Limits and colimits of partial groups

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Fusion and transporter categories

Let G be a finite group, $S \in \mathrm{Syl}_p(G)$ and $\Delta \subseteq Subgr(G)$.

Definition (Puig, 1976)

The *fusion category* $\mathcal{F}_S(G)$ is the category with

- $Ob(\mathcal{F}_S(G)) = \{\text{subgroups of S}\}$,
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The *transporter category* $\mathcal{T}_\Delta(G)$ is the category with

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When $\Delta \subseteq Ob(\mathcal{F}_S(G))$, there is an obvious functor $\mathcal{T}_\Delta(G) \longrightarrow \mathcal{F}_S(G)$.

From transporter systems to partial groups

- Algebraic topologists studied fusion and transporter systems, abstract generalizations of fusion and transporter categories.
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- Their goal being a proof of existence and uniqueness of a centric linking system (i.e. a transporter system with some specific properties) over any fusion system with the *saturation* property.
- In 2013 A. Chermak proved both existence and uniqueness by translating the categorical language of transporter systems into the language of other algebraic structures, namely partial groups.
- He translated the composition of morphisms in a transporter system into the product of elements of a partial group.

$$X_0 \xrightarrow{g_1} X_1 \xrightarrow{g_2} \dots \xrightarrow{g_n} X_n$$

Partial groups

Definition

Let $\mathcal{L}$ be a non-empty set and $W(\mathcal{L}) = W$ be the free monoid on $\mathcal{L}$; also

$$D = D(\mathcal{L}) \subseteq W \text{ (domain)} \ , \quad \Pi : D \to \mathcal{L} \text{ (multivariable product)}$$

and $i : \mathcal{L} \to \mathcal{L}$ an involutory bijection.
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4. by extending i to $W(\mathcal{L})$ defining $(x_1, \ldots, x_n)i = ((x_n)i, \ldots, (x_1)i)$, if $w \in D$ then we have $(w)i \circ w \in D$ and $((w)i \circ w)\Pi = 1 = (\emptyset)\Pi$, where $\emptyset$ is the empty word.

The category *Part* of partial groups

Definition

Given partial groups $(\mathcal{L}, D, \Pi, i)$ and $(\mathcal{L}', D', \Pi', i')$ and a set-wise map $\beta : \mathcal{L} \to \mathcal{L}'$, consider the componentwise extension $\beta^* : W(\mathcal{L}) \to W(\mathcal{L}')$. Then β is a morphism of partial groups if:

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(a) $(D)\beta^* \subseteq D'$;

(b) we have $\Pi\beta = \beta^*\Pi'$, i.e. a commutative diagram

$$\begin{array}{ccc} D & \xrightarrow{\Pi} & \mathcal{L} \\ \downarrow{\scriptstyle \beta^*} & /\!/ & \downarrow{\scriptstyle \beta} \\ D' & \xrightarrow{\Pi'} & \mathcal{L}'. \end{array}$$

Thus we obtain a category *Part* with objects partial groups and morphisms as defined above.

An example

Example

Let G be a finite group with $S \in \mathrm{Syl}_p(G)$;
$S \in \Delta \subseteq Subgr(S)$ closed under G-conjugation and overgroups.
Set

$$\mathcal{L} = \mathcal{L}_\Delta(G) := \{g \in G \mid S \cap S^g \in \Delta\}.$$

The product Π and the inversion i are defined by restriction of those in G, $D(\mathcal{L})$ by a technical property we require on $\mathcal{L}$.

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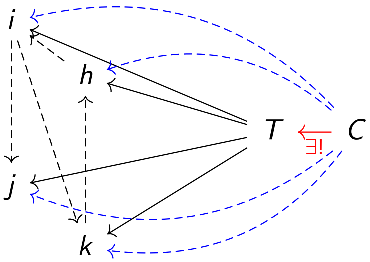
There is an abstract generalization of the partial groups $\mathcal{L}_\Delta(G)$, namely *localities*, corresponding to transporter systems.

Limits and colimits

$\mathcal{I}$ is a small category, $\mathcal{C}$ any category, $D : \mathcal{I} \longrightarrow \mathcal{C}$ a diagram.

A limit of D is a universal terminal object.



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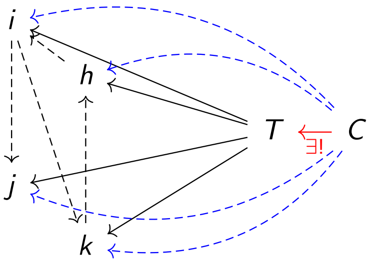

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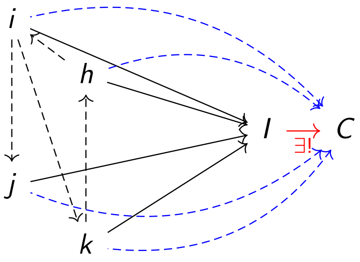
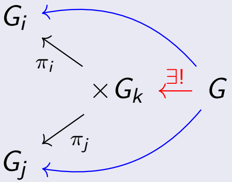


Few examples: I

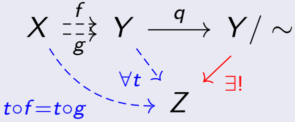
Products in *Grp*

Products are limits over diagrams with no morphisms. In *Grp* they correspond to direct products.



Coequalizers in *Set*

Coequalizers are colimits over the diagram $\bullet \rightrightarrows \bullet$ In *Set* we have



where $\sim$ is the equivalence relation generated by pairs (xf, xg).

Adjoint functors

Definition (Sketch)

Consider two functors $\mathcal{C} \underset{U}{\overset{F}{\rightleftarrows}} \mathcal{C}'$ between locally small categories.

We say that F is a left adjoint to U (equivalently, that U is a right adjoint to F), and write $F \dashv U$, if there are *natural* bijections

$$Hom_{\mathcal{C}'}(FX, Y) \overset{1:1}{\longleftrightarrow} Hom_{\mathcal{C}}(X, UY).$$

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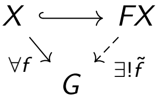
For example, let U be the forgetful and F the free-construction functor; then $F \dashv U : Grp \to Set$. Indeed, for any set X



which provides the bijection $Hom_{Grp}(FX, G) \cong Hom_{Set}(X, UG)$.

Some category theory facts

Lemma

Suppose to have functors $F \vdash U : \mathcal{C} \longrightarrow \mathcal{C}'$; then F preserves colimits and U preserves limits.

For example, the forgetful functor $U : Grp \longrightarrow Set$ is a right adjoint, so for groups G_i,

$$U\left(\times\, G_i\right) = \times\, U(G_i).$$

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Theorem

If $\mathcal{C}$ is a locally small category, every (co)limit is the (co)equalizer of a (co)product.

Limits and colimits in *Part*

Forgetful and free-construction functors in *Part*

Similar to what happens in *Grp*, we have forgetful and free-construction functors in *Part*:

$$U : Part \longrightarrow Set^*, \qquad F : Set^* \longrightarrow Part \qquad \text{such that } F \dashv U.$$

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Set^* is the category of pointed sets.

- Objects are pairs (X, x) with X a non-empty set and $x \in X$.
- Morphisms $f : (X, x) \to (Y, y)$ are set-wise maps such that $f : x \mapsto y$.

In our setting, *Set* and Set^* play analogous roles.

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A limit in *Part* is built the same way as limits in *Grp*.

- Take the set-wise limit as underlying set.
- Endow it with a proper partial group structure.

Colimits in *Part*

Coproducts in *Part* behave as in *Set**.

$$\mathcal{L} = \coprod \mathcal{L}_i := \frac{\bigsqcup \mathcal{L}_i}{\sim_0}, \qquad \text{"} D(\mathcal{L}) = \bigcup D(\mathcal{L}_i) \pmod{\sim_0}\text{"}$$

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$$\mathcal{M} \underset{g}{\overset{f}{\rightrightarrows}} \mathcal{L} \xrightarrow{\ q\ } coeq(f,g) = \mathcal{L}/\sim$$

Who is $\sim$? How do we control it?

Coequalizers in *Part*: a first attempt

Let's try to make the set-wise coequalizer $\mathcal{L}/\sim_0$ (i.e., $\sim_0$ is generated by the pairs (xf, xg) for $x \in \mathcal{M}$) also the coequalizer in *Part*.
q has to be a morphism in *Part*, so we need a commutative diagram

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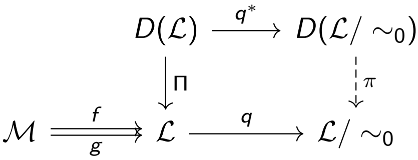

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- Problem! There is no guarantee $(xf, yf)\Pi \sim_0 (xf, yg)\Pi$ (counterexample, S.).

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Final result

Part is complete and cocomplete.

For the future

- Obtain a generalization of Chermak's elementary expansions, a construction for expanding the sets Δ of localities (i.e. transporter systems), including the partial groups $\mathcal{L}_\Delta(G)$.
- Develop a theory of generators and relations for partial groups and localities.
- Detect a suitable notion of morphisms of localities (i.e. of transporter systems).

Thank you