then there exist $y_{k+1} \in P_{k+1}, \dots, y_n \in P_n$ such that $(y_i \to y_j)^{\varepsilon_i^j}$ for all $i < j$ and $k < j$.*

Proof. Take $x_{k+1} \in P_{k+1}, \dots, x_n \in P_n$. Consider the following structure

$$\mathbf{A} = ((y_1^A, \dots, y_n^A, x_{k+1}^A, \dots, x_n^A), \to^{\mathbf{A}})$$

where $(y_i^A \to^{\mathbf{A}} y_j^A) \Leftrightarrow (y_i \to y_j)$ if $i < j \leq k$, $(y_i^A \to^{\mathbf{A}} y_j^A) \iff (\varepsilon_i^j = 1)$ if $1 \leq i < j \leq n$ and $k < j$. We also have $x_i^A \perp^{\mathbf{A}} y_i^A$ for $i > k$ and $(x_i^A \to^{\mathbf{A}} x_j^A \Leftrightarrow x_i \to x_j)$ for $k < i < j$.

We put edges between x_i^A and y_j^A in order for them to respect the parity condition. Remark that there is more than one way to do this, for instance one can ask that when $k < i < j$, $(x_i^A \to^{\mathbf{A}} y_j^A) \Leftrightarrow (x_i^A \to^{\mathbf{A}} x_j^A)$ and $(x_j^A \to^{\mathbf{A}} y_i^A) \Leftrightarrow (y_j^A \to^{\mathbf{A}} y_i^A)$. The remaining edges can be added arbitrarily because they concern columns with only one vertex.

We make sure that $\mathbf{A} \in \mathcal{S}$. Indeed, noting that since there is one point in the first k columns, and two in the remaining ones, it suffices to check the parity condition in the last $n-k$ columns. Take $k < j < i \leq n$. We know that the edges between x_i^A and y_j^A and the edge between x_i^A and x_j^A go in the same direction. Similarly, the edge between x_j^A and y_i^A and the edge between y_j^A and y_i^A also go in the same direction. Therefore the parity condition must be respected.

Remark that $((y_1^A, \dots, y_k^A, x_{k+1}^A, \dots, x_n^A), \to^{\mathbf{A}})$ and $((y_1, \dots, y_k, x_{k+1}, \dots, x_n), \to^{\mathbb{S}})$ are isomorphic, hence $\mathbf{A}$ embeds in $\mathbb{S}$ in a way that extends this isomorphism. The image of $(y_{k+1}^A, \dots, y_n^A)$ is as wanted.

□

The fundamental observation for the proof of Proposition 5.3.1 is that if we take $x_1, \dots, x_n \in \mathbb{S}$ all in different columns,

$$\overline{\mathrm{Aut}(\mathbb{S}) \cdot (<^*, R^*)} = \bigsqcup_{\sigma \in S_n,\ (\varepsilon_i^j)_{1 \leq i < j \leq n}} U_{(x_{\sigma(i)})_{i=1}^n, (\varepsilon_i^j)_{1 \leq i < j \leq n}}.$$

We will show that for any two families $\varepsilon = \left(\varepsilon_i^j\right)_{i<j\leq n}$, $\alpha = \left(\alpha_i^j\right)_{i<j\leq n}$ and $\sigma \in S_n$ there is a $g \in G$ such that

$$U_{(x_i)_{i=1}^n,\varepsilon} = g \cdot U_{(x_{\sigma(i)})_{i=1}^n,\alpha}.$$

This means that all sets of this form have the same measure, hence we will have the result because there are $n!2^{\binom{n}{2}}$ such sets.

First, we construct $g' \in G$ such that

$$g' \cdot U_{(x_{\sigma(i)})_{i=1}^n,\alpha} = U_{x_1,\ldots,x_n,\beta}$$

for some $\beta = (\beta_i^j)_{1\leq i<j\leq n}$.

We want to prove that there is $g' \in G$ such that $g' \cdot x_i \in (x_{\sigma(i)})^{\perp}$. By Lemma 5.3.6, there exists $x'_1, \ldots, x'_n \in \mathbb{S}$ such that $x_{\sigma(i)} \perp x'_i$ and $x_i \to x_j$ iff $x'_i \to x'_j$. Remark that by construction, there is a partial automorphism τ that sends $x_{\sigma(i)}$ to x'_i. By homogeneity, there is g' an automorphism of $\mathbb{S}$ that extends τ. We remark that

$$g' \cdot U_{(x_i)_{i=1}^n,\alpha} = U_{(x'_{\sigma(i)})_{i=1}^n,\alpha}$$

and as we observed before, $U_{(x'_{\sigma(i)})_{i=1}^n,\alpha}$ does not depend on x'_i, but on their columns. Thus, there exist a family $\beta = (\beta_i^j)_{1\leq i<j\leq n}$ such that

$$U_{(x'_{\sigma(i)})_{i=1}^n,\alpha} = U_{(x_i)_{i=1}^n,\beta}.$$

Next, we construct $h \in G$ such that

$$U_{(x_i)_{i=1}^n,\varepsilon} = h \cdot U_{(x_i)_{i=1}^n,\beta}.$$

Assume that there are $k < l$ such that $\beta_i^j = \varepsilon_i^j$ for all $(i,j) \neq (k,l)$ and $\beta_k^l \neq \varepsilon_k^l$. Remark that taking care of this case will be enough to prove the result : If α and β disagree in more than one coordinate, iterating this process still allows to modify coordinates one at a time.

Let us take $x'_k \perp x_k$ such that for all $i \in [n]\backslash\{k,l\}$, $x'_k \to x_i$ iff $x_k \to x_i$ and $x'_k \to x_l$ iff $x_l \to x_k$. This is possible using Lemma 5.3.6 where $\{y_1,\ldots,y_{n-1}\} = \{x_1,\ldots,x_n\}\backslash\{x_k\}$ and $P_n = x_k^{\perp}$. We define $x'_l \perp x_l$ similarly.

We take $h \in G$ such that $h(x_i) = x_i$ for all $i \in [n]\backslash\{k,l\}$, $h(x'_k) = x_k$ and $h(x'_l) = x_l$. By homogeneity, such a h exists: indeed, by the parity condition, we have $(x_k \to x_l) \Leftrightarrow (x'_k \to x'_l)$. Let us prove that h gives the result.

Take $E \in U_{x_1,\ldots,x_n,\beta}$. We will prove that

$$h \cdot E \in U_{(x_i)_{i=1}^n,\varepsilon}.$$

For all $i < j$ we want to prove that

$$R^{h\cdot E}(x_i,x_j) \Leftrightarrow (x_i \to x_j)^{\varepsilon_i^j},$$

and since

$$R^{h\cdot E}(x_i,x_j) \Leftrightarrow R^E(h^{-1}(x_i),h^{-1}(x_j)),$$

we prove

$$R^E(h^{-1}(x_i), h^{-1}(x_j)) \Leftrightarrow (x_i \to x_j)^{\varepsilon_i^j}.$$

If $\{i,j\} \cap \{k,l\} = \varnothing$, the result is obvious.

If $j = k$ and $i < k$, we have:

$$\begin{aligned} R^{h\cdot E}(x_i, x_k) &\Leftrightarrow R^E(h^{-1}(x_i), h^{-1}(x_k)) \\ &\Leftrightarrow (x_i \to h^{-1}(x_k))^{\beta_i^k} \\ &\Leftrightarrow (x_i \to x_k')^{\beta_i^k} \\ &\Leftrightarrow (x_i \to x_k)^{\beta_i^k}, \end{aligned}$$

and since $\beta_i^k = \varepsilon_i^k$, we have

$$R^{h\cdot E}(x_i, x_k) \Leftrightarrow (x_i \to x_k)^{\varepsilon_i^k}.$$

The other cases where $|\{i,j\} \cap \{k,l\}| = 1$ are similar.

Finally, if $(i,j) = (k,l)$, we have:

$$\begin{aligned} R^{h\cdot E}(x_k, x_l) &\Leftrightarrow R^E(h^{-1}(x_k), h^{-1}(x_l)) \\ &\Leftrightarrow (x_k \to h^{-1}(x_l))^{\beta_k^l} \\ &\Leftrightarrow (x_k \to x_l')^{\beta_k^l} \\ &\Leftrightarrow (x_k \to x_l)^{\varepsilon_k^l}. \end{aligned}$$

The last equivalence is a direct consequence of the definition of x_l' and the fact that $\beta_k^l = (1 - \varepsilon_k^l)$. □

5.3.2 Proof of Proposition 5.3.2

We prove the result by induction on the number k of columns.

By homogeneity, for any column $(a_1^j)^\perp$ and $\sigma \in S_{i_j}$ there exists $g \in G$ such that

$$g \cdot V_{(a_1^j,\ldots,a_{i_j}^j)} = V_{(a_{\sigma(1)}^j,\ldots,a_{\sigma(i_j)}^j)},$$

thus

$$\mu\left(V_{(a_1^j,\ldots,a_{i_j}^j)}\right) = \frac{1}{i_j!}.$$

This proves the initial case.

Let us now assume that for all $(a_1^1, \ldots, a_{i_1}^1, \ldots, a_1^{k-1}, \ldots, a_{i_{k-1}}^{k-1})$ such that $a_i^j \perp a_{i'}^{j'}$ iff $j = j'$, we have

$$\mu\left(V_{(a_1^1,\ldots,a_{i_1}^1),\ldots,(a_1^{k-1},\ldots,a_{i_{k-1}}^{k-1})}\right) = \frac{1}{\prod_{j=1}^{k-1} i_j!}.$$

We consider $(a_1^k, \ldots, a_{i_k}^k)$ all in the same column and not in any $(a_1^i)^\perp$ for $i < k$. Remark that

$$V_{(a_1^1,\ldots,a_{i_1}^1),\ldots,(a_1^k,\ldots,a_{i_k}^k)} = V_{(a_1^1,\ldots,a_{i_1}^1),\ldots,(a_1^{k-1},\ldots,a_{i_{k-1}}^{k-1})} \cap V_{(a_1^k,\ldots,a_{i_k}^k)}.$$

We want to prove that the ordering of $(a_1^k)^\perp$ is independent from the ordering of the other columns.

Enumerate as $(V_1, \dots, V_\tau)$ all the different sets of the form $V_{(a^1_{\sigma_1(1)},\dots,a^1_{\sigma_1(i_1)}),\dots,(a^{k-1}_{\sigma_{k-1}(1)},\dots,a^{k-1}_{\sigma_{k-1}(i_{k-1})})}$ where σ_j is a permutation of $\{1,\dots,i_j\}$. Thus $\tau = \prod_{j=1}^{k-1} i_j!$.

For all $l \in \{1,\dots,\tau\}$, we define

$$\mu_{V_l}(\cdot) = \frac{\mu(\cdot \cap V_l)}{\mu(V_l)}.$$

This is the conditional probability of μ given V_l. We remark that:

$$\mu = \sum_{l=1}^{\tau} \mu(V_l)\mu_{V_l}.$$

Denote $\mathrm{LO}((a_1^k)^\perp)$ the space of linear orderings on $(a_1^k)^\perp$. There is a restriction map r from $\mathcal{M}$ to $\mathrm{LO}((a_1^k)^\perp)$. We denote $V^r_{(a_1^k,\dots,a^k_{i_k})}$ the image of $V_{(a_1^k,\dots,a^k_{i_k})}$ by r. Let ν be, the pushforward of μ on $\mathrm{LO}({a_1^1}^\perp)$ by r, and let ν_{V_l} be the pushforward of μ_{V_l} by the same map. We have:

$$\nu = \sum_{l=1}^{\tau} \mu(V_l)\nu_{V_l}.$$

Observe that the initial step of the induction implies that ν is the uniform measure on $\mathrm{LO}((a_1^k)^\perp)$

We denote $\mathrm{Stab}^{\mathrm{set}}_{(a_1^k)^\perp}$ the setwise stabilizer of $(a_1^k)^\perp$, $\mathrm{Stab}^{\mathrm{pw}}_{(a_1^1,\dots,a^1_{i_1},\dots,a_1^{k-1},\dots,a^{k-1}_{i_{k-1}})}$ the pointwise stabilizer of $(a_1^1,\dots,a^1_{i_1},\dots,a_1^{k-1},\dots,a^{k-1}_{i_{k-1}})$ and set $H = \mathrm{Stab}^{\mathrm{set}}_{(a_1^k)^\perp} \cap \mathrm{Stab}^{\mathrm{pw}}_{(a_1^1,\dots,a^1_{i_1},\dots,a_1^{k-1},\dots,a^{k-1}_{i_{k-1}})}$. We remark that ν_{V_l} is H-invariant for all $l \in \{1,\dots,\tau\}$.

Since $\mathrm{LO}({a_1^1}^\perp)$ is compact, by Theorem 5.3.4, if we prove that ν is H-ergodic, then we have the result. Indeed, then ν is an extreme point of the H-invariant measures and all the ν_{V_l} are equal to ν, thus for any l we have

$$\begin{aligned}
\mu\left(V_{(a_1^k,\dots,a^k_{i_k})} \cap V_l\right) &= \mu_{V_l}\left(V_{(a_1^k,\dots,a^1_{i_k})}\right)\mu(V_l)\\
&= \nu_{V_l}\left(V^r_{(a_1^k,\dots,a^1_{i_k})}\right)\mu(V_l)\\
&= \nu\left(V^r_{(a_k^1,\dots,a^1_{i_k})}\right)\mu(V_l)\\
&= \frac{1}{i_k!}\frac{1}{\prod_{j=1}^{k-1} i_j!}
\end{aligned}$$

and this equality finishes the induction.

It only remains to prove the ergodicity of ν. The following lemma will allow us to conclude.

Lemma 5.3.7. *Let K be a group acting on a set $\mathcal{N}$ with no finite orbits. Denote* $\mathrm{LO}(\mathcal{N})$ *the space of linear orderings on $\mathcal{N}$. Then the uniform measure λ on* $\mathrm{LO}(\mathcal{N})$ *is K-ergodic.*

Proof. Suppose A is a Borel subset of $\mathrm{LO}(\mathcal{N})$ such that for all $g \in K$ $\lambda(A \triangle g \cdot A) = 0$. We want to show that $\lambda(A) \in \{0,1\}$. Let $\varepsilon > 0$. There is a cylinder, i.e. a set depending only on a finite set of $\mathcal{N}$, $B = B(b_1, \ldots, b_k)$ such that $\mu(B \triangle A) \leq \varepsilon$. Using Neumann's Lemma, we get that there exists $g \in K$ such that $\{b_1, \ldots, b_k\} \cap g \cdot \{b_1, \ldots, b_k\} = \varnothing$.

Moreover, since ν is uniform, the orderings of two disjoint sets of points are independent. Indeed, taking $(a_1, \ldots, a_i)$ and $(c_1, \ldots, c_{i'})$ two disjoint families of points. Note that $\lambda(V_{(a_1,\ldots,a_i)} \cap V_{(c_1,\ldots,c_{i'})})$ is equal to the number of way to insert $(c_1, \ldots, c_{i'})$ in $(a_1, \ldots, a_i)$ respecting both orderings times the weight of a given ordering of $(a_1, \ldots, a_i, c_1, \ldots, c_{i'})$. We therefore have

$$\begin{aligned}
\lambda(V_{(a_1,\ldots,a_i)} \cap V_{(c_1,\ldots,c_{i'})}) &= \binom{i+i'}{i} \frac{1}{(i+i')!} \\
&= \frac{1}{i!} \frac{1}{i'!}.
\end{aligned}$$

This means that B and $g \cdot B$ are independent. We can now write:

$$\begin{aligned}
\left|\lambda(A) - \lambda(A)^2\right| =& \left|\lambda(A \cap g \cdot A) - \lambda(A)^2\right| \\
\leq& |\lambda(A \cap g \cdot A) - \lambda(B \cap g \cdot A)| + |\lambda(B \cap g \cdot A) - \lambda(B \cap g \cdot B)| \\
&+ \left|\lambda(B \cap g \cdot B) - \lambda(B)^2\right| + \left|\lambda(B)^2 - \nu(A)^2\right| \\
\leq& 4\varepsilon.
\end{aligned}$$

The last inequality comes from the following inequalities

$$\begin{aligned}
&|\lambda(A \cap g \cdot A) - \lambda(B \cap g \cdot A)| \leq \lambda((A \triangle B) \cap g \cdot A) \leq \varepsilon, \\
&|\lambda(B \cap g \cdot A) - \lambda(B \cap g \cdot B)| \leq \lambda(g \cdot (A \triangle B) \cap B) \leq \varepsilon, \\
&\lambda(B \cap g \cdot B) = \lambda(B)^2 \\
&\text{and} \\
&\left|\lambda(B)^2 - \lambda(A)^2\right| = (\lambda(A) + \lambda(B))|\lambda(A) - \lambda(B)| \leq 2\varepsilon.
\end{aligned}$$

This proves that λ is K-ergodic.

□

We only have to prove that H has no finite orbits on $(a_1^1)^\perp$. It suffices to remark that for all $a \in \mathbb{S}$, $(u_1, \ldots, u_i) \in \mathbb{S}$, there are infinitely many $b \in a^\perp$ such that $a \to u_j$ iff $b \to u_j$ for all $1 \leq j \leq i$.

Indeed, take $k \in \mathbb{N}$. Consider the structure $((a_1, \ldots, a_k, v_1, \ldots, v_i), \to)$, where $a_l \perp a_j$, $a_l \to v_k$ iff $a \to u_k$ and $v_m \to v_{m'}$ iff $u_m \to u_{m'}$ for all $l, j \leq k$ and $m, m' \leq i$. It is obvious that this structure verifies the parity condition. Therefore in $\mathbb{S}$ we can find k copies of a in its column for any $k > 0$.

This is enough to conclude that ν is indeed H-ergodic.

□

5.3.3 Proof of Theorem 1

In what follows, we will show that

$$\mu\left(U \cap V\right) = \mu\left(U\right)\mu\left(V\right)$$

for all $U = U_{(x_i)_{i=1}^n,(\varepsilon_i^j)_{1\le i<j\le n}}$ and $V = V_{(a_1^1,\ldots,a_{i_1}^1),\ldots,(a_1^k,\ldots,a_{i_k}^k)}$. It will follow that $\mu = \mu_0$.

Let us take a certain set $\{x_1, \ldots, x_n\}$ where none of the x_i are in the same column. We denote m the number of sets U as above associated to this family. We consider $(U_i)_{i=1}^m$ the disjoint sets of $\mathcal{M}$ corresponding to the ways of defining a relation R and an order on the columns $x_1^\perp, \ldots, x_n^\perp$, i.e. $U_i = U_{(x_{\sigma(i)})_{i=1}^n,\varepsilon}$ for some $\sigma \in S_n$ and $\varepsilon \in 2^{\binom{n}{2}}$. Proposition 5.3.1 tells us that:

$$\forall i, j \in \{1, \ldots, m\},\ \mu(U_i) = \mu(U_j).$$

We remark that this quantity is $\frac{1}{m}$. We now define, for all $i \in \{1, \ldots, m\}$,

$$\mu_{U_i}(\cdot) = \frac{\mu\left(\cdot \cap U_i\right)}{\mu\left(U_i\right)}.$$

This is the conditional probability of μ given U_i. Denote H the subgroup of G that stabilizes $x_i^\perp$ for all $1 \le i \le n$ and each $\sim_{x_j^\perp}$-equivalence class in $x_i^\perp$ for $i \neq j$. Remark that H stabilizes U_i, by construction, hence μ_{U_I} is H-invariant.

A simple but fundamental remark is that since $\bigsqcup_{i=1}^m U_i = \mathcal{M}$ and all the U_i have the same measure under μ, we have

$$\mu = \frac{1}{m}\sum_{i=1}^m \mu_{U_i}.$$

Let $\mathrm{LO}_p(\mathbb{S})$ denote the space of partial orders that are total on each column and do not compare elements of different columns. There is a restriction map from $\mathcal{M}$ to $\mathrm{LO}_p(\mathbb{S})$. We consider λ the pushfoward of μ on $\mathrm{LO}_p(\mathbb{S})$ by this map. Similarly, we consider λ_{U_i} the pushfoward of μ_{U_i} on $\mathrm{LO}_p(\mathbb{S})$. We have

$$\lambda = \frac{1}{m}\sum_{i=1}^m \lambda_{U_i}.$$

The rest of the proof is similar to the proof of Proposition 5.3.2: we prove that λ is H-ergodic. Take A a Borel subset of $\mathrm{LO}_p(\mathbb{S})$ such that for all $h \in H$, $\lambda(A \triangle h \cdot A) = 0$. For any $\varepsilon > 0$, there is a cylinder B that depends only on finitely many points $(b_1, \ldots, b_k)$ such that $\lambda(A \triangle B) \le \varepsilon$. We now want to find an element $g \in H$ such that B and $g \cdot B$ are λ-independent.

Take $\{b_1, \ldots, b_k\} \subset \mathbb{S}$. Remark that there is $\{b_1', \ldots, b_k'\} \subset \mathbb{S}$ disjoint from $\{b_1, \ldots, b_k\}$ such that $b_l \perp b_l'$ and $b_l \sim_{x_j^\perp} b_l'$ for all $1 \le l \le k$ and $1 \le j \le n$. Therefore there is an element of H that sends $\{b_1, \ldots, b_k\}$ to $\{b_1, \ldots, b_k\}$ and is therefore as wanted.

Just as in the proof of Proposition 5.3.2, we have:

$$\begin{aligned}\left|\lambda(A)-\lambda(A)^2\right| =& \left|\lambda(A\cap g\cdot A)-\lambda(A)^2\right| \\ \leq& \left|\lambda(A\cap g\cdot A)-\lambda(B\cap g\cdot A)\right| + \left|\lambda(B\cap g\cdot A)-\lambda(B\cap g\cdot B)\right| \\ &+ \left|\lambda(B\cap g\cdot B)-\lambda(B)^2\right| + \left|\lambda(B)^2-\lambda(A)^2\right| \\ \leq& 4\varepsilon.\end{aligned}$$

Thus $\lambda(A)\in\{0,1\}$.

Since $\mathrm{LO}_p(\mathbb{S})$ is compact, we have the result: λ is an extreme point of the H-invariant measures and all the λ_{U_i} are equal. Therefore we have,

$$\begin{aligned}\mu(V\cap U_i) &= \mu_{U_i}(V)\mu(U_i) \\ &= \lambda_{U_i}(V)\mu(U_i) \\ &= \lambda(V)\mu(U_i) \\ &= \mu(V)\mu(U_i)\end{aligned}$$

for all $i\in\{1,\ldots,m\}$, and $V=V_{(a^1_1,\ldots,a^1_{i_1}),\ldots,(a^k_1,\ldots,a^k_{i_k})}$. This finishes the proof of Theorem 1.

CHAPTER 6

Minimal model-universal flow for locally compact groups

This is joint work with Andy Zucker, it follows closely [JZ1].

6.1 Introduction

Topological dynamics can be understood as an attempt to describe the actions of topological groups on compact spaces. Part of this process consists of understanding which objects are generic or universal among the actions of a given group. A famous instance of this is the universal minimal flow, whose existence was proven by Ellis ([E2]). This is a minimal flow which maps onto any other minimal flow; by understanding the properties of this one object, we can better understand the collection of all minimal flows. In this paper, we prove the existence of a minimal flow which is universal in a different sense, in that it contains a copy of any measured free action. Similarly, this "universal minimal model" can help shed light on the dynamical properties of a given locally compact group.

Let G be a locally compact non-compact Polish group. Recall that a *G-flow* is a continuous G-action on a compact space. A G-flow is said to be *minimal* if every orbit is dense. If Y is a G-flow, then $P_G(Y)$ denotes the G-invariant regular Borel probability measures on Y.

By a *G-system*, we will mean a Borel G-action on a standard Lebesgue space (X, μ) which preserves μ. We say that a G-system (X, μ) is *free* if the set

$$\mathrm{Free}(X) := \{x \in X : \forall g \in (G \setminus \{1_G\})\ gx \neq x\}$$

has measure 1. Because G is locally compact, this set is Borel. Therefore when dealing with free G-systems, we will often just assume that $X = \mathrm{Free}(X)$.

If (X, μ) and (Y, ν) are G-systems, we say that Y is a *factor* of (X, μ) if there is a Borel, G-invariant subset $X' \subseteq X$ with $\mu(X') = 1$ and a Borel, G-equivariant map $f \colon X' \to Y$ with $\nu = f_*\mu$.

If we can find f as above which is also injective, then we say that (X, μ) and (Y, ν) are *isomorphic* G-systems. We will denote this $(X, \mu) \cong (Y, \nu)$.

A compact metric G-flow Y is *weakly model-universal* if for every free G-system (X, μ), there is $\nu \in P_G(Y)$ with (Y, ν) a factor of (X, μ). We say that Y is *model-universal* if for every free G-system (X, μ), there is $\nu \in P_G(Y)$ with $(X, \mu) \cong (Y, \nu)$.

The main theorem of this paper is the construction of a minimal, model-universal flow for every locally compact, non-compact Polish group.

Theorem 6.1.1. *Let G be a locally compact, non-compact Polish group. Then there exists a minimal model-universal flow for G.*

This result extends a result of Weiss [W], who proved Theorem 6.1.1 in the case of countable groups. Weiss uses the slightly different terminology *universal minimal model* to describe this object. More recently, Zucker [Z4] gives a new proof of Weiss's result, and it is this proof that we generalize. We remark that Theorem 6.1.1 cannot extend to all Polish groups. As an example, the group $\mathrm{Aut}(\mathbb{Q})$ admits no non-trivial minimal flows, while the shift action on $[0,1]^{\mathbb{Q}}$ equipped with the product Lebesgue measure is a free G-system. It would be interesting to understand the precise class of Polish groups for which Theorem 6.1.1 is true.

The universality of the flow from Theorem 6.1.1 is quite different from that of the universal minimal flow $M(G)$. Indeed, $M(G)$ is universal in the sense that it surjects onto every minimal flow, while we construct a minimal flow into which every free G-system can be injected. Another difference is uniqueness; while $M(G)$ is unique up to isomorphism, it is shown in [Z4] that when G is countable, there are continuum many minimal model-universal flows up to isomorphism. Unfortunately, the proof of this requires some machinery for countable groups that is not yet known to generalize to locally compact groups. However, we strongly suspect that for any locally compact G, minimal model-universal flows are not unique.

Another result of [Z4] that we do not address here is whether a minimal, model-universal flow can be free. If Y is a minimal flow and $\nu \in P_G(Y)$ is such that (Y, ν) is a free G-system, then Y must be *essentially free*, meaning that $\mathrm{Free}(Y) \subseteq Y$ is dense G_δ. However, Weiss's construction of a minimal model-universal flow left open the question of whether such an object could be free. The first construction of a free minimal model-universal flow was given by Elek [E1], and in [Z4], an easy method of transforming any minimal model-universal flow into a free one is provided. The method is roughly as follows: start with Z a minimal model-universal flow, where we note that the construction from [Z4] gives a zero-dimensional flow. Then construct an *almost one-one* extension $\pi \colon Y \to Z$ with Y free and so that for every $z \in \mathrm{Free}(Z)$, $|\pi^{-1}(\{z\})| = 1$. Then $\pi \colon \pi^{-1}(\mathrm{Free}(Z)) \to \mathrm{Free}(Z)$ is a G-equivariant homeomorphism, and thus the map $\pi^{-1} \colon \mathrm{Free}(Z) \to Y$ will show that Y is also model-universal. Unfortunately, it is essential that Z be zero-dimensional for this to work, and our construction here does not produce zero-dimensional flows (indeed this is impossible when G is connected). A simple example is provided by Antonyan [A2]; let $G = \mathbb{Z}/2\mathbb{Z}$ and set $Z = [-1,1]^\omega$. G acts on Z by negating every coordinate. Hence $\mathrm{Free}(Z) = Z \setminus \{0\}$, where $0 = (0,0,0,...)$. However, Antonyan shows that $Z \setminus \{0\}$ does not embed as a G-subspace of any free G-flow. Therefore any soft method of transforming a minimal model-universal flow into a free one must use a different method to work for all locally compact groups.

One immediate consequence of Theorem 6.1.1 is a result that was suggested by Angel, Kechris and Lyons in [AKL]. Recall that a topological group G is said to be *uniquely ergodic* if all minimal G-flows admit exactly one G-invariant probability measure.

Theorem 6.1.2. *Let G be a locally compact non-compact Polish group. Then there is a minimal G-flow with multiple invariant probability measures. In particular, G is not uniquely ergodic.*

Proof. Let Z be a minimal model-universal flow for G. It suffices to show that Z admits an invariant probability measure that is not ergodic. Take (X, μ) a free G-system (see [AEG], Proposition 1.2). Then G acts on $X \times 2$ by acting on the first coordinate. Letting $\delta_{1/2}$ be the $(1/2, 1/2)$-measure on 2, then $(X \times 2, \mu \times \delta_{1/2})$ is a free, non-ergodic G-system. So letting $\nu \in P_G(Z)$ be chosen so that $(Z, \nu) \cong (X \times 2, \mu \times \delta_{1/2})$, we see that ν is not ergodic. $\square$

Aknowledgements: The authors would like to thank Todor Tsankov for helpful discussions and the referee for their careful reading of an earlier draft, which caught several minor errors.

6.2 Preliminaries

Fix a non-compact, locally compact Polish group G, on which we fix a right-invariant compatible metric d. Throughout we assume that $\{g \in G : d(g, 1_G) \leq 1\}$ is compact.

Definition 6.2.1. *If (X, d_X) is a compact metric space and (Y, d_Y) is another metric space,* $\mathrm{Lip}(Y, X)$ *denotes the space of Lipschitz functions (we always mean 1-Lipschitz unless specified otherwise) from Y to X with the topology of pointwise convergence. If $f \in \mathrm{Lip}(G, X)$ and $g, h \in G$, we set*

$$(g \cdot f)(h) = f(hg).$$

This action turns $\mathrm{Lip}(G, X)$ *into a G-flow. Given a subflow $Y \subseteq \mathrm{Lip}(G, X)$ and any $B \subseteq G$, we set*

$$S_B(Y) := \{y|_B : y \in Y\}.$$

Notice that $S_B(Y) \subseteq \mathrm{Lip}(B, X)$ is a compact metric space; when $B \subseteq G$ is pre-compact, we will use the uniform metric

$$d_B(u, v) := \sup\{d_X(u(g), v(g)) : g \in B\}.$$

If $u \in S_B(Y)$ and $g \in G$, we define $g \cdot u \in S_{Bg^{-1}}(Y)$ via $(g \cdot u)(h) = u(hg)$. If $A \subseteq G$ is another subset, we set

$$A|B := \{g \in G : A \subseteq Bg^{-1}\}.$$

In particular, if $g \in A|B$ and $u \in S_B(Y)$, then $(g \cdot u)|_A \in S_A(Y)$.

Remark. *If (X, d_X) and (Y, d_Y) are metric spaces, we will equip $X \times Y$ with the metric $d_{X \times Y}((x_0, y_0), (x_1, y_1)) = \max\{d_X(x_0, x_1), d_Y(y_0, y_1)\}$.*

We now spend some time understanding minimality in subflows of $\mathrm{Lip}(G, X)$.

Definition 6.2.2. *Let (X, d_X) be a compact metric space. If $\epsilon > 0$ and $S \subseteq X$, we say that $S \subseteq X$ is* ϵ-dense *if for every $x \in X$, there is $s \in S$ with $d_X(x, s) < \epsilon$.*

Now fix a pre-compact $D \subseteq G$ and $\epsilon > 0$. We say that a subflow $Y \subseteq \mathrm{Lip}(G, X)$ is (D, ϵ)-minimal *if for any $y \in Y$, we have that $\{(g \cdot y)|_D : g \in G\} \subseteq S_D(Y)$ is ϵ-dense.*

We remark that this notion is monotone; if Y is (D, ϵ)-minimal, then also Y is (D_0, ϵ_0)-minimal for any $D_0 \subseteq D$ and $\epsilon_0 \geq \epsilon$.

Proposition 6.2.3. *With notation as in Definition 6.2.2, the following are equivalent.*

1. *The subflow $Y \subseteq \mathrm{Lip}(G, X)$ is (D, ϵ)-minimal.*

2. *There is a pre-compact open $E \subseteq G$ so that for any $u \in S_E(Y)$, we have that $\{(g \cdot u)|_D : g \in D|E\} \subseteq S_D(Y)$ is ϵ-dense.*

Proof. Item 2 certainly implies item 1. Conversely, if Item 2 fails, let $E_n \subseteq G$ be an exhaustion of G by pre-compact open sets, and find $u_n \in S_{E_n}(Y)$ and $v_n \in S_D(Y)$ with $d_D((g \cdot u_n)|_D, v_n) \geq \epsilon$ for every $g \in D|E_n$. Passing to a subsequence, let $u_n \to y \in Y$ and $v_n \to v \in S_D(Y)$. Towards a contradiction, suppose there were $g \in G$ with $d_D((g \cdot y)|_D, v) < \epsilon$. Then for some $N < \omega$, we have $g \in D|E_n$ for every $n \geq N$. But then we must have $d_D((g \cdot u_n)|_D, v_n) < \epsilon$ for some n, a contradiction. Hence Y is not (D, ϵ)-minimal. $\square$

Proposition 6.2.4. *With notation as in Definition 6.2.2, the subflow $Y \subseteq \mathrm{Lip}(G, X)$ is minimal iff it is (D, ϵ)-minimal for every pre-compact $D \subseteq G$ and $\epsilon > 0$.*

Proof. If Y is minimal, then fix pre-compact $D \subseteq G$, $\epsilon > 0$, and $y \in Y$. If $u \in S_D(Y)$, find $z \in Y$ with $z|_D = u$. Find $g_n \in G$ with $g_n \cdot y \to z$. It follows that we must have $(g_n \cdot y)|_D \to u$, implying that $d_D((g_n \cdot y)|_D, u) < \epsilon$ for some $n < \omega$. Hence Y is (D, ϵ)-minimal.

Now assume that Y is (D, ϵ)-minimal for every pre-compact $D \subseteq G$ and $\epsilon > 0$. Let $E_n \subseteq G$ be an exhaustion of G by pre-compact open sets, and let $\epsilon_n \to 0$. Fix $y, z \in Y$. We can then find for each $n < \omega$ some $g_n \in G$ with $d_{E_n}((g_n \cdot y)|_{E_n}, z|_{E_n}) < \epsilon_n$. It follows that $g_n \cdot y \to z$, showing that Y is minimal. $\square$

The remainder of this section looks at other important flows closely related to $\mathrm{Lip}(G, X)$.

Definition 6.2.5. *We let 2^G denote the space of closed subsets of G with the Fell topology. If $B \subseteq G$, we set $\mathrm{Meets}(B) := \{F \in 2^G : F \cap B \neq \emptyset\}$ and $\mathrm{Avoids}(B) := \{F \in 2^G : F \cap B = \emptyset\}$. By considering the sets $\mathrm{Meets}(U)$ and $\mathrm{Avoids}(C)$ for $U \subseteq G$ open and $C \subseteq G$ compact, we obtain a sub-basis for the Fell topology. The action we take is not quite the obvious one: given $g \in G$ and $F \in 2^G$, we set $g \cdot F = Fg^{-1}$. We do this for the following reason: letting $\iota \colon 2^G \to \mathrm{Lip}(G, [0, 1])$ denote the map $\iota(S)(g) = \min(d(g, S), 1)$, then ι is an injective G-map.*

Definition 6.2.6. *Fix $D \subseteq G$ any symmetric subset. Then a set $S \subseteq G$ is D-*spaced *if $Dg \cap Dh = \emptyset$ for any $g \neq h \in S$. Notice that $S \subseteq G$ is a maximal D-spaced set iff S is D-spaced and $D^2S = G$.*

Now suppose $D \subseteq G$ is a compact symmetric neighborhood of the identity. We let $G_D \subseteq 2^G$ denote the collection of $S \subseteq G$ which are $\mathrm{Int}(D)$*-spaced and with $D^2S = G$. In particular, G_D contains every maximal D-spaced set and every maximal* $\mathrm{Int}(D)$*-spaced set.*

Proposition 6.2.7. *For every $D \subseteq G$ a compact symmetric neighborhood of the identity, the G-flow G_D is weakly model-universal.*

Proof. This fact follows from a theorem of Kechris and a theorem of Slutsky. Suppose $G \times X \to X$ is a standard Borel G-space. A *cross-section* is any Borel $C \subseteq X$ so that $G \cdot C = X$. If $D \subseteq G$ is a compact neighborhood of the identity, we say that a cross-section C is *D-lacunary* if whenever $x \neq y \in C$, we have $Dx \cap Dy = \emptyset$. A *maximal D-lacunary cross-section* is a D-lacunary cross-section C such that for any $x \notin C$, $C \cup \{x\}$ is not D-lacunary. Notice that if C is a maximal D-lacunary cross-section and D is symmetric, then $D^2C = X$.

Kechris in [K2] proves the following.

Theorem 6.2.8. *Let G be a locally compact Polish group and D a compact neighborhood of the identity. Then any standard Borel G-space X admits a D-lacunary cross section.*

The measurable version of this result is a classical result of Feldman, Hahn, and Moore [FHM]; however, we appeal to the Borel version as we will need the following strengthening due to Slutsky ([S], Theorem 2.4).

Theorem 6.2.9. *Let G be a locally compact Polish group, and let X be a standard Borel G-space. Then if D is a compact symmetric neighborhood of the identity and $C \subseteq X$ is a D-lacunary cross-section of X, there is a maximal D-lacunary cross section $C' \supseteq C$.*

Now fix (X, μ) a free G-system. Applying Theorem 6.2.9, let $C \subseteq X$ be a maximal D-lacunary cross section. We define a Borel G-map $f : X \to G_D$ by setting

$$f(x) = C_x := \{g \in G : gx \in C\}.$$

As C is a maximal D-lacunary cross-section, we have $C_x \in G_D$, and certainly f is G-equivariant. To see that f is Borel, suppose $B \subseteq G$ is Borel, and consider $\mathrm{Meets}(B) \subseteq 2^G$. Then $f(x) \in \mathrm{Meets}(B)$ iff $Bx \cap C \neq \emptyset$. The set $Y := \{(g, x) : g \in B \text{ and } gx \in C\} \subseteq G \times X$ is Borel, and $f^{-1}(\mathrm{Meets}(B)) = \pi_X[Y]$. We note that since C is D-lacunary, the projection π_X is countable-to-one on Y, showing that $f^{-1}(\mathrm{Meets}(B))$ is Borel. □

6.3 A (non minimal) model-universal flow

In this section, we prove

Theorem 6.3.1. *The G-flow* $\mathrm{Lip}(G, [0, 1])^\omega$ *is model-universal.*

The proof is an adaptation of the proof of a result of Varadarajan.

Theorem 6.3.2 (Varadarajan [V]). *Let G be a locally compact Polish group, and let $G \times X \to X$ be a Borel action of G on a standard Borel space X. Then there is a compact metric G-flow Y and a G-equivariant Borel injection $X \hookrightarrow Y$.*

Write $\mathrm{Lip} := \mathrm{Lip}(G, [0,1])$, and let λ denote the left Haar measure on G.

Definition 6.3.3. *Suppose $f\colon X \to [0,1]$ is Borel. We let $f_x\colon G \to [0,1]$ be the Borel function given by $f_x(g) := f(gx)$. If $\phi\colon G \to \mathbb{R}^+$ is in $L^1(G, \lambda)$ with $\|\phi\|_1 \leq 1$, we let $(\phi * f)_x\colon G \to [0,1]$ be defined via*

$$(\phi * f)_x(g) = \int_G \phi(h) f(h^{-1}gx)d\lambda(h)$$

Lemma 6.3.4. *With notation as in Definition 6.3.3, suppose $\lambda(\mathrm{Supp}(\phi)) \leq L$ and that ϕ is K-Lipschitz. Then $(\phi * f)_x$ is $2L \cdot K$-Lipschitz.*

Proof. Fix $g_0, g_1 \in G$. By considering the change of variables $h \to g_i h$, we see that

$$|(\phi * f)_x(g_0) - (\phi * f)_x(g_1)| \leq \int_G |\phi(g_0 h) - \phi(g_1 h)| f(h^{-1}x)d\lambda(h)$$

The right hand side is identically zero whenever $h \notin g_0^{-1} \cdot \mathrm{Supp}(\phi) \cup g_1^{-1} \cdot \mathrm{Supp}(\phi)$. For h inside this set, the integrand is at most $K \cdot d(g_0, g_1)$. $\square$

In particular, suppose ϕ is such that $(\phi * f)_x \in \mathrm{Lip}$ for every $x \in X$. Then the map $\phi * f\colon X \to \mathrm{Lip}$ given by $\phi * f(x) = (\phi * f)_x$ is Borel and G-equivariant.

Recall that a sequence $(\phi_n)_n$ from $L^1(G, \lambda)$ is an *approximate identity* if $\phi_n \geq 0$, ϕ_n is symmetric, $\|\phi_n\|_1 = 1$, $\mathrm{Supp}(\phi_n)$ is compact, and $\mathrm{Supp}(\phi_n) \to \{1_G\}$.

Fact 6.3.5 ([F1], Proposition 2.44). *Suppose $f\colon X \to [0,1]$ is Borel, and let $(\phi_n)_n$ be an approximate identity. Then for any $x \in X$ and any compact $K \subseteq G$, we have that $\|(\phi_n * f)_x \cdot \chi_K - f_x \cdot \chi_K\|_1 \to 0$.*

We can now work towards our proof of Theorem 6.3.1. Let $\{f_k : k < \omega\}$ be a sequence of characteristic functions of Borel subsets of X which generate X. Let $(\phi_n)_n$ be an approximate identity with each ϕ_n C_n-Lipschitz for some $C_n \in \mathbb{R}^+$. Using Lemma 6.3.4, choose constants $c_n > 0$ so that $(c_n\phi_n * f_k)_x \in \mathrm{Lip}$ for every $n, k < \omega$ and $x \in X$. We define the map $\gamma\colon X \to \mathrm{Lip}^{\omega\times\omega} \cong \mathrm{Lip}^\omega$ by setting $\gamma(x)(n,k) = (c_n\phi_n * f_k)_x$. Then γ is Borel and G-equivariant, and we need only check that it is injective. Suppose $\gamma(x) = \gamma(y)$. Using Fact 6.3.5, this implies that for each $k < \omega$, $(f_k)_x(g) = (f_k)_y(g)$ for λ-almost every $g \in G$. So for most $g \in G$, this is true for every $k < \omega$. Fix such a $g \in G$. But then $gx = gy$, since the f_k separate points. In particular, also $x = y$.

6.4 A minimal model-universal flow

In this section we prove our main theorem.

Definition 6.4.1. *Suppose $D \subseteq G$ is a compact symmetric neighborhood of the identity. We call $S_0, S_1 \subseteq G$* D-apart *if $DS_0 \cap DS_1 = \emptyset$.*

Suppose (X, d_X) is a compact metric space. We call a subflow $Y \subseteq \mathrm{Lip}(G, X)$ D-irreducible *if whenever $S_0, S_1 \subseteq G$ are D-apart and whenever $y_0, y_1 \in Y$, there is $z \in Y$ with $z|_{S_i} = y_i|_{S_i}$.*

We note that if $D \supseteq \{g \in G : d(g, 1_G) \leq 1\}$, then $\mathrm{Lip}(G, [0,1]^n) \cong \mathrm{Lip}^n$ is D-irreducible. We first give an argument for $n = 1$. Let $S_0, S_1 \subseteq G$ be D-apart, and fix $y_0, y_1 \in \mathrm{Lip}$. We define $y \in \mathrm{Lip}$ via

$$y(g) := \max\left(\sup_{h \in S_0 \cup S_1} (y_i(h) - d(g,h));\, 0\right)$$

where $i \in \{0, 1\}$ is understood depending on the membership of h. Then y is clearly Lipschitz, and because S_0 and S_1 are D-apart, we have $y|_{S_i} = y_i|_{S_i}$ for each $i \in \{0, 1\}$. To conclude that Lip^n is D-irreducible, we observe that if $Y_i \subseteq \mathrm{Lip}(G, X_i)$ is D-irreducible for $i < n$, then so is $\prod_{i<n} Y_i \subseteq \mathrm{Lip}(G, \prod_{i<n} X_i)$.

Definition 6.4.2. *Suppose $D \supseteq \{g \in G : d(g, 1_G) \leq 1\}$ is a compact symmetric neighborhood of 1_G, and let $Y \subseteq \mathrm{Lip}^n$ be a D-irreducible subflow.*

Fix $E \subseteq G$ another compact symmetric neighborhood of 1_G with $D^2 \subseteq E$. Fix some $u \in S_E(Y)$. Suppose $F \subseteq G$ is yet another compact symmetric neighborhood of 1_G with $E^3 \subseteq F$. We define the flow

$$\Theta(Y, u, F) \subseteq \mathrm{Lip}^n$$

to consist of those functions $f \in \mathrm{Lip}^n$ so that all of the following hold:

1. *There is $T \in G_F$ so that $(g \cdot f)|_E = u$ for each $g \in T$.*
2. *There is $y \in Y$ with $f(g) = y(g)$ for any $g \notin E^2 T$.*
3. *For every $g \in G$, we have $(g \cdot f)|_D \in S_D(Y)$.*

If $f_n \in \Theta(Y, u, F)$, where items 1 and 2 are witnessed by $T_n \in G_F$ and $y_n \in Y$, respectively, then suppose $f_n \to f \in \mathrm{Lip}^n$. To show that $f \in \Theta(Y, u, F)$, we first note that item 3 is a closed condition. Then pass to a subsequence with $T_n \to T \in G_F$ and $y_n \to y \in Y$. Then T and y will witness that items 1 and 2 hold for f, showing that $\Theta(Y, u, F)$ is closed. To see that $\Theta(Y, u, F)$ is G-invariant, take $f \in \Theta(Y, u, F)$ and $g \in G$. Item 3 is clear for $g \cdot f$, and if $T \in G_F$ and $y \in Y$ witness items 1 and 2 for f, then $g \cdot T = Tg^{-1}$ and $g \cdot y$ will be witnesses for $g \cdot f$.

We remark that since Y is D-irreducible, the flow $\Theta(Y, u, F)$ is non-empty; the next proposition shows this and more.

Proposition 6.4.3. *In the setting of Definition 6.4.2, suppose in addition that Y is weakly model-universal. Then so is $\Theta(Y, u, F)$.*

Proof. First consider the restriction map $S_{E^3}(Y) \to S_{E^3 \setminus E^2}(Y) \times S_E(Y)$. Since Y is D-irreducible, this map is surjective. As this is a continuous surjection between compact metric spaces, let $\eta\colon S_{E^3\setminus E^2}(Y) \times S_E(Y) \to S_{E^3}(Y)$ be a Borel section.

We now define a map $\theta\colon Y \times G_F \to \Theta(Y,u,F)$, where given $y \in Y$ and $T \in G_F$, $\theta(y,T)$ is defined as follows.

1. If $g \notin E^2T$, then $\theta(y,T)(g) = y(g)$.
2. If $g = hk$ for some $h \in E^3$ and $k \in T$, we set $\theta(y,T)(g) = \eta((k \cdot y)|_{E^3\setminus E^2}, u)(h)$.

Then θ is a Borel G-equivariant map. We check that $\theta(y,T) \in \Theta(Y,u,F)$. By construction, items 1 and 2 from Definition 6.4.2 are satisfied. For item 3, consider $g \in G$. Since $D^2 \subseteq E$, either $Dg \cap E^2T = \varnothing$ or $Dg \subseteq E^3k$ for some $k \in T$. If $Dg \cap E^2T = \varnothing$, then $(g \cdot \theta(y,T))|_D = (g \cdot y)|_D \in S_D(Y)$. If $Dg \subseteq E^3k$ for some $k \in T$, then

$$(g \cdot \theta(y,T))\,|_D = \left(gk^{-1} \cdot \eta\left((k \cdot y)|_{E^3\setminus E^2}, u\right)\right)\Big|_D \in S_D(Y).$$

Lastly, to see that $\theta(y,T) \in \mathrm{Lip}^n$, we note that $D \supseteq \{g \in G : d(g, 1_G) \leq 1\}$, and we have just seen that item 3 holds.

Since $Y \times G_F$ is weakly model-universal, then so is $\Theta(Y,u,F)$. $\square$

We will sometimes want to refer to the θ constructed here as the various parameters change. In this case, we will refer to the map as $\theta\langle Y,u,F\rangle$.

Proposition 6.4.4. *In the setting of Definition 6.4.2, $\Theta(Y,u,F)$ is F^6-irreducible.*

Proof. Let $f_0, f_1 \in \Theta(Y,u,F)$, where the membership of f_i is witnessed by $T_i \in G_F$ and $y_i \in Y$. Suppose $S_0, S_1 \subseteq G$ are F^6-apart. Set $T = (F^5S_0 \cap T_0) \cup (F^5S_1 \cap T_1)$. Then T is an $\mathrm{Int}(F)$-spaced set, so let $U \supseteq T$ be a maximal $\mathrm{Int}(F)$-spaced set, and let $V = (U \setminus F(S_0 \cup S_1)) \cup T$. So in particular, $V \cap FS_i = T_i \cap FS_i$.

We claim that $V \in G_F$. Since $V \subseteq U$, V is $\mathrm{Int}(F)$-spaced. To see that $F^2V = G$, let $g \in G$. If $g \notin F^3(S_0 \cup S_1)$, then $g \in \mathrm{Int}(F)^2h$ for some $h \in U$, and since $h \notin F(S_0 \cup S_1)$, we have $h \in V$. If $g \in F^3S_i$, then $g \in F^2h$ for some $h \in (T_i \cap F^5S_i) \subseteq T \subseteq V$.

Since Y is D-irreducible and $D \subseteq F$, let $y \in Y$ be chosen with $y|_{F^5S_i} = y_i|_{F^5S_i}$. Using V and y, we define the $f \in \Theta(Y,u,F)$ which will satisfy $f|_{S_i} = f_i|_{S_i}$. We set $(g \cdot f)|_E = u$ for each $g \in V$, and we set $f(g) = y(g)$ whenever $g \notin E^2V$. It remains to define f on $(E^2 \setminus E)g$ for $g \in V$. If $g \in V$ and $E^2g \cap S_i \neq \varnothing$, we set $f|_{E^2g} = f_i|_{E^2g}$. If $g \in V$ and $E^2g \cap (S_0 \cup S_1) = \varnothing$, then we use the D-irreducibility of Y to find any $v_g \in S_{E^3}(Y)$ with $v_g|_{E^3\setminus E^2} = (g \cdot f)|_{E^3\setminus E^2}$ and $v_g|_E = u$, and we set $(g \cdot f)(h) = v_g(h)$ for $h \in E^3$.

We verify that f is as desired. Since $V \cap FS_i = T_i \cap FS_i$, we have $f|_{S_i} = f_i|_{S_i}$. To check that $f \in \Theta(Y,u,F)$, items 1 and 2 are witnessed by V and y. For item 3, let $g \in G$. Then either $Dg \cap E^2V = \varnothing$, in which case $(g \cdot f)|_D = (g \cdot y)|_D \in S_D(Y)$, or $Dg \subseteq E^3h$ for some $h \in V$. When $Dg \subseteq E^3h$ for some $h \in V$, then either $E^2h \cap (S_0 \cup S_1) = \varnothing$, in which case $(g \cdot f)|_D = (gh^{-1}) \cdot v_h|_D \in S_D(Y)$, or $E^2h \cap S_i \neq \varnothing$ for some $i \in \{0,1\}$, in which case $(g \cdot f)|_D = (g \cdot f_i)|_D \in S_D(Y)$. $\square$

We can now undertake the main construction. We set $\mathrm{Lip} = \mathrm{Lip}(G, [0,1])$. So note in particular that $\mathrm{Lip}^n = \mathrm{Lip}(G, [0,1]^n)$, and we can treat Lip^0 as the trivial (singleton) G-flow. We will inductively construct the following objects:

- $D_n, E_n, F_n \subseteq G$ compact symmetric neighborhoods of the identity,
- A subflow $Y_n \subseteq \mathrm{Lip}^n$ so that $Y_n \times \mathrm{Lip} \subseteq \mathrm{Lip}^{n+1}$ is D_n-irreducible.

The sets $D_n, E_n, F_n \subseteq G$ will have the following properties:

1. $D_n^2 \subseteq E_n$, $E_n^3 \subseteq F_n$, $F_n^6 \subseteq D_{n+1}$
2. $\bigcup_n D_n = G$
3. Suppose $K_n < \omega$ is such that $S_{D_n}(Y_n \times \mathrm{Lip})$ can be covered by K_n-many balls of radius $1/2^n$. Then there is a D_n^2-spaced set $\{g_{n,i} : i < K_n\} \subseteq G$ so that $D_n g_{n,i} \subseteq E_n$ for each $i < K_n$.
4. There is a E_n^4-spaced set $\{h_{n,i} : i < 2^n\} \subseteq G$ so that $E_n^4 h_{n,i} \subseteq \mathrm{Int}(F_n)$ for each $i < 2^n$.

We remark that the construction of the sets $D_n, E_n, F_n \subseteq G$ requires that G be non-compact, especially in regards to items 3 and 4. Start by setting $D_0 = \{g \in G : d(g, 1_G) \leq 1\}$ and $Y_0 = \mathrm{Lip}^0$.

Suppose we have defined $Y_0, ..., Y_n$; $D_0, ..., D_n$; $E_0, ..., E_{n-1}$; and $F_0, ..., F_{n-1}$. In particular, we have arranged so far that $Y_n \times \mathrm{Lip} \subseteq \mathrm{Lip}^{n+1}$ is D_n-irreducible. Find $E_n \subseteq G$ satisfying items (1) and (3).

Lemma 6.4.5. *There is $u_n \in S_{E_n}(Y_n \times \mathrm{Lip})$ so that*

$$\{(g \cdot u_n)|_{D_n} : g \in D_n | E_n\} \subseteq S_{D_n}(Y_n \times \mathrm{Lip})$$

is $(1/2^n)$-dense.

Proof. By assumption, $Y_n \times \mathrm{Lip}$ is D_n-irreducible. Suppose $K_n < \omega$ is as in item 3, and let $\{f_i : i < K_n\} \subseteq S_{D_n}(Y_n \times \mathrm{Lip})$ be chosen so that every $f \in S_{D_n}(Y_n \times \mathrm{Lip})$ satisfies $d_{D_n}(f, f_i) < 1/2^n$ for some $i < K_n$. Let $\{g_{n,i} : i < K_n\} \subseteq G$ be as guaranteed by item 3. Then $Dg_{n,i}$ and $Dg_{n,j}$ are D_n-apart whenever $i \neq j < K_n$. Using D_n-irreducibility, we can find $u_n \in S_{E_n}(Y_n \times \mathrm{Lip})$ so that $(g_{n,i} \cdot u_n)|_{D_n} = f_i$ for every $i < K_n$. Then u_n is as desired. ☐

Now find $F_n \subseteq G$ satisfying items (1) and (4), and form

$$Y_{n+1} := \Theta(Y_n \times \mathrm{Lip}, u_n, F_n) \subseteq \mathrm{Lip}^{n+1}.$$

Then find $D_{n+1} \subseteq G$ satisfying item (1). By Proposition 6.4.4, $Y_{n+1} \times \mathrm{Lip}$ is D_{n+1}-irreducible.

We can regard each Y_n as a subflow of Lip^ω by adding a tail of constant zero functions. Conversely, if $m < n \leq \omega$, we let $\pi^n_m : [0,1]^n \to [0,1]^m$ denote projection onto the first m coordinates. So if $Z \subseteq \mathrm{Lip}^\omega$ is a subflow, we let $\pi^\omega_n \circ Z$ denote its projection to a subflow of Lip^n. We now consider the space $\mathrm{Sub}(\mathrm{Lip}^\omega)$ of subflows of Lip^ω equipped with the Vietoris topology. In this topology, we have $Z_n \to Z$ iff for each compact $D \subseteq G$ and $m < \omega$, we have $S_D(\pi^n_m \circ Z_n) \to^n S_D(\pi^\omega_m \circ Z)$ in the space $K(\mathrm{Lip}(D, [0,1]^m))$, the space of compact subsets of $\mathrm{Lip}(D, [0,1]^m)$ also equipped with the Vietoris topology.

Lemma 6.4.6. *In the space* $\mathrm{Sub}(\mathrm{Lip}^\omega)$*, we have* $Y_n \to Z$ *for some minimal flow* Z.

Proof. For notation, set $S(k,m,n) := S_{D_k}(\pi^n_m \circ Y_n)$. Notice that when $k \leq n$, we have that $S(k,m,n+1) \subseteq S(k,m,n)$ by item 3 in Definition 6.4.2. So it follows that $S(k,m,n) \to^n S(k,m)$ for some compact $S(k,m) \subseteq \mathrm{Lip}(D_k,[0,1]^m)$. Also notice that $\pi^{m+1}_m \circ S(k,m+1) = S(k,m)$. Furthermore, if $\ell \geq k$ and $m < \omega$, let

$$\rho^l_k \colon \mathrm{Lip}(D_\ell,[0,1]^m) \to \mathrm{Lip}(D_k,[0,1]^m)$$

denote the restriction map. We note that whenever $k \leq \ell \leq n$, we have $\rho^l_k[S(\ell,m,n)] = S(k,m,n)$, so also $\rho^l_k[S(\ell,m)] = S(k,m)$. It follows that

$$Z := \varprojlim_k \varprojlim_m S(k,m) \subseteq \mathrm{Lip}^\omega$$

satisfies $Y_n \to Z$ in $K(\mathrm{Lip}^\omega)$. Since $\mathrm{Sub}(\mathrm{Lip}^\omega)$ is a closed subspace of $K(\mathrm{Lip}^\omega)$, it follows that $Z \in \mathrm{Sub}(\mathrm{Lip}^\omega)$.

To see that Z is minimal, it suffices to argue that $\pi^\omega_m \circ Z$ is minimal for each $m < \omega$. To do this, we use Proposition 6.2.4. Fix $z \in \pi^\omega_m \circ Z$ and $k < \omega$. We will argue that $\{(g \cdot z)|_{D_k} : g \in G\}$ is dense in $S(k,m)$, thus handling all $\epsilon > 0$ simultaneously.

Let $n \geq k$. The construction of Y_{n+1} yields (keeping in mind that $F^6_n \subseteq D_{n+1}$) that for any $v \in S(n+1,m,n+1)$, we have that $\{(g \cdot v)|_{D_n} : g \in D_n|D_{n+1}\} \subseteq S(n,m,n)$ is $(1/2^n)$-dense. Since $z|_{D_{n+1}} \in S(n+1,m)$, we have that $\{(g \cdot z)|_{D_n} : g \in D_n|D_{n+1}\} \subseteq S(n,m)$ is $(1/2^n)$-dense. By restricting to D_k, we see that $\{(g \cdot z)|_{D_k} : g \in D_n|D_{n+1}\} \subseteq S(k,m)$ is also $(1/2^n)$-dense. Letting n grow, we see that $\{(g \cdot z)|_{D_k} : g \in G\} \subseteq S(k,m)$ is dense as desired. □

Set $X = \mathrm{Lip}^\omega \times \prod_n G_{F_n}$. Weakly model-universal flows are closed under products, and if any member of the product is model-universal, then the product is as well ([Z4], Proposition 6). So X is model-universal. We will often write elements of X as tuples $((f_n)_n, (S_n)_n))$, where $f_n \in \mathrm{Lip}$ and $S_n \in G_{F_n}$. We will find a Borel G-invariant set $W \subseteq X$ with the following two properties:

1. For every measure $\mu \in P_G(X)$, $\mu(W) = 1$.
2. There is a Borel, G-equivariant injection $\phi \colon W \to Z$.

This will show that Z is model-universal. We set

$$W = \{((f_n)_n, (S_n)_n) : \forall k < \omega\ \exists m < \omega\ \forall n \geq m\ (E_k \cap E^3_n S_n = \emptyset)\}.$$

In particular, membership in W only depends on the sets S_n. Fix $\mu \in P_G(X)$; we will show that $\mu(W) = 1$. Fix $n < \omega$, and let $\nu \in P_G(G_{F_n})$ be the projection of μ onto G_{F_n}.

Lemma 6.4.7. *If* $k \leq n$*, we have*

$$\nu(\{S \in G_{F_n} : E_k \cap E^3_n S \neq \emptyset\}) \leq 1/2^n$$

Proof. Recall that by item (4) of the properties of the sets D_n, E_n, F_n, there is an E_n^4-spaced set $\{h_{n,i} : i < 2^n\} \subseteq G$ so that $E_n^4 h_{n,i} \subseteq \mathrm{Int}(F_n)$ for each $i < 2^n$. Note that we have:

$$\begin{aligned} h_{n,i}^{-1} \cdot \{S \in G_{F_n} : E_k \cap E_n^3 S \neq \emptyset\} &= \{S h_{n,i} \in G_{F_n} : E_k \cap E_n^3 S \neq \emptyset\} \\ &= \{T \in G_{F_n} : E_n^3 E_k h_{n,i} \cap T \neq \emptyset\} \\ &\subseteq \{T \in G_{F_n} : E_n^4 h_{n,i} \cap T \neq \emptyset\}. \end{aligned}$$

Note that if $g_i \in E_n^4 h_{n,i} \subseteq \mathrm{Int}(F_n)$ and $g_j \in E_n^4 h_{n,j} \subseteq \mathrm{Int}(F_n)$, then we have

$$1_G \in \mathrm{Int}(F_n) g_i \cap \mathrm{Int}(F_n) g_j \neq \emptyset.$$

It follows that the collection

$$\{h_{n,i}^{-1} \cdot \{S \in G_{F_n} : E_k \cap E_n^3 S\} : i < 2^n\}$$

is pairwise disjoint. Since ν is G-invariant, we are done. $\square$

We can now apply the Borel-Cantelli lemma to conclude that $\mu(W) = 1$.

We now turn to defining $\phi \colon W \to Z$. First let $j \colon \omega \to \omega \times 2$ be an infinite-to-one surjection. We define for each $n < \omega$ a Borel G-equivariant map $\phi_n \colon W \to Y_n$ so that $\phi(w) = \lim_n \phi_n(w)$. Start by letting ϕ_0 denote the only map to Y_0. Suppose ϕ_n is defined. Let $w = ((f_n)_n, (S_n)_n) \in W$ be given. For notation, we set $f_{(n,0)} = f_n$ and $f_{(n,1)} = \iota(S_n)$. We set

$$\theta_n := \theta\langle Y_n \times \mathrm{Lip}, u_n, F_n\rangle$$

i.e. $\theta_n \colon (Y_n \times \mathrm{Lip}) \times G_{F_n} \to Y_{n+1}$ denotes the map defined in the proof of Proposition 6.4.3. We set

$$\phi_{n+1}(w) = \theta_n((\phi_n(w), f_{j(n)}), S_n).$$

Lemma 6.4.8. *For every $w \in W$, the sequence $\phi_n(w)$ is convergent.*

Proof. Fix $w := ((f_n)_n, (S_n)_n) \in W$, and write $\phi_n(w) = (\alpha_{n,m})_{m<\omega}$, with $\alpha_{n,m} \in \mathrm{Lip}$. When $m > n$, we set $\alpha_{n,m}$ to the constant zero function. Fix $m < \omega$ and $g \in G$. We will show that as $n \to \infty$, eventually $\alpha_{n,m}(g)$ is constant. Suppose $k < \omega$ is such that $g \in E_k$. Since $w \in W$, we eventually have that $E_n^3 S_n \cap E_k = \emptyset$. This implies that $\phi_{n+1}(w)|_{E_k} = (\phi_n(w), f_{j(n)})|_{E_k}$, so in particular $\alpha_{n,m}(g) = \alpha_{n+1,m}(g)$ whenever n is suitably large. $\square$

We can now define $\phi \colon W \to Z$ by setting

$$\phi(w) = \lim_n \phi_n(w)$$

Then ϕ is Borel and G-equivariant. To see that ϕ is injective, suppose $w, w' \in W$, where $w = ((f_n)_n, (S_n)_n)$ and $w' = ((f'_n)_n, (S'_n)_n)$. Find $n < \omega$ and $i < 2$ with $f_{(n,i)} \neq f'_{(n,i)}$. In particular, there is some $k < \omega$ with $f_{(n,i)}|_{E_k} \neq f'_{(n,i)}|_{E_k}$. Because $j \colon \omega \to \omega \times 2$ is an infinite-to-one surjection, we can find $N < \omega$ with $j(N) = (n, i)$ which is as large as desired, in particular large enough so that $E_m^3 S_m \cap E_k = \emptyset$ and $E_m^3 S'_m \cap E_k = \emptyset$ for every $m \geq N$. Writing $\phi(w) = (\alpha_n)_n$ and $\phi(w') = (\alpha'_n)_n$, we have that $\alpha_N|_{E_k} = f_{j(N)}|_{E_k}$ and $\alpha'_N|_{E_k} = f'_{j(N)}|_{E_k}$. Hence $\phi(w) \neq \phi(w')$ as desired.

This concludes the proof that Z is a minimal, model-universal G-flow.

Bibliography

[A1] N. Ackerman, *Representations of* $\mathrm{Aut}(M)$*-invariant measures: Part 1*, 2015. Preprint `arXiv:1509.06170`.

[A2] N. Antonyan, *Equivariant embeddings and compactifications of free G-spaces*, Int. J. Math. Math. Sci. **1** (2003), 1–14. MR1959644

[AEG] S. Adams, G. A. Elliott, and T. Giordano, *Amenable actions of groups*, Trans. Amer. Math. Soc. **344** (1994), no. 2, 803–822. MR1250814

[AFP] A. V. Arkhangel'skiĭ, V. V. Fedorchuk, and L. S. Pontryagin, *General topology i: Basic concepts and constructions dimension theory*, 1st ed., Encyclopaedia of Mathematical Sciences 17, Springer-Verlag Berlin Heidelberg, 1990.

[AKL] O. Angel, A. S. Kechris, and R. Lyons, *Random orderings and unique ergodicity of automorphism groups*, J. Eur. Math. Soc. (JEMS) **16** (2012), no. 10, 2059–2095. MR3274785

[BIT] I. Ben Yaacov, T. Ibarlucía, and T. Tsankov, *Eberlein oligomorphic groups*, Trans. Amer. Math. Soc. **370** (2018), no. 3, 2181–2209.

[BMT] I. Ben Yaacov, J. Melleray, and T. Tsankov, *Metrizable universal minimal flows of Polish groups have a comeagre orbit*, Geom. Funct. Anal. **27** (2017), no. 1, 67–77. MR3613453

[BPT] M. Bodirsky, M. Pinsker, and T. T., *Decidability of definability*, J. Symbolic Logic **78** (2013), 1036–1054.

[C1] P. J. Cameron, *Permutation groups*, London Mathematical Society Student Texts, vol. 45, Cambridge University Press, Cambridge, 1999. MR1721031

[C2] G. L. Cherlin, *The classification of countable homogeneous directed graphs and countable homogeneous n-tournaments*, Mem. Amer. Math. Soc 131 **621** (1998).

[CT] H. Crane and H. Towsner, *Relatively exchangeable structures*, J. Symb. Log. **83** (2018), no. 2, 416–442.

[E1] G. Elek, *Free minimal actions of countable groups with invariant probability measures*, Ergodic Theory Dyn. Sys. (to appear).

[E2] R. Ellis, *Lectures on topological dynamics*, W. A. Benjamin, Inc., New York, 1969. MR0267561

[EHKN] D. . Evans, J. Hubička, M. Konečný, and J. Nešetřil, *EPPA for two-graphs and antipodal metric spaces*, Proc. Amer. Math. Soc. **148** (2020), no. 5, 1901–1915. MR4078076

[ET] D. M. Evans and T. Tsankov, *Free actions of free groups on countable structures and property (T)*, Fund. Math. **232** (2016), no. 1, 49–63.

[F1] G. Folland, *A course in abstract harmonic analysis*, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1995. MR1397028

[F2] H. Furstenber, *Recurrence in ergodic theory and combinatorial number theory*, Princeton University Press, Princeton, N.J. (1981).

[FHM] J. Feldman, P. Hahn, and C. Moore, *Orbit structure and countable sections for actions of continuous groups*, Adv. in Math. **28** (1978), no. 3, 186–230. MR492061

[G] A. Gut, *Probability: a graduate course*, Second, Springer Texts in Statistics, Springer, New York, 2013. MR2977961

[GW1] E. Glasner and B. Weiss, *Minimal actions of the group* $S(\mathbb{Z})$ *of permutations of the integers*, Geom. Funct. Anal. **12** (2002), no. 5, 964–988.

[GW2] ______, *Spatial and non-spatial actions of Polish groups*, Ergodic Theory Dynam. Systems **25** (2005), no. 5, 1521–1538.

[H1] A. Haar, *Der Massbegriff in der Theorie der kontinuierlichen Gruppen*, Ann. of Math. (2) **34** (1933), no. 1, 147–169. MR1503103

[H2] W. Hodges, *Model theory*, Encyclopedia of Mathematics and its Applications, vol. 42, Cambridge University Press, Cambridge, 1993. MR1221741

[H3] W. Hodges, *Model theory*, Encyclopedia of Mathematics and its Applications, vol. 42, Cambridge University Press, Cambridge, 1993.

[HS1] N. Hindman and D. Strauss, *Algebra in the Stone-Čech Compactification*, 2nd Edition, De Gruyter (2012).

[HS2] ______, *Topological properties of some algebraically defined subsets of* $\beta\mathbb{N}$, Topol. Appl. **220** (2017), 43–49.

[J] C. Jahel, *Unique ergodicity of the automorphism group of the semigeneric directed graph*, Ergodic Theory and Dynamical Systems, accepted.

[JLNVTW] J. Jasiński, C. Laflamme, L. Nguyen Van Thé, and R. Woodrow, *Ramsey precompact expansions of homogeneous directed graphs*, Electron. J. Combin. **21** (2014), no. 4, Paper 4.42, 31. MR3292279

[JT] Jahel, C. and Tsankov T., *Invariant measures on products and on the space of linear orders*, arXiv:2007.00281 (2020).

[JZ1] C. Jahel and A. Zucker, *Minimal model-universal flows for locally compact Polish groups*, arXiv:2006.01710 (2020).

[JZ2] Jahel, C. and Zucker, A., *Topological dynamics of Polish group extensions*, arXiv:1902.04901 (2019).

[K1] O. Kallenberg, *Foundations of modern probability*, Second, Probability and its Applications (New York), Springer-Verlag, New York, 2002.

[K2] A. Kechris, *Countable sections for locally compact group actions*, Ergodic Theory Dynam. Systems **12** (1992), no. 2, 283–295. MR1176624

[KPT] A. S. Kechris, V. G. Pestov, and S. Todorcevic, *Fraïssé limits, Ramsey theory, and topological dynamics of automorphism groups*, Geom. Funct. Anal. **15** (2005), no. 1, 106–189. MR2140630 (2007j:37013)

[KR] A. S. Kechris and C. Rosendal, *Turbulence, amalgamation and generic automorphisms of homogeneous structures*, Proc. Lond. Math. Soc. **94** (2007), no. 2, 302–350.

[KS] A. S. Kechris and M. Sokić, *Dynamical properties of the automorphism groups of the random poset and random distributive lattice*, Fund. Math. **218** (2012), no. 1, 69–94.

[M1] S. M., *Relational quotients*, Fund. Math. **221** (2013), 7417–7436.

[M2] C. McDiarmid, *On the method of bounded differences*, Surveys in combinatorics, 1989 (Norwich, 1989), 1989, pp. 148–188. MR1036755

[M3] ______, *On the method of bounded differences*, Surveys in combinatorics, 1989 (Norwich, 1989), 1989, pp. 148–188. MR1036755

[MNT] J. Melleray, L. Nguyen Van Thé, and T. Tsankov, *Polish groups with metrizable universal minimal flows*, Int. Math. Res. Not. IMRN **5** (2016), 1285–1307. MR3509926

[N] L. Nguyen Van Thé, *More on the Kechris-Pestov-Todorcevic correspondence: precompact expansions*, Fund. Math. **222** (2013), no. 1, 19–47. MR3080786

[NR] J. Nešetřil and V. Rödl, *On a probabilistic graph-theoretical method*, Proc. Amer. Math. Soc. **72** (1978), no. 2, 417–421. MR507350

[NVT] L. Nguyen Van Thé, *More on the Kechris-Pestov-Todorcevic correspondence: precompact expansions*, Fund. Math. **222** (2013), no. 1, 19–47.

[P1] V. Pestov, *Amenability versus property* (T) *for non-locally compact topological groups*, Trans. Amer. Math. Soc. **10** (2018), 7417–7436.

[P2] V. G. Pestov, *On free actions, minimal flows, and a problem by Ellis*, Trans. Amer. Math. Soc. **350** (1998), no. 10, 4149–4165. MR1608494

[P3] R. R. Phelps, *Lectures on Choquet's theorem*, Second, Lecture Notes in Mathematics, vol. 1757, Springer-Verlag, Berlin, 2001.

[P4] B. Poizat, *A course in model theory*, Universitext, Springer-Verlag, New York, 2000. An introduction to contemporary mathematical logic, Translated from the French by Moses Klein and revised by the author. MR1757487

[PS] M. Pawliuk and M. Sokić, *Amenability and unique ergodicity of automorphism groups of countable homogeneous directed graphs*, Ergodic Theory Dynam. Systems **40** (2020), no. 5, 1351–1401. MR4082266

[RN] C. Ryll-Nardzewski, *On stationary sequences of random variables and the de Finetti's equivalence*, Colloq. Math. **4** (1957), 149–156.

[R] S. Rideau, *Imaginaries and invariant types in existentially closed valued differential fields*, J. Reine Angew. Math. **750** (2019), 157–196. MR3943320

[S] K. Slutsky, *Lebesgue orbit equivalence of multidimensional Borel flows: a picturebook of tilings*, Ergodic Theory Dynam. Systems **37** (2017), no. 6, 1966–1996. MR3681992

[T1] T. Tsankov, *Unitary representations of oligomorphic groups*, Geom. Funct. Anal. **22** (2012), no. 2, 528–555.

[T2] ———, *Automorphism groups and their actions*, Université Paris Diderot, 2014. Available at http://math.univ-lyon1.fr/~tsankov/.

[VN] J. Von Neumann, *Zur allgemeinen theorie des maßes*, Fund. Math. **13** (1929), 73–115.

[V] V. Varadarajan, *Groups of automorphisms of Borel spaces*, Trans. Amer. Math. Soc. **109** (1963), 191–220. MR159923

[W] B. Weiss, *Minimal models for free actions* **567** (2012), 249–264. MR2931921

[Z1] A. Zucker, *Amenability and unique ergodicity of automorphism groups of Fraïssé structures*, Fund. Math. **226** (2014), no. 1, 41–62.

[Z2] ———, *Topological dynamics of automorphism groups, ultrafilter combinatorics, and the generic point problem*, Trans. Amer. Math. Soc. **368** (2016), no. 9, 6715–6740. MR3461049

[Z3] ———, *New directions in the abstract topological dynamics of polish groups*, PhD. thesis, Carnegie Mellon University (2018).

[Z4] ———, *A note on minimal models for pmp actions*, Proc. Amer. Math. Soc. **148** (2020), no. 3, 1161–1168. MR4055943

Des progrès sur le problème d'unique ergodicité

Résumé. Cette thèse est à l'intersection de la dynamique, de la combinatoire et de la théorie des probabilités. Mon travail se concentre sur une spécialisation de la notion de moyennabilité : l'unique ergodicité. Il s'agit d'une qualification pour décrire les groupes pour lesquelles les actions minimales admettent exactement une mesure invariante. Je m'intéresse particulièrement à des groupes d'automorphismes de limites de Fraïssé qui ont cette propriété. Les groupes d'automorphismes de limites de Fraïssé ont plusieurs propriétés qui les rendent très intéressants. Premièrement, il sont assez "gros", c'est à dire non localement compact. Le chapitre 6 traite le cas des groupes localement compacts. Une autre propriété intéressante est la compréhension que nous avons de la topologie de certains espaces sur lesquels agissent ces groupes, ceci étant directement lié au fait qu'on a à faire à des groupes d'automorphismes. Mon travail consiste principalement à exploiter notre compréhension de ces actions pour construire des mesures qui nous permettent de comprendre toutes les mesures invariantes pour certains groupes.

Mots-clés: Moyennabilité, Unique ergodicité, Probabilités, Dynamique des groupes topologiques, Limites de Fraïssé.

Abstract. This thesis is at the intersection of dynamics, combinatorics and probability theory. My work focuses on a specialization of the notion of amenability: unique ergodicity. This notion refers to those groups whose minimal actions admit a unique invariant measure. I am especially interested in automorphism groups of Fraïssé limits. Those groups have several interesting properties. One of those properties is that they are somewhat "big", meaning not locally compact. The case of locally compact groups is discussed in Chapter 6. Another interesting property of automorphism groups is the understanding we have of some of their actions. This is deeply linked to the fact that we are dealing with automorphism groups. My work essentially relies on the understanding of those actions to construct measures that allow us to understand all the invariant measures of some groups.

Keywords: Amenability, Unique ergodicity, Dynamics of topological groups, Fraïssé limits.

Image de couverture: Réalisée par Marie Callier.