Completeness and cocompleteness for partial groups

Edoardo Salati

Technische Universität Dresden

Algebra Seminar - Universidad del País Vasco

October 13^{th}, 2021

The idea behind partial groups

- For the past 20-30 years there has been much interest in certain categories (fusion and transporter systems).
- Partial groups were thought, by A. Chermak, as a translation of the composition of (invertible) morphisms in such categories into a product defined on elements of a set.
- Problem: in general, in categories not all morphisms can be composed. Codomain and domain of consecutive morphisms must agree.

$$X_0 \xrightarrow{g_1} X_1 \xrightarrow{g_2} \dots \xrightarrow{g_n} X_n$$

- Thus, we cannot pretend to have a full operation, defined on all pairs.

Partial groups

Definition

Let $\mathcal{L}$ be a non-empty set and $W(\mathcal{L}) = W$ be the free monoid on $\mathcal{L}$; also

$$D = D(\mathcal{L}) \subseteq W \text{ (domain)} , \quad \Pi : D \to \mathcal{L} \text{ (multivariable product)}$$

and $i : \mathcal{L} \to \mathcal{L}$ an involutory bijection. Extend i to a map $i : W(\mathcal{L}) \longrightarrow W(\mathcal{L})$ by $(x_1, \ldots, x_n)i = ((x_n)i, \ldots, (x_1)i)$.
Then the quadruple $(\mathcal{L}, D, \Pi, i)$ is a partial group provided:

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Then the quadruple $(\mathcal{L}, D, \Pi, i)$ is a partial group provided:

1. $\mathcal{L} \subseteq D$ and moreover $u \circ v \in D \implies u, v \in D$;
2. $\Pi|_{\mathcal{L}} = id_{\mathcal{L}}$;
3. if $u \circ v \circ w \in D$, then $u \circ (v)\Pi \circ w \in D$ and $(u \circ v \circ w)\Pi = (u \circ (v)\Pi \circ w)\Pi$;
4. if $w \in D$, then $(w)i \circ w \in D$ and $((w)i \circ w)\Pi = 1 = (\emptyset)\Pi$, where $\emptyset$ is the empty word.

Morphisms of partial groups

Definition

Given partial groups $(\mathcal{L}, D, \Pi, i)$ and $(\mathcal{L}', D', \Pi', i')$ and a set-wise map $\beta : \mathcal{L} \to \mathcal{L}'$, consider the componentwise extension $\beta^* : W(\mathcal{L}) \to W(\mathcal{L}')$. Then β is a morphism of partial groups if:

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(a) $(D)\beta^* \subseteq D'$;

(b) we have $\Pi\beta = \beta^*\Pi'$, i.e. a commutative diagram

$$\begin{array}{ccc} D & \xrightarrow{\Pi} & \mathcal{L} \\ \downarrow{\scriptstyle \beta^*} & /\!/ & \downarrow{\scriptstyle \beta} \\ D' & \xrightarrow{\Pi'} & \mathcal{L}'. \end{array}$$

The category *Part*

We obtain a category *Part* with objects partial groups and morphisms as defined in the previous slide.

Few properties

Let $\mathcal{L} = (\mathcal{L}, D, \Pi, i)$ be a partial group.

- If $w \in D$, then any word w^* obtained from w by adding 1s is in D.
- $\mathcal{L}$ is a group iff $D = W(\mathcal{L})$.
- *Grp* $\hookrightarrow$ *Part* is a full embedding of categories.
- If $f : \mathcal{L} \longrightarrow \mathcal{M}$ is a morphism in *Part*, then $(1_{\mathcal{L}})f = 1_{\mathcal{M}}$, so partial groups are naturally pointed.
- The trivial group $\{1\}$ is a 0-object in *Part*.

Some category theory facts

Theorem

If $\mathcal{C}$ is a locally small category, every (co)limit is the (co)equalizer of a (co)product.

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Lemma

Suppose to have functors $F \vdash U : \mathcal{C} \longrightarrow \mathcal{C}'$; then F preserves colimits and U preserves limits.

For example, the forgetful functor $U : Grp \longrightarrow Set$ is a right adjoint to the free construction functor, so for groups G_i,

$$U(\times G_i) = \times U(G_i).$$

Note! Clearly, we have a forgetful functor $U : Part \longrightarrow Set$.

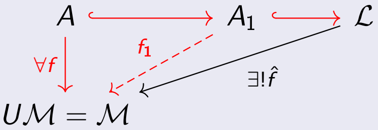
Free partial groups: an example (I)

Let Set^* be the category of pointed sets.

- Objects are pairs (X, x) with X a non-empty set and $x \in X$.
- Morphisms $f : (X, x) \to (Y, y)$ are set-wise maps such that $f : x \mapsto y$.

Free partial groups

$(\mathcal{L}, D(\mathcal{L}), \Pi, i)$ is the free partial group over the set A (equivalently, over the pointed set A_1) iff for every partial group $\mathcal{M}$ and every set-map f (pointed set-map f_1) there exists a unique extension $\hat{f}$, which is a morphism of partial groups and is such that the diagram below is commutative.



Free partial groups: an example (II)

- Let $A := \{a\}$, $A_1 := (\{a, 1\}, 1)$. The free partial group over A (or over the pointed set A_1) is given by $\mathcal{L} := \{a, 1, a^{-1}\}$.
- $D(\mathcal{L}) \subseteq W(\mathcal{L})$ is the smallest subset making $\mathcal{L}$ a partial group; it is the set of strings which, after removing the occurrences of 1, are alternating strings of a and a^{-1}.
- Π is defined as the symbol a or a^{-1} with the highest number of occurrences or, else, 1.

Forgetful and free-construction functors in *Part*

Similar to what happens in *Grp*, we have forgetful and free-construction functors in *Part*:

$$U : Part \longrightarrow Set^*, \qquad F : Set^* \longrightarrow Part \qquad \text{such that } F \dashv U.$$

Limits in *Part*

Since the forgetful functor U is a right adjoint, it preserves limits. Thus, a limit in *Part* is built the same way as a limit in *Grp*.

- Take the set-wise limit as underlying set.
- Endow it with a suitable partial group structure.

The partial group structure is inherited naturally "componentwise".

Example

$\mathcal{N} := \mathcal{L} \times \mathcal{M} = (\mathcal{L} \times \mathcal{M}, D(\mathcal{N}), \Pi, i)$ where

$$D(\mathcal{N}) = \{((l_1, m_1), \dots, (l_k, m_k)) \in W(\mathcal{N}) \mid (l_1, \dots, l_k) \in D(\mathcal{L}) \text{ and } (m_1, \dots, m_k) \in D(\mathcal{M})\}$$

and Π and i are defined componentwise.

Colimits in *Set** and *Grp*

In *Set**

$$\coprod (X_i, x_i) = \left(\frac{\bigsqcup X_i}{\sim_0}, \{x_i\} \right)$$

where $\sim_0$ is the equivalence relation identifying exactly all the x_i.

$$\operatorname{colim} X_i = \frac{\coprod X_i}{\sim}$$

where $\sim$ is the smallest eq. rel. identifying elements "according" to the morphisms of the diagram.

In *Grp*

$$\coprod X_i = * X_i := \langle U(X_i) \mid R_i \rangle$$

where the R_i are the defining relations in X_i.

$$\operatorname{colim} X_i = \frac{\coprod X_i}{R_\sim}$$

where $R_\sim$ is the smallest, *normal* subgroup containing the relations "arising" from the morphisms of the diagram.

Colimits in *Part*

Coproducts in *Part* behave as in *Set**.

$$\mathcal{L} = \coprod \mathcal{L}_i := \frac{\bigsqcup \mathcal{L}_i}{\sim_0}, \qquad "D(\mathcal{L}) = \bigcup D(\mathcal{L}_i) \quad (mod \ \ \sim_0)"$$

where $\sim_0$ is the equivalence relation identifying the units.

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Instead, coequalizers in *Part* behave similarly as in *Grp*.

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In *Part* we don't have a substructure analogous to normal subgroups of groups and affording quotients!

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$$\mathcal{M} \underset{g}{\overset{f}{\rightrightarrows}} \mathcal{L} \xrightarrow{q} coeq(f, g) = \mathcal{L}/\sim$$

Who is $\sim$? How do we control it?

Coequalizers in *Part*: a first attempt

Let's try to make the set-wise coequalizer $\mathcal{L}/\sim_0$ (i.e., $\sim_0$ is generated by the pairs (xf, xg) for $x \in \mathcal{M}$) also the coequalizer in *Part*.
q has to be a morphism in *Part*, so we need a commutative diagram

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D(\mathcal{M}) & \underset{g^*}{\overset{f^*}{\rightrightarrows}} & D(\mathcal{L}) & \xrightarrow{q^*} & D(\mathcal{L}/\sim_0) \\
\downarrow \Pi_{\mathcal{M}} & & \downarrow \Pi & & \downarrow \pi \\
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\end{array}$$

Thus, define
$D(\mathcal{L}/\sim_0) := q^*(D(\mathcal{L}))$.

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- Consider $(xf, yf), (xf, yg), (xg, yg) \in D(\mathcal{L})$. They all represent the same string in $D(\mathcal{L}/\sim_0)$.
- $(xf, yf)\Pi = (x, y)f^*\Pi = ((x, y)\Pi_{\mathcal{M}})f \sim_0 ((x, y)\Pi_{\mathcal{M}})g = (xg, yg)\Pi$.

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- $(xf, yf)\Pi = (x, y)f^*\Pi = ((x, y)\Pi_{\mathcal{M}})f \sim_0 ((x, y)\Pi_{\mathcal{M}})g = (xg, yg)\Pi$.
- Problem! There is no guarantee $(xf, yf)\Pi \sim_0 (xf, yg)\Pi$ (counterexample, S.).

Coequalizers in *Part*

We need a relation $\sim$ on $\mathcal{L}$ such that:

1. $\sim$ contains $\sim_0$, the relation generated by the pairs (xf, xg).
2. $u = (u_i), v = (v_i) \in D(\mathcal{L})$ with $u_i \sim v_i \; \forall i$, then $u\Pi \sim v\Pi$.

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One can then prove that:

- a relation $\sim$ satisfying (1) and (2) exists;
- (1) and (2) are stable under taking intersections;
- there exists a smallest equivalence relation $\mathcal{R}$ satisfying (1) and (2);
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Final result

Part is complete and cocomplete.

For the future

- Develop a theory of generators and relations for partial groups.
- Detect a suitable notion of morphisms of localities, a subclass of partial groups satisfying additional axioms.

Thank you